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SENIOR HIGH SCHOOL

DEPARTMENT OF EDUCATION
SCHOOLS DIVISION OF NEGROS ORIENTAL
REGION VII
Kagawasan Ave., Daro, Dumaguete City, Negros Oriental

BASIC CALCULUS

Quarter 3 – Module 8
Implicit Differentiation and Related Rates

ALTERNATIVE DELIVERY MODE
ADM

Basic Calculus – Grade 11
Alternative Delivery Mode
Quarter 3 – Module 8: Implicit Differentiation and Related Rates
Second Edition, 2021

Published by the Department of Education
Secretary: Leonor Magtolis Briones
Undersecretary: Diosdado M. San Antonio

Development Team of the Module

Writer: Duke Princeton D. Mariño

Editors: Ronald G. Tolentino & Gil S. Dael

Reviewer: Littie Beth S. Bernadez

Layout Artist: Radhiya A. Ababon

Management Team: Senen Priscillo P. Paulin CESO V, Elisa L. Bagiuo EdD, Joelyza M. Arcilla EdD, CESE, Rosela R. Abiera, Marcelo K. Palispis JD, EdD, Maricel S. Rasid, Nilita L. Ragay EdD, Elmar L. Cabrera

Printed in the Philippines by ________________________

Department of Education –Region VII Schools Division of Negros Oriental

Office Address: Kagawasan, Ave., Daro, Dumaguete City, Negros Oriental
Tel #: (035) 225 2376 / 541 1117
E-mail Address: negros.oriental@deped.gov.ph

Introductory Message

This Self-Learning Module (SLM) is prepared so that you, our dear learners, can continue your studies and learn while at home. Activities, questions, directions, exercises, and discussions are carefully stated for you to understand each lesson.

Each SLM is composed of different parts. Each part shall guide you step-by-step as you discover and understand the lesson prepared for you.

Pre-tests are provided to measure your prior knowledge on lessons in each SLM. This will tell you if you need to proceed on completing this module or if you need to ask your facilitator or your teacher's assistance for better understanding of the lesson. At the end of each module, you need to answer the post-test to self-check your learning. Answer keys are provided for each activity and test. We trust that you will be honest in using these.

In addition to the material in the main text, Notes to the Teacher are also provided to our facilitators and parents for strategies and reminders on how they can best help you on your home-based learning.

Please use this module with care. Do not put unnecessary marks on any part of this SLM. Use a separate sheet of paper in answering the exercises and tests. And read the instructions carefully before performing each task.

If you have any questions in using this SLM or any difficulty in answering the tasks in this module, do not hesitate to consult your teacher or facilitator.

Thank you.

What I Need to Know

This module was designed to provide you with fun and meaningful opportunities for guided and independent learning at your own pace and time. You will be enabled to process the contents of the learning resource while being an active learner.

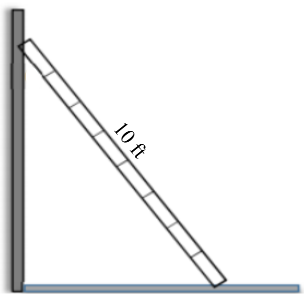
The module is intended for you to solve problems (including logarithmic, and inverse trigonometric functions) using implicit differentiation and situational problems involving related rates.

What I Know

Pretest

A. Find the derivative, $\frac{dy}{dx}$ of the following functions, if it exists.
   1. $y = \ln(7x^2 - 3x + 1)$
   2. $y = tan^{-1}(2x - 3cosx)$
   3. $y^2 - 4x^3 + xy = 0$

B. Solve the following problems.
   1. A 10-foot ladder is standing vertically against the side of a house. The base of the ladder is pulled away from the side of the house at the rate of 1 foot per second. How fast will the top of the ladder by falling down the side of the house 1 second after the base begins being pulled away from the house? After 3 seconds? After 8 seconds?



Source: https://che.gg/3Ghp6Do

   2. The width of a rectangle is increasing at a rate of 2 cm/sec, while the length increases at 3 cm/sec. At what rate is the area increasing when w = 4 cm and l = 5 cm (Pryor 2022)?

