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DEPARTMENT OF EDUCATION
SCHOOLS DIVISION OF NEGROS ORIENTAL
REGION VII

Kagawasan Ave., Daro, Dumaguete City, Negros Oriental

BASIC CALCULUS
Quarter 3 – Module 7
Chain Rule
and Implicit Differentiation

ALTERNATIVE DELIVERY MODE

ADM

Basic Calculus – Grade 11
Alternative Delivery Mode
Quarter 3 – Module 7: Chain Rule and Implicit Differentiation
Second Edition, 2021

Introductory Message

This Self-Learning Module (SLM) is prepared so that you, our dear learners, can continue your studies and learn while at home. Activities, questions, directions, exercises, and discussions are carefully stated for you to understand each lesson.

Each SLM is composed of different parts. Each part shall guide you step-by-step as you discover and understand the lesson prepared for you.

Pre-tests are provided to measure your prior knowledge on lessons in each SLM. This will tell you if you need to proceed on completing this module or if you need to ask your facilitator or your teacher's assistance for better understanding of the lesson. At the end of each module, you need to answer the post-test to self-check your learning. Answer keys are provided for each activity and test. We trust that you will be honest in using these.

In addition to the material in the main text, Notes to the Teacher are also provided to our facilitators and parents for strategies and reminders on how they can best help you on your home-based learning.

Please use this module with care. Do not put unnecessary marks on any part of this SLM. Use a separate sheet of paper in answering the exercises and tests. And read the instructions carefully before performing each task.

If you have any questions in using this SLM or any difficulty in answering the tasks in this module, do not hesitate to consult your teacher or facilitator.

Thank you.

What I Need to Know

This module was designed to provide you with fun and meaningful opportunities for guided and independent learning at your own pace and time. You will be enabled to process the contents of the learning resource while being an active learner.

The module is intended for you to illustrate the Chain Rule of Differentiation; solve problems using the Chain Rule; and ilustrate Implicit Differentiation.

What I Know

TASK 1:

QUICK RECALL! Let us determine how much you already know about differentiation. Use your notebooks/answer sheets for your answers.

INSTRUCTION: Identify what Rule of Differentiation should be used in the following:

1. $y = 8 + \pi$

2. $y = 3x^2 + 2x - 1$

3. $y = (x^3 + 2x)(2x-1)$

4. $y = \dfrac{(x^3 - 2x)}{(2x-1)}$

5. $y = \dfrac{1}{(2x-1)}$

Lesson 1

THE CHAIN RULE OF DIFFERENTIATION

What's In

> ### CHAIN RULE THEOREM:
>
> Let f be a function differentiable at c and let g be a function differentiable at $(f(c))$. Then the composition
>
> $g \circ f$ is differentiable at c and
>
> $$Dx \, (g \circ f)(c) \; = \; g'(f(c)) \cdot f'c$$

There is also another notation which can be easier to work with when using the Chain Rule.

Let u be a function of *x, (u(x))*

and let y be a function of *u, (y(u))*

then, the derivative of *y (u(x)) = f(x)*

is given by

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

In words, the derivative of a composition of functions is the derivative of the outer function evaluated at the inner function, times the derivative of the inner function.

Example 1: Find the derivative of $f(x) \; = \; (3x^2 - 2x + 4)^2$

Solution:

Rewrite $f(x) = y$

$$y = (3x^2 - 2x + 4)^2$$

Let $u = 3x^2 - 2x + 4$

So, $y = f(u) = u^2$

where $u = 3x^2 - 2x + 4$ is a differentiable function of x.

Note that $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = 6x - 2$

Thus,

$$f'(x) = y' = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y' = (2u)(6x - 2)$$

$$y' = 2(3x^2 - 2x + 4)(6x - 2).$$

Example 2: Solve for $f'(x)$ if $f(x) = (6x^2 + 7x)^4$

Solution:

Let: $u = 6x^2 + 7x \qquad y = f(u) = u^4$

Then, $\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = 12x + 7$

Thus,

$$f'(x) = y' = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y' = 4u^3(12x - 7)$$

$$y' = 4(6x^2 + 7x^3)(12x - 7).$$

Example 3: Find $\frac{d}{dx}(\cos x^2)$.

