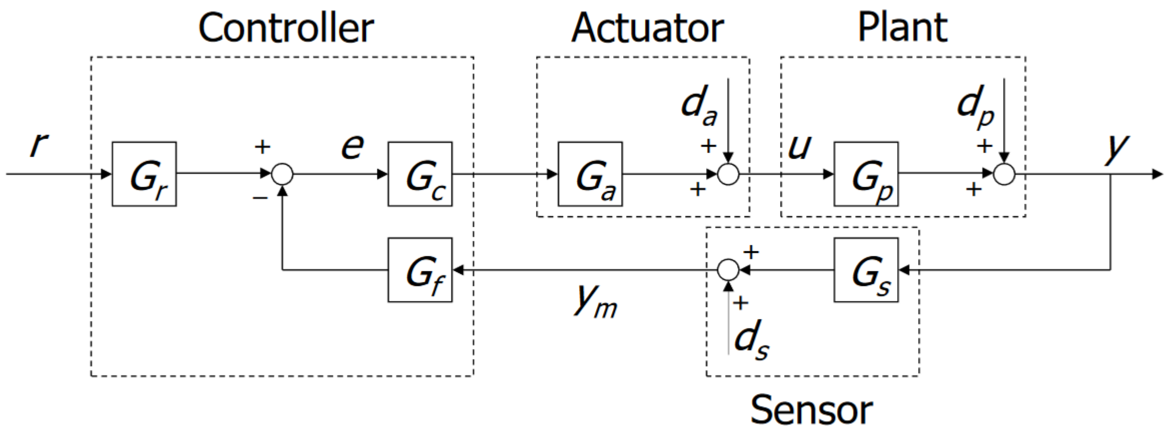
1 Feedback Control System Definition

For this course, we will consider the model below to represent the system under study:



With the following characteristics:

- Plant G_p with disturbance d_p
- Actuator G_a with disturbance d_a
- Sensor G_s with disturbance d_s
- Feedback Controller G_c

2 Step response of prototype 2$^{\text{nd}}$ order system

For a prototype second order system with Sensitivity and Complementary sensitivity functions of the form:

$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}} \quad ; \quad S(s) = \frac{s\left(\frac{2\zeta}{\omega_n} + \frac{s}{\omega_n^2}\right)}{1 + \frac{2\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}} \tag{2.0.1}$$

From the transient response indices for maximum overshoot $\hat{s}$, rise time t_r and settling time $t_{s,\alpha\%}$, we can compute the system's damping coefficient, as well as constraints on the natural frequency and resonance peaks T_p and S_p:

- $\zeta = \dfrac{|\log(\hat{s})|}{\sqrt{\pi^2 + \log^2(\hat{s})}}$ (2.0.2)
- $\omega_{n,tr} = \dfrac{1}{t_r\sqrt{1-\zeta^2}} \cdot (\pi - acos(\zeta)) \quad ; \quad \omega_{n,t,s\alpha\%} = \dfrac{\log(\frac{100}{\alpha})}{t_{s,\alpha\%}\zeta}$ (2.0.3)
- $T_p = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} \quad ; \quad S_p = \dfrac{2\zeta\sqrt{2 + 4\zeta^2 + 2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2} + 4\zeta^2 - 1}$ (2.0.4)

3 Steady-state Response to Polynomial Reference Inputs

Requirements on steady-state tracking error and steady-state errors in the presence of polynomial disturbances can be translated into constraints on the Sensitivity function. Therefore, by first writing $S(s) = s^{\nu+p}S^*(s)$, we can apply the final value theorem to each expression as described in the following subsections.

3.1 Tracking Error

3.1.1 Problem formulation

The tracking error is defined as:

$$e_r(t) = y_d(t) - y_r(t) = K_d r(t) - y_r(t) \tag{3.1.1}$$

Applying the final value theorem leads to:

$$\begin{aligned}
|e_r^\infty| &= \lim_{t\to\infty} |e_r(t)| = \lim_{s\to 0} s\,|e_r(s)| = \lim_{s\to 0} s\,|K_d r(s) - y_r(s)| = \\
&= \lim_{s\to 0} s\,|G_{re}(s) r(s)| = \lim_{s\to 0} s\,|S(s) K_d r(s)| = \\
&= \lim_{s\to 0} s \left| s^{\nu+p} S^*(s) K_d \frac{R_0}{s^{h+1}} \right| = \begin{cases} 0, & \text{if } \nu + p > h \\ |S^*(0) K_d R_0|, & \text{if } \nu + p = h \end{cases}
\end{aligned} \tag{3.1.2}$$

The given system type is then $\nu + p$.

