11

SENIOR HIGH SCHOOL

DEPARTMENT OF EDUCATION
SCHOOLS DIVISION OF NEGROS ORIENTAL
REGION VII
Kagawasan Ave., Daro, Dumaguete City, Negros Oriental

BASIC CALCULUS

Quarter 3 – Module 3
Continuity of Functions

ALTERNATIVE DELIVERY MODE
ADM

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Alternative Delivery Mode
Quarter 3 – Module 3: Continuity of Functions
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Introductory Message

This Self-Learning Module (SLM) is prepared so that you, our dear learners, can continue your studies and learn while at home. Activities, questions, directions, exercises, and discussions are carefully stated for you to understand each lesson.

Each SLM is composed of different parts. Each part shall guide you step-by-step as you discover and understand the lesson prepared for you.

Pre-tests are provided to measure your prior knowledge on lessons in each SLM. This will tell you if you need to proceed on completing this module or if you need to ask your facilitator or your teacher's assistance for better understanding of the lesson. At the end of each module, you need to answer the post-test to self-check your learning. Answer keys are provided for each activity and test. We trust that you will be honest in using these.

In addition to the material in the main text, Notes to the Teacher are also provided to our facilitators and parents for strategies and reminders on how they can best help you on your home-based learning.

Please use this module with care. Do not put unnecessary marks on any part of this SLM. Use a separate sheet of paper in answering the exercises and tests. And read the instructions carefully before performing each task.

If you have any questions in using this SLM or any difficulty in answering the tasks in this module, do not hesitate to consult your teacher or facilitator.

Thank you.

What I Need to Know

This module was designed to provide you with fun and meaningful opportunities for guided and independent learning at your own pace and time. You will be enabled to process the contents of the learning resource while being an active learner.

The module is intended for you to illustrate continuity of a function at a number and on an interval; determine whether a function is continuous at a number or not; and solves problems involving continuity of a function.

What I Know

Graphing functions can be tedious and, for some functions, impossible. Calculus gives us a way to test for continuity using limits instead. Learn about continuity in calculus and see examples of testing for continuity in both graphs and equations.

At the basic level, students tend to describe continuous functions as those whose graphs can be traced without lifting your pencil. While it is generally, practical way to define continuity. Many graphs and functions are continuous, or connected, in some places, and discontinuous or broken, in other places. There are even functions containing too many variables to be graphed by hand. Therefore, it's necessary to have a more precise definition of continuity, one that doesn't rely on our ability to graph and trace a function.

PRE-ASSESSMENT:
I. Evaluate the limits of the following functions, if they exist.

a. 50	b. $-\frac{3}{5}$	c. $-\frac{5}{4}$	d. -15
e. $\frac{2}{5}$		f. 0	

1. $\lim\limits_{x \to 2}(8 - 3x + 12x^2)$
2. $\lim\limits_{t \to -3} \frac{6+4t}{1+t^2}$
3. $\lim\limits_{x \to -5} \frac{x^2-25}{x^2+2x-15}$
4. $\lim\limits_{z \to 8} \frac{2z^2-17z+8}{8-z}$
5. $\lim\limits_{y \to 7} \frac{y^2-4y-21}{3y^2-17y-28}$

II. Choose the letter that corresponds to the correct answer:

6. Which of the following choices makes the concept "A function f is said to be continuous from the left at x = c if $f(c) = \lim_{x \to c^-} f(x)$ " true.

A. $5 = \lim_{x \to 4^-} (x + 1)$

B. $5 = \lim_{x \to 4^-} (2x + 1)$

C. $5 = \lim_{x \to 4^-} (x + 4)$

D. $4 = \lim_{x \to 4^-} (x - 4)$

7. Which of the following makes the concept "A function f is said to be continuous from the right at x = c if $f(c) = \lim_{x \to c^+} f(x)$ " true.

