11

SENIOR HIGH SCHOOL

DEPARTMENT OF EDUCATION
SCHOOLS DIVISION OF NEGROS ORIENTAL
REGION VII
Kagawasan Ave., Daro, Dumaguete City, Negros Oriental

BASIC CALCULUS

Quarter 3 – Module 1
The Limit of a Function and Limit Laws

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ADM

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Quarter 3 – Module 1: The Limit of a Function and Limit Laws
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Introductory Message

This Self-Learning Module (SLM) is prepared so that you, our dear learners, can continue your studies and learn while at home. Activities, questions, directions, exercises, and discussions are carefully stated for you to understand each lesson.

Each SLM is composed of different parts. Each part shall guide you step-by-step as you discover and understand the lesson prepared for you.

Pre-tests are provided to measure your prior knowledge on lessons in each SLM. This will tell you if you need to proceed on completing this module or if you need to ask your facilitator or your teacher's assistance for better understanding of the lesson. At the end of each module, you need to answer the post-test to self-check your learning. Answer keys are provided for each activity and test. We trust that you will be honest in using these.

In addition to the material in the main text, Notes to the Teacher are also provided to our facilitators and parents for strategies and reminders on how they can best help you on your home-based learning.

Please use this module with care. Do not put unnecessary marks on any part of this SLM. Use a separate sheet of paper in answering the exercises and tests. And read the instructions carefully before performing each task.

If you have any questions in using this SLM or any difficulty in answering the tasks in this module, do not hesitate to consult your teacher or facilitator.

Thank you.

What I Need to Know

This module was designed to provide you with fun and meaningful opportunities for guided and independent learning at your own pace and time. You will be enabled to process the contents of the learning resource while being an active learner.

The module is intended for you to illustrate the limit of a function using table of values and the graph of a function and the limit laws; distinguish between $\lim_{x \to c} f(x)$ *and* $f(c)$; and apply the limits laws in evaluating the limit of algebraic functions (polynomial, rational, and radical)

What I Know

PRE-ASSESSMENT

Multiple Choice. Read and analyze each item carefully and write the letter of the correct answer on your activity sheets/notebook.

1. Given the table of values below. Which among the statements **DOES NOT** correctly describe the values in the table?

x	***f(x)***		***x***	***f(x)***
0.5	-3.5		1.5	-2.5
0.88	-3.12		1.17	-2.83
0.996	-3.004		1.003	-2.997
0.9999	-3.0001		1.0001	-2.9999

   A. The table shows that as the value of ***x*** increases the value of ***f(x)*** also increases.
   B. The table shows that as the value of ***x*** decreases the value of ***f(x)*** also decreases.
   C. The table shows that as ***x*** approaches **1** from left or right, ***f(x)*** approaches **-3.**
   D. The table shows that as the value of ***x*** decreases the value of ***f(x)*** increases.

For items 2-4, refer to the given. Let $f(x) = \frac{x^3 - 3x^2 + x - 3}{x - 3}$.

2. Given the table of values below, what is the value of ***f(x)*** if ***x*** = 2.99999?

x	***f(x)***
2.7	8.29
2.85	9.12250000000001
2.995	9.97002499999939
2.99999	

A. 6.0111 B. 9.99993876 C. 10.0001 D. 10.51

3. Given the table of values below, what is the value of ***f(x)*** if ***x*** = 3.00001?

x	***f(x)***
3.5	13.25
3.1	10.61
3.001	10.006000999997
3.00001	

A. 10.00006 B. 10.51 C. 13.25 D. 13.99

4. What do the tables in items 2 and 3 show based on their limit values?
   A. The tables show that as x approaches 3 from the left or right, *f(x)* approaches to the same value of 10.
   B. The tables show that as x approaches -3 from the left and 3 from the right, *f(x)* approaches to the same value.
   C. The tables show that as the value of x increases, *f(x)* decreases.
   D. The tables show that as the value of x decreases, *f(x)* increases.
5. Does the limit exist for this function as x approaches to 3?
   A. No, only left hand limit exist. C. Yes. Both hand limit exist. .
   B. No, only right hand limit exist. D. No limit exist.
6. Which of the following expressions follow the limit of a constant?
   A. $\lim_{x \to 2} x = 2$ C. $\lim_{x \to 2} x^2 = 4$
   B. $\lim_{x \to c} -3.14 = c, c \in \mathbb{R}$ D. $\lim_{x \to c} 7 = 7, c \in \mathbb{R}$
7. Which of the following illustrates the Multiplication Theorem?
   A. $\lim_{x \to c} (f(x))^P = \lim_{x \to c} (f(x))^P = L^P$.
   B. $\lim_{x \to c} f(x) = L$
   C. $\lim_{x \to c} (f(x) \cdot g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = L \cdot M$.
   D. $\lim_{x \to c} k \cdot f(x) = k \cdot \lim_{x \to c} f(x) = k \cdot L$.
8. What theorem states that the limit of a sum/difference of functions is the sum/difference of the limits of the individual functions?
   A. The Summing Up Theorem C. The Addition/Subtraction Theorem
   B. Limitless Theorem D. The Add On Theorem

9. How is the Division Theorem written in symbols if it states that the limit of a quotient of functions is equal to the quotient of the limits of the individual functions, provided the denominator limit is not equal to 0?

