Modern Design of Control Systems

Notes and Summary

Gregorio Valenti

Polytechnic of Turin, September 2024 - February 2025

1 Feedback Control System Definition

For this course, we will consider the model below to represent the system under study:

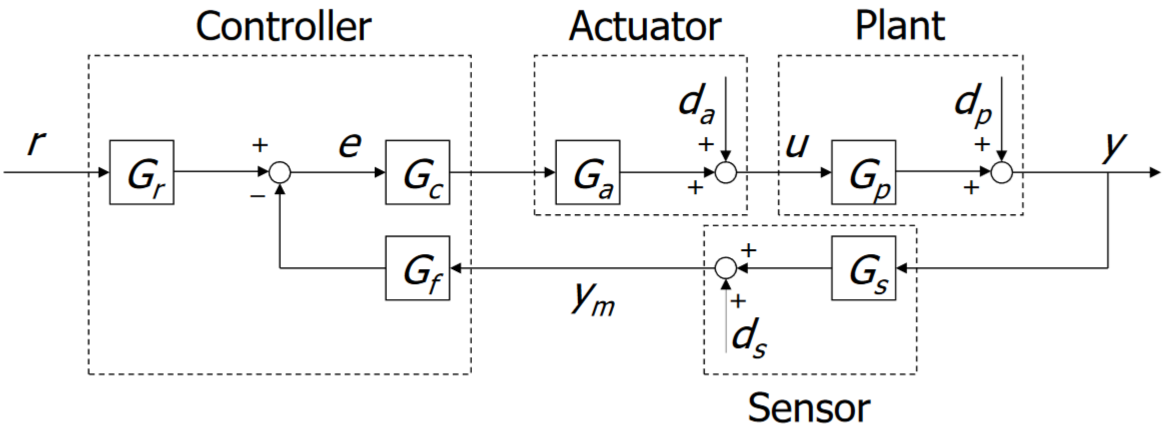


Figure 1: Feedback Control System

With the following characteristics:

- **Plant** G_p with disturbance d_p
- **Actuator** G_a with disturbance d_a
- **Sensor** G_s with disturbance d_s
- **Feedback Controller** G_c

2 Step response of prototype 2nd order system

For a prototype second order system with **Sensitivity** and **Complementary sensitivity** functions of the form:

$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}} \quad ; \quad S(s) = \frac{s\left(\frac{2\zeta}{\omega_n} + \frac{s}{\omega_n^2}\right)}{1 + \frac{2\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}} \tag{2.0.1}$$

From the transient response indices for maximum overshoot $\hat{s}$, rise time t_r and settling time $t_{s,\alpha\%}$, we can compute the system's damping coefficient, as well as constraints on the natural frequency and resonance peaks T_p and S_p:

- $$\zeta = \frac{|\log(\hat{s})|}{\sqrt{\pi^2 + \log^2(\hat{s})}} \tag{2.0.2}$$

- $$\omega_{n,tr} = \frac{1}{t_r\sqrt{1-\zeta^2}} \cdot (\pi - acos(\zeta)) \quad ; \quad \omega_{n,t,s\alpha\%} = \frac{\log(\frac{100}{\alpha})}{t_{s,\alpha\%}\zeta} \tag{2.0.3}$$

- $$\omega_c = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2} \tag{2.0.4}$$

- $$T_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad ; \quad S_p = \frac{2\zeta\sqrt{2+4\zeta^2+2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2}+4\zeta^2-1} \tag{2.0.5}$$

3 Steady-state Response to Polynomial Reference Inputs

Requirements on steady-state tracking error and steady-state errors in the presence of polynomial disturbances can be translated into constraints on the Sensitivity function. Therefore, by first writing $S(s) = s^{\nu+p}S^*(s)$, we can apply the final value theorem to each expression as described in the following subsections.

3.1 Tracking Error

3.1.1 Problem formulation

The **tracking error** is defined as:

$$e_r(t) = y_d(t) - y_r(t) = K_d r(t) - y_r(t) \tag{3.1.1}$$

Applying the **final value theorem** leads to:

$$\begin{aligned} |e_r^\infty| &= \lim_{t\to\infty} |e_r(t)| = \lim_{s\to 0} s\,|e_r(s)| = \lim_{s\to 0} s\,|K_d r(s) - y_r(s)| = \\ &= \lim_{s\to 0} s\,|G_{re}(s) r(s)| = \lim_{s\to 0} s\,|S(s) K_d r(s)| = \\ &= \lim_{s\to 0} s\left|s^{\nu+p} S^*(s) K_d \frac{h! R_0}{s^{h+1}}\right| = \begin{cases} 0, & \text{if } \nu + p > h \\ |S^*(0) K_d h! R_0|, & \text{if } \nu + p = h \end{cases} \end{aligned} \tag{3.1.2}$$

The given system type is then $\nu + p$.

