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SENIOR HIGH SCHOOL

DEPARTMENT OF EDUCATION
SCHOOLS DIVISION OF NEGROS ORIENTAL
REGION VII
Kagawasan Ave., Daro, Dumaguete City, Negros Oriental

BASIC CALCULUS

Quarter 3 – Module 4
Introduction to Derivatives

ALTERNATIVE DELIVERY MODE

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Secretary: Leonor Magtolis Briones
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Development Team of the Module

Writer: Gil S. Dael

Editor: Ronald G. Tolentino

Reviewer: Littie Beth S. Bernadez

Layout Artist: Radhiya A. Ababon

Management Team:	Senen Priscillo P. Paulin CESO V	Elisa L. Bagiuo EdD
	Joelyza M. Arcilla EdD, CESE	Rosela R. Abiera
	Marcelo K. Palispis JD, EdD	Maricel S. Rasid
	Nilita L. Ragay EdD	Elmar L. Cabrera

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Department of Education –Region VII Schools Division of Negros Oriental

Office Address: Kagawasan, Ave., Daro, Dumaguete City, Negros Oriental
Tel #: (035) 225 2376 / 541 1117
E-mail Address: negros.oriental@deped.gov.ph

Introductory Message

This Self-Learning Module (SLM) is prepared so that you, our dear learners, can continue your studies and learn while at home. Activities, questions, directions, exercises, and discussions are carefully stated for you to understand each lesson.

Each SLM is composed of different parts. Each part shall guide you step-by-step as you discover and understand the lesson prepared for you.

Pre-tests are provided to measure your prior knowledge on lessons in each SLM. This will tell you if you need to proceed on completing this module or if you need to ask your facilitator or your teacher's assistance for better understanding of the lesson. At the end of each module, you need to answer the post-test to self-check your learning. Answer keys are provided for each activity and test. We trust that you will be honest in using these.

In addition to the material in the main text, Notes to the Teacher are also provided to our facilitators and parents for strategies and reminders on how they can best help you on your home-based learning.

Please use this module with care. Do not put unnecessary marks on any part of this SLM. Use a separate sheet of paper in answering the exercises and tests. And read the instructions carefully before performing each task.

If you have any questions in using this SLM or any difficulty in answering the tasks in this module, do not hesitate to consult your teacher or facilitator.

Thank you.

What I Need to Know

This module was designed to provide you with fun and meaningful opportunities for guided and independent learning at your own pace and time. You will be enabled to process the contents of the learning resource while being an active learner.

The module is intended for you to illustrate the tangent line to the graph of a function at a given point; apply the definition of the derivative of a function at a given number; and relate the derivative of a function to the slope of the tangent line.

What I Know

The tangent problem is one of the problems that gave the growth of Calculus. This problem involves limits which had been discussed in the prior modules. The special type of limit we shall devote this module to is called the derivative.

To give you an introduction to the concept of derivative, we shall do so by finding the tangent line to a curve at a point. After which, the formal definition of the concept of derivative will be given.

PRE-ASSESSMENT:

I. Match the statements given in Column I with those equations given in Column II. Write the letter of your answer in your activity sheet/notebook.

Column I

1. An equation of the line through $(2, -3)$ with slope 6 is ____
2. An equation of the line through $(1,7)$ with slope $\frac{2}{3}$ is _____
3. An equation of the line through $(2,1)$ and $(1,6)$ is _____
4. An equation of the line with slope 3 and y-intercept -2 is ___
5. An equation of the line with x-intercept 1 and y-intercept of -3 is ____