A. $5 = \lim_{x \to 4^+} (x + 1)$

B. $5 = \lim_{x \to 4^+} (2x + 1)$

C. $5 = \lim_{x \to 4^+} (x + 4)$

D. $4 = \lim_{x \to 4^+} (x - 4)$

8. Polynomial functions are continuous everywhere, like $f(x) = x^2 + x - 7$.

A. True

B. False

C. Sometimes True

D. Sometimes False

9. The roots of polynomial function $f(x) = x^3 - x + 1$ cannot be found using factoring and synthetic division but we can use Intermediate Value Theorem (IVT).

A. True

B. False

C. Sometimes True

D. Sometimes False

10. Consider $f(x) = x^3 - x + 4$. The interval containing a root of f(x) is

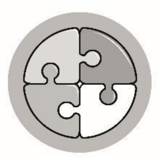
A. [1 , 3]

B. [– 1 , 2]

C. [– 2 , 2]

D. [– 3 , – 2]

Lesson Continuity of a Function



What's In

Activity 1. Consider f defined as follows:

$$f(x) = \begin{cases} \dfrac{(x+2)(x-2)}{x-2}, & x \neq 2 \\ 1, & x = 2. \end{cases}$$

Then, do the following:

1. Sketch the graph of f.
2. Find $f(2)$.
3. Determine:
   a) $\lim_{x \to 2^-} f(x)$
   b) $\lim_{x \to 2^+} f(x)$

What's New

Activity 2. Based on Activity 1, answer the following:

1. What is the $\lim_{x \to 2} f(x)$?
2. Is $\lim_{x \to 2} f(x) = f(2)$? If not, what should be done so that the equality is achieved?

What is It

Definition:

A function $f(x)$ is said to be **continuous** at $x = a$ if

i. f(a) exists

ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$.

A function is said to be continuous on the interval $[a, b]$ if it is continuous at each point in the interval.

Note that this definition is also implicitly assuming that both $f(a)$ and $\lim_{n \to a} f(x)$ exist. If either of these do not exist, the function will not be continuous at $x = a$.

Hence, it can be turned into the following facts:

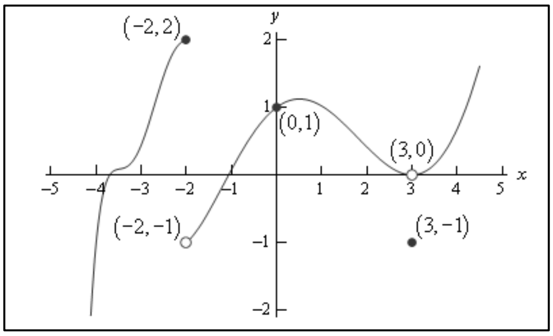
Fact 1

If f(x) is continuous ar $x = a$ then,

i. $\lim_{x \to a^-} f(x) = f(a)$ ii. $\lim_{x \to a^+} f(x) = f(a)$ iii. $\lim_{x \to a} f(x) = f(a)$

However, the definition doesn't really tell just what it means for a function to be continuous. Thus, to understand just what it means for a function to be continuous, refer to the next examples.

Example 1: Given the graph of $f\ (x)$, show below, determine if $f\ (x)$ is continuous at $x = -2,\ x = 0, x = 3.$



To determine if the function is continuous at a certain point, get both the limit at that point and the function value at that point. If both the limit at that point and the function value at that point are equal, then the function is continuous at that point, otherwise it is not continuous.

First, we consider, $x = -2$. We have,

$$\lim_{x\to -2^-} f(x) = 2 \text{ and } \lim_{x\to -2^+} f(x) = -1.$$

Since Left Hand Limit and Right Hand Limit is not equal, therefore, $\lim_{x\to -2} f(x)$ does not exist.

Since, the function value and the limit are not the same,therefore the function is not continuous at this point. This type of discontinuity in a graph is called a **jump discontinuity**. Jump discontinuities happen when the graph has a break in it as this graph does and the values of the function to either side of the break are finite.

