11

SENIOR HIGH SCHOOL

DEPARTMENT OF EDUCATION
SCHOOLS DIVISION OF NEGROS ORIENTAL
REGION VII
Kagawasan Ave., Daro, Dumaguete City, Negros Oriental

BASIC CALCULUS

Quarter 3 – Module 6
Extreme Value Theorem and Optimization Problems

ALTERNATIVE DELIVERY MODE
ADM

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Quarter 3 – Module 6: Extreme Value Theorem and Optimization Problems
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Introductory Message

This Self-Learning Module (SLM) is prepared so that you, our dear learners, can continue your studies and learn while at home. Activities, questions, directions, exercises, and discussions are carefully stated for you to understand each lesson.

Each SLM is composed of different parts. Each part shall guide you step-by-step as you discover and understand the lesson prepared for you.

Pre-tests are provided to measure your prior knowledge on lessons in each SLM. This will tell you if you need to proceed on completing this module or if you need to ask your facilitator or your teacher's assistance for better understanding of the lesson. At the end of each module, you need to answer the post-test to self-check your learning. Answer keys are provided for each activity and test. We trust that you will be honest in using these.

In addition to the material in the main text, Notes to the Teacher are also provided to our facilitators and parents for strategies and reminders on how they can best help you on your home-based learning.

Please use this module with care. Do not put unnecessary marks on any part of this SLM. Use a separate sheet of paper in answering the exercises and tests. And read the instructions carefully before performing each task.

If you have any questions in using this SLM or any difficulty in answering the tasks in this module, do not hesitate to consult your teacher or facilitator.

Thank you.

What I Need to Know

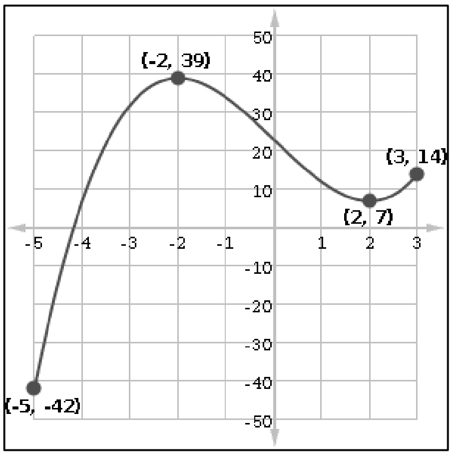
This module was designed to provide you with fun and meaningful opportunities for guided and independent learning at your own pace and time. You will be enabled to process the contents of the learning resource while being an active learner.

The module is intended for you to illustrate the Extreme Value Theorem; and solve optimization problems that yield polynomial functions.

What I Know

Task 1

A. Direction: Identify the absolute maximum and minimum points in the graph below. Write your answers in your notebook.



(Ault 2017)

1. Maximum point:
   Minimum point:

B. Direction: Read each question carefully. Choose the letter of the BEST answer. Write it on your notebook.

2. Which theorem states that if $f(x)$ is continuous on a closed interval [a,b], then $f(x)$ achieves its maximum and minimum values on [a,b]?
   A. Extreme Value Theorem
   B. Limit Theorem
   C. Intermediate Value Theorem
   D. Fermat's Theorem

3. Which theorem is sometimes referred to as the *Bolzano-Weierstrass* Theorem?
   A. Extreme Value Theorem
   B. Limit Theorem
   C. Intermediate Value Theorem
   D. Fermat's Theorem

4. Which do you call the process of maximizing or minimizing a function f(x) subject to some constraints set on x?
   A. Maximization
   B. Minimization
   C. Optimization
   D. None of these

5. Which of the following is **not** a step in optimization?
   A. Read the problem carefully.
   B. Identify the quantity to be optimized.
   C. Identify the constraints set on the input values.
   D. None of these

Lesson 1 Illustrating the Extreme Value Theorem and Solving Optimization Problems

The **Extreme Value Theorem** is sometimes referred to as the *Bolzano-Weierstrass* Theorem. It states that,

if f(x) is continuous on a closed interval [a,b], then f(x) achieves its maximum and minimum values on [a,b]".