Column II

A. $y = \frac{2}{3}x + \frac{19}{3}$

B. $y = \frac{3}{4}x - 3$

C. $y = 3x - 3$

D. $y = -5x + 11$

E. $y = 3x - 2$

F. $y = 6x - 15$

II. Choose the letter of the correct answer.

6. Find the distance between $P(1,1)$ and $Q(4,5)$. Call it $|PQ|$.

A. $|PQ|$ =5 B. $|PQ|$ =6 C. $|PQ|$ =4 D. $|PQ|$ =7

7. Find the distance between $R(6,-2)$ and $S(-1,3)$. Call it $|RS|$.

A. $|RS| = \sqrt{71}$ B. $|RS| = \sqrt{72}$ C. $|RS| = \sqrt{73}$ D. $|RS| = \sqrt{74}$

8. Find the distance between $U(2,5)$ and $V(4,-7)$. Call it $|UV|$.

A. $|UV| = 2\sqrt{37}$ B. $|UV| = 3\sqrt{37}$ C. $|UV| = 2\sqrt{33}$ D. $|UV| = 4\sqrt{33}$

9. Using items 6 and 8, compare: $|PQ|$ ___$|RS|$.

A. $<$ B. $>$ C. $=$ D. $\neq$

10. Using items 6 and 8, compare: $|UV|$ ___$|PQ|$.

A. $<$ B. $>$ C. $=$ D. $\neq$

What's In

The Tangent Problem

How should we define the tangent line to a curve at its point?

In a circle, a tangent is a line that intersects it at one point; while a secant is a line that intersects it at exactly at two points (Alexander and Koeberlein 2014, 288-289). Figure 1 shows a tangent line, f, to a circle.

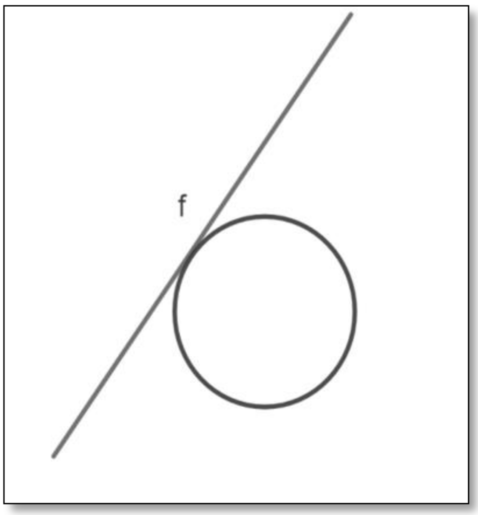


Figure 1

At this point, let us first consider Figure 2 with lines l, m, and n passing through a point P on a curve f. These lines are tangent at apoint of a curve.

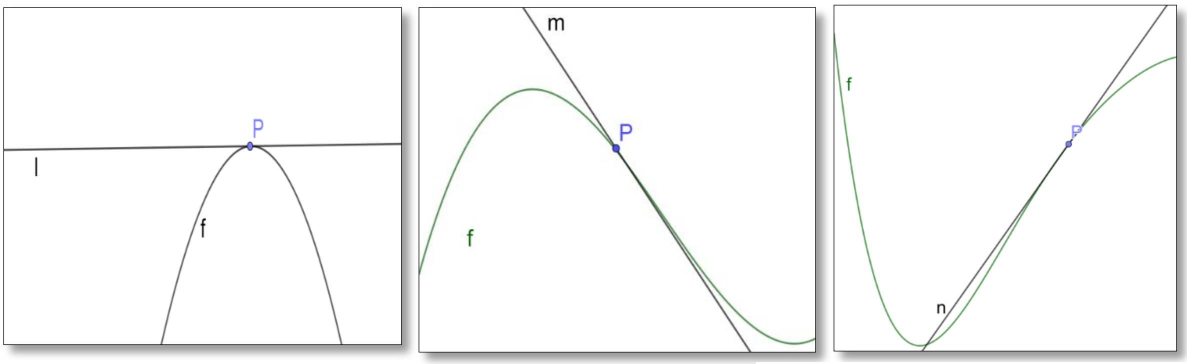


Figure 2.a **Figure 2.b** **Figure 2.c**

Figure 2 Tangent line at a point of a curve

Given these three figures, how should we define a tangent line to a curve at its point? We might do it by considering the definition of the tangent line to a circle. If we pattern the definition of tangent line to a curve at a point to that of a circle's, then we can consider the following as probable definitions of a tangent line to a curve (Larson and Edwards 2010, 96):

a. A tangent line to a curve at a point of the curve touches, but does not cross the curve at the point of intersection; or
b. A tangent line to a curve at a point of the curve intersects the curve at exactly one point.

TASK 1

If we use these as definitions of a tangent line to a curve at a point, then some of the tangent lines in Figure 2 will fail to become tangent lines.