Now, consider $x = 0$.

$$f\ (0) = 1 \qquad\qquad \lim_{x\to 0} f(x) = 1$$

Since the function and limit have the same value, then the function is continuous at this point.

Finally, consider $x = 3$.

$$f(3) = -1 \qquad\qquad \lim_{x\to 3} f(x) = 0$$

Since f(3) is equal to 1, and $\lim_{x\to 3} f(x) = 0$, implies that the value of the function at 3 is not equal to the limit of the function at 3, therefore the function is not continuous at x=3. This type of discontinuity is called a **removable discontinuity**. Removable discontinuities are those where there is a hole in the graph as there is in this case

Hence, a function is continuous on an interval if the graph can be drawn from start to finish without ever once picking up the pen. The graph in the last example has only two discontinuities since there are only two places where there is a need to pick up the pen in sketching it.

Thus, a function is continuous if its graph has no breaks or holes.

However, functions will not be continuous if division by zero or logarithms of zero is possible.

Example 2. Determine where the function below is continuous or not.

$$h(t) = \frac{4t + 10}{x^2 - 2t - 15}$$

Rational functions are always continuous except when division by zero is possible. So, all that we need to do is determine when the denominator is zero. That's easy enough to determine by setting the denominator equal to zero and solving for the value of the variable.

$$t^2 - 2t - 15 = (t - 5)(t + 3) = 0$$

So, the function will not be continuous at $t = -3 \ and \ t = 5$.

Fact 2

If $f(x)$ is continuous at $x = b \ and \ \lim_{x \to a} g(x) = b$ then,

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

Example 3. Evaluate the following limit.

$$\lim_{x \to 0} e^{\sin x}$$

Since we know that exponentials are continuous everywhere, we can use the fact above. Therefore,

$$\lim_{x \to 0} e^{\sin x} = e^{\lim_{x \to 0} \sin x} = e^0 = 1$$

Quick Review: Recall the Three Conditions of Continuity

A function ***f(x)*** is said to be continuous at **x = c** if the following three conditions are satisfied:

i. *f(c) exists;*

ii. $\lim_{x\to c} f(x)\ \ exists;$

iii. $f(c) = \lim_{x\to c} f(x).$

Example 4: Determine if f(x) = $x^3 + x^2 - 2$ is continuous or not at x = 1.
Solution. We have to check the three conditions for continuity of a function.

1) If x = 1, then f(1) = 0: since 1 is in the domain of *f(x).*
2) $\lim_{x\to c} f(x) = \lim_{x\to 1}(x^3 + x^2 - 2) = 1^3 + 1^2 - 2 = 0$ (a real number so $\lim_{x\to c} f(x)$ exists)
3) f(1) = 0 = $\lim_{x\to 1}(x^3 + x^2 - 2)$

Since the three conditions are satisfied, therefore, f is continuous at x = 1.

Example 5. Determine if $f(x) = -\sqrt{x}$ is continuous or not at x = – 1.

Solution: We have to check the three conditions for continuity of a function.

1) If x = – 1, then f(-1) = $-\sqrt{-1}$ is an imaginary number. So, the first condition is not satisfied.

   Since we cannot take the square root of a negative number *i* then *f(x)* is not defined at *x = -1*

   Therefore, *f* is not continuous at x = –1.

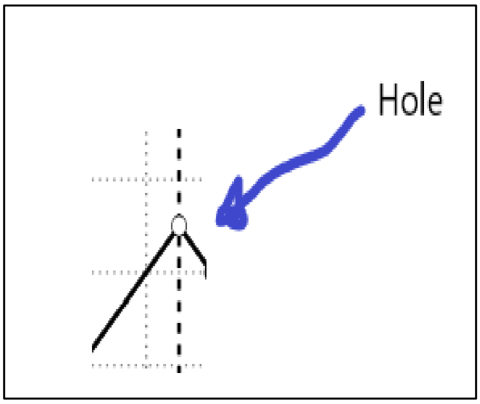
What is Continuity?

