Coding Assignment 4

The equations of structural dynamics, reduced to the one-dimensional setting, lead to the so-called *wave equation*. Various interpretations are possible. For the purposes of this assignment, we will use this equation to describe the axial motion of an elastic rod of length l. The equations are written as

$$\rho u_{,tt} = \sigma_{,x} + f \tag{1}$$

where $\sigma = Eu_{,x}$, ρ is the mass density of the material, and E is Young's modulus. Boundary and initial conditions may be specified in a manner similar to what was shown for the general case. In other words,

$$\begin{aligned} u(l,t) &= g(t) \quad t \in (0,T) \\ -Eu_{,x}(0,t) &= h(t) \quad t \in (0,T) \\ u(x,0) &= u_0(x) \quad x \in (0,l) \\ \dot{u}(x,0) &= \dot{u}_0(x) \quad x \in (0,l). \end{aligned}$$

Using separation of variables results in the corresponding eigenvalue problem governing the spectral behavior of (1):

$$\lambda\rho u + (Eu_{,x})_{,x} = 0, \quad x \in (0,l). \tag{2}$$

In particular we will consider two cases of (2):

1. In the *fixed-fixed* case (i.e., $u(0) = 0$ and $u(l) = 0$) the exact solution for the n^{th} natural frequency ω_n is given by

$$\omega_n = \frac{n\pi}{l}\sqrt{\left(\frac{E}{\rho}\right)}.$$

2. In the *free-fixed* case (i.e., $u(l) = 0$) the exact solution for the n^{th} natural frequency ω_n is given by

$$\omega_n = (n - \frac{1}{2})\frac{\pi}{l}\sqrt{\left(\frac{E}{\rho}\right)}.$$

Note that the term $\sqrt{\left(\frac{E}{\rho}\right)}$ is the propagation speed (velocity) of longitudinal waves along the rod. Feel free to play around with E and ρ as it will not affect the fundamental behavior of the spectrum.

1. To familiarize yourself with (1) derive the entries of the element mass, stiffness,

and forcing matrices and vectors. In other words, derive

$$
\begin{aligned}
m^e_{ab} &= \int_{\Omega^e} N_a \rho N_b d\Omega \\
k^e_{ab} &= \int_{\Omega^e} N_{a,x} E N_{b,x} d\Omega \\
f^e_a &= \int_{\Omega^e} N_a f d\Omega - N_a(0)\delta_{e_1} h - \sum_{b=1}^{n_{en}} (k^e_{ab} g^e_b + m^e_{ab} \ddot{g}^e_b).
\end{aligned}
$$

You may consult Chapter 7 of the Hughes book for help. Turn in your completed derivations.

2. Assuming ρ is constant evaluate the element mass matrix, $\mathbf{m}^e$, for piecewise linear shape functions. Use these expressions to debug your mass matrix computations in the coding assignments below. Turn in the matrix expressions and your derivations.

3. Modify your one-dimensional FEA code to be able to solve simple eigenvalue problems. You should already be able to produce appropriate stiffness matrices $\mathbf{K}$ but you will need to add to your code the ability to compute mass matrices $\mathbf{M}$. Once you have computed $\mathbf{K}$ and $\mathbf{M}$ you will compute the eigenvalues and eigenvectors using the generalized eigensolver of your choice (I recommend Matlab).

4. For $p = 1, 2, 3$ B-spline basis functions construct a mesh with exactly $N = 1000$ dofs. Solve the *fixed-fixed* eigenvalue problem. Plot the error in the frequency spectra for $p = 1, 2, 3$ B-spline basis functions. In other words, plot the error in the computed frequency (i.e., $\frac{\omega_n^h}{\omega_n}$) versus the normalized mode number $\frac{n}{N}$. On the plot the x-axis is the normalized mode number while the y-axis is the normalized discrete frequency error. On a single plot show the first 10 computed mode shapes. Then, animate a mode shape of your choosing by multiplying the mode shape by $\cos(2\pi t)$ where $t \in [0, T]$. In other words, show the dynamic behavior of the mode shape over a specific time period. Turn in an electronic version of the movie to me via email. It's fun to watch them wiggle around.

5. Repeat the above for the *free-fixed* case.