1. Can you identify which among the lines ($l, m,$ or n) fails to satisfy the said definitions?
2. Can you give your own definition of the tangent line to a curve so that each lines in Figure 2 will all satisfy?

What's New

What should be clear to you by now is the fact that when the definition of a tangent line to a circle is used for complicated curves, it may be lacking.

For us to have a concrete discussion let us consider the following problem: Suppose we want the equation of the tangent line to the curve $y = x^2$ at the point $P(2,4)$.

If we know its slope m, then we can easily solve for the equation of the tangent line. What we shall do, then, is to approximate the value of m by choosing a nearby point $Q(x, x^2)$ on the curve and compute the slope $m_{\overline{PQ}}$ of the secant line $\overline{PQ}$. Thus,

$$m_{\overline{PQ}} = \frac{y_2-y_1}{x_2-x_1} = \frac{x^2-4}{x-2}.$$

As an example, if we take $Q(3,9)$, then by the equation above,

$$m_{\overline{PQ}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{x^2 - 4}{x - 2} = \frac{9 - 4}{3 - 2} = \frac{5}{1} = 5.$$

With $P(2,4)$ and $m_{\overline{PQ}} = 5$ and using the point-slope form, the equation of the secant line is

$$y - y_1 = m(x - x_1)$$
$$y - 4 = 5(x - 2)$$
$$y = 5(x - 2) + 4.$$

If graphed, we can easily see that this secant line does not look very much like a tangent line (See Figure 3).

Let us refine the procedure by taking points that are closer to the point of tangency $P(2,4)$. Do this by summarizing your calculations in the table below.

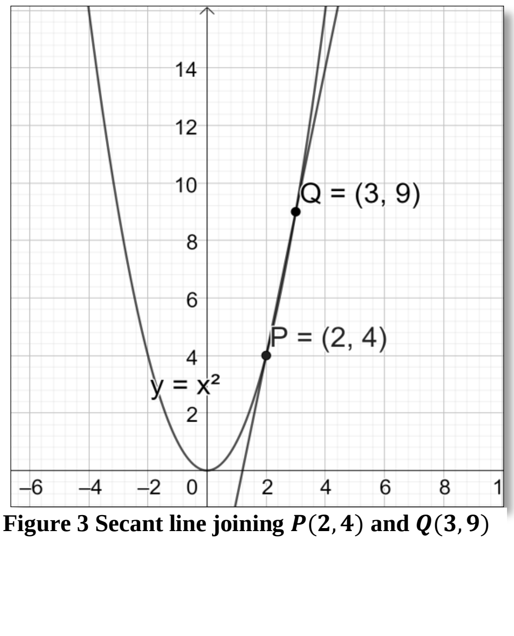


Figure 3 Secant line joining $P(2,4)$ and $Q(3,9)$

TASK 2

A. Complete the table below by solving for slope $m_{\overline{PQ}}$ for $P(2,4)$ and for the given coordinates of Q. Then, find the equation of the secant line.

Q	$m_{\overline{PQ}}$	Equation of the secant line
$(3,9)$	5	$y = 5(x-2)+4$
$(2.1, 4.41)$		
$(2.01, 4.0401)$		
$(2.001, 4.004001)$		

What is It

Suppose we continue the procedure as we have done in Task 2. Only that, by this time, we will pick an unspecified point Q to be in this form

$$\big(1+h, f(1+h)\big) = (1+h, (1+h)^2)$$

for some values of h close to 1 (but $h \neq 1$). We choose $h \neq 1$ so that $Q \neq P$.

Because if $h = 1$, then Q becomes

$$\big(1+h, f(1+h)\big) = (1+h, (1+h)^2) = (1+1, (1+1)^2) = (2,4)$$

which is just point P. It must be that $h \neq 1$.

With that, the slope of the secant line is:

$$\begin{aligned} m_{\overline{PQ}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(1+h) - 4}{(1+h) - 2} \\ &= \frac{(1+h)^2 - 4}{h-1} \\ &= \frac{(1+2h+h^2) - 4}{h-1} \\ &= \frac{h^2 + 2h - 3}{h-1} \\ &= \frac{(h+3)(h-1)}{h-1} \\ &= h+3. \end{aligned}$$

(Note: Since $h \neq 1$, then the cancellation for it in the second to the last line is valid since $h - 1 \neq 0$)

Notice that as h approaches to 1, the slope of the secant line approaches to 4, which we define as the slope of the tangent line.