Solution:

Let $u = x^2$ so that $y = \cos u$

Then, $\frac{dy}{du} = -\sin u$ and $\frac{du}{dx} = 2x$

Thus,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (-\sin u)\, 2x$$

$$= 2x\,(-\sin x^2)$$

$$= -2x \sin x^2.$$

Example 4: : Solve for $f'(x)$ if $f(x) = (2x^4 - 3x + 4)^3$

Solution:

Rewrite $f(x) = y$

$$y = (2x^4 - 3x + 4)^3$$

Let $u = 2x^4 - 3x + 4$

So, $y = f(u) = u^3$

where $u = 2x^4 - 3x + 4$ is a differentiable function of x.

Note that $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 8x^3 - 3$

Thus,

$$f'(x) = y' = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y' = (2u^2)(8x^3 - 3)$$

$$y' = 2(2x^4 - 3x + 4)^2(8x^3 - 2).$$

(Arceo 2016)

What's New

TASK 2: Recall the discussion on composite functions. Then, answer the following:

1. Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$
   Find: a. $(f \circ g)$
   b. $(g \circ f)$

What is It

TASK 3: Suppose y is a function of u:

$$y = f(u) \qquad\qquad (1)$$

If, in turn, u is a function x, then
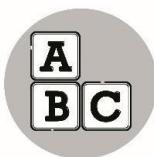
$$u = g(x) \qquad\qquad (2)$$

Note that we can combine equation 1 and equation 2 to define y as a function of x. We have

$$f = f(u) = f(g(x)).$$

This is a composite function we have previously defined.

Given a function y below, define y to be a function of u in the second column and, in turn, define u as a function of x in the third column such that $y = f(u) = f(g(x))$. No. 1 is done for you.

	$y = f(u)$	$u = g(x)$
$y = (2x^3 - 5x^2 + 4)^5$	$y = u^5$	$u = (2x^3 - 5x^2 + 4)$
$y = \dfrac{1}{3x^2 + 1}$		
$y = \left(\dfrac{3x + 1}{4x - 2}\right)^3$		
$y = (3x + 2)^2$		

What's More

LET'S PRACTICE! Here are another sample problems made just for you. Observe the process to fully grasp the concept.

1. $y = (4x^3 - x^2 + 1)^4$

   Solution: We will consider y as a function of u, where u is a function of x. We have,

   $$y = u^4 \qquad where\ u = 4x^3 - x^2 + 1$$

   Note that we will have the following values
   $$\frac{dy}{du} = 4u^3 \qquad and \qquad \frac{du}{dx} = 12x^2 - 2x$$

   Thus, from the chain rule
   $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3(12x^2 - 2x)$$
   $$= 4(4x^3 - x^2 + 1)^3(12x^2 - 2x)$$

2. $y = \sqrt{5x - 3}$
   Solution: We will consider y as a function of u, where u is a function of x. We have,
   $$y = \sqrt{u} = u^{\frac{1}{2}} \qquad where\ u = 5x - 3$$
   Note the following values
   $$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2u^{\frac{1}{2}}} \qquad and \qquad \frac{dy}{dx} = 5.$$

From the chain rule, we get

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2u^{\frac{1}{2}}}(5)$$

$$= \frac{1}{2(5x - 3)^{\frac{1}{2}}}(5)$$

$$= \frac{5}{2(5x - 3)^{\frac{1}{2}}}$$

$$= \frac{5}{2\sqrt{5x - 3}}$$

3. $y = \dfrac{1}{(3 - x)^4}$

   Solution: We will consider y as a function of u, where u is a function of x. We have,

   $$y = \frac{1}{u^4} = u^{-4} \qquad where\ u = 3 - x$$

   We note of the following values

   $$\frac{dy}{du} = -4u^{-5} \qquad and \qquad \frac{dy}{dx} = -1$$

   Thus, from the chain rule:

   $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -4u^{-5}(-1)$$

   $$= -4(3 - x)^{-5}(-1)$$

   $$= \frac{4}{(3 - x)^5}$$

TASK 4:

An alternative form of the Chain Rule of differentiation can be expressed in the following form:

$$\frac{d}{dx}\big(f(g(x))\big) = (f \circ g)'(x) = f'(g(x))g'(x).$$

Given the following table of values, find the indicated derivatives in a and b.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
3	5	-2	5	7
5	3	-1	12	4

a. $f'(3)$ where $F(x) = f(g(x))$.
b. $g'(3)$ where $G(x) = g(f(x))$.

Implicit Differentiation

What's In

TASK 1: On your activity notebook, find the derivatives of the following problem.