Lesson 1 Solving Problems Using Implicit Differentiation

What's In

The majority of differentiation problems in basic calculus involves functions *y* written explicitly as functions of the independent variable *x*. This means that we can write the function in the form *y = f(x)*. For such a function, we can find the derivative directly. For example, if

$$y = 4x^5 + cos\ (2x\ -\ 7),$$

then the derivative of *y* with respect to *x* is

$$\frac{dy}{dx} = 20x^4\ -\ 2sin\ (2x\ -\ 7).$$

However, some functions *y* are written implicitly as functions of *x*. This means that the expression is not given directly in the form *y = f(x).* A familiar example of this is the equation

$$x^2 + y^2 = 5,$$

which represents a circle of radius $\sqrt{5}$ with its center at the origin (0, 0). We can think of the circle as the union of the graphs of two functions, namely the function represented by the upper semi-circle and the function represented by the lower semi-circle. Suppose that we wish to find the slope of the line tangent to the circle at the point *(−2, 1).*

The solution is to find the the derivative at the point *(−2, 1*). Since the equation of the circle is not complicated, one way to do this is to write *y* explicitly in terms of *x*. Thus, from $x^2 + y^2 = 5$, we obtain $y = \pm\sqrt{5 - x^2}$. The positive square root represents the upper semi-circle while the negative square root represents the bottom semi-circle. Since the point *(−2, 1)* is on the upper semi-circle, the slope of the tangent line is now obtained by differentiating the function

$$y = \sqrt{5 - x^2} = (5 - x^2)^{\frac{1}{2}}$$

and evaluating the derivative at *x* = *−2*. Thus, using the Chain Rule,

$$\frac{dy}{dx} = \frac{1}{2}(5 - x^2)^{\frac{-1}{2}}(-2x) = \frac{-x}{\sqrt{5 - x^2}}$$

Therefore, the slope of the tangent line at the point *(−2, 1)* is the value of the above derivative evaluated at $x = -2$, namely

$$\frac{dy}{dx} = \frac{2}{\sqrt{5-(-2)^2}} = \frac{2}{\sqrt{1}} = 2$$

In the above example, we obtained the required derivative because we were able to write y explicitly in terms of the variable x. That is, we were able to transform the original equation into an equation of the form $y = f(x)$, with the variable y on one side of the equation, and the other side consisting of an expression in terms of x.

(Arceo 2016)

What's New

We have seen that functions are not always given in the form $y = f(x)$ but in a more complicated form that makes it difficult or impossible to express y explicitly in terms of x. Such functions are called implicit functions, and y is said to be defined implicitly. In this lesson, we explain how these can be differentiated using a method called implicit differentiation.

Differentiating quantities involving only the variable x with respect to x is not a problem; for instance, the derivative of x is just 1. But if a function y is defined implicitly, then we need to apply the Chain Rule in getting its derivative. So, while the derivative of x^2 is $2x$, the derivative of y^2 is

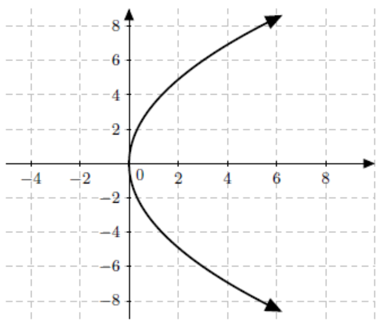
$$2y\frac{dy}{dx}.$$

More generally, if we have the expression f(y), where y is a function of x, then

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \cdot \frac{dy}{dx}.$$

In order to master implicit differentiation, students need to review and master the application of the Chain Rule.

Consider a simple expression such as $y^2 = 4x$. Its graph is a parabola with vertex at the origin and opening to the right.



If we consider only the upper branch of the parabola, then y becomes a function of x. We can obtain the derivative dy/dx by applying the Chain Rule. When differentiating terms involving y, we are actually applying the Chain Rule, that is, we first differentiate with respect to y, and then multiply by dy/dx.

Differentiating both sides with respect to x, we have

$$y^2 = 4x$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4x)$$

$$2y\frac{dy}{dx} = 4.$$

Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}.$$

Notice that the derivative contains y. This is typical in *implicit differentiation.*

(Arceo 2016)

What is It

Let us now apply implicit differentiation to some common problems.

Let us start with our original problem involving the circle finding the derivatives using implicit differentiation.