General Case

Let us now generalize the procedure we have done above.

Suppose we have a curve C defined by the equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$. We find so by considering a nearby point $Q(x, f(x))$, where $x \neq a$, and solve for the slope of the secant line $\overline{PQ}$:

$$m_{\overline{PQ}} = \frac{f(x) - f(a)}{x - a}.$$

Then, we let Q approach P along the curve C, by letting x approach a. If $m_{\overline{PQ}}$ approaches a number, say m, then we can define the tangent t to be the line through P with slope m.

The formal definition is given below.

Definition 1. (Stewart 2016, 106)

The ***tangent line*** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line trough P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example 1. Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(2,4)$.

Solution:

By Definition 1, with $a = 2$ and $f(x) = x^2$, the slope is

$$\begin{aligned} m &= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x+2)(x-2)}{x - 2} \\ &= \lim_{x \to 2} (x + 2) \\ &= 2 + 2 \\ &= 4. \end{aligned}$$

Using the point-slope form of the equation of a line, the equation of the tangent line at $(2,4)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 4 = 4(x - 2)$$

$$y = 4(x - 2) + 4$$

$$y = 4x - 8 + 4$$

$$y = 4x - 4$$

$$y = 4(x - 1).$$

Let us consider an example employing this formula.

Example 2. Find an equation of the tangent line to the hyperbola $y = \frac{3}{x}$ at the point $(3,1)$.

Solution:

Let $f(x) = \frac{3}{x}$. Then, by Equation 1, the slope of the tangent line at $(3,1)$ is

$$
\begin{aligned}
m &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{\frac{3}{3+h} - 1}{h} \\
&= \lim_{h \to 0} \frac{\frac{3-(3+h)}{3+h}}{h} \\
&= \lim_{h \to 0} \frac{\frac{3-3-h}{3+h}}{h} \\
&= \lim_{h \to 0} \frac{\frac{-h}{3+h}}{h} \\
&= \lim_{h \to 0} \frac{-h}{h(3+h)} \\
&= \lim_{h \to 0} \frac{-1}{3+h} \\
&= \frac{\lim_{h \to 0} -1}{\lim_{h \to 0} 3+h} \\
&= \frac{-1}{3+0} \\
&= \frac{-1}{3}.
\end{aligned}
$$

Thus, at point $(1,3)$, the equation of the tangent line is $y - 1 = -\frac{1}{3}(x - 3)$.

Simplifying we get,

$$y - 1 = -\frac{1}{3}(x - 3)$$
$$3(y - 1) = -(x - 3)$$
$$3y - 3 = -x + 3$$
$$x + 3y - 3 - 3 = 0$$
$$x + 3y - 6 = 0.$$

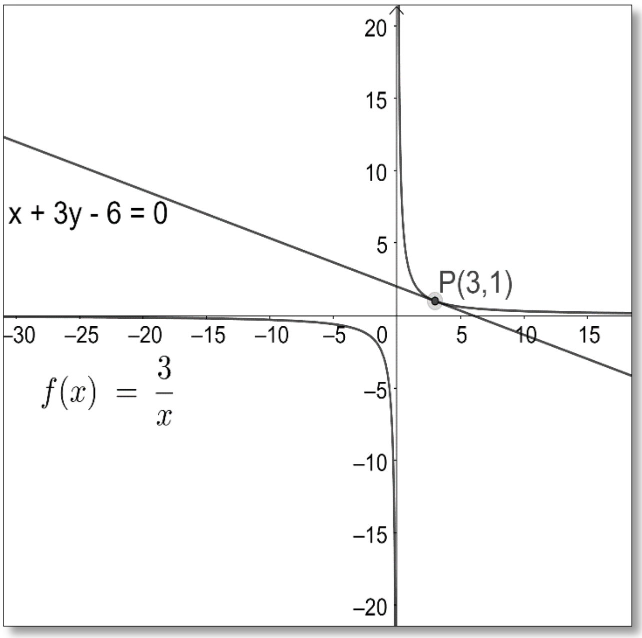


Figure 4 shows the hyperbola and its tangent.

