Coding Assignment 1

Consider our model boundary value problem,

$$\begin{aligned} u_{,xx} + f &= 0 \\ u(1) &= 0 \\ -u_{,x}(0) &= 0 \end{aligned}$$

1. Employing a piecewise linear finite element space with equally spaced nodes, implement a finite element code that solves the Galerkin finite element equations for $n = 10, 100, 1000, 10,000$ for the following cases:

   (a) $f(x) = c$, a constant.

   (b) $f(x) = x$.

   (c) $f(x) = x^2$.

   Note that I will just note whether you attempted to solve the $n = 10,000$ problem. Try it and see how your code behaves and if you can figure out how you might be able to overcome the issues that arise (Hint: Leverage the structure of $\mathbf{K}$). Your code must use the element point-of-view and implement a global assembly routine where local element stiffness matrices and force vectors are assembled into the global system $\mathbf{Kd} = \mathbf{F}$. Plot the exact and approximate solution for each $f(x)$ and n. Describe the behavior of the solution at the nodes and the interior of the elements for each $f(x)$. To what can you attribute the differences?

2. Let the global error between the computed solution and the exact solution be denoted by $e = ||u - u^h|| = (\int_0^1 |u - u^h|^2 dx)^{\frac{1}{2}}$. Compute and report the global error for each f and n (12 numbers). Note that the global error gives a measure of how close your computed solution matches the exact solution. You should observe this error diminish as the number of elements increases.

   Hint: To accurately compute the error integral use the following 3-point Gauss quadrature rule over each element domain:

$$\begin{aligned} \left(\int_0^1 |u - u^h|^2 dx \right)^{\frac{1}{2}} &= \left(\sum_e \int_{-1}^1 |u(\xi) - u^h(\xi)|^2 \frac{\partial x(\xi)}{\partial \xi} d\xi \right)^{\frac{1}{2}} \\ &\approx \left(\sum_e \sum_{i=1}^3 |u(\xi_i) - u^h(\xi_i)|^2 \frac{\partial x(\xi_i)}{\partial \xi} w_i \right)^{\frac{1}{2}} \end{aligned}$$

   where $\xi_1 = -\sqrt{\frac{3}{5}}$, $\xi_2 = 0$, and $\xi_3 = \sqrt{\frac{3}{5}}$ and $w_1 = \frac{5}{9}$, $w_2 = \frac{8}{9}$, and $w_3 = \frac{5}{9}$. We will return to numerical integration in more detail later in the class.

3. Employing the data for $n = 10, 100, 1000, 10,000$ generate a log-log plot for e versus h for each $f(x)$ (1 plot). Determine the slope of the graph for the

two cases when $f(x)$ is non-constant. The slope of this graph is the rate of convergence of the method or how fast the approximate solution is converging to the exact solution as the mesh is refined. How does the rate of convergence depend on n?