A. $\lim_{x \to c} \frac{f(x)}{g(c)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{L}{M}$, provided $M = 0$.

B. $\lim_{x \to c} \frac{f(x)}{g(c)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{L}{M}$, provided $M > 0$.

C. $\lim_{x \to c} \frac{f(x)}{g(c)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{L}{M}$, provided $M < 0$.

D. $\lim_{x \to c} \frac{f(x)}{g(c)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{L}{M}$, provided $M \neq 0$.

10. The Radical/ Root Theorem $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = \sqrt[n]{L}$, in words it states that:

A. If ***n*** is a negative integer, the limit of the ***n*th** root of a function is just the ***n*th** root of the limit of the function, provided the ***n*th** root of the limit is a real number. Thus, it is important to keep in mind that if n is even, the limit of the function must be positive.
B. If ***n*** is a positive integer, the limit of the ***n*th** root of a function is just the ***n*th** root of the limit of the function, provided the ***n*th** root of the limit is a real number. Thus, it is important to keep in mind that if ***n*** is even, the limit of the function must be positive.
C. If ***n*** is a positive integer, the limit of the ***n*th** root of a function is just the limit of the function, provided the ***n*th** root of the limit is a negative number.
D. If ***n*** is a even integer, the limit of the ***n*th** root of a function is just the ***n*th** root of the limit of the function, provided the ***n*th** root of the limit is a real number. Thus, it is important to keep in mind that if ***n*** is odd, the limit of the function must be negative.

Lesson 1

Illustration of Limit of a Function using the Table of Values and Graph and Distinction between $\lim_{x \to c} f(x)$ and $f(c)$

What's In

PRIOR-KNOWLEDGE

Complete the table of values below given the functions

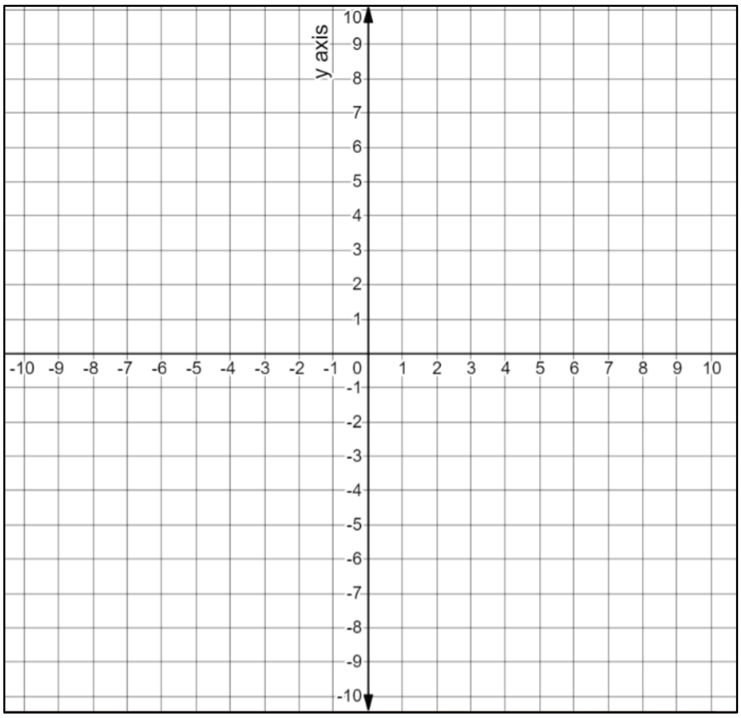
f(x) = 1 + 3x

x	-3	-2	-1	0	1	2	3
f(x)							

and *g(x) = x²– 1*

x	-3	-2	-1	0	1	2	3
f(x)							

Now using the table above, sketch the graph of the two functions.



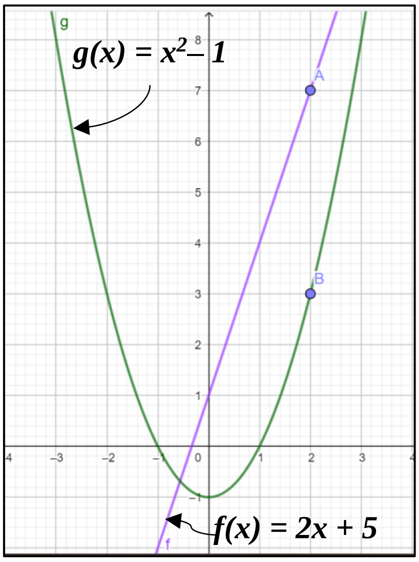


Figure 1. Graph of f(x) and g(x)

From the graph of the two functions as shown at the left, how will you reflect on the following?

a. What happened to *f(x) and g(x)* if *x* increases in value approaching 0? What value is the function approaching then?