3.1.2 Conclusions

Considering the specification $|e_r^\infty| \le \rho_r$, we obtain the following result:

- case $\rho_r = 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity greater than $h \implies$ We have no constraint on $|S^*(0)|$
- case $\rho_r > 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity $h \implies$ We have the following constraint:

$$|S^*(0)| \le \frac{\rho_r}{K_d R_0} \tag{3.1.3}$$

3.2 Error due to Generic Polynomial Disturbances $d(t)$

3.2.1 Problem formulation

The output error due to the generic disturbance $d(t)$ is the contribution of the disturbance to the output $y(t)$:

$$e_d(t) = y_d(t) \tag{3.2.1}$$

3.2.2 Polynomial disturbance $d_a(t)$

Applying the final value theorem leads to:

$$\begin{aligned}
\left|e_{d_a}^\infty\right| &= \lim_{t\to\infty} |e_{d_a}(t)| = \lim_{s\to 0} s\,|e_{d_a}(s)| = \lim_{s\to 0} s\,|y_{d_a}(s)| = \lim_{s\to 0} s\,|S(s) G_p(s) d_a(s)| = \\
&= \lim_{s\to 0} \left| s^{\nu+p+1} S^*(s) G_p(s) \frac{D_{a0}}{s^{h+1}} \right| = \lim_{s\to 0} \left| s^{\nu+1} S^*(s) K_p \frac{D_{a0}}{s^{h+1}} \right| = \begin{cases} 0, & \text{if } \nu > h \\ |S^*(0) K_p D_{a0}|, & \text{if } \nu = h \end{cases}
\end{aligned} \tag{3.2.2}$$

We then obtain two cases as before:

- case $\rho_a = 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity greater than $h \implies$ We have no constraint on $|S^*(0)|$
- case $\rho_a > 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity $h \implies$ We have the following constraint:

$$|S^*(0)| \le \frac{\rho_a}{K_p D_{a0}} \tag{3.2.3}$$

3.2.3 Polynomial disturbance $d_p(t)$

Applying the final value theorem leads to:

$$\begin{aligned}
\left|e_{d_p}^\infty\right| &= \lim_{t\to\infty} \left|e_{d_p}(t)\right| = \lim_{s\to 0} s\,\left|e_{d_p}(s)\right| = \lim_{s\to 0} s\,\left|y_{d_p}(s)\right| = \lim_{s\to 0} s\,|S(s) d_p(s)| = \\
&= \lim_{s\to 0} \left| s^{\nu+p+1} S^*(s) \frac{D_{p0}}{s^{h+1}} \right| = \begin{cases} 0, & \text{if } \nu + p > h \\ |S^*(0) D_{p0}|, & \text{if } \nu + p = h \end{cases}
\end{aligned} \tag{3.2.4}$$

We then obtain two cases as before:

- case $\rho_p = 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity greater than $h \implies$ We have no constraint on $|S^*(0)|$
- case $\rho_p > 0 \implies S(s)$ must have a zero at $s = 0$ with multiplicity $h \implies$ We have the following constraint:

$$|S^*(0)| \leq \frac{\rho_p}{D_{p0}} \tag{3.2.5}$$

4 Steady-state Response to Sinusoidal Disturbances

4.1 Output disturbance $d_p(t)$

The focus is on the class of sinusoidal signals of the following type:

$$d_p = a_p \sin(\omega_p t) \quad \forall \omega_p \leq \omega_p^+ \quad \text{given } a_p \text{ and } w_p^+ \tag{4.1.1}$$

The output at steady-state is required to be bounded by a given constant:

$$\left| e_{d_p}^\infty \right| = \left| y_{d_p}^\infty \right| \leq \rho_p \quad \text{with } \rho_p > 0 \tag{4.1.2}$$

The specification leads to a frequency domain constraint on the Sensitivity function $S(s)$ which can be computed as follows:

$$\begin{aligned} \left| e_{d_p}^\infty \right| = \left| y_{d_p}^\infty \right| &= |a_p S(j\omega_p) \sin(\omega_p t + \varphi_p)| \leq a_p \, |S(j\omega_p)| \leq \rho_p \\ \implies |S(j\omega_p)| &\leq \frac{\rho_p}{a_p} = M_S^{LF} \quad \forall \omega_p \leq \omega_p^+ \end{aligned} \tag{4.1.3}$$

4.2 Sensor Noise $d_s(t)$

The focus is on the class of sinusoidal signals of the following type:

$$d_s = a_s \sin(\omega_s t) \quad \forall \omega_s \geq \omega_s^- \quad \text{given } a_s \text{ and } w_s^- \tag{4.2.1}$$

The output at steady-state is required to be bounded by a given constant:

$$\left| e_{d_s}^\infty \right| = \left| y_{d_s}^\infty \right| \leq \rho_s \quad \text{with } \rho_s > 0 \tag{4.2.2}$$

The specification leads to a frequency domain constraint on the Complementary Sensitivity function $T(s)$ which can be computed as follows:

$$\begin{aligned} \left| e_{d_s}^\infty \right| = \left| y_{d_s}^\infty \right| &= \left| a_s T(j\omega_s) \frac{1}{G_s} \sin(\omega_s t + \varphi_s) \right| \leq a_s \left| T(j\omega_s) \frac{1}{G_s} \right| \leq \rho_s \\ \implies |T(j\omega_s)| &\leq \frac{\rho_s G_s}{a_s} = M_T^{HF} \quad \forall \omega_s \leq \omega_s^- \end{aligned} \tag{4.2.3}$$

5 Weighting Functions Construction

5.1 Rational Approximation of Frequency Constraints

- Rational functions of the Laplace variable s are used to approximate the frequency domain constraints on $S(s)$ and $T(s)$.
- The parameters of the approximating functions (steady-state gain zeros and poles) can be moved to get the desired result.
- Butterworth polynomials can be used either as denominator or numerator of the approximating rational function to effectively retain constraints on different frequency ranges.