In calculus, a function is continuous at x= a if and only if – all three of the following conditions are met:

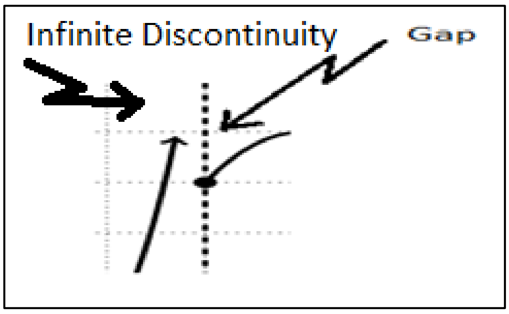
1. The function is defined at *x*= *a*; that is, $f(a)$ equals to areal number
2. The limit of the function as *x* approaches to *a* exist.
3. The limit of the function as *x* approaches to *a* is equal to the function value at *x = a.*

There are three types of discontinuities:

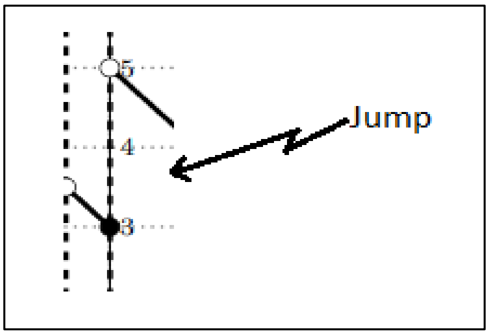
1. **Removable (point) discontinuity** - the graph has a hole at a single x-value.



2. **Infinite discontinuity** – the function goes toward positive or negative infinity.

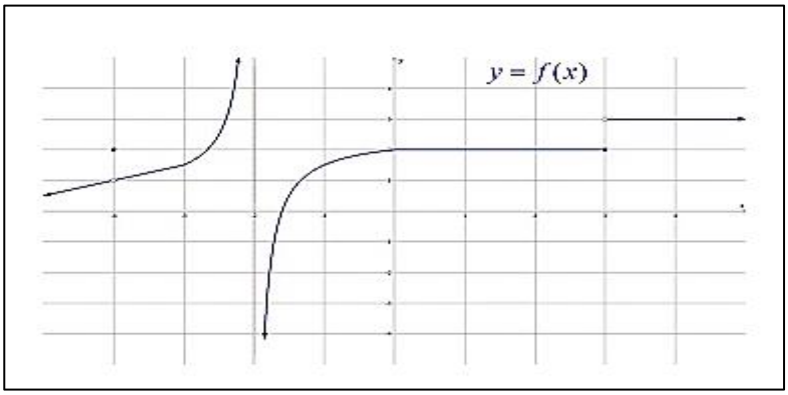


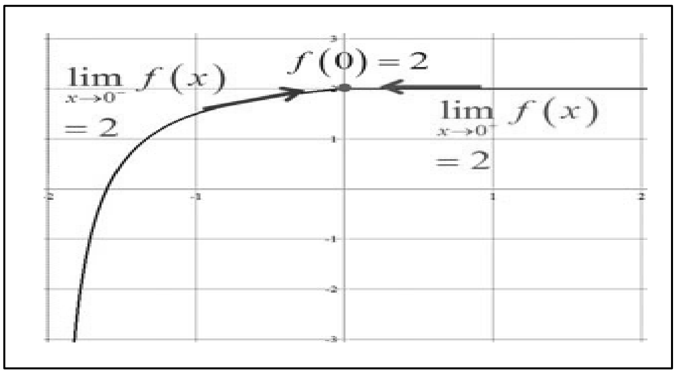
3. **Jump discontinuity** – the graph jumps from one place to another.



Both infinite and jump discontinuities fail condition #2 (limit does not exist), but how they fail is different. Recall for a limit to a exist, the left and right limits must exist (be finite) and be equal. Infinite discontinuities have infinite left and right limits. Jump discontinuities have infinite left and right limits that are not equal.