3.1.2 Conclusions

Considering the specification $|e_r^\infty| \leq \rho_r$, we obtain the following result:

- case $\rho_r = 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity greater than $h \implies$ We have no constraint on $|S^*(0)|$
- case $\rho_r > 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity $h \implies$ We have the following constraint:
$$|S^*(0)| \leq \frac{\rho_r}{K_d h! R_0} \tag{3.1.3}$$

3.2 Error due to Generic Polynomial Disturbances $d(t)$

3.2.1 Problem formulation

The **output error** due to the generic disturbance $d(t)$ is the contribution of the disturbance to the output $y(t)$:

$$e_d(t) = y_d(t) \tag{3.2.1}$$

3.2.2 Polynomial disturbance $d_a(t)$

Applying the **final value theorem** leads to:

$$\begin{aligned} \left|e_{d_a}^\infty\right| &= \lim_{t\to\infty} |e_{d_a}(t)| = \lim_{s\to 0} s\,|e_{d_a}(s)| = \lim_{s\to 0} s\,|y_{d_a}(s)| = \lim_{s\to 0} s\,|S(s) G_p(s) d_a(s)| = \\ &= \lim_{s\to 0}\left|s^{\nu+p+1} S^*(s) G_p(s) \frac{h! D_{a0}}{s^{h+1}}\right| = \lim_{s\to 0}\left|s^{\nu+1} S^*(s) K_p \frac{h! D_{a0}}{s^{h+1}}\right| = \begin{cases} 0, & \text{if } \nu > h \\ |S^*(0) K_p h! D_{a0}|, & \text{if } \nu = h \end{cases} \end{aligned} \tag{3.2.2}$$

We then obtain two cases as before:

- case $\rho_a = 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity greater than $h \implies$ We have no constraint on $|S^*(0)|$
- case $\rho_a > 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity $h \implies$ We have the following constraint:
$$|S^*(0)| \leq \frac{\rho_a}{K_p h! D_{a0}} \tag{3.2.3}$$

3.2.3 Polynomial disturbance $d_p(t)$

Applying the **final value theorem** leads to:

$$\begin{aligned} \left|e_{d_p}^\infty\right| &= \lim_{t\to\infty} \left|e_{d_p}(t)\right| = \lim_{s\to 0} s\left|e_{d_p}(s)\right| = \lim_{s\to 0} s\left|y_{d_p}(s)\right| = \lim_{s\to 0} s\,|S(s) d_p(s)| = \\ &= \lim_{s\to 0}\left|s^{\nu+p+1} S^*(s) \frac{h! D_{p0}}{s^{h+1}}\right| = \begin{cases} 0, & \text{if } \nu + p > h \\ |S^*(0) h! D_{p0}|, & \text{if } \nu + p = h \end{cases} \end{aligned} \tag{3.2.4}$$

We then obtain two cases as before:

- case $\rho_p = 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity greater than $h \implies$ We have no constraint on $|S^*(0)|$
- case $\rho_p > 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity $h \implies$ We have the following constraint:
$$|S^*(0)| \leq \frac{\rho_p}{h!D_{p0}} \tag{3.2.5}$$

4 Steady-state Response to Sinusoidal Disturbances

4.1 Output disturbance $d_p(t)$

The focus is on the class of sinusoidal signals of the following type:

$$d_p = a_p \sin(\omega_p t) \quad \forall \omega_p \leq \omega_p^+ \quad \text{given } a_p \text{ and } w_p^+ \tag{4.1.1}$$

The output at steady-state is required to be bounded by a given constant:

$$\left|e_{d_p}^{\infty}\right| = \left|y_{d_p}^{\infty}\right| \leq \rho_p \quad \text{with } \rho_p > 0 \tag{4.1.2}$$

The specification leads to a **frequency domain constraint** on the Sensitivity function $S(s)$ which can be computed as follows:

$$\begin{aligned}\left|e_{d_p}^{\infty}\right| = \left|y_{d_p}^{\infty}\right| &= |a_p S(j\omega_p)\sin(\omega_p t + \varphi_p)| \leq a_p\,|S(j\omega_p)| \leq \rho_p \\ &\implies |S(j\omega_p)| \leq \frac{\rho_p}{a_p} = M_S^{LF} \quad \forall \omega_p \leq \omega_p^+\end{aligned} \tag{4.1.3}$$

4.2 Sensor Noise $d_s(t)$

The focus is on the class of sinusoidal signals of the following type:

$$d_s = a_s \sin(\omega_s t) \quad \forall \omega_s \geq \omega_s^- \quad \text{given } a_s \text{ and } w_s^- \tag{4.2.1}$$

The output at steady-state is required to be bounded by a given constant:

$$\left|e_{d_s}^{\infty}\right| = \left|y_{d_s}^{\infty}\right| \leq \rho_s \quad \text{with } \rho_s > 0 \tag{4.2.2}$$