1. $y = 8 + \pi$ 4. $y = \dfrac{(x^3 - 2x)}{(2x - 1)}$

2. $y = 3x^2 + 2x - 1$ 5. $y = \dfrac{1}{(2x - 1)}$

3. $y = (x^3 + 2x)(2x-1)$

INTRODUCTION

In the past lessons, we discussed functions in the form of $y = f(x)$ which express y explicitly in terms of x and differentiated according to the rules for the types of functions involved. However, there are functions that are not given in the form $y = f(x)$ but in a more complicated form in which is difficult to express y explicitly in terms of x. These functions are called implicit functions and are said to determine y as an implicit function of x, that is, of the form $f(x) = 0$.

To find the derivative of an implicit function, $f(x) = 0$ with respect to x, y is considered as an unknown but differentiable function of x, and every time we differentiate y, we differentiate is as usual but multiply the result by $\dfrac{dy}{dx}$.

Consider this expression,

$$y^2 + x^3 - y^3 + 6 = 3y$$

with respect of x.

We differentiate each term with respect to x:

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x^3) - \frac{d}{dx}(y^3) + \frac{d}{dx}(6) = \frac{d}{dx}(3y)$$

Differentiating functions of x with respect to x is straightforward. But differentiating a function of y with respect to x we find,

$$2y\frac{dy}{dx} + 3x^2 - 3y^2\frac{dy}{dx} + 0 = 3\frac{dy}{dx}$$

that is

$$2y\frac{dy}{dx} + 3x^2 - 3y^2\frac{dy}{dx} + 0 = 3\frac{dy}{dx}$$

We arrange this to collect all terms with $\frac{dy}{dx}$ together.

$$3x^2 = 3\frac{dy}{dx} - 2y\frac{dy}{dx} + 3y^2\frac{dy}{dx}$$

then,

$$3x^2 = (3 - 2y + 3y^2)\frac{dy}{dx}$$

so,

$$\frac{dy}{dx} = \frac{3x^2}{3 - 2y + 3y^2}$$

.

> **REMEMBER:**
> Implicit differentiation takes four steps:
> Step 1: Differentiate both sides of the equation with respect to x.
> Step 2 Collect the terms with $\frac{dy}{dx}$ on one side of the equation.
> Step 3: Factor out $\frac{dy}{dx}$.
> Step 4: Solve for $\frac{dy}{dx}$ by dividing.

Example 1: Differentiate the implicit function,

$$x^2 - xy + y^2 = 7$$

Solution

We have,

$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$ *differentiate both sides with respect to x*

$2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$ *treating xy as a product and y^2 as a power*

$2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} - x\frac{dy}{dx} = y - 2x$ *collect the terms with $\frac{dy}{dx}$ on one side of the equation*

$(2y - x)\frac{dy}{dx} = y - 2x$ *factor out $\frac{dy}{dx}$*

$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$ *solve for $\frac{dy}{dx}$ by dividing.*

Example 2: Find $\frac{dy}{dx}$ if $2y = x^2 + \sin y$

Solution

$$2y = x^2 + \sin y \qquad \textit{differentiate both sides with respect to x}$$

$$\frac{d}{dx}2(y) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin y)$$

$$2\frac{dy}{dx} = 2x + \cos y \frac{dy}{dx} \qquad \textit{Collect terms with } \frac{dy}{dx}$$

$$2\frac{dy}{dx} - \cos y \frac{dy}{dx} = 2x$$

$$(2 - \cos y)\frac{dy}{dx} = 2x \qquad \textit{Factor out } \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y} \qquad \textit{and divide.}$$

What's New

TASK 2: Solve for *y* in terms of *x* in the following equation:

1. $x^2 + y^2 = 1$
2. $yx + y + 1 = x$

What is It

d $\frac{dy}{dx}$ given *xy = 1* by rewriting the equation in terms of *y*.

What's More

TASK 4: Find $\frac{dy}{dx}$ for *xy = 1,* now, by implicit differentiation. Verify your answer against that in TASK 3.

What I Have Learned

Complete the following statements.