Let us go through some examples using this graph to represent the function of *f(x)*.





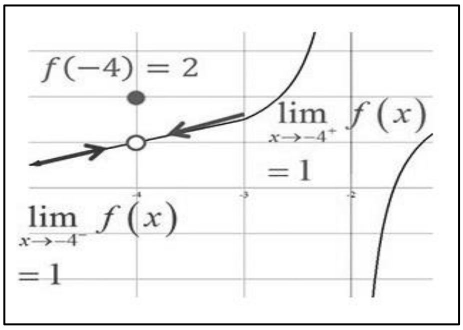
Example 6: Is *f(x)* continuous at $x = 0$

To check for continuity at $x = 0$, we check the three conditions:

1. Is the function defined at $x = 0$? Yes, $f(0) = 2$
2. Does the limit of the function as x approaches 0 exist? Yes
3. Does the limit of the function as x approaches 0 equal the function value at $x = 0$? Yes

Since all three conditions are met, *f(x)* is continuous at $x = 0$.

Example 7: Is f(x) continuous at x = -4?



To check for continuity at x = -4, we check the same three conditions:

1. The function is defined; $f(-4) = 2$.

2. The limit exists, because the limit of *f* from left at x = – 4 is equal to the limit of *f* from right at x = – 4. The limit of *f(x)* as x approaches to 4 is 1.
3. The function value and the limit are not equal.

   Thus, there is a point discontinuity at x = 4.

Continuity on an Interval

A function can be continuous on an interval. This simply means that it is continuous at every point on the interval. Equivalently, if we are able to draw the entire graph of the function on an interval without lifting our tracing pen, or without being interrupted by a hole, gap or jump in the middle of the graph, then we can conclude that the function is continuous on that interval.

Consider the graph of the function below;

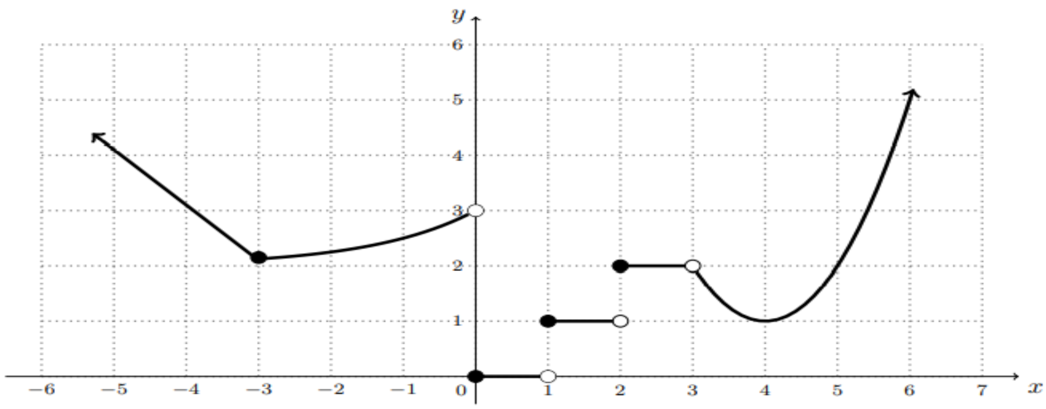


Figure 1.1

One-Sided Continuity

1. A function f is said to be **continuous from the left** at x = c if
$$f(c) = \lim_{x\to c^-} f(x)$$
Example 8: Let c = – 3

   Figure 1.1 shows that if c = – 3 then f(-3) = 2 while the $\lim_{x\to -3^-} f(x) = 2$ means that as x values get closer and closer to – 3 from left, the values of f(x) get closer and closer to 2. Thus, both f(-3) and $\lim_{x\to -3^-} f(x)$ are equal to 2. Therefore, the function f is continuous from the left at x = – 3.