The specification leads to a **frequency domain constraint** on the Complementary Sensitivity function $T(s)$ which can be computed as follows:

$$\begin{aligned}\left|e_{d_s}^{\infty}\right| = \left|y_{d_s}^{\infty}\right| &= \left|a_s T(j\omega_s)\frac{1}{G_s}\sin(\omega_s t + \varphi_s)\right| \leq a_s \left|T(j\omega_s)\frac{1}{G_s}\right| \leq \rho_s \\ &\implies |T(j\omega_s)| \leq \frac{\rho_s G_s}{a_s} = M_T^{HF} \quad \forall \omega_s \leq \omega_s^-\end{aligned} \tag{4.2.3}$$

5 Weighting Functions Construction

5.1 Rational Approximation of Frequency Constraints

- Rational functions of the Laplace variable s are used to approximate the frequency domain constraints on $S(s)$ and $T(s)$.
- The parameters of the approximating functions (steady-state gain zeros and poles) can be moved to get the desired result.
- **Butterworth polynomials** can be used either as denominator or numerator of the approximating rational function to effectively retain constraints on different frequency ranges.

Polynomial Order	Polynomial Structure
0	1
1	$1+\frac{s}{\omega_a}$
2	$1+\frac{2\zeta}{\omega_a}s+\left(\frac{s}{\omega_a}\right)^2$
3	$1+\frac{2}{\omega_a}s+2\left(\frac{s}{\omega_a}\right)^2+\left(\frac{s}{\omega_a}\right)^3$

Table 1: Butterworth Polynomials

The **key property** of Butterworth polynomials is that, when used in the numerator or denominator of a rational function, they increase or decrease the frequency response magnitude by 3 dB respectively, at the frequency ω_a, regardless of the polynomial's order.

The goal is to obtain the **rational functions $W_S^{-1}(s)$ and $W_T^{-1}(s)$** such that the constraints derived above are satisfied.

- Considering low frequencies we have:

$$\left|W_S^{-1}(j\omega)\right| \leq M_S^{LF} \quad \forall\omega_p \leq \omega_p^+ \quad ; \quad \max_{\omega}\left|W_S^{-1}(\infty)\right| \leq S_{p0} \tag{5.1.1}$$

- Considering high frequencies we have:

$$\left|W_T^{-1}(j\omega)\right| \leq M_T^{HF} \quad \forall\omega_s \geq \omega_s^- \quad ; \quad \max_{\omega}\left|W_T^{-1}(j\omega)\right| \leq T_{p0} \implies \left|W_T^{-1}(0)\right| = T_{p0} \tag{5.1.2}$$

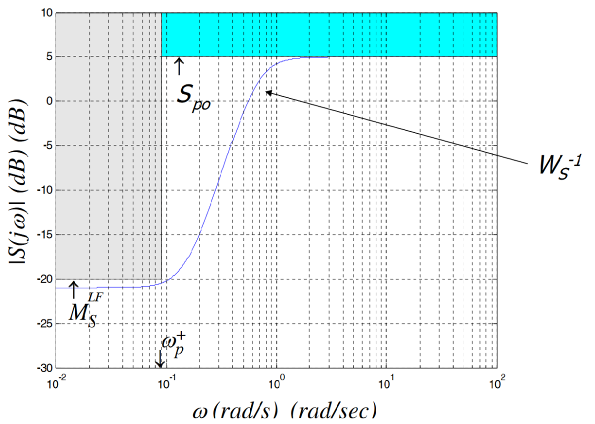


Figure 2: Weighting Function W_S^{-1}

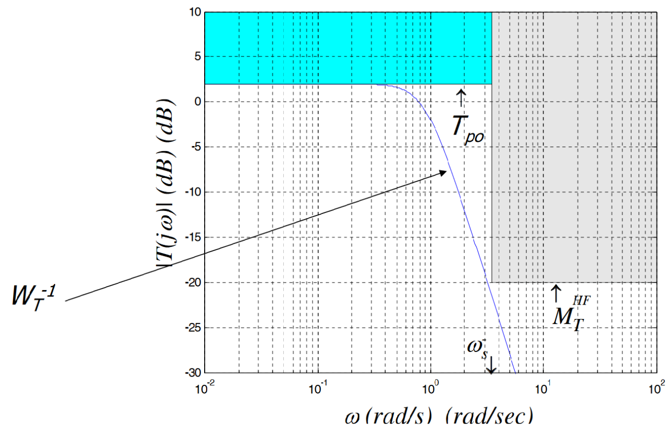


Figure 3: Weighting Function W_T^{-1}

6 Performance Specification as H_∞ Norm Constraints

6.1 H_∞ Norm Definition

By defining:

- $W_S(s)$ as the inverse of the rational approximation of the frequency domain constraints on the Sensitivity function S(s)
- $W_T(s)$ as the inverse of the rational approximation of the frequency domain constraints on the Complementary Sensitivity function T(s)

Design constraints obtained from the considered performance requirements can be written in the following compact form:

$$|W_S(j\omega)S(j\omega)| \leq 1 \quad ; \quad |W_T(j\omega)T(j\omega)| \leq 1 \quad \forall\omega \tag{6.1.1}$$