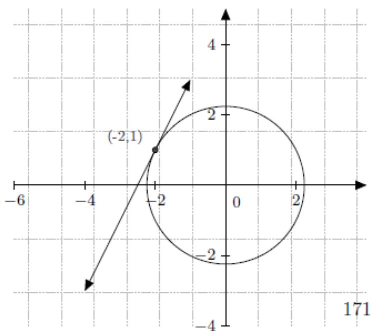
EXAMPLE 1: Find the slope of the tangent line to the circle $x^2 + y^2 = 5$ at the point $(-2, 1)$.
Solution.

$$x^2 + y^2 = 5$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(5)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$



Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

Substituting $x = -2$ and $y = 1$, we find that the slope is

$$\frac{dy}{dx} = 2.$$

EXAMPLE 2: Find $\frac{dy}{dx}$ *for* $y^3 + 4y^2 + 3x^2y + 10 = 0$.

Solution. Differentiating both sides of the equation gives

$$\frac{d}{dx}(y^3 + 4y^2 + 3x^2y + 10) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(y^3) + \frac{d}{dx}(4y^2) + \frac{d}{dx}(3x^2y) + \frac{d}{dx}(10) = 0$$

$$3y^2\frac{dy}{dx} + 8y\frac{dy}{dx} + 3x^2\frac{dy}{dx} + 6xy + 0 = 0$$

We collect the terms involving $\frac{dy}{dx}$ and rearrange to get

$$\frac{dy}{dx}(3y^2 + 8y + 3x^2) + 6xy = 0$$

Thus,

$$\frac{dy}{dx} = \frac{-6xy}{3y^2+8y+3x^2}.$$

Note that the derivative of the term $3x^2y$ is obtained by applying the Product Rule. We consider $3x^2$ as one function and y as another function.

Implicit differentiation can be applied to any kind of function, whether they are polynomial functions, or functions that involve trigonometric and exponential, quantities.

EXAMPLE 3: Find $\frac{dy}{dx}$ *for* $x + y^3 = e^{xy^4}$.

Solution. Differentiating both sides with respect to x gives

$$\frac{d}{dx}(x + y^3) = \frac{d}{dx}\left(e^{xy^4}\right)$$

$$1 + 3y^2\frac{dy}{dx} = e^{xy^4}\frac{d}{dx}(xy^4)$$

$$1 + 3y^2\frac{dy}{dx} = e^{xy^4}\left[x\frac{d}{dx}(y^4) + y^4\frac{d(x)}{dx}\right]$$

$$1 + 3y^2\frac{dy}{dx} = e^{xy^4}\left[x\left(4y^3\frac{dy}{dx}\right) + y^4(1)\right]$$

$$1 + 3y^2\frac{dy}{dx} = e^{xy^4}\left(4xy^3\frac{dy}{dx} + y^4\right)$$

$$1 + 3y^2\frac{dy}{dx} = \left(e^{xy^4}\right)(4xy^3)\frac{dy}{dx} + e^{xy^4}(y^4)$$

$$3y^2\frac{dy}{dx} - \left(e^{xy^4}\right)(4xy^3)\frac{dy}{dx} = e^{xy^4}y^4 - 1$$

$$\frac{dy}{dx}\left(3y^2 - 4e^{xy^4}xy^3\right) = e^{xy^4}y^4 - 1$$

$$\frac{dy}{dx} = \frac{e^{xy^4}y^4 - 1}{3y^2 - 4e^{xy^4}xy^3}$$

DERIVATIVES OF THE NATURAL LOGARITHMIC AND INVERSE TANGENT FUNCTIONS

The derivatives of some inverse functions can be found by implicit differentiation. Take, for example, the natural logarithmic function

$$y = lnx.$$

Note that it is the inverse function of the exponential function. To find $\frac{dy}{dx}$, we firstrewrite this into

$$e^y = x$$

and then differentiate implicitly:

$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$e^y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