What the theorem implies is, for as long as function is continuous on an interval, we are sure to find a maximum value and minimum value in that interval. If a function can be easily graphed in an interval in such a way that we can see its entire graph, then we can point to its maximum and minimum values as easily as a kid points to balloons. But this is not true except for basic and simple functions. In most instances, to see the maximum or minimum values of a function requires designing an algorithm. This problem is a standard exercise for students studying applied mathematics or computer science (Cliff Notes 2021).

What's In

Task 2

Direction: Study the table below. Follow the steps in obtaining the extreme value of functions differentiable on [a,b]. Complete the provided solution in the given problem.

Steps to obtain the absolute extreme value of functions differentiable on [a,b]
a. Find the critical numbers of the function in (a,b) and evaluate it at these numbers. b. Evaluate the function at the endpoints of [a.b] c. Compare the results of steps a and b. The largest is the absolute maximum and the smallest is the absolute minimum value of "*f*" on [a,b]

6(x-2)(x+1)=0

1. Find the critical numbers, $x_1 =$ ___ and $x_2 =$ ___ .
2. Using the critical numbers, evaluate the function $f(x) = 2x^3 - 3x^2 - 12x$ at these numbers.

$f(x_1) = 2x^3 - 3x^2 - 12x$ = _____
$f(x_2) = 2x^3 - 3x^2 - 12x$ = _____

b. Evaluate the function at the endpoints of the interval [-3,5].

3. Evaluate the function $f(x) = 2x^3 - 3x^2 - 12x$ at [-3,5].

$f(-3) = 2x^3 - 3x^2 - 12x$ = _____
$f(5) = 2x^3 - 3x^2 - 12x$ = _____

c. Choose the absolute maximum (MAX) and absolute minimum (MIN) values.

4. Identify the Maximum and Minimum values.

MAX = max {-20, 7, -45, 115 } = ____
MIN = min {-20, 7, -45, 115 } = ____

What's New

Task 3

Direction: Match the words in Column A to its corresponding definition in Column B. Write the letter of your choice in your notebook.

Column A

1. It is a process of maximizing or minimizing a function *f(x)* to some constraints set on *x*.
2. These are the inputs to the function.
3. It is the function which is to be either maximized or minimized over a set of feasible values for *x*.
4. This refers to the set containing all feasible values for the decision variable *x*.
5. It is the value of *x* that allows the objective function to attain either its maximum (minimum) value over the feasible region.

Column B

A. Optimization
B. Objective Function
C. Decision variables
D. Feasible region
E. Optimal solution

What is It

Optimization is a process of maximizing or minimizing a function f(x) subject to some constraints set on x. An optimization problem is formally defined in this manner.

P: maximize *f(x)*
Subject to $x \epsilon X$

There are technical terms that often appear in optimization problems. These are *objective function*, *decision variables*, *feasible region*, and *optimal solution.*

The function *f* is called the *objective function*, it is the function which is to be either maximized or minimized over a set of feasible values for x.

Decision variables are the inputs to the function, in this particular case, the possible values of *x* that can either maximize or minimize the objective function.

Feasible region refers to the set containing all feasible values for the decision variable *x*. It is also called the *constraint region.*

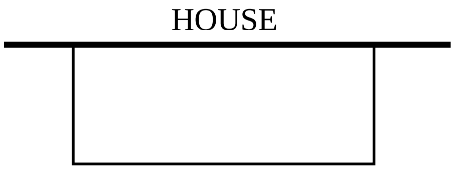
The optimal solution to the problem is the value of x that allows the objective function to attain either its maximum or minimum value over the feasible region. For maximization if x* is the optimal solution then $f(x*) \geq f(x)$ for all $x \in X$.

Steps to solve optimization problems:

a. Read the problem carefully.
b. Identify the quantity to be optimized.
c. Identify the constraints set on the input values.
d. Define the objective function, the function that will compute the quantity to be optimized.
e. Find the maximum or minimum value subject to constraints.

Example 1. Applying optimization to find the largest area of a fence using the first derivative test.