Polynomial Order	**Polynomial Structure**
0	1
1	$1+\dfrac{s}{\omega_a}$
2	$1+\dfrac{2\zeta}{\omega_a}s+\left(\dfrac{s}{\omega_a}\right)^2$
3	$1+\dfrac{2}{\omega_a}s+2\left(\dfrac{s}{\omega_a}\right)^2+\left(\dfrac{s}{\omega_a}\right)^3$

Table 1: Butterworth Polynomials

The key property of Butterworth polynomials is that, when used in the numerator or denominator of a rational function, they increase or decrease the frequency response magnitude by 3 dB respectively, at the frequency ω_a, regardless of the polynomial's order.

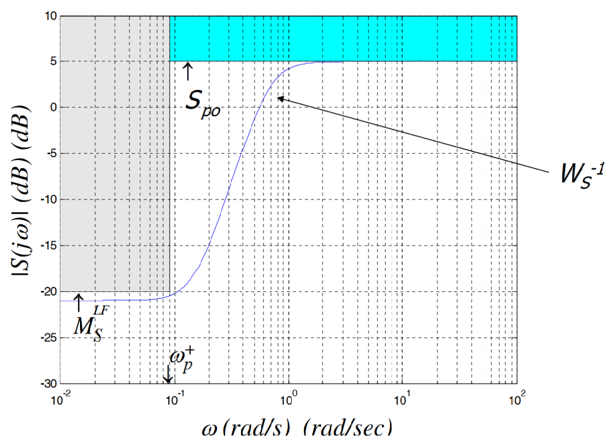
The goal is to obtain the rational functions $W_S^{-1}(s)$ and $W_T^{-1}(s)$ such that the constraints derived above are satisfied.

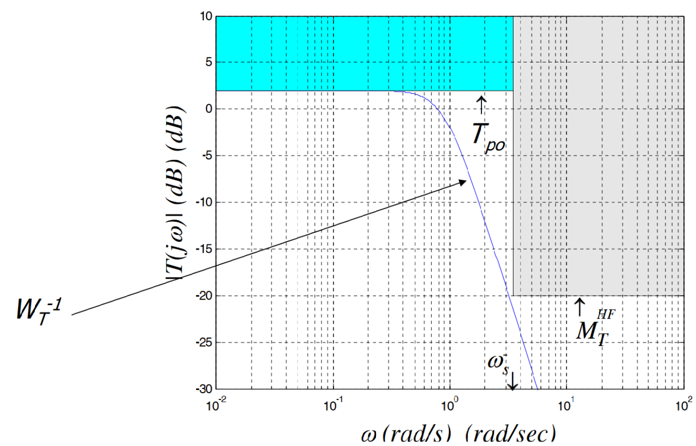
- Considering low frequencies we have:

$$\left|W_S^{-1}(j\omega)\right| \leq M_S^{LF} \quad \forall \omega_p \leq \omega_p^+ \quad ; \quad \max_{\omega}\left|W_S^{-1}(\infty)\right| \leq S_{p0} \tag{5.1.1}$$

- Considering high frequencies we have:

$$\left|W_T^{-1}(j\omega)\right| \leq M_T^{HF} \quad \forall \omega_s \geq \omega_s^- \quad ; \quad \max_{\omega}\left|W_T^{-1}(j\omega)\right| \leq T_{p0} \implies \left|W_T^{-1}(0)\right| = T_{p0} \tag{5.1.2}$$





6 Performance Specification as H_∞ Norm Constraints

6.1 H_∞ Norm Definition

By defining:

- $W_S(s)$ as the inverse of the rational approximation of the frequency domain constraints on the Sensitivity function S(s)
- $W_T(s)$ as the inverse of the rational approximation of the frequency domain constraints on the Complementary Sensitivity function T(s)

Design constraints obtained from the considered performance requirements can be written in the following compact form:

$$|W_S(j\omega)S(j\omega)| \leq 1 \quad ; \quad |W_T(j\omega)T(j\omega)| \leq 1 \quad \forall \omega \tag{6.1.1}$$

The H_∞ norm of a SISO LTI system with transfer function $H(s)$ is defined as:

$$\|H(s)\|_\infty \triangleq \max_{\omega}|H(j\omega)| \tag{6.1.2}$$

It is now possible to rewrite the design constraints obtained above in terms of the weighted H_∞ norm of $S(s)$ and $T(s)$:

$$\|W_S(s)S(s)\|_\infty \leq 1 \quad ; \quad \|W_T(s)T(s)\|_\infty \leq 1 \tag{6.1.3}$$