Figure 4 . The hyperbola and its tangent

Derivative

The type of limit in the form of

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

occurs when we calculate a rate of change in any of the science or engineering. Since this type of limit occurs so widely, there is given a special name and notation. This is introduced below.

Definition 2. (Stewart 2016, 109)

The ***derivative of a function*** f at a number a, denoted by $f'(a)$ (read as "f prime of a"), is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If we write $x = a + h$, then, we have, $h = x - a$. Then, h approaches 0 if and only if x approaches a. Thus, we can alternatively state the definition of the derivative as:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \qquad \text{(Equation 2)}$$

Example 3. Find the derivative of the function $f(x) = x^2 - 4x + 5$ at the number a.

Solution:

From Definition 2,

$$\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{[(a+h)^2 - 4(a+h) + 5] - [a^2 - 4a + 5]}{h} \\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 4a - 4h + 5 - a^2 + 4a - 5}{h} \\
&= \lim_{h \to 0} \frac{2ah + h^2 - 4h}{h} \\
&= \lim_{h \to 0} \frac{h(2a + h - 4)}{h} \\
&= \lim_{h \to 0} 2a + h - 4 \\
&= \lim_{h \to 0} 2a + \lim_{h \to 0} h - \lim_{h \to 0} 4 \\
&= 2a + 0 - 4 \\
&= 2a - 4.
\end{aligned}$$

We have defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has the slope m given by the equation in Definition 1 or Equation 1. This is the same as the derivative $f'(a)$ as given by Definition 2. Thus, we can state the following (Stewart 2016, 110):

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative f at a.

Using the point-slope form of the equation of the line, the equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is:

$$y - f(a) = f'(a)(x - a) \qquad \textit{(Equation 3)}$$

Example 4. Find an equation of the tangent line to the parabola $y = x^2 - 4x + 5$ at the point $(2,1)$.

Solution:

The derivative of $f(x) = x^2 - 4x + 5$ at the number a is

$$f'(a) = 2a - 4,$$

from Example 3. At $(2,1)$ $[where\ \ a = 2\ and\ f(a) = 1]$,

$$f'(2) = 4 - 4 = 0.$$

From Equation 3, the equation of the tangent line at $(2,1)$ is $y - 1 = 0(x - 2)$ or, by simplifying, $y = 1$.

What's More

TASK 3: Directions: Answer the following exercises. Show complete solution for full credit.

Exercise 1. A curve has an equation of $y = f(x)$

a. Write an expression for the slope of the secant line through the point $P(3, f(3))$ and $Q(x, f(x))$
b. Write an expression for the slope of the tangent line at P.

Exercise 2. Find the slope of the tangent line to the parabola $y = 3x - x^2$ at the point (3, 0)

a. Using Definition 1
b. Using Equation 1

Exercise 3. Find $f'(a)$ for $f(x) = x^{-2}$.

TASK 4: Answer the following exercises.

1. Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.
2. Find the equation of the tangent line at the point $(1,5)$.
3. Graph the curve and the tangent line.

What I Have Learned

TASK 5

In your answer sheet, explain the relationship between the slope of a tangent line and the value of the derivative at a point.

What I Can Do

TASK 6

If an equation of the tangent line to the curve $y = f(x)$ at the point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

Rubric for Problem Solving (adapted from rubistar.4teachers.com and

CATEGORY	5	4	3	2
Mathematical Concepts	Explanation shows complete understanding of the mathematical concepts used to solve the problem(s).	Explanation shows substantial understanding of the mathematical concepts used to solve the problem(s).	Explanation shows some understanding of the mathematical concepts needed to solve the problem(s).	Explanation shows very limited understanding of the underlying concepts needed to solve the problem(s) OR is not written.
Mathematical Errors	90-100% of the steps and solutions have no mathematical errors.	Almost all (85-89%) of the steps and solutions have no mathematical errors.	Most (75-84%) of the steps and solutions have no mathematical errors.	More than 75% of the steps and solutions have mathematical errors.
Neatness and Organization	The work is presented in a neat, clear, organized fashion that is easy to read.	The work is presented in a neat and organized fashion that is usually easy to read.	The work is presented in an organized fashion but may be hard to read at times.	The work appears sloppy and unorganized. It is hard to know what information goes together.
Completion	All problems are completed.	All but one of the problems are completed.	All but two of the problems are completed.	Several of the problems are not completed.