A house owner wishes to enclose his front yard with a fence. He keeps a wire fencing 150 meters long. He wishes to build it in such a way that the side nearest to the front of his house is open. Find the dimensions of the fence so that it encloses the largest possible area. The front yard is rectangular in shape.



Solution: The quantity to be optimized is area A. The appropriate equation to work on is

$$A = xy$$

Since the length of wire available is 150 meters, the *restriction or constraint* set on our enclosure is

$$2x + y = 150$$

Combining these two statements gives us an optimization problem.

Maximize $A = xy$
Subject to $2x + y = 150$

Write y in terms of x using the constraint equation.

$$y = 150 - 2x$$

This gives A as a function of x

$$A = x(150 - 2x) = 150x - 2x^2$$

We find the optimal value by obtaining absolute maximum of this function.

a. Find the critical numbers and evaluate the function A on these numbers.

A' = 150 - 4x
0 = 150 – 4x
$x = \frac{75}{2}$

$$\longrightarrow \quad \boldsymbol{A\left(\frac{75}{2}\right) = 150\left(\frac{75}{2}\right) - 2\left(\frac{75}{2}\right)^2 = 2\left(\frac{75}{2}\right)^2}$$

$= 2812.5\ \text{m}^2$

b. Evaluate the function at the endpoints of the closed interval [0, 150]. If x=0, this gives us no area for the enclosure. On the other hand, if x = 150, the area is -22,500 and we also get no enclosure.

c. Choose the maximum value.

MAX = max{0, -22500, 2812.5} = 2812.5 m^2

What are the dimensions of the enclosure? We already know that $x = \frac{75}{2}$. To solve *y*

$$y = 150 - 2x = 150 - 75 = 75$$

The dimension of the enclosure should be 75 meters by 37.5 meters.

Example 2. Optimization to find the largest area using the second derivative test

Solve the problem in Example 1 using the second derivative.

Solution

The objective function we found in the previous problem is a quadratic function whose graph opens downward. The only maximum value it will take is the one which coincides with the vertex. The vertex of a parabola which opens downward is one where $f'(x) = 0$ and $f"(x) < 0$.Since $A = 150x - 2x^2$

$$A' = 150 - 4x = 0 \ \rightarrow x = \frac{75}{2}$$

$$A" = -4 < 0$$

Since $A" = -4$ for all values of x, the relative maximum occurs only when $x = \frac{75}{2}$. The maximum value of the function is simply

$$A(\frac{75}{2}) = 2812.5\ m^2$$

The dimensions of the enclosure are still 75 meters by 37.5 meters.

What's More

Task 4

Direction: Read and analyze the problem carefully. Solve the problem using optimization. Show your solution.

1. Divide 25 into two parts whose product is a maximum.

What I Have Learned

Task 5

Direction: In your notebook, complete the following statements.

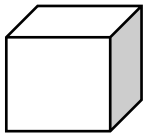
1. I have learned that ____
2. I have realized that ____
3. I will apply what I have learned ______________________________________

What I Can Do

Task 6

Direction: Read and analyze the problem carefully. Solve the problem using optimization. Show your solution.

1. A closed box with a square base is to have a volume of 64 ft^3. Find the dimensions so that the total area will be a minimum.



Rubric for Solving Problems

CATEGORY	5	4	3	2
Mathematical Concepts	Explanation shows complete understanding of the mathematical concepts used to solve the problem(s).	Explanation shows substantial understanding of the mathematical concepts used to solve the problem(s).	Explanation shows some understanding of the mathematical concepts needed to solve the problem(s).	Explanation shows very limited understanding of the underlying concepts needed to solve the problem(s) OR is not written.
Mathematical Errors	90-100% of the steps and solutions have no mathematical errors.	Almost all (85-89%) of the steps and solutions have no mathematical errors.	Most (75-84%) of the steps and solutions have no mathematical errors.	More than 75% of the steps and solutions have mathematical errors.
Neatness and Organization	The work is presented in a neat, clear, organized fashion that is easy to read.	The work is presented in a neat and organized fashion that is usually easy to read.	The work is presented in an organized fashion but may be hard to read at times.	The work appears sloppy and unorganized. It is hard to know what information goes together.
Completion	All problems are completed.	All but one of the problems are completed.	All but two of the problems are completed.	Several of the problems are not completed.