However, from $e^y = x$. Hence after substituting this to $\frac{dy}{dx} = \frac{1}{e^y}$ we see that

$$y = ln\,x \longrightarrow \frac{dy}{dx} = \frac{1}{x}.$$

Now, we do a similar process as above to find the derivative of the inverse tangent function. Let's consider

$$y = tan^{-1}x$$

Applying the tangent function to both sides gives

$$tan\,y = x$$

Now, we apply implicit differentiation. So,

$$\frac{d}{dx}(tan\,y) = \frac{d}{dx}(x)$$

$$sec^2\,y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{sec^2\,y}$$

Next, we have to find a way so that by the use of $tany = x,$ we may be able to write $\frac{1}{sec^2 y}$ as a function of *x*. The relationship between *tan y* and *sec*^{2} *y* is straightforward from one of the trigonometric identities:

$$sec^2\, y - tan^2 y = 1$$

Therefore, *sec*^{2} *y = 1 + tan*^{2} *y = 1+x*^{2}. Finally, substituting this into *sec*^{2}*-tan*^{2}*y=1* we get the following result:

$$y = tan^{-1}\, x \longrightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

We summarize these two derivatives with consideration to Chain Rule.

Derivatives of the Natural Logarithmic and Inverse Tangent Functions

Suppose *u* is a function of *x*. Then

- $\frac{d}{dx}(ln\, u) = \frac{1}{u} \cdot \frac{du}{dx}$
- $\frac{d}{dx}(tan^{-1}\, u) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$

EXAMPLE 4: Find $\frac{dy}{dx}$ for cos $(y^2 - 3) = \tan^{-1}(x^3) + \ln y$.

Solution. Differentiating both sides gives

$$\frac{d}{dx}\big(cos\,(y^2-3)\big) = \frac{d}{dx}(tan^{-1}(x^3) + ln\, y)$$

$$-sin\,(y^2-3) \cdot 2y \cdot \frac{dy}{dx} = \frac{1}{1+(x^3)^2} \cdot 3x^2 + \frac{1}{y} \cdot \frac{dy}{dx}$$

Collecting terms with $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(-2y\, sin\,(y^2-3) - \frac{1}{y}\right) = \frac{3x^2}{1+x^6}$$

$$\frac{dy}{dx} = \frac{\frac{3x^2}{1+x^6}}{-2y\, sin\,(y^2-3) - \frac{1}{y}}$$

DERIVATIVES OF OTHER TRANSCENDENTAL FUNCTIONS

We have so far learned the derivatives of the following transcendental functions:

$a.\ \ \frac{d}{dx}(e^x) = e^x$ $b.\ \ \frac{d}{dx}(ln\, x) = \frac{1}{x}$ $c.\ \ \frac{d}{dx}(tan^{-1}\, x) = \frac{1}{1+x^2}$

We now explore the derivatives of b_x, $\log_b x$, and other inverse trigonometric functions.

I. Let $b > 0$, $b \neq 1$. To find the derivative of $y = b^x$, we first rewrite it in another form:

$$y = b^x = e^{\ln b^x} = e^{x \ln b}$$

Hence, using Chain Rule, $\frac{dy}{dx} = e^{x \ln b} \cdot \ln b$.

Therefore, using $y = b^x = e^{lnb^x} = e^{xlnb}$,

$$y = b^x \longrightarrow \frac{dy}{dx} = b^x \cdot \ln b$$

II. Let $b > 0$, $b \neq 1$. To find the derivative of

$$y = \log_b x$$

we may now use implicit differentiation. We first rewrite $y = \log_b x$, into

$$b^y = x$$

Therefore,

$$\frac{d}{dx}(b^y) = \frac{d}{dx}(x)$$
$$b^y \cdot \ln b \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{b^y \cdot \ln b}$$

Substituting ,

$$y = \log_b x \longrightarrow \frac{dy}{dx} = \frac{1}{x \ln b}$$

III. We only show how to find the derivative of $sin^{-1} x$. Let $x \in [-1, 1]$, and consider $y = sin^{-1} x$. Applying the sine function to both sides of the equation gives

$$\sin y = x$$

Implicitly differentiating $\sin y = x$, we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$
$$\cos y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$

We now find a way to express *cos y* in terms of *x* using equation *sin y = x*. However, we know from trigonometry that

$$\cos^2 y + \sin^2 y = 1$$
$$\cos y = \pm\sqrt{1 - \sin^2 y}$$

Recall that in Precalculus, the range of $y = sin^{-1}x$ has been restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (the fourth and first quadrants). Therefore, $cosy > 0$ and so we only choose $cosy = +\sqrt{1 - sin^2 y}$. Finally, substituting this into $\frac{dy}{dx} = \frac{1}{cosy}$ gives

$$y = \sin^{-1} x \quad \rightarrow \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