____

b. What happened to the function as the value of x decreases approaching 0? What value is it approaching then?

____

c. Are the two values similar? So, what do you call this value of f(x)?

____

What's New

Activity 1. Try This!

Evaluate the following functions. You can refer to figure 1 above for the graph of the functions f(x) and g(x).

1. $f(x) = 1 + 3x, \quad at\ x = 2$

2. $g(x) = x^2 - 1, \quad at\ x = 2$

What is It

1.1. ILLUSTRATING THE LIMIT OF A FUNCTION USING TABLE OF VALUES AND GRAPH

Consider a function ***f*** of a single variable ***x***. Consider a constant ***c*** which the variable ***x*** will approach (***c*** may or may not be in the domain of ***f***). The limit, to be denoted by ***L***, is the unique real value that ***f(x)*** will approach as ***x*** approaches ***c***. In symbols, we write this process as $\lim\limits_{x \to c} f(x) = L$. This is read as, **"The limit of f(x) as x approaches to c is L."**

To illustrate, let us consider $\lim\limits_{x \to 2}(1 + 3x)$. Here, $f(x) = (1 + 3x)$ and the constant c, which x will approach is *2*. To evaluate the given limit, we will make use of the table to help us keep track of the effect that the approach of x toward 2 will have on f(x).

x	***f(x)***
1	4
1.4	5.2
1.7	6.1
1.9	6.7
1.95	6.85
1.997	6.991
1.9999	6.9997
1.9999999	6.9999997

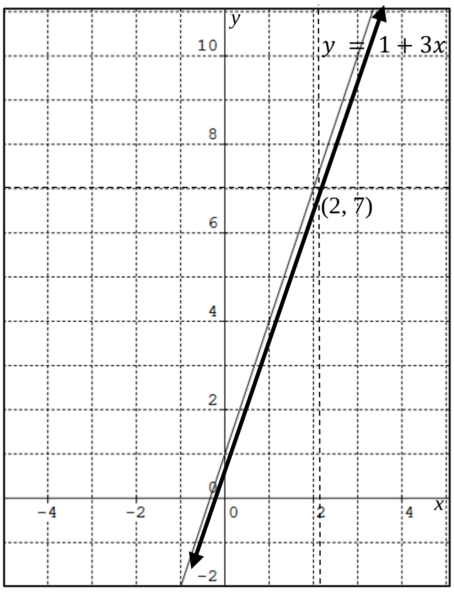
Now, consider approaching 2 from its right or through values greater than but close to 2.

x	***f(x)***
3	10
2.5	8.5
2.2	7.6
2.1	7.3
2.03	7.09
2.009	7.027
2.0005	7.0015
2.0000001	7.0000003

Observe that as the values of x get closer and closer to 2, the values of *f(x)* get closer and closer to 7. This behavior can be shown no matter what set of values, or what direction, is taken in approaching 2. In symbols, it is written as

$$\lim_{x \to 2}(1 + 3x) = 7.$$

Consider again f(x) = 1+3x. Its graph is the straight line with slope 3 and intercepts (0,1) and $\left(\frac{-1}{3}, 0\right)$. Look at the graph in the vicinity of x = 2. You can easily see the points (from the table of values above (1,4), (1.4, 5.2), (1.7,6.1), and so on, approaching the level where f(x) = 7. The same can be seen from the right . Hence, the graph clearly confirms that $\lim_{x\to 2}(1+3x)$= 7.



Example 1. Investigate $\lim_{x\to -1}(x^2+1)$ by constructing tables of values.

Here, c = -1 and *f(x)* = $x^2 + 1$.
We start again by approaching -1 from the left.

x	*f(x)*
-1.5	3.25
-1.2	2.44
-1.01	2.0201
-1.0001	2.00020001

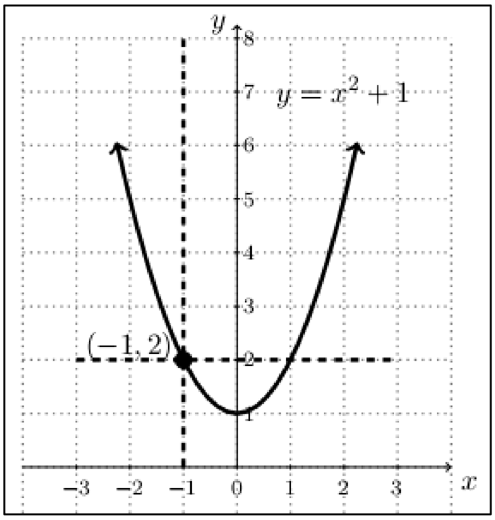
Now, approach -1 from the right

x	*f(x)*
-0.5	1.25
-0.8	1.64
-0.99	1.9801
-0.9999	1.99980001

The tables show that as x approaches -1, f(x) approaches 2.
Therefore,

$$\lim_{x\to -1}(x^2+1) = 2.$$

The graph of f(x)= $x^2 + 1$, is given below:



It can be seen from the graph that as values of *x* approach to -1, the values of *f(x)* approach 2.