Assessment

A. Direction: Identify the absolute maximum and minimum points in each of the graph. Write your answers in your notebook.

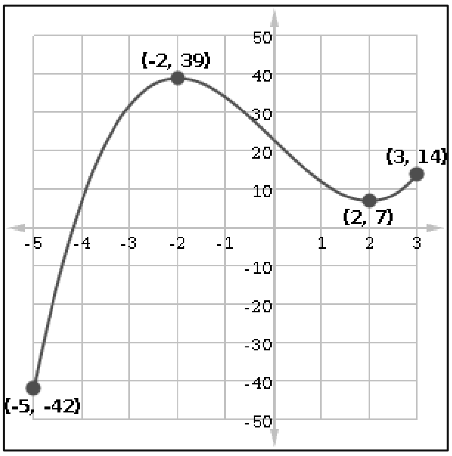


Photo source: *https://magoosh.com/hs/ap/ap-calculus-review-finding-absolute-extrema/*

B. Direction: Read each question carefully. Choose the letter of the BEST answer. Write it on your notebook.

1. What theorem states that if $f(x)$ is continuous on a closed interval [a,b], then $f(x)$ achieves its maximum and minimum values on [a,b]?
   A. Extreme Value Theorem C. Intermediate Value Theorem
   B. Limit Theorem D. Fermat's Theorem

2. What theorem is sometimes referred to as the *Bolzano-Weierstrass* Theorem?
   A. Extreme Value Theorem C. Intermediate Value Theorem
   B. Limit Theorem D. Fermat's Theorem

3. What do you call the process of maximizing or minimizing a function f(x) subject to some constraints set on x?
   A. Maximization C. Optimization
   B. Minimization D. None of these

4. Which of the following is **not** a step in optimization?
   A. Read the problem carefully.
   B. Identify the quantity to be optimized.
   C. Identify the constraints set on the input values.
   D. None of these

Answer Key

Task 1

A. 1. Absolute max:(-2, 39)
   Absolute min: (-5, -42)

B. 2. A
   3. A
   4. C
   5. D

Task 2

a. 1. $x_1 = 2$, $x_2 = -1$
   2. -20,7

b. -45, 115

c. Max = 115, Min = -45

Task 3

1. A
2. C
3. B
4. D
5. E

Task 4

Solution:

Let x – 1st part
y – 2nd part

$x + y = 25$

$x = 25 - y$

$P = xy$

$P = (25 - y)(y)$

$P = 25y - y^2$

$\frac{dP}{dy} = 25 - 2y = 0$

$25 = 2y$

$y = 12.5$

$x = 25 - 12.5 = 12.5$

$\frac{d^2P}{dy^2} = -2 < 0$

So, a relative maximum P occurs on y = 12.5 and x = 12.5

Task 5

Answers may vary.

Task 6

Solution:

$V = 64\ ft^3$

$A = 2x^2 + 4xy$

$V = x^2y = 64$

$y = \frac{64}{x^2}$

$A = 2x^2 + 4x\left(\frac{64}{x^2}\right)$

$A = 2x^2 + \frac{256}{x}$

$\frac{dA}{dx} = 4x - \frac{256D_x(x)}{x^2}$

$0 = 4x - \frac{256}{x^2}$

$D_x\left(\frac{c}{v}\right) = \frac{-cD_x(v)}{v^2}$

$0 = 4x^3 - 256$

$4x^3 = 256$

$x^3 = \frac{256}{4}$

$x^3 = 64$

$x = 4\ ft$

see continuation above this box

$y = \frac{64}{16}$

$y = 4\ ft$

Continuation:

$\frac{d^2A}{dx^2} = 4 + \frac{2(256)}{x^3}$

$\frac{d^2A}{dx^2} > 0$

, So a relative minimum area occurs when x is 4 ft.

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