The **H_∞ norm** of a SISO LTI system with transfer function $H(s)$ is defined as:

$$\|H(s)\|_\infty \triangleq \max_{\omega}|H(j\omega)| \tag{6.1.2}$$

It is now possible to rewrite the design constraints obtained above in terms of the weighted H_∞ norm of $S(s)$ and $T(s)$:

$$\|W_S(s)S(s)\|_\infty \leq 1 \quad ; \quad \|W_T(s)T(s)\|_\infty \leq 1 \tag{6.1.3}$$

7 Unstructured Uncertainty Modelling

Mathematical models cannot exactly describe a physical process, irrespective of their complexity. Thus, model uncertainty has to be taken into account when a mathematical model is used to analyse the behaviour of a system, or to design a feedback control system. **Unstructured uncertainty** comes into play when complete ignorance regarding the order and the phase behaviour of the system is assumed, and parametric uncertainty can also be described by means of unstructured model sets.

7.1 Uncertainty Model Sets

The following four uncertainty model sets will be considered, in which:

- $G_p(s)$ is the transfer function of the generic member of the uncertainty set.
- $G_{pn}(s)$ is the transfer function of the **nominal model**.
- $\Delta(s)$ represents any possible transfer function whose H_∞ norm is less than 1.
- $W_u(s)$ is a **weighting function** which accounts for the size of the uncertainty.

Particular focus will be on the second set.

7.1.1 Additive Uncertainty

The additive uncertainty model set is defined as:

$$M_a = \{G_p(s) : G_p(s) = G_{pn}(s) + W_u(s)\Delta(s), \|\Delta(s)\|_\infty \leq 1\} \tag{7.1.1}$$

where, by construction, the following condition must be satisfied by the weighting function $W_u(s)$:

$$\left\| \frac{G_p(s) - G_{pn}(s)}{W_u(s)} \right\|_\infty = \|\Delta(s)\|_\infty \leq 1 \tag{7.1.2}$$

which is equivalent to:

$$|G_p(j\omega) - G_{pn}(j\omega)| \leq |W_u(j\omega)| \quad \forall\omega \tag{7.1.3}$$

Below is an example of the frequency response of $W_u(j\omega)$ for an additive uncertainty model set:

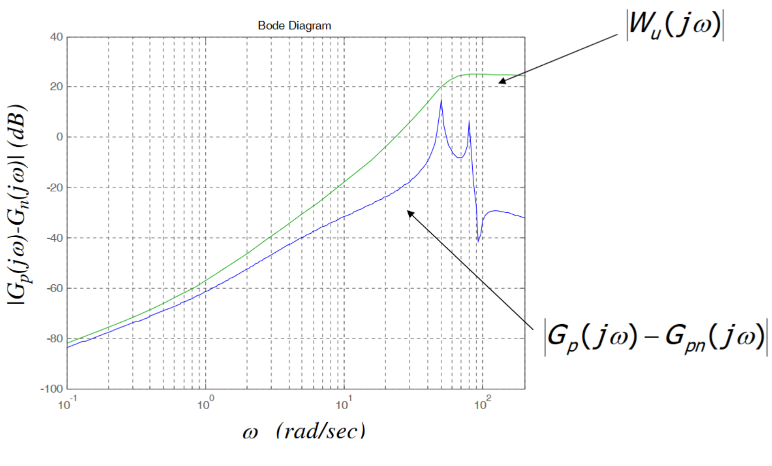


Figure 4: Weighting Function W_u considering M_a

7.1.2 Multiplicative Uncertainty

The multiplicative uncertainty model set is defined as:

$$M_m = \{G_p(s) : G_p(s) = G_{pn}(s)\left[1 + W_u(s)\Delta(s)\right], \|\Delta(s)\|_\infty \leq 1\} \tag{7.1.4}$$

where, by construction, the following condition must be satisfied by the weighting function $W_u(s)$:

$$\left\| \left(\frac{G_p(s)}{G_{pn}(s)} - 1 \right) \frac{1}{W_u(s)} \right\|_\infty = \|\Delta(s)\|_\infty \leq 1 \tag{7.1.5}$$

which is equivalent to:

$$\left|\frac{G_p(j\omega)}{G_{pn}(j\omega)} - 1\right| \leq |W_u(j\omega)| \quad \forall\omega \tag{7.1.6}$$

Below is an example of the frequency response of $W_u(j\omega)$ for a multiplicative uncertainty model set:

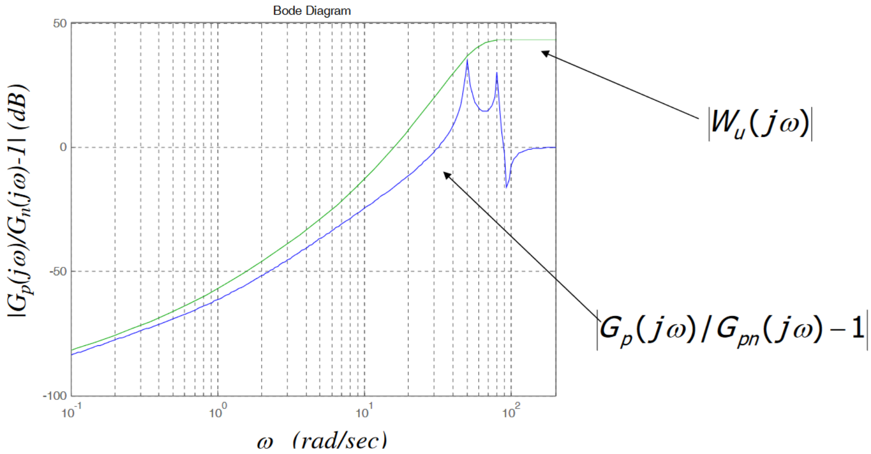


Figure 5: Weighting Function W_u considering M_m

Now we consider the problem of properly selecting the **nominal model G_{pn}** in order to **minimize** the size W_u of the unstructured uncertainty of the given model: Let's consider, without loss of generality, the following nominal model:

$$G_{pn}(s) = \frac{K_n}{s-p} \tag{7.1.7}$$

where K_n is a constant value to be computed. The weighting function $W_u(s)$ must satisfy the following condition:

$$\|\Delta(s)\|_\infty = \sup_\omega \left|\left(\frac{G_p(j\omega)}{G_{pn}(j\omega)} - 1\right)\frac{1}{W_u(j\omega)}\right| \leq 1 \tag{7.1.8}$$

which is equivalent to:

$$|W_u(j\omega)| \geq \left|\frac{G_p(j\omega)}{G_{pn}(j\omega)} - 1\right| = \left|\frac{K}{K_n} - 1\right| \quad \forall\omega,\ \forall K \tag{7.1.9}$$

K_n can be selected in order to minimize the size of the uncertainty in the following way:

$$|W_u(j\omega)| \geq \min_{K_n}\max_{K}\left|\frac{K}{K_n} - 1\right| \quad \forall\omega \tag{7.1.10}$$

It can be easily shown that:

$$\min_{K_n}\max_{K}\left|\frac{K}{K_n} - 1\right| = \min_{K_n}\max\left\{\left|\frac{\overline{K} - K_n}{K_n}\right|, \left|\frac{\underline{K} - K_n}{K_n}\right|\right\} \tag{7.1.11}$$

The solution is found at the intersection of the two functions to maximise (between curly braces in 7.1.11), and it is the following:

$$K_n = \frac{\overline{K} + \underline{K}}{2} \tag{7.1.12}$$

7.1.3 Inverse Additive Uncertainty

The inverse additive uncertainty model set is defined as:

$$M_{ia} = \left\{G_p(s) : G_p(s) = \frac{G_{pn}(s)}{1 + W_u(s)\Delta(s)G_{pn}(s)}, \|\Delta(s)\|_\infty \leq 1\right\} \tag{7.1.13}$$

7.1.4 Inverse Multiplicative Uncertainty

The inverse multiplicative uncertainty model set is defined as:

$$M_{im} = \left\{G_p(s) : G_p(s) = \frac{G_{pn}(s)}{1 + W_u(s)\Delta(s)}, \|\Delta(s)\|_\infty \leq 1\right\} \tag{7.1.14}$$

7.2 Conclusion and Remarks

An important remark is that **unstructured uncertainty model sets can only provide a conservative description of parametric uncertainties** since, as shown in the above example, a complex function $\Delta(s)$ is used to account for the source of uncertainty, which is a real number. The unstructured uncertainty model set describes, at each frequency ω, the uncertainty as a disk of radius $|W_u(j\omega)L_n(j\omega)|$.

8 Unstructured Uncertainty Modelling and Robustness

The aim is to study the stability of a feedback control system under the assumption that G_p is an uncertain system described by a given uncertainty model set. The following discussion will consider a multiplicative uncertainty model set M_m.

8.1 Robust Stability

The feedback control system displayed in Figure 1 is **robustly stable** if and only if it is internally stable for each G_p which belongs to M_m. As a result, the following condition on the weighting function W_u must be satisfied:

$$\|W_u T_n\|_\infty < 1 \tag{8.1.1}$$

where T_n is the **nominal complementary sensitivity function**. Robust stability conditions for every described uncertainty model set are shown in the table below:

Uncertainty Model Set	Robust Stability Condition
M_m	$\|W_u T_n\|_\infty < 1$
M_a	$\|W_u G_c S_n\|_\infty < 1$
M_{ia}	$\|W_u G_{pn} S_n\|_\infty < 1$
M_{im}	$\|W_u S_n\|_\infty < 1$

In other words, we have to guarantee that the uncertainty does not change the number of encirclements of the point -1 in the Nyquist plot of the loop function:

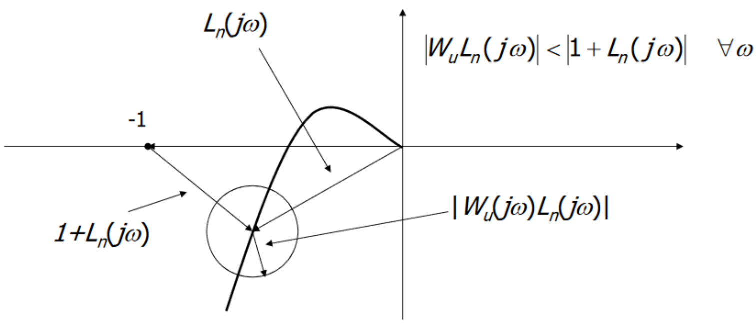


Figure 6: Nyquist Plot for Robust Stability Visualization

8.2 Nominal Performance

The nominal performance conditions, derived previously in subsection 6.1, are requirements affecting the sensitivity and complementary sensitivity functions as follows:

$$\|W_S S_n\|_\infty < 1 \iff |1 + L_n(j\omega)| > |W_S(j\omega)| \quad \forall\omega \tag{8.2.1}$$

$$\|W_T T_n\|_\infty < 1 \tag{8.2.2}$$

If we use the Nyquist Plot as before for visualization, the concept is very similar, but the "uncertainty disk" this time is centered around -1.

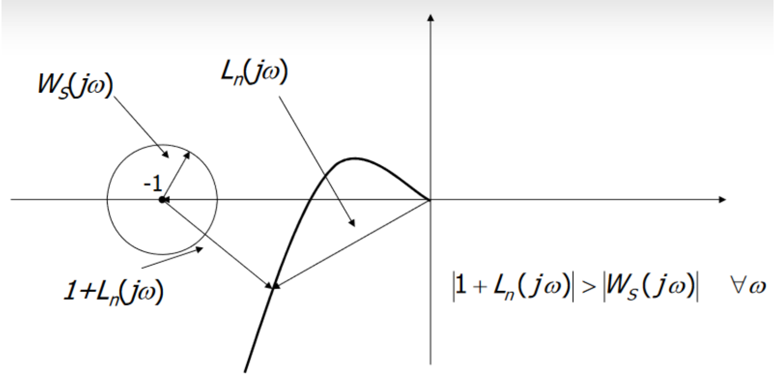


Figure 7: Nyquist Plot for Nominal Performance Visualization

8.3 Robust Performance

The feedback system guarantees **robust performance** if and only if performance requirements are satisfied for each G_p which belongs to the given uncertainty set. Considering the case in which performance requirements affect only the sensitivity function and uncertainty is described by means of a multiplicative uncertainty model set, the following condition must be satisfied (restricted to the case of $W_T = 0$):

$$\| |W_S S_n| + |W_u T_n| \|_\infty < 1 \tag{8.3.1}$$

The Nyquist Plot below allows for an easy visualization of the concept of Robust Performance, which is satisfied if and only if the two "uncertainty disks" do not overlap.

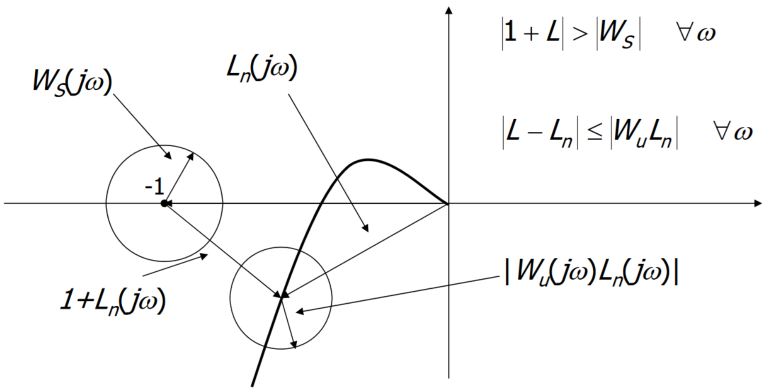


Figure 8: Nyquist Plot for Robust Performance Visualization

9 H_∞ Design for Robust Control

9.1 Problem Definition

Given definition 6.1.2 for SISO LTI systems, the H_∞ norm minimization approach, called H_∞ control, refers to a general formulation of the control problem which is based on the following block diagram representation of a general feedback system:

- $\boldsymbol{M}$ is the **generalized plant**
- $\boldsymbol{G_c}$ is the controller
- $\boldsymbol{u}$ are the control inputs
- $\boldsymbol{v}$ are the controller inputs
- $\boldsymbol{w}$ are the external inputs
- $\boldsymbol{z}$ are the external outputs

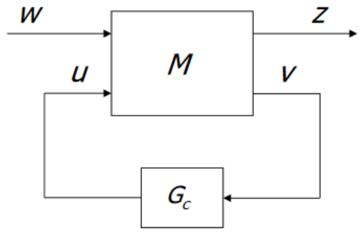


Figure 9: H_∞ Feedback Control System Block Diagram.

The external inputs and output signals of the generalized plant are not necessarily physical variables of the control system and must be carefully selected in order to take into account the stability and performance requirements of the considered control problem.

