Basic constrained gradient-based optimization

```
clear
x0 = rand(1,3);
penalty = 2.^[10:20];
for ii = 1:length(penalty)
    disp("Penalty = " + num2str(penalty(ii)))
    options = optimoptions("fminunc","Display","iter-detailed");
    objFun = @(x) computeLagrangian(x, penalty(ii));
    solVector = fminunc(objFun, x0, options);
    x0 = solVector;
    disp("    Unconstrained solution: ")
    disp("r_1 = " + num2str(solVector(1)) + " || r_2 = " + num2str(solVector(2)) + " || theta =
    disp("%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
end
```

```
Penalty = 1024
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
     0           4         -132.906                         49.9
     1          12         -133.032      0.0020032            40
     2          16         -134.237              1          9.03
     3          20         -134.318              1          7.97
     4          24         -134.572              1          12.8
     5          32         -136.466       0.213174          50.3
     6          36         -136.793              1            45
     7          40         -138.176              1          21.6
     8          44         -138.457              1          4.78
     9          48         -138.472              1          1.98
    10          52         -138.474              1        0.0365
    11          56         -138.474              1       0.00865
    12          60         -138.474              1      7.63e-05
    13          64         -138.474              1      1.91e-06

Optimization completed: The first-order optimality measure, 3.745774e-08, is less
than options.OptimalityTolerance = 1.000000e-06.
    Unconstrained solution:
r_1 = 0.073978 || r_2 = -0.080944 || theta = 0.46882
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Penalty = 2048
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
     0           4         -121.364                          185
     1          16         -130.336    0.000438598          32.1
     2          20         -130.533              1          17.5
     3          24         -130.592              1          8.59
     4          28          -130.62              1          5.65
     5          32         -130.654              1          4.85
     6          36         -130.725              1          9.55
     7          40         -130.803              1          11.6
     8          44         -130.867              1          8.44
     9          48         -130.905              1           4.1
    10          84         -137.641        17.3358          42.3
    11         100         -137.641          0.001          42.5
    12         108         -142.127             10          14.2
    13         112           -142.2              1          3.21
    14         116         -142.208              1           0.8
    15         120         -142.209              1       0.00978
    16         124         -142.209              1      0.000156

Optimization completed: The first-order optimality measure, 8.369223e-07, is less
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than options.OptimalityTolerance = 1.000000e-06.
    Unconstrained solution:
r_1 = 0.074277 || r_2 = 0.30852 || theta = 3.2284
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Penalty = 4096
                                                        First-order 
 Iteration  Func-count       f(x)        Step-size       optimality
     0           4         -140.749                         56.6
     1          20         -141.468    0.000225393          4.49  
     2          24         -141.476              1             4  
     3          28         -141.482              1          3.56  
     4          32         -141.496              1          4.39  
     5          36         -141.502              1          2.97  
     6          40         -141.504              1         0.746  
     7          44         -141.504              1        0.0622  
     8          48         -141.504              1       0.00258  
     9          52         -141.504              1      0.000267  
    10          56         -141.504              1      7.63e-06  

Optimization completed: The first-order optimality measure, 1.324285e-07, is less 
than options.OptimalityTolerance = 1.000000e-06.
    Unconstrained solution:
r_1 = 0.087709 || r_2 = 0.29343 || theta = 3.1838
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Penalty = 8192
                                                        First-order 
 Iteration  Func-count       f(x)        Step-size       optimality
     0           4         -140.825                           55
     1          20         -140.994    0.000207343          39.6  
     2          24         -141.163              1          2.12  
     3          32         -141.168             10          1.37  
     4          36         -141.171              1         0.138  
     5          40         -141.171              1       0.00661  
     6          44         -141.171              1      5.72e-06  

Optimization completed: The first-order optimality measure, 1.021538e-07, is less 
than options.OptimalityTolerance = 1.000000e-06.
    Unconstrained solution:
r_1 = 0.094001 || r_2 = 0.28559 || theta = 3.1623
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Penalty = 16384
                                                        First-order 
 Iteration  Func-count       f(x)        Step-size       optimality
     0           4         -140.846                           54
     1          24         -141.006    5.35499e-05          6.32  
     2          28         -141.008              1          1.09  
     3          36         -141.009             10         0.971  
     4          40          -141.01              1          1.41  
     5          44          -141.01              1          0.47  
     6          48          -141.01              1        0.0444  
     7          52          -141.01              1       0.00077  
     8          56          -141.01              1      6.65e-05  
     9          60          -141.01              1       3.8e-06  

Optimization completed: The first-order optimality measure, 6.907372e-08, is less 
than options.OptimalityTolerance = 1.000000e-06.
    Unconstrained solution:
r_1 = 0.097037 || r_2 = 0.28162 || theta = 3.1519
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Penalty = 32768
                                                        First-order 
 Iteration  Func-count       f(x)        Step-size       optimality
     0           4         -140.851                         53.5
     1          24         -140.929    3.50041e-05          8.07  
     2          28          -140.93              1          0.55  
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     3          36         -140.931             10          0.501
     4          40         -140.931              1          0.455
     5          44         -140.931              1         0.0911
     6          48         -140.931              1        0.00128
     7          52         -140.931              1       1.14e-05

Optimization completed: The first-order optimality measure, 2.098964e-07, is less
than options.OptimalityTolerance = 1.000000e-06.
    Unconstrained solution:
r_1 = 0.098528 || r_2 = 0.27963 || theta = 3.1467
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Penalty = 65536
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
     0           4         -140.852                          53.3
     1          28         -140.892    1.52483e-05           0.278
     2          36         -140.892        9.69499           0.48
     3          40         -140.892              1           0.68
     4          44         -140.892              1           1.09
     5          48         -140.892              1          0.591
     6          52         -140.892              1         0.0888
     7          56         -140.892              1        0.00442
     8          60         -140.892              1       0.000269
     9          64         -140.892              1       1.91e-05

Optimization completed: The first-order optimality measure, 3.515795e-07, is less
than options.OptimalityTolerance = 1.000000e-06.
    Unconstrained solution:
r_1 = 0.099266 || r_2 = 0.27863 || theta = 3.1441
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Penalty = 131072
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
     0           4         -140.853                          53.1
     1          28         -140.872    6.82629e-06           5.55
     2          32         -140.872              1          0.139
     3          44         -140.872             91          0.173
     4          48         -140.872              1          0.107
     5          52         -140.872              1         0.0123
     6          56         -140.872              1       4.96e-05

Optimization completed: The first-order optimality measure, 9.164427e-07, is less
than options.OptimalityTolerance = 1.000000e-06.
    Unconstrained solution:
r_1 = 0.099634 || r_2 = 0.27813 || theta = 3.1429
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Penalty = 262144
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
     0           4         -140.853                            53
     1          28         -140.862     4.7281e-06           12.7
     2          32         -140.862              1         0.0694
     3          48         -140.862            820           0.17
     4          52         -140.862              1        0.00298
     5          56         -140.862              1       1.14e-05

Optimization completed: The first-order optimality measure, 2.117170e-07, is less
than options.OptimalityTolerance = 1.000000e-06.
    Unconstrained solution:
r_1 = 0.099817 || r_2 = 0.27788 || theta = 3.1422
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Penalty = 524288
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
     0           4         -140.853                            53
```