1. I have learned that _____
   _____
   _____.

2. I have realized that _____
   _____
   _____.

3. I will apply what I have learned _____
   _____
   _____.

What I Can Do

A. Given that $f'(0) = 2$, $g(0) = 0$ and $g'(0) = 3$, find $(f \circ g)'(0)$.

B. Use Implicit Differentiation to find $\frac{dy}{dx}$ if $5y^2 + \sin y = x^2$.

C. **RUBRIC**

CATEGORY	5	4	3	2
Mathematical Concepts	Explanation shows complete understanding of the mathematical concepts used to solve the problem(s).	Explanation shows substantial understanding of the mathematical concepts used to solve the problem(s).	Explanation shows some understanding of the mathematical concepts needed to solve the problem(s).	Explanation shows very limited understanding of the underlying concepts needed to solve the problem(s) OR is not written.
Mathematical Errors	90-100% of the steps and solutions have no mathematical errors.	Almost all (85-89%) of the steps and solutions have no mathematical errors.	Most (75-84%) of the steps and solutions have no mathematical errors.	More than 75% of the steps and solutions have mathematical errors.
Neatness and Organization	The work is presented in a neat, clear, organized fashion that is easy to read.	The work is presented in a neat and organized fashion that is usually easy to read.	The work is presented in an organized fashion but may be hard to read at times.	The work appears sloppy and unorganized. It is hard to know what information goes together.
Completion	All problems are completed.	All but one of the problems are completed.	All but two of the problems are completed.	Several of the problems are not completed.

Assessment

A. In your activity notebook, answer the following problem using the Chain rule of Differentiation.

1. $y = (6x + 5)^{\frac{5}{3}}$
2. $f(x) = (6x^2 + 7x)^4$
3. $g(t) = (4t^2 - 3t + 2)^{-2}$
4. $f(z) = \sqrt[3]{1 - 8z}$
5. $y = (3x^2 - 7x)^8$

B. Find the indicated derivative. Write your answer on your activity paper/notebook.

1. $x^2 - x + y^2 = 2xy$
2. $x^2 - y^2 + xy = x^2y$
3. $x^2 + 2xy + y^2 = 0$
4. $x^3 + y^2 = 3 + xy$
5. $4x^2 - 9y^2 - x + y = 1$

Answer Key

What I Know
1. Derivative of Constant
2. Derivative of Sum and Difference
3. Derivative of a Product
4. derivative of a Quotient
5. derivative of a reciprocal

LESSON 1

TASK 2: a. $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 + x + 3 = x + \sqrt{x} + 3$

b. $(g \circ f)(x) = g(f(x)) = g(x^2 + 3) = \sqrt{x^2 + 3}$

TASK 3:

	$y = f(u)$	$u = g(x)$
$y = (2x^3 - 5x^2 + 4)^5$	$y = u^5$	$u = (2x^3 - 5x^2 + 4)$
$y = \dfrac{1}{3x^2 + 1}$	$y = u^{-1}$	$u = 3x^2 + 1$
$y = \left(\dfrac{3x + 1}{4x - 2}\right)^3$	$y = u^3$	$u = \dfrac{3x + 1}{4x - 2}$
$y = (3x + 2)^2$	$y = u^2$	$u = 3x + 2$

TASK 4:   a. -7   b. -8

WHAT I CAN DO:

a. $(f \circ g)'(0) = \dfrac{f'(g(0)) \cdot g'(0)}{} = 2 \cdot 3 = 6$

b. $\dfrac{dy}{dx} = \dfrac{2x}{10y + \cos y}$

ASSESSMENT

A. 1. $y' = 10(6x + 5)^{2/3}$

2. $f(x)' = (24x^2 + 28x)^3(12x + 7)$

3. $f(t)' = -\dfrac{2(8t-3)}{(4t^2 - 3t + 2)^3}$

4. $f(z)' = -\dfrac{8}{\sqrt[3]{(1-8z)^2}}$

5. $y = (24x^2 - 56x)^7(6x - 7)$

B. 1. $\dfrac{dy}{dx} = \dfrac{2x - 2y - 1}{2x - 2y}$

2. $\dfrac{dy}{dx} = \dfrac{2xy - y - 2x}{-2y + x - x^2}$

3. $\dfrac{dy}{dx} = -1$

4. $\dfrac{dy}{dx} = \dfrac{y - 3x^2}{2y - x}$

5. $\dfrac{dy}{dx} = \dfrac{1 - 8x}{18y - 1}$

LESSON 2

TASK 1
1. $y' = 0$
2. $y' = 6x + 2$
3. $y' = 8x^3 - 3x^2 + 8x - 2$
4. $y' = \dfrac{4x^3 - 3x^2 + 2}{(2x-1)^2}$
5. $y' = \dfrac{-2}{4x^2 - 4x + 1}$

TASK 2

1. $x^2 + y^2 = 1$

$y^2 = 1 - x^2$

$y = \pm\sqrt{1 - x^2}$

2. $yx + y + 1 = x$

$$yx + y = x - 1$$
$$y(x + 1) = x - 1$$
$$y = \frac{x - 1}{x + 1}$$

TASK 3:

$$y = \frac{1}{x} = x^{-1}$$
$$\frac{dy}{dx} = -1(x^{-2})$$
$$= -\frac{1}{x^2}$$

References

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