9.2 Controller Design

In the H_∞ design, the controller is obtained by solving the following optimization problem:

$$G_c(s) = \arg \min_{G_c \in G_c^{stab}} \|T_{wz}(s)\|_\infty \tag{9.2.1}$$

where G_c^{stab} is the class of all the controllers which provide internal stability of the nominal feedback system, and T_{wz} is the closed loop transfer function between the input w and the output z.
Therefore, the design of the controller is performed in three steps:

- select the transfer function T_{wz}
- build the generalized plant M corresponding to the selected T_{wz}.
- compute $G_c(s)$ by solving 9.2.1

9.2.1 Generalized Plant for Robust Stability

Let's consider the problem of designing a controller G_c to robustly stabilize an uncertain system described by means of the unstructured multiplicative model set.
The condition for robust stability is 8.1.1, and a controller that satisfies this condition can be found by choosing the following transfer function T_{wz}:

$$T_{wz}(s) = W_2 T_n \tag{9.2.2}$$

where $W_2(s) = W_u(s)$.

9.2.2 Generalized Plant for Nominal Performance

Let's consider the problem of designing a controller G_c to satisfy the nominal performance conditions.
The conditions for nominal performance are 8.2.1 and 8.2.2, and a controller that satisfies them can be found by choosing the following transfer function T_{wz} as a **stack** of two transfer functions:

$$T_{wz}(s) = \begin{bmatrix} W_1 S_n \\ W_2 T_n \end{bmatrix} \tag{9.2.3}$$

where $W_1(s) = W_S(s)$ and $W_2(s) = W_T(s)$.

9.2.3 Generalized Plant for Robust Stability and Nominal Performance

Finally, let's consider the problem of designing a controller G_c that satisfies both conditions.
Since the complementary sensitivity function must satisfy the following two frequency domain constraints:

$$|T_n(j\omega)| \leq \left|W_T^{-1}(j\omega)\right| \quad , \quad |T_n(j\omega)| \leq \left|W_u^{-1}(j\omega)\right| \quad \forall\omega \tag{9.2.4}$$

T_{wz} can be chosen as in 9.2.3, where in this case $W_1(s) = W_S(s)$ and $W_2(s)$ is such that for each ω:

$$|W_2(j\omega)| = \max\left(|W_u(j\omega)|, |W_T(j\omega)|\right) \tag{9.2.5}$$

For better visualization assume, without loss of generality, that $G_a = G_s = G_f = 1$.
The generalized plant M corresponding to the selected transfer function T_{wz} is the following:

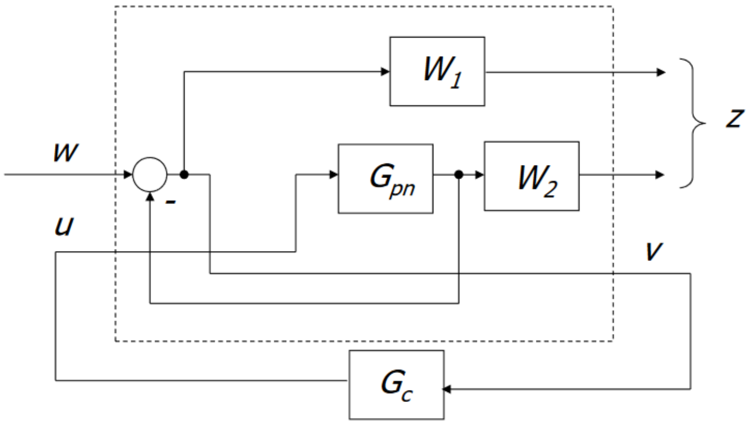
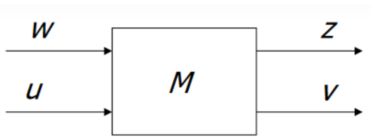


Figure 10: Generalized Plant Block Diagram.

Control problems involving constraints on the H_∞ norm on more than one closed loop transfer functions are called **mixed sensitivity problems**.
The controller design problem obtained by applying the "stacking procedure" to a mixed sensitivity problem is referred to as a **stacked mixed sensitivity problem**.

9.3 LMI Optimization Approach

As previously stated, the H_∞ controller is designed by solving equation 9.2.1.
Among the approaches proposed in the literature which solve such an optimization problem, this document will present one based on the solution of a suitably constrained optimization problem, where the constraints are in the form of linear matrix inequalities (LMI).
The LMI approach is based on a state-space description of the generalized plant M:



$$\begin{cases} \dot{x_M} = Ax_M + B_1 w + B_2 u \\ z = C_1 x_M + D_{11} w + D_{12} u \\ v = C_2 x_M + D_{21} w + D_{22} u \end{cases} \tag{9.3.1}$$

where $\boldsymbol{X_M}$ **is the state of the generalized plant.**
The LMI optimization problem can be solved under the following assumptions:

A1. The matrix triplet (A, B_2, C_2) is stabilizable and detectable.