We summarize these three derivatives of the other inverse trigonometric functions:

Summary of Derivatives of Transcendental Functions

a. $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$

b. $\frac{d}{dx}(b^u) = b^u \cdot \ln b \ \frac{du}{dx}$

c. $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$

d. $\frac{d}{dx}(\log_b u) = \frac{1}{u \ln b}\frac{du}{dx}$

e. $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}\frac{du}{dx}$

f *x.* *f.* $\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}\frac{du}{dx}$

g. $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}\frac{du}{dx}$

h. $\frac{d}{dx}\left(\cot^{-1}(x)\right) = -\frac{1}{1+x^2}\frac{du}{dx}$

i. $\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{x\sqrt{x^2-1}}\frac{du}{dx}$

j. $\frac{d}{dx}\left(\csc^{-1}(x)\right) = -\frac{1}{x\sqrt{x^2-1}}\frac{du}{dx}$

EXAMPLE 5: Find $\frac{dy}{dx}$ of $3^y = \sec^{-1} x - \cos^{-1}(y^2 + 1)$.

Solution: We use implicit differentiation here.

$$3^y \ln 3 \frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}} - \left(-\frac{1}{\sqrt{1 - (y^2+1)^2}} \cdot 2y \frac{dy}{dx}\right)$$

$$\left(3^y \ln 3 - \frac{1}{\sqrt{1 - (y^2+1)^2}} \cdot 2y\right)\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x\sqrt{x^2 - 1}}}{3^y \ln 3 - \frac{1}{\sqrt{1 - (y^2+1)^2}} \cdot 2y}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x\sqrt{x^2 - 1}}}{3^y \ln 3 - \frac{2y}{\sqrt{1 - (y^2+1)^2}}}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x\sqrt{x^2 - 1}}}{\frac{3^y \ln 3\left(\sqrt{1 - (y^2+1)^2}\right) - 2y}{\sqrt{1 - (y^2+1)^2}}}$$

$$\frac{dy}{dx} = \frac{\sqrt{1 - (y^2+1)^2}}{\left(x\sqrt{x^2 - 1}\right)\left(3^y \ln 3\left(\sqrt{1 - (y^2+1)^2}\right) - 2y\right)}$$

(Arceo 2016)

What's More

Activity 1: Lets try it out…

- Find $\frac{dy}{dx}$ of $\ln(3xy) = x + x^5$

Lesson 2	Related Rates

What's In

Imagine a water droplet falling into a still pond, producing ripples that propagate away from the center. Ideally, the ripples are concentric circles which increase in radius (and also in area) as time goes on. Thus, the radius and area of a single ripple are changing at rates that are related to each other. This means that if we know how fast the radius is changing, we should be able to determine how fast the area is changing at any point in time, and vice versa (Arceo 2016).

The Ripple Effect | Northstar Church
northstar.church

In this topic, we shall find the rate at which a particular quantity is changing by to other quantities whose rates of change we know of.

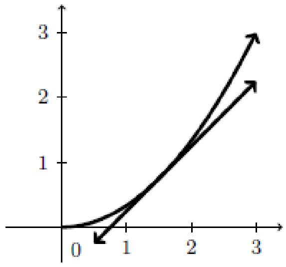
What's New

Introduction

This portion culminates the chapter on derivatives. The discussion on related rates concerns quantities which change (increase/decrease) with time, and which are related by an equation. Differentiating this equation with respect to time gives an equation of relationship between the rates of change of the quantities involved. Therefore, if we know the rates of change of all but one quantity, we are able to solve this using the aforementioned relationship between the rates of change.

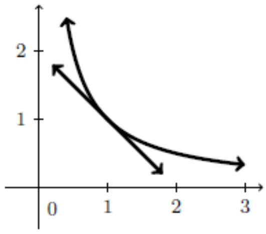
We first need to recall that aside from being the slope of the tangent line to a function at a point, the derivative is also interpreted as a rate of change. The sign of the derivative indicates whether the function is increasing or decreasing.

Suppose the graph of a differentiable function is increasing. This means that as *x* increases, the *y*-value also increases. Hence, its graph would typically start from the bottom left and increase to the top right of the frame. Refer to the figure below.