Example 2. Investigate $\lim_{x \to 0} |x|$ through a table of values.

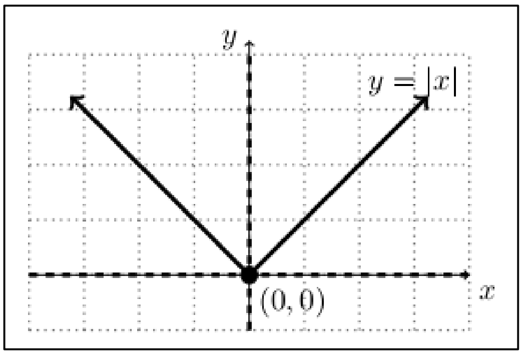
Approaching 0 from the left and from the right, we get the following tables:

x	$\lvert x \rvert$		x	$\lvert x \rvert$
-0.3	0.3		0.3	0.3
-0.01	0.01		0.01	0.01
-0.00009	0.00009		0.00009	0.00009
-0.00000001	0.00000001		0.00000001	0.00000001

Hence,

$$\lim_{x \to 0} |x| = 0.$$

In this example, f(x) = $|x|$. The graph is presented below:



It can be seen from the graph that as values of *x* approach to 0, the values of *f(x)* approach 0 also.

Example 3. Investigate $\lim_{x\to 1}(\frac{x^2-5x+4}{x-1})$ by constructing tables of values. Here, c = 1 and $f(x) = \frac{x^2-5x+4}{x-1}$. Take note that 1 is not in the domain of f, but this is not a problem. In evaluating a limit, remember that we only need to go very close to 1, we will not go to 1 itself.

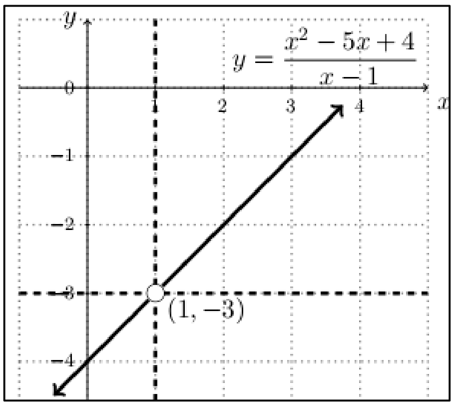
We now approach 1 from left.

x	***f(x)***
1.5	-2.5
1.17	-2.83
1.003	-2.997
1.0001	-2.9999

Then, we approach 1 from the right.

x	***f(x)***
0.5	-3.5
0.88	-3.12
0.996	-3.004
0.9999	-3.0001

The tables show that as x approaches to 1, f(x) approaches to -3. In symbols, $\lim_{x\to 1}(\frac{x^2-5x+4}{x-1}) = -3$. Since $f(x) = \frac{x^2-5x+4}{x-1}$, the graph shows this way



Take note that $f(x) = \frac{x^2-5x+4}{x-1}$, then $f(x) = \frac{(x-4)(x-1)}{x-1}$. Thus, f(x) = x - 4, provided $x \neq 1$.Hence, the graph of f(x) is also the graph of y = x-1, excluding the point where x = 1.

1.2. DISTINGUISHING BETWEEN $\lim\limits_{x \to c} f(x)$ AND $f(c)$

Is $\lim\limits_{x \to c} f(x)$ always equal to $f(c)$? To answer this question, consider the table of values of the function $f(x)$ below.

The table of values as x approaches to the left or to the right of 2 is presented below.

x	*f(x)*
1	4
1.4	5.2
1.7	6.1
1.9	6.7
1.95	6.85
1.997	6.991
1.9999	6.9997
1.9999999	6.9999997

x	*f(x)*
3	10
2.5	8.5
2.2	7.6
2.1	7.3
2.03	7.09
2.009	7.027
2.0005	7.0015
2.00000001	7.00000003

We can conclude that $\lim\limits_{x \to 2}(1+3x) = 7$. While, $f(2) = 7$. So, in this example, $\lim\limits_{x \to 2} f(x)$ *and* $f(2)$ are equal. Notice that the same holds for the following examples as discussed.

$\lim\limits_{x \to c} f(x)$	**f(c)**
$\lim\limits_{x \to -1} x^2 + 1 = 2$	f(-1)=2
$\lim\limits_{x \to 0} \lvert x \rvert = 0$	f(0)=0

This, however, is not always the case. Let us consider the function.

$$f(x) = \begin{cases} |x| & if\ x \neq 0 \\ 2 & if\ x \neq 0 \end{cases}$$

Does this in any way affect the existence of the limit? Not at all. This example shows that $\lim\limits_{x \to c} f(x)$ and ***f(c)*** may be distinct.