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     1          32         -140.858    1.85698e-06           1.39
     2          36         -140.858              1         0.0347
     3          52         -140.858            820          0.388
     4          56         -140.858              1          0.213
     5          60         -140.858              1         0.0131
     6          64         -140.858              1       0.000639

Optimization stopped because the norm of the current step, 3.742823e-07, is
less than options.StepTolerance = 1.000000e-06.
    Unconstrained solution:
r_1 = 0.099909 || r_2 = 0.27775 || theta = 3.1419
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Penalty = 1048576
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
     0           4         -140.853                            53
     1          32         -140.855    8.78939e-07           4.15
     2          36         -140.855              1         0.0174
     3          52         -140.855            820         0.0338
     4          56         -140.855              1        0.00575
     5          60         -140.855              1       0.000204

Optimization stopped because the norm of the current step, 1.351273e-07, is
less than options.StepTolerance = 1.000000e-06.
    Unconstrained solution:
r_1 = 0.099954 || r_2 = 0.27769 || theta = 3.1418
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Function definitions

Lagrangian evaluation

```
function LAG = computeLagrangian(x, penalty)
r1 = x(1);
r2 = x(2);
theta = x(3);

f = -1*computeFreq(r1, r2, pi*theta);
g = constraintFunctions(r1, r2, theta);

LAG_F = f;
LAG_G = sum(g.^2);
LAG = LAG_F + penalty/2 * LAG_G;
end
```

Function evaluation (additive inverse is objective function)

```
function freq = computeFreq(r1,r2,theta)
    %surface fit for frequency on radial disk with 2 supports
    %r1 refers to the distance from center of the first support (0.1 - 0.9)
    %r2 is the distance from center to the second support (0.1 - 0.9)
    %theta is the angle between the supports (from center) (0 - pi)
    freq = (140.93-r1*25)+(-7.458*r1+9.1185)*theta+(-170)*r2+ ...
           (5.783*r1-10.367)*theta^2+(-8.1)*theta*r2+(117)*r2^2+ ...
           (4.2)*theta^3+(-28.075*r1+31.5125)*theta^2*r2+(15.63*r1-2.26)*theta*r2^2+ ...
           (-.7)*theta^4+(-.35)*theta^3*r2+(21.77*r1-27.862)*theta^2*r2^2;
end
```

Constraint functions

```
function g = constraintFunctions(r1, r2, theta)
    g(1) = max(0, r1 - 0.9); % Radius < 0.9
    g(2) = max(0, r2 - 0.9); % Radius < 0.9
    g(3) = max(0, 0.1 - r1); % Radius > 0.1
    g(4) = max(0, 0.1 - r2); % Radius > 0.1
    g(5) = max(0, 0.1 - sqrt(r1^2 + r2^2 - 2*r1*r2*cos(pi*theta))); % Minimum distance between
    g(6) = max(0, -theta); % Theta > 0
    g(7) = max(0, theta - 1); % Theta < 1*pi
end
```