Observe that the tangent line to the graph at any point slants to the right and therefore, has a positive slope. This, in fact, describes increasing differentiable functions: A differentiable function is increasing on an interval if its derivative is positive on that interval.

Similarly, a differentiable function is decreasing on an interval if and only if its derivative is negative on that interval.



Remark

Let *x* be a differentiable function which represents a quantity that changes with time *t*, then

- $\frac{dx}{dt}$ is the rate of change of *x* with respect to *t*;
- $\frac{dx}{dt}$ is positive if and only if *x* increases with time; and
- $\frac{dx}{dt}$ is negative if and only if *x* decreases with time.

The unit of measurement of $\frac{dx}{dt}$ is $\frac{unit\ of\ measurement\ of\ x}{unit\ of\ measurement\ of\ t}$.

(Arceo 2016)

What is It

A related rates problem concerns the relationship among the rates of change of several variables with respect to time, given that each variable is also dependent on the others. In particular, if y is dependent on x, then the rate of change of y with respect to t is dependent on the rate of change of x with respect to t, that is, $\frac{dy}{dt}$ is dependent on $\frac{dx}{dt}$.

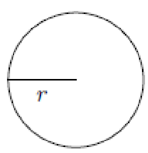
Suggestions in solving problems involving related rates:

1. If possible, provide an illustration for the problem that is valid for anytime t.
2. Identify those quantities that change with respect to time, and represent them with variables. (Avoid assigning variables to quantities which are constant, that is, which do not change with respect to time. Label them right away with the values provided in the problem.)
3. Write down any numerical facts known about the variables. Interpret each rate of change as the derivative of a variable with respect to time. Remember that if a quantity decreases over time, then its rate of change is negative.
4. Identify which rate of change is being asked, and under what particular conditions this rate is being computed.
5. Write an equation showing the relationship of all the variables by an equation that is valid for any time t.
6. Differentiate the equation in (5) implicitly with respect to t.
7. Substitute into the equation, obtained in (6), all values that are valid at the particular time of interest. Sometimes, some quantities still need to be solved by substituting the particular conditions written in (4) to the equation in (6). Then, solve for what is being asked in the problem.
8. Write a conclusion that answers the question of the problem. Do not forget to include the correct units of measurement.

Example 1: A water droplet falls onto a still pond and creates concentric circular ripples that propagate away from the center. Assuming that the area of a ripple is increasing at the rate of 2π cm^2/s, find the rate at which the radius is increasing at the instant when the radius is 10 cm.

Solution:

1. Illustrate



2. Let r and A be the radius and area, respectively, of a circular ripple at any time t.
3. The given rate of change is $\frac{dA}{dt} = 2\pi$.
4. We are asked to find $\frac{dr}{dt}$ at the instant when r = 10.

5. The relationship between A and r is given by the formula for the area of a circle:

$$A = \pi r^2$$

6. We now differentiate implicitly with respect to time

$$\frac{dA}{dt} = \pi(2r)\frac{dr}{dt}$$

7. Substituting

$$2\pi = \pi(2)(10)\frac{dr}{dt}$$

$$2\pi = 20\pi\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{2\pi}{20\pi}$$

$$\frac{dr}{dt} = \frac{1}{10}$$

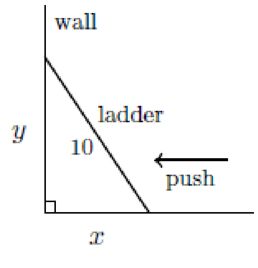
8. Conclusion:

Therefore, the radius of a circular ripple is increasing at the rate of $\frac{1}{10} cm/s$.

Example 2: A ladder 10 meters long is leaning against a wall. If the bottom of the ladder is being pushed horizontally towards the wall at 2 m/s, how fast is the top of the ladder moving when the bottom is 6 meters from the wall?

Solution:

We first illustrate the problem.



Let x be the distance between the bottom of the ladder and the wall. Let y be the distance between the top of the ladder and the ground (as shown). Note that the length of the ladder is not represented by a variable as it is constant.

We are given that $\frac{dx}{dt} = -2$ (Observe that this rate is negative since the quantity x decreases with time.)

We want to find $\frac{dy}{dt}$ at the instant when $x = 6$.