Furthermore, consider the third example where $f(x) = \begin{cases} x+1 & if\ x < 4 \\ (x-4)^2 + 3 & if\ x \geq 4 \end{cases}$

We have,

$\lim_{x \to c} f(x)$	***f(c)***
$\lim_{x \to c} f(x)\ DNE$	*f(4)=2*

Once again, we see, that $\lim_{x \to c} \boldsymbol{f(x)}$ and ***f(c)*** are not the same.

What's More

Activity

Complete the following tables to investigate $\lim_{x \to 1}(x^2 - 2x + 4)$.

x	***f(x)***
0.5	
0.7	
0.95	
0.995	
0.9995	
0.99995	

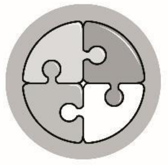
x	***f(x)***
1.6	
1.35	
1.05	
1.005	
1.0005	
1.00005	

Answer the following questions.

1. What observations do you get from the result in the activity above?
2. Given the table of values. What does this show base on their limit values?
3. Describe the values of *x* in the two tables.

Lesson 2

Illustration of Limit Laws and their Applications in Evaluating the Limit of Functions



What's In

PRIOR-KNOWLEDGE

Lesson 1 showed us how limits can be determined through either a table of values or the graph of a function. One might ask: Must one always construct a table or graph the function to determine the limit? Filling in a table of values sometimes requires very tedious calculations. Likewise, a graph may be difficult to sketch. However, these should not be reasons for a student to fail to determine a limit.

In this lesson, we will learn how to compute the limit of a function using Limit Theorems.

What's New

Activity 1: Construct the table of values to evaluate the following:

1. $\lim_{x \to 1} x =$
2. $\lim_{x \to 2} x^2 =$
3. $\lim_{x \to 4} \sqrt{x} =$
4. $\lim_{x \to -2} 5 =$

What is It

2.1 ILLUSTRATION OF LIMIT LAWS

The limit laws enable us to directly evaluate limits, without need for a table or a graph.

Let c is a constant, and f and g are functions which may or may not have c in their domains.

LIMIT OF A CONSTANT

i. The limit of a constant is itself. If k is any constant, then,

$$\lim_{x \to c} k = k.$$

Example: 1.a. $\lim_{x \to c} 2 = 2$

1.b. $\lim_{x \to c} -3.14 = -3.14$

1.c. $\lim_{x \to c} 789 = 789$

ii. The limit of x as x approaches c is equal to c. This may be thought of as the substitution law because x is simply substituted by c.

$$\lim_{x \to c} x = c$$

Example: 2.a. $\lim_{x \to 9} x = 9$

2.b. $\lim_{x \to 0.005} x = 0.005$

2.c. $\lim_{x \to -10} x = -10$

For the remaining theorems, we will assume that the limits of f and g both exist as x approaches c and that they are L and M, respectively. In other words,

$$\lim_{x \to c} f(c) = L, \text{ and } \lim_{x \to c} g(c) = M$$

iii. *The Constant Multiple Theorem:* This says that the limit of a multiple of a function is simply that multiple of the limit of the function.

$$\lim_{x \to c} k \cdot f(x) = k \cdot \lim_{x \to c} f(x) = k \cdot L.$$

For example, let $\lim_{x \to c} f(x) = 4.$ Then

a. $\lim_{x \to c} 8 \cdot f(x) = 8 \cdot \lim_{x \to c} f(x) = 8 \cdot 4 = 32.$

b. $\lim_{x \to c} -11 \cdot f(x) = -11 \cdot \lim_{x \to c} f(x) = -11 \cdot 4 = -44.$

c. $\lim_{x \to c} \frac{3}{2} \cdot f(x) = \frac{3}{2} \cdot \lim_{x \to c} f(x) = \frac{3}{2} \cdot 4 = 6.$

iv. *The Addition Theorem:* This says that the limit of a sum of functions is the sum of the limits of the individual functions. Subtraction is also included in this law, that is, the limit of a difference of functions is the difference of their limits. In symbols,

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M.$$

$$\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M.$$

For example, if $\lim_{x\to c} f(x) = 4$ and $\lim_{x\to c} g(x) = -5$, then

a. $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 4 + (-5) = -1.$

b. $\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = 4 - (-5) = 9.$

v. *The Multiplication Theorem:* This is similar to the Addition Theorem, with multiplication replacing addition as the operation involved. Thus, the limit of a product of functions is equal to the product of their limits. In symbols,

$$\lim_{x\to c}(f(x)\cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = L \cdot M.$$

For example, let $\lim_{x\to c} f(x)$=4 and $\lim_{x\to c} g(x) = -5$. Then

$$\lim_{x\to c}(f(x)\cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = 4 \cdot (-5) = -20.$$

v.a. Remark 1: The Addition and Multiplication Theorems may be applied to sums, differences and products of more than functions.