Assessment

I. Directions: Match the given function and point in Column A with the corresponding derivative at that point, $f'(a)$ in Column B.

Column A Function $f(x)$ and Point a	Column B Derivative $f'(a)$
(1) $f(x) = 8x;\ a = -3$	(A) -4
(2) $f(x) = x^2;\ a = 3$	(B) 8
(3) $f(x) = 4x^2 + 2x;\ a = -2$	(C) 6
(4) $f(x) = \frac{1}{\sqrt{x}};\ a = {}^1\!/_4$	(D) -2
(5) $f(x) = \frac{1}{x^2};\ a = 1$	(E) 4
	(F) -14

II. Directions: Match the given function and point in Column P with the corresponding equation of the tangent line to it at that point in Column R.

Column P Function $f(x)$ and Point a	Column R Equation of the tangent line to $f(x)$ and Point a
(6) $f(x) = 8x;\ a = -3$	(G) $y = x - 3$
(7) $f(x) = x^2;\ a = 3$	(H) $y = -4x + 3$
(8) $f(x) = 4x^2 + 2x;\ a = -2$	(I) $y = -2x + 3$
(9) $f(x) = \frac{1}{\sqrt{x}};\ a = {}^1\!/_4$	(J) $y = 8x$
(10) $f(x) = \frac{1}{x^2};\ a = 1$	(K) $\mathrm{y = 3(2x - 3)}$
	(L) $y = -2(7x + 8)$

Additional Activities

A Derivative Formula

Let $f(x) = ax^2 + bx + c$, where $a, b,$ and c are constants.

(a) Find $f'(x)$.
(b) Use the result in part (a) to find $f'(x)$ if $f(x) = 4x^2 - 3x + 10$.

Answer Key

WHAT I KNOW

Pre-Assessment

I. 1. F
2. A
3. D
4. E
5. C

II. 6. A
7. D
8. A
9. A
10. B

WHAT'S IN

TASK 1

1. Figure 2b does not satisfy the first definition. Figure 2c does not satisfy the second definition.
2. Answers may vary.

WHAT'S NEW

TASK 2

Q	$m_{\overline{PQ}}$	Equation of the secant line
(3,9)	5	$y = 5(x-2)+4$
(2.1, 4.41)	4.1	$y = 4.1(x-2)+4$
(2.01, 4.0401)	4.01	$y = 4.01(x-2)+4$
(2.001, 4.004001)	4.001	$y = 4.001(x-2)+4$

WHAT'S MORE

TASK 3

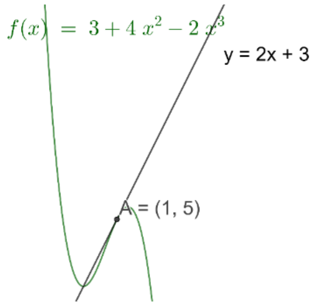
Exercise 1. a. $m_{\overline{PQ}} = \frac{f(x)-f(3)}{x-3}$ b. $m = \lim_{x\to 3}\frac{f(x)-f(3)}{x-3}$

Exercise 2. a. $m = \lim_{x\to 3}\frac{(3x-x^2)-0}{x-3} = -3$ b. $m = \lim_{h\to 0}\frac{[3(a+h)-(a+h)^2]-(3a-3a^2)}{h} = -3$

Exercise 3. $f'(a) = -\frac{2}{a^3}$

TASK 4

1. a. $m = 8a - 6a^2$
b. $m = 2$ c.



TASK 5

They are the same.

TASK 6

$f(2) = 3, \quad f'(2) = 3$

ASSESSMENT

I. 1. B
2. C
3. F
4. A
5. D

II. 6. J
7. K
8. L
9. H
10. I

ASSESSMENT

(a) $f'(x) = 2ax + b$
(b) $f'(x) = 8x - 3$

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For inquiries or feedback, please write or call:

Department of Education – Schools Division of Negros Oriental
Kagawasan, Avenue, Daro, Dumaguete City, Negros Oriental

Tel #: (035) 225 2376 / 541 1117
Email Address: negros.oriental@deped.gov.ph
Website: lrmds.depednodis.net