Tissue Biomechanics UE 317.032

Dynamic mechanical analysis and rheological models – Background theory

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Bones, tendons, ligaments, and other tissues are load-bearing structures, they are of great biomechanical importance and studying their mechanical is of interest to many researchers. Biologicals tissue behave at the very least as linear viscoelastic materials. A compelling theory as to what is responsible for biological tissue viscoelastic behavior, is the formation and breaking of hydrogen bonds (Figure 1).