Remark 2: The Constant Multiple Theorem is a special case of the Multiplication Theorem. Indeed, in the Multiplication Theorem, if the first function f(x) is replaced by a constant k, the result is the Constant Multiple Theorem.

vi. *The Division Theorem:* This says that the limit of a quotient of functions is equal to the quotient of the limits of the individual functions, provided the denominator limit is not equal to 0. In symbols,

$$\lim_{x\to c}\frac{f(x)}{g(c)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}, \text{ provided M} \neq 0.$$

For example,

a. If $\lim_{x\to c} f(x) = 4$ and $\lim_{x\to c} g(x) = -5$

$$\lim_{x\to c}\frac{f(x)}{g(c)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = = -\frac{4}{5}.$$

b. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = -5$

$$\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = = -\frac{0}{5} = 0.$$

c. If $\lim_{x\to c} f(x) = 4$ and $\lim_{x\to c} g(x) = 0$, it is not possible to evaluate $\lim_{x\to c}\frac{f(x)}{g(c)}$, or we may say that the limit DNE (does not exist).

vii. *The Power Limit*. This theorem states that the limit of an integer power *p* of a function is just that power of the limit of a function. In symbols,

$$\lim_{x\to c}(f(x))^P = \lim_{x\to c}(f(x))^P = L^P$$

a. If $\lim_{x\to c} f(x) = 4$, then
$\lim_{x\to c}(f(x))^3 = 4^3 = 64.$

b. If $\lim_{x\to c} f(x) = 4$, then
$\lim_{x\to c}(f(x))^{-2} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$

viii. *The Radical/ Root Theorem*. This theorem states that if *n* is a positive integer, the limit of the *n*th root of a function is just the *n*th root of the limit of the function, provided the *n*th root of the limit is a real number. Thus, it is important to keep in mind that if n is even, the limit of the function must be positive. In symbols,

$$\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}.$$

For example,

a. If $\lim_{x\to c} f(x)=4$, then $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = 2.$

b. If $\lim_{x\to c} f(x)=-4$, then it is not possible to evaluate $\lim_{x\to c}\sqrt[n]{f(x)}$ because $\lim_{x\to c}\sqrt[n]{f(x)}$ and this is not a real number.

2.2 APPLICATION OF LIMIT LAWS IN EVALUATING THE LIMIT OF ALGEBRAIC FUNCTIONS

Recall the limit theorems. These theorems will be used in evaluating algebraic functions and illustrated in the following examples.

Limits of Polynomial Functions

We start with evaluating of polynomial functions.

Example 1. Determine $\lim_{x\to 1}(2x+1)$

Solution: From the theorems above,

$$\begin{aligned}\lim_{x\to 1}(2x+1) &= \lim_{x\to 1} 2x + \lim_{x\to 1} 1 && \text{(Addition)}\\ &= (2\lim_{x\to 1} x) + 1 && \text{(Constant Multiple)}\\ &= 2(1) + 1 && (\lim_{x\to c} x = c)\\ &= 2 + 1\\ &= 3.\end{aligned}$$

Example 2. Determine $\lim_{x\to -1}(2x^3 - 4x^2 + 1)$

Solution: From the theorems above,

$$\begin{aligned}\lim_{x\to -1}(2x^3 - 4x^2 + 1) &= \lim_{x\to -1} 2x^3 - \lim_{x\to -1} 4x^2 + \lim_{x\to -1} 1 && \text{(Addition)}\\ &= 2\lim_{x\to -1} x^3 - 4\lim_{x\to -1} x^2 && \text{(Constant Multiple)}\\ &= 2(-1)^3 - 4(-1)^2 + 1 && \text{(Power)}\\ &= -2-4+1\\ &= -5.\end{aligned}$$

Example 3. Evaluate $\lim_{x\to 0}(3x^4 - 2x - 1)$.

Solution. From the theorems above,

$$\begin{aligned}\lim_{x\to 0}(3x^4 - 2x - 1) &= \lim_{x\to 0} 3x^4 - \lim_{x\to 0}(2x - \lim_{x\to 0} 1 && \text{(Addition)}\\ &= 3\lim_{x\to 0} x^4 - 2\lim_{x\to 0}(x - 1 && \text{(Constant Multiple)}\\ &= 3(0)^4 - 2(0) - 1 && \text{(Power)}\\ &= 0 - 0 - 1\\ &= -1\end{aligned}$$

Limits of Rational Functions

We will now apply the limit theorems in evaluating rational functions. In evaluating the limits of such functions, recall from Theorem 1 the Division Rule, and all the rules stated in Theorem 1 which have been useful in evaluating limits of polynomial functions, such as the Additional and Product Rules.

Example 4. Evaluate $\lim_{x\to 1}\frac{1}{x}$.