2. A function f is said to be **continuous from the right** at x = c if
$$f(c) = \lim_{x\to c^+} f(x)$$
Example 9. Let c = 0

   Figure 1.1 shows that if c = 0 then f(0) = 0 because of the solid dot. It can never be 3 because there is a hole. On the other hand the $\lim_{x\to 0^+} f(x) = 0$ means that as x values get closer and closer to 0 from the right, the values of f(x) get closer and closer to 0 also. Therefore, the function f is continuous from the right at x = 0.

Now, if a function is given without its corresponding graph, we must find other means to determine if the function is continuous or not on an interval. Below are the definitions that will help us:

CONTINUITY OF POLYNOMIAL, ABSOLUTE VALUE, RATIONAL AND SQUARE ROOT FUNCTIONS

1. Polynomial functions are continuous everywhere.
   Examples:
   a. f(x) = 2x3 – 7x2 + 3x – 1
   b. f(x) = 5x2 + 4x – 9
   c. f(x) = x + 1

2. The absolute value function f(x) = $|x|$ is continuous everywhere.
   Example: f(x) = $|x + 2|$

3. Rational functions are continuous on their respective domains.
   Examples:
   a. $f(x) = \frac{3}{x}$, domains: all real numbers except 0. So, f(x) is continuous everywhere except at x = 0.
   b. $f(x) = \frac{5}{x-2}$, domain: all real numbers except 2. So, *f(x)* is continuous everywhere except at *x = 2*.

4. The square root function f(x) = $\sqrt{x}$ is continuous on [0,∞) because for every real number [domain of f(x)] assigned to x there is a corresponding real value to y or f(x).
   Example:

   f(x) = $\sqrt{x-2}$, domain of f(x): [2, ∞)

   Conclusion: f(x) is continuous on [2, ∞).

The **mathematical terms** we use in the conclusion for the continuity of a function are:

1. **Continuous everywhere** if f is continuous at every real number. In this case, we also say f is continuous on R.
2. **On (a,b)** if f is continuous at every point x in (a,b). The values a and b are not included.
3. **On [a,b)** if f is continuous on (a,b) and from the right at a. The value a is included while b is not.
4. **On (a,b]** if f is continuous on (a,b) and from the left at b. The value a is not included while b is included.

5. **On [a,b]** if f is continuous on (a,b] and on [a,b). The extreme values a and b are included.
6. **On (a, ∞)** if f is continuous at all x > a.
7. **On [a, ∞)** if f is continuous on (a, ∞) and from the right at a.
8. **On (1 – ∞)** if f is continuous at all x < b.
9. **On (– ∞,b]** if f is continuous on (– ∞,b) and from the left at b.

Example 10. Determine the largest interval over which the function $f(x) = \sqrt{x + 2}$ is continuous.

Solution.

Observe that the function $f(x) = \sqrt{x + 2}$ has function values only if (x + 2) ≥ 0, this means that the value of f(x) is a real number if x + 2 is 0 or positive. That is, if x ∈ [–2, +∞) in words if x is an element of the interval [–2, +∞).

Suppose c is any number element of the interval (–2, +∞) then

$$f(c) = \sqrt{c + 2} = \lim_{x \to c^+} \sqrt{x + 2}$$

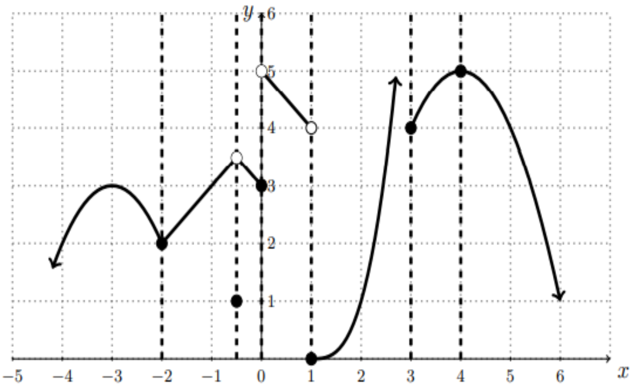
means both f(c) and $\lim_{x \to c^+} \sqrt{x + 2}$ exist (condition 1 and 2 are satisfied) and $\lim_{x \to c^+} \sqrt{x + 2}$ is equal to f(c). Moreover, f is continuous from the right at –2 because $f(-2) = \sqrt{-2 + 2} = 0 = \lim_{x \to -2^+} \sqrt{x + 2}$ (condition 3 is also satisfied).