Observe the wall, the ground and the ladder determine a right triangle. Hence, the relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 100$$

Differentiating both sides with respect to time t gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Substituting $x = 6$

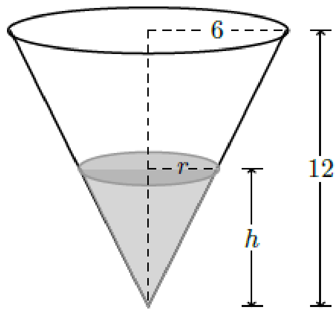
$$6^2 + y^2 = 100 \rightarrow y = 8$$
$$y^2 = 100 - 6^2$$
$$y^2 = 100 - 3$$
$$y^2 = 64$$
$$y = 8$$

Substituting all the values:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$2(6)(-2) + 2(8)\frac{dy}{dt} = 0$$
$$\frac{dy}{dt} = \frac{3}{2}$$
$$= 1.5m/s.$$

Example 3: Water is pouring into an inverted cone at the rate of 8 cubic meters per minute. If the height of the cone is 12 meters and the radius of its base is 6 meters, how fast is the water level rising when the water is 4-meter deep?

Solution:



We first illustrate the problem.

Let V be the volume of the water inside the cone at any time t. Let h, r be the height and radius, respectively, of the cone formed by the volume of water at any time t. We are given $\frac{dV}{dt} = 8$ and we wish to find $\frac{dh}{dt}$ when $h = 4$.

Now, the relationship between the three defined variables is given by the volume of the cone:

$$V = \frac{\pi}{3}r^2h$$

Observe that the rate of change of r is neither given nor asked. This prompts us to find a relationship between r and h. From the illustration, we see that by the proportionality relations in similar triangles, we obtain

$$\frac{r}{h} = \frac{6}{12} \longrightarrow r = \frac{h}{2}$$

Thus, expressing V as a function of h,

$$\begin{aligned} V &= \frac{\pi}{3} r^2 h \\ &= \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h \\ &= \frac{\pi}{12} h^3 \end{aligned}$$

Differentiating both sides with respect to t,

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{12} h^3\right)$$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

Thus, after substituting all given values, we obtain

$$8 = \frac{\pi}{4}(4)^2 \frac{dh}{dt}$$

$$\begin{aligned} \frac{dh}{dt} &= \frac{32}{16\pi} \\ &= \frac{2}{\pi}\ m/s \end{aligned}$$

Finally, we conclude that the water level inside the cone is rising at the rate of $\frac{2}{\pi} m/s$.

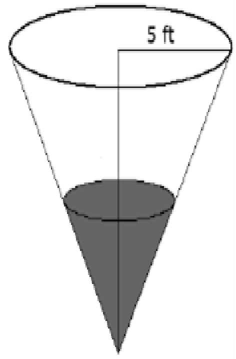
(Arceo 2016)

What's More

Activity 2: Lets try it out…

Solve the situational problem.

1. A tank, shaped like a cone shown on the picture, is being filled up with water. The top of the tank is a circle with radius 5ft, its height is 15 ft. Water is added to the tank at the rate of $V'(t) = 2\pi \frac{ft^3}{min}$. How fast is the water level rising when the water level is 6 ft high? (The volume of a cone with height h and base radius r is $V = \frac{\pi r^2 h}{3}$)



What I Have Learned

Activity 3: Lets reflect…

Write a short reflection in your notebook.

1. I have learned that______________________.
2. I will apply________________________.

What I Can Do

Activity 4: Lets create…

Create one situational problem using implicit differentiation and related rates, then show the process in solving that problem.

RUBRIC for Solving Problems

CATEGORY	5	4	3	2
Mathematical Concepts	Explanation shows complete understanding of the mathematical concepts used to solve the problem(s).	Explanation shows substantial understanding of the mathematical concepts used to solve the problem(s).	Explanation shows some understanding of the mathematical concepts needed to solve the problem(s).	Explanation shows very limited understanding of the underlying concepts needed to solve the problem(s) OR is not written.
Mathematical Errors	90-100% of the steps and solutions have no mathematical errors.	Almost all (85-89%) of the steps and solutions have no mathematical errors.	Most (75-84%) of the steps and solutions have no mathematical errors.	More than 75% of the steps and solutions have mathematical errors.
Neatness and Organization	The work is presented in a neat, clear, organized fashion that is easy to read.	The work is presented in a neat and organized fashion that is usually easy to read.	The work is presented in an organized fashion but may be hard to read at times.	The work appears sloppy and unorganized. It is hard to know what information goes together.
Completion	All problems are completed.	All but one of the problems are completed.	All but two of the problems are completed.	Several of the problems are not completed.

