Solution to BVP

$$\frac{d^2}{\mathrm{d}x^2}u + f = 0;\ u(1) = 0; -\frac{d}{\mathrm{d}x}u(0) = 0$$

$$\frac{d^2}{\mathrm{d}x^2}u + a\frac{d}{\mathrm{d}x}u + bu = R(x);$$
$$a = b = 0$$

Let m_1 and m_2 be the roots of $m^2 + am + b = 0$. Then we find that $m_1 = m_2 = 0$. For this case, the solution has the form:

$$u = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + x e^{m_1 x}\int e^{-m_1 x}R(x)\mathrm{dx} - e^{m_1 x}\int x e^{-m_1 x}R(x)\mathrm{dx}$$

Substituting our solution for m_1 we find:

$$u = c_1 + c_2\, x + x\int R(x)\mathrm{d}x - \int x\, R(x)\mathrm{d}x$$

Noting that the first derivative of this solution is

$$\mathrm{du} = c_2 + \int R(x)\mathrm{d}x$$

We then proceed with evaluating the provided cases:

Case 1: $f(x) = k,\ (constant)$

$$\frac{d^2}{\mathrm{d}x^2}u + k = 0;$$

The additive inverse of $f(x)$ replaces $R(x)$ in the equations above.

$$u = -\frac{k\, x^2}{2} + c_2\, x + c_1$$

$$\mathrm{du} = c_2 - k\, x$$

We apply the boundary condition $-\frac{d}{\mathrm{d}x}u(0) = 0$ and solve for c_2

$$0 = c_2$$

We the turn to the other boundary condition $u(1) = 0$ and solve for c_1

$$0 = c_1 + c_2 - \frac{k}{2}$$

$$0 = c_1 - \frac{k}{2}$$

Finally we substitute c_1 and c_2 back into our equation to find our exact solution.

$$u = \frac{k}{2} - \frac{k\,x^2}{2}$$

Case B $f(x) = x$

$\frac{d^2}{\mathrm{d}x^2} u + x = 0;$

The additive inverse of $f(x)$ replaces $R(x)$ in the equations above.

$$u = -\frac{x^3}{6} + c_2\,x + c_1$$

$$\mathrm{du} = c_2 - \frac{x^2}{2}$$

We apply the boundary condition $-\frac{d}{\mathrm{d}x}u(0) = 0$ and solve for c_2

$$0 = c_2$$

We the turn to the other boundary condition $u(1) = 0$ and solve for c_1

$$0 = c_1 + c_2 - \frac{1}{6}$$

$$0 = c_1 - \frac{1}{6}$$

Finally we substitute c_1 and c_2 back into our equation to find our exact solution.

$$u = \frac{1}{6} - \frac{x^3}{6}$$

Case C $f(x) = x^2$

$$u = -\frac{x^4}{12} + c_2\,x + c_1$$

$$\mathrm{du} = c_2 - \frac{x^3}{3}$$

We apply the boundary condition $-\frac{d}{\mathrm{d}x}u(0) = 0$ and solve for c_2

$$0 = c_2$$

We the turn to the other boundary condition $u(1) = 0$ and solve for c_1

$$0 = c_1 + c_2 - \frac{1}{12}$$

$$0 = c_1 - \frac{1}{12}$$

Finally we substitute c_1 and c_2 back into our equation to find our exact solution.

$$u = \frac{1}{12} - \frac{x^4}{12}$$