Therefore, for all x ∈ [–2, +∞), the function $f(x) = \sqrt{x + 2}$ is continuous.

What I Have Learned

I. Based on the graph, choose the appropriate answer in the box for the given interval. Write your answer in your notebook or pad/bond paper.

f(x) is continuous on	f(x) is discontinuous on



1. ________________ (-∞, -2).
2. ________________ [-2, 0]
3. ________________ (0, 1)
4. ________________ [0, 1]
5. ________________ (1,3)
6. ________________ [1,3]

II. Provide the missing words/mathematical terms for each of the following solutions of the given problems.

1. Determine the largest interval in which $f(x) = x^4 - x^3 + 1$ is continuous.
   Solution:
   Since f is a _______(1a)_________ function, then it is ______(1b)_____ everywhere. Hence, f is _____(1c)____ on __(1d)__.

Assessment

I. Problem Solving.

1. Is $f(x) = \dfrac{x^2 - x - 2}{x + 1}$ continuous at x = 2? At x = -1?
2. Is $g(x) = \begin{cases} x^2 + 3x, & x \leq 0 \\ x, & 0 < x \leq 2 \\ 3x^2, & x > 2 \end{cases}$ continuous at x = 0? At x = 2?
3. Find values for the constants a and b so that the function

$$h(x) = \begin{cases} ax + 5 & x \leq 1 \\ -2x^2 - 9 & -1 < x \leq 5 \\ 3x + b & x > 5 \end{cases}$$

   is continuous everywhere.

4. Determine if $f(x) = x^3 + x^2 - 2$ is continuous or not at $x = 1$.
5. Determine if $f(x) = \frac{x^2-x-2}{x-2}$ is continuous or not at $x = 0$.

II. Determine if the following functions are continuous on the interval *I*.

1. $f(x) = x^3 - x + 1$; I = (2, +∞)

III. Find the distinct interval of length 1 containing a root or solution of $f(x) = x^3 - 3x + 5$ using IVT.

Answer Key

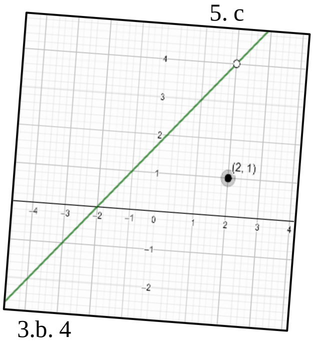
What I know: Pretest:

I.	II.
1. a	1. a
2. b	2. a
3. c	3. a
4. d	4. a
5. e	5. c

What's In

Activity 1

1.



2. 1 3 a. 4 3.b. 4

Activity 2

1. 2
2. Redefine $f(2)$ to be equal to 4 so that $\lim_{x \to 2} f(x) = f(2)$.

What I have learned

I. 1. f(x) is continuous on

2. f(x) is discontinuous on

3. f(x) is continuous on

4. f(x) is discontinuous on

5. f(x) is continuous on

6. f(x) is discontinuous on

II.

1a. Polynomial

1b. Continuous

1c. Continuous

1d. $(-\infty,+\infty)$ or R

Assessment:

1. Limit exists, at a point discontinuity x = -1.
2. Limit does not exist. A jump discontinuity at x = 2.
3. For the limit to exist and equal the function, we need -59 = 15 + b, so b = -74,a=16.
4. Continuous at x = 1.
5. Continuous at x = 0.

References

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