A2. $D_{22} = 0$

9.3.1 Internal stability of the generalized plant M

Considering the mixed sensitivity problem, the generalized plant M can be internally stabilized by an LTI controller G_c having input v and output u **if and only if** W_1 and W_2 are stable transfer functions.
Now, assume that the performance requirements are such that the weighting function W_1 has $\nu + p$ poles at $s = 0$. In order to satisfy assumption A_1 we replace W_1 in the generalized plant M with a new weighting function W_1^* obtained as follows:

$$W_1^*(s) = W_1 \frac{s^{\nu+p}}{(s+\lambda^*)^{\nu+p}} \tag{9.3.2}$$

where $\lambda^* > 0$, and $s = -\lambda^*$ is a low frequency pole, for which a reasonable choice is $\lambda^* \leq 0.01\omega_c$.
The controller obtained with such a modified weighting function has, if any, at most $\nu + p$ poles at about $s = -\lambda^*$. Each pole at about $s = -\lambda^*$ in the controller must be replaced with a pole at $s = 0$.

We consider now the transfer function W_2, which we choose as described in 9.2.5. This transfer function has a gain and two complex conjugate zeroes, therefore it is not proper.
We modify W_2 in one of the two following ways:

$$W_2^*(s) = \left(\frac{W_2}{1 + \frac{2\zeta}{\omega_a}s + \left(\frac{s}{\omega_a}\right)^2} \right) \tag{9.3.3}$$

$$W_2^*(s) = \left(\frac{W_2}{\left(1 + \frac{s}{p}\right)\left(1 + \frac{s}{p}\right)} \right) \tag{9.3.4}$$

where ζ and ω_a are the damping factor and the frequency of the complex conjugate poles of W_T^{-1} respectively, when considering the first solution, while p is the frequency of a pole, chosen to cancel the complex conjugate zeros considering the second solution.
In conclusion, W_2^* will be equal to $\frac{1}{T_p}$, where T_p was defined in 2.0.5, thus the resulting modified weighting function will be proper.
On Matlab, the following command can be used twice to perform 9.3.4:

$$M = \text{sderiv}(M, 2, [1/p, 1]); \tag{9.3.5}$$

9.3.2 Matlab Commands

This subsections aims to list and illustrate the Matlab commands that should be used to perform the LMI optimization technique to obtain the H_∞ controller.

- $[A_m, B_m, C_m, D_m] = \text{linmod}('\text{generalized\_plant}');$
  Given a Simulink model of the generalized plant M named 'generalized_plant.mdl', **linmod** provides the matrices of the state-space description of M.

- $M = \text{ltisys}(A_m, B_m, C_m, D_m);$
  Given the matrices of the state-space realization of M, **ltisys** computes the generalized model M in a specific form needed by the LMI Control Toolbox to solve the optimization problem.

- $[gopt, G_{cmod}] = \text{hinflmi}(M, R, G, tol, options);$
  Given the generalized plant M obtained with the command **ltisys**, and the parameters listed and explained below, **hinflmi** provides the optimal value of γ (gopt) and the controller G_c in a specific compact form.

  - **M** is the system matrix
  - **R** is a 1x2 vector specifying the dimensions of D22.
  - **G** is the user-specified target for the closed-loop performance. Set $G = 0$ to compute G_{opt}, and set $G = gamma$ to test whether the performance $gamma$ is achievable.
  - **tol** is the relative accuracy required on G_{opt}. Default is $1e - 2$.
  - **options** is an optional 3-entry vector of control parameters for the numerical computations.

  Our chosen values will be the following:
  $[gopt, G_{cmod}] = \text{hinflmi}(M, [1, 1], 0, 0.01, [0, 0, 0]);$

- $[A_c, B_c, C_c, D_c] = \text{ltiss}(G_{cmod});$
  Given G_{cmod} computed by **hinflmi**, extracts the state-space matrices from the system matrix representation of the LTI system.

- $G_{cmod} = ss(A_c, B_c, C_c, D_c);$
  Given the state-space matrices computed above, computes the transfer function G_{cmod} of the controller.

9.3.3 Computation of the Final Controller Transfer Function

The transfer function G_{cmod} computed above will likely be of relatively high order, and some modifications must be performed in order to simplify the obtained controller:

- If the transfer function presents a zero and a pole at very low frequencies compared to the crossover frequency ω_c, these elements can effectively **cancel** each other.
- If the transfer function presents a high frequency pole relatively to ω_c, such pole can be **removed** by multiplying for a suitably chosen zero.
- Depending on the system type obtained from the translation of the requirements, a set number of low frequency poles will have to be **substituted** by poles at the origin.

A general formula to be considered for the simplification of the obtained controller is presented below:

$$G_c(s) = G_{cmod} \frac{s + p_{LF}}{s + z_{LF}} \left\{ 1 + \frac{s}{p_{HF}} \right\} \frac{s + p_{VLF}}{s} \tag{9.3.6}$$

where p_{LF} and z_{LF} are a pole and a zero at low frequencies which will **cancel** each other, p_{HF} is the high frequency pole to be **removed**, p_{VLF} is a pole at very low frequencies to be **substituted** with a pole at the origin.