Assessment

Activity 5: Lets do this…

A. Find $\frac{dy}{dx}$ for the following:
   1. $y = \log_2 x$
   2. $y = 5^x$
   3. $y^3 - 6y + x^2 = 0$
   4. $y^2 - 8y + x^2 = 5$
   5. $y = \cos^{-1}(2x + 1)$

B. Find the equation of the tangent line to $x^2 - 3xy + y^2 = -1$ at (2,1)

C. Solve the following situational problems.

   1. An 8-foot ladder is leaning against a wall. The top of the ladder is sliding down the wall at the rate of 2 feet per second. How fast is the bottom of the ladder moving along the ground at the point in time when the bottom of the ladder is 4 feet from the wall?

   2. Water is being poured at the rate of $2\pi\ m^3/min.$ into an inverted conical tank that is 12-meter deep with a radius of 6 meters at the top. If the water level is rising at the rate of $\frac{1}{6}m/min$ and there is a leak at the bottom of the tank, how fast is the water leaking when the water is 6-meter deep?

Additional Activities

Activity 6: Challenge yourself…

A. Consider $xy^2 + x^2y = 6$.
   1. Find $\frac{dy}{dx}$.
   2. Find the slope of the tangent at the point (1,2).

B. Solve.
   1. A large red balloon is rising at the rate of 20 ft/sec. The balloon is 10 ft above the ground at the point in time that the back end of a green car is directly below the bottom of the balloon. The car is traveling at 40 ft/sec. What is the rate of change of the distance between the bottom of the balloon and the point on the ground directly below the back of the car one second after the back of the car is directly below the balloon?

Answer Key

Pretest

A. 1. $\frac{dy}{dx} = \frac{(14x-3)}{7x^2-3x+1}$ ·
2. $\frac{dy}{dx} = \frac{(2+3sinx)}{1+(2x - 3cos\,x)^2}$ ·
3. $\frac{dy}{dx} = \frac{12x^2 - y}{2y+x}$

B. 1. After 1 sec- the ladder is sliding down at approximately 0.101ft/sec or -0.101ft/sec
After 3 secs- the ladder is sliding down at approximately 0.314ft/sec or - 0.314ft/sec
After 8 secs- the ladder is sliding down at approximately 1.333ft/sec or -1.333ft/sec
2. The rate is increasing at a rate of $22\text{cm}^2/\text{sec}$

Lesson 1

What's More: Activity 1

1. $\frac{dy}{dx} = \frac{y(5x^5+x-1)}{x}$

Lesson 2

What's More: Activity 2

1. $\frac{1}{2}\,ft/min$

What I Have Learned: Activity 3

(Answers may vary)

What I Can Do: Activity 4

(Answers may vary)

Assessment

A. 1. $\frac{dy}{dx} = \frac{1}{x\,ln\,2}$
2. $\frac{dy}{dx} = 5^x \cdot ln\,5$
3. $\frac{dy}{dx} = -\frac{2x}{3(y^2-2)}$
4. $\frac{dy}{dx} = \frac{-x}{y-4}$
5. $\frac{dy}{dx} = \frac{-2}{\sqrt{1-(2x+1)^2}}$

B. $x - 4y + 2 = 0$

C. 1. Approximately 3.464ft/sec
2. $\frac{\pi}{2}\frac{m^3}{min}$

Additional Activities

A. 1. $\frac{dy}{dx} = -\frac{y^2+2xy}{2xy+x^2}$
2. $-\frac{8}{5}$

B. 1. 44 ft/sec

References

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For inquiries or feedback, please write or call:

Department of Education – Schools Division of Negros Oriental
Kagawasan, Avenue, Daro, Dumaguete City, Negros Oriental

Tel #: (035) 225 2376 / 541 1117
Email Address: negros.oriental@deped.gov.ph
Website: lrmds.depednodis.net