Solution: First, note that $\lim_{x\to 1} x = 1$. Since the limit of the denominator is nonzero, we can apply the Division Rule. Thus,

$$\begin{aligned}\lim_{x\to 1}\frac{1}{x} &= \frac{\lim_{x\to 1} 1}{\lim_{x\to 1} x} && \text{(Division)}\\ &= \frac{1}{1}\\ &= 1.\end{aligned}$$

Example 5. Evaluate $\lim\limits_{x \to 2} \frac{x}{x+1}$

Solution: Start checking the limit of the polynomial function in the denominator.

$$\lim_{x \to 2}(x - 1) = \lim_{x \to 2} x - \lim_{x \to 2} 1$$
$$= 2 - 1$$
$$= 1$$

Since the limit of the denominator is not zero, it follows that

$$\lim_{x \to 2} \frac{x}{x - 1} = \frac{\lim\limits_{x \to 2} x}{\lim\limits_{x \to 2}(x - 1)}$$
$$= \frac{2}{1} \qquad \text{(Division)}$$
$$= 2$$

Example 6. Evaluate $\lim\limits_{x \to 1} \frac{(x-3)(x^2-2)}{x^2+1}$. First, note that

$$\lim_{x \to 1}(x^2 + 1) = \lim_{x \to 1} x^2 + \lim_{x \to 1} 1$$
$$= 1 + 1$$
$$= 2 \neq 0.$$

Thus, using the theorem,

$$\lim_{x \to 1} \frac{(x-3)(x^2-2)}{x^2+1} = \frac{\lim\limits_{x \to 1}(x-3)(x^2-2)}{\lim\limits_{x \to 1}(x^2+1)} \qquad \text{(Division)}$$
$$= \frac{\lim\limits_{x \to 1}(x-3) \cdot \lim\limits_{x \to 1}(x^2-2)}{2} \qquad \text{(Multiplication)}$$
$$= \frac{(\lim\limits_{x \to 1} x - \lim\limits_{x \to 1} 3)(\lim\limits_{x \to 1} x^2 - \lim\limits_{x \to 1} 2)}{2} \qquad \text{(Addition)}$$
$$= \frac{(1-3)(1^2-2)}{2}$$
$$= 1.$$

Limits of Radical Functions

We will now evaluate limits of radical functions using limit theorems.

Example 7. Evaluate $\lim\limits_{x \to 1} \sqrt{x}$.

Solution. Note that $\lim\limits_{x \to 1} x = 1 > 0$. Therefore, by the radical/root rule,

$$\lim_{x \to 1} \sqrt{x} = \sqrt{\lim_{x \to 1} x} = \sqrt{1} = 1$$

Example 8. Evaluate $\lim\limits_{x \to 0} \sqrt{x + 4}$.

Solution. Note that $\lim\limits_{x \to 0}(x + 4) = 4 > 0$. Hence, by radical/root rule,

$$\lim_{x \to 0} \sqrt{x + 4} = \sqrt{\lim_{x \to 0}(x + 4)}$$
$$= \sqrt{4}$$
$$= 2$$

Example 9. Evaluate $\lim_{x \to -2} \sqrt[3]{x^2 + 3x - 6}$

Solution. Since the index of the radical sign in odd, we do not have to worry that the limit of the radicand is negative. Therefore, the radical/root rule implies that

$$\lim_{x \to -2} \sqrt[3]{x^2 + 3x - 6} = \sqrt[3]{\lim_{x \to -2} (x^2 + 3x - 6)}$$
$$= \sqrt[3]{4 - 6 - 6}$$
$$= \sqrt[3]{-8}$$
$$= -2$$

What's More

Activity.

Use the limit theorems to evaluate the following, if the limit exist.

1. $\lim_{x \to 12} x$
2. $\lim_{x \to 2} 5$
3. $\lim_{x \to -3} (4x + 2)$
4. $\lim_{x \to 3} 2x - 4$
5. $\lim_{x \to 2} (8 - 3x + 12x^2)$
6. $\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}$

What I Have Learned

Generalization

Directions: Reflect the learning that you gained after taking up the two lessons in this module by completing the given statements below. Do this on your activity notebook. Do not write anything on this module.

What were your thoughts or ideas about the topic before taking up the lesson?

I thought that ____

____.

What new or additional ideas have you had after taking up this lesson?

I learned that (write as many as you can)

____

____.

How are you going to apply your learning from this lesson?

I will apply

____

____.

What I Can Do

Application (Performance Task)

In a short sized bondpaper folded crosswise, make a booklet then write the **8 Limit Laws** and give 2 examples of each. Each of the examples is worth 5 points. Examples should not be copied from the given in the previous activities. Be guided with the rubrics in scoring your output.

CREATIVE PROJECT ASSESSMENT RUBRIC

CATEGORY	5	4	3	2
Required Elements	Goes over and above all the required elements stated in the directions/instructions	Includes all of the required elements as stated in the directions/instructions	Missing one or more of the required elements as stated in the directions/instructions	Missing one or more of the required elements as stated in the directions/instructions
Creativity	Exceptionally clever and unique in showing deep understanding	Thoughtfully and uniquely presented; clever at times in showing understanding of the material	A few original touches enhance the project to show some understanding of the material	Shows little creativity, originality and/or effort in understanding the material
Neatness and Attractiveness	Exceptionally attractive and particularly neat in design and lay out	Attractive and particularly neat in design and lay out	Acceptably attractive but may be messy at times and/or show lack of organization	Distractingly messy or very poorly designed. Does not show pride in work.
Mathematical Grammar and Solution	No mathematical mistakes in the project	A few mathematical mistakes which are not distracting in the project	Several mathematical mistakes which are distracting in the project	Many mathematical mistakes throughout the project. Clearly not proofread.

Source: West Mark School. Retrieved from https://www.westmarkschool.org/uploaded/photos/1617/Summer_Reading/Creative_Project_Assessment_Rubric.pdf

Assessment

Multiple Choice. Read and understand each statement. Write the letter of the correct answer on your activity sheet/notebook.

1. Which defintion below illustrates the Constant Multiple Theorem which defines the limit of a multiple of a function is simply that multiple of the limit of the function?

A. $\lim_{x \to c} \left(f(x)\right)^P = \lim_{x \to c} \left(f(x)\right)^P = L^P.$

B. $\lim_{x \to c} f(x) = L$

C. $\lim_{x \to c} (f(x) \cdot g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = L \cdot M.$

D. $\lim_{x \to c} k \cdot f(x) = k \cdot \lim_{x \to c} f(x) = k \cdot L.$

2.

3.

5. What is the limit of $\frac{x^2-5x+4}{x-1}$ as x approaches 1 from left.

A. 3 B. 1 C. -1 D. -3

6. What is the limit $\frac{x^2-5x+4}{x-1}$ as x approaches 1 from right.

A. 3 B. 1 C. -1 D. -3

7. Given the table of values below. What does this show base on their limit values?

x	***f(x)***		***x***	***f(x)***
0.5	-3.5		1.5	-2.5
0.88	-3.12		1.17	-2.83
0.996	-3.004		1.003	-2.997
0.9999	-3.0001		1.0001	-2.9999

A. The table shows that as the value of ***x*** increases the value of ***f(x),*** decreases.
B. The table shows that as the value of ***x*** decreases the value of ***f(x)***, increases.
C. The table shows that as ***x*** approached ***1*** from left or right, ***f(x)*** approaches ***-3.***
D. The table shows that as the value of ***x*** decreases the value of ***f(x)*** increases.

8. This symbol, $\lim_{x \to c} f(x) = L$ is read as

A. The limit of ***x*** to ***c*** as it approaches to ***f(x)*** is ***L***.
B. The limit of ***f(x)*** as ***x*** approaches to ***c*** is ***L***.
C. The limit of ***L*** as ***x*** approaches to ***c*** is ***f(x).***
D. The limit of ***L*** as ***f(x)*** approaches to ***c*** is ***x***.

9. Evaluate $\lim_{x \to \frac{1}{3}} \frac{9x^2-1}{3x-1} =$

A. ∞ B. -∞ C. 2 D. 0

10. Evaluate $\lim_{x \to 0} \frac{x^3-8}{x^2-4} =$

A. 4 B. 2 C. 1 D. 0

11.

12. The Power Limit states that

A. the limit of an integer power ***p*** of a function is just that power of the limit of a function.
B. the limit of an integer power ***p*** of a function is just the limit of the power of a function.
C. the limit of an integer power ***p*** of a function is just that power raised to another power of the limit of a function.
D. the limit of a function power ***p*** of an integer is just that power of the limit of a function.

13. Which among the following shows the limit of a constant given the symbol?

A. $\lim_{x \to 12} 12 = 12$
B. $\lim_{x \to c} -3.14 = c$
C. $\lim_{x \to 64} c = 64$
D. $\lim_{x \to c} 7 = 7$

14. The limit of ***x*** as ***x*** approaches to ***c*** is equal to ***c***. This may be thought of as the substitution law because *x* is simply substituted by ***c***. $\lim_{x \to c} x = c$ as illustrated in the example below.

A. $\lim_{x \to 5} x = 5$
B. $\lim_{x \to 0.005} 0.005 = x$
C. $\lim_{x \to -10} -10 = -10$
D. $\lim_{5 \to x} x = 0$

15. Evaluate $\lim_{x \to \frac{1}{3}} (\frac{9x^2-1}{3x-1})$.

A. ∞ B. -∞ C. 2 D. 0

Answer Key

PRE-ASSESSMENT

1. D
2. B
3. A
4. A
5. C
6. D
7. C
8. C
9. D
10. B

Lesson 1.

What's More

Activity.

x	***f(x)***
0.5	3.25
0.7	3.09
0.95	3.0025
0.995	3.000025
0.9995	3.00000025
0.99995	3.0000000025

x	***f(x)***
1.6	3.36
1.35	3.1225
1.05	3.0025
1.005	3.000025
1.0005	3.00000025
1.00005	3.0000000025

Lesson 2.

What's More

Activity.

1. 12
2. 5
3. -10
4. 2
5. 50
6. $\frac{1}{4}$

Assessment

1. D
2. C
3. D
4. D
5. D
6. D
7. C
8. A
9. C
10. B
11. B
12. D
13. C
14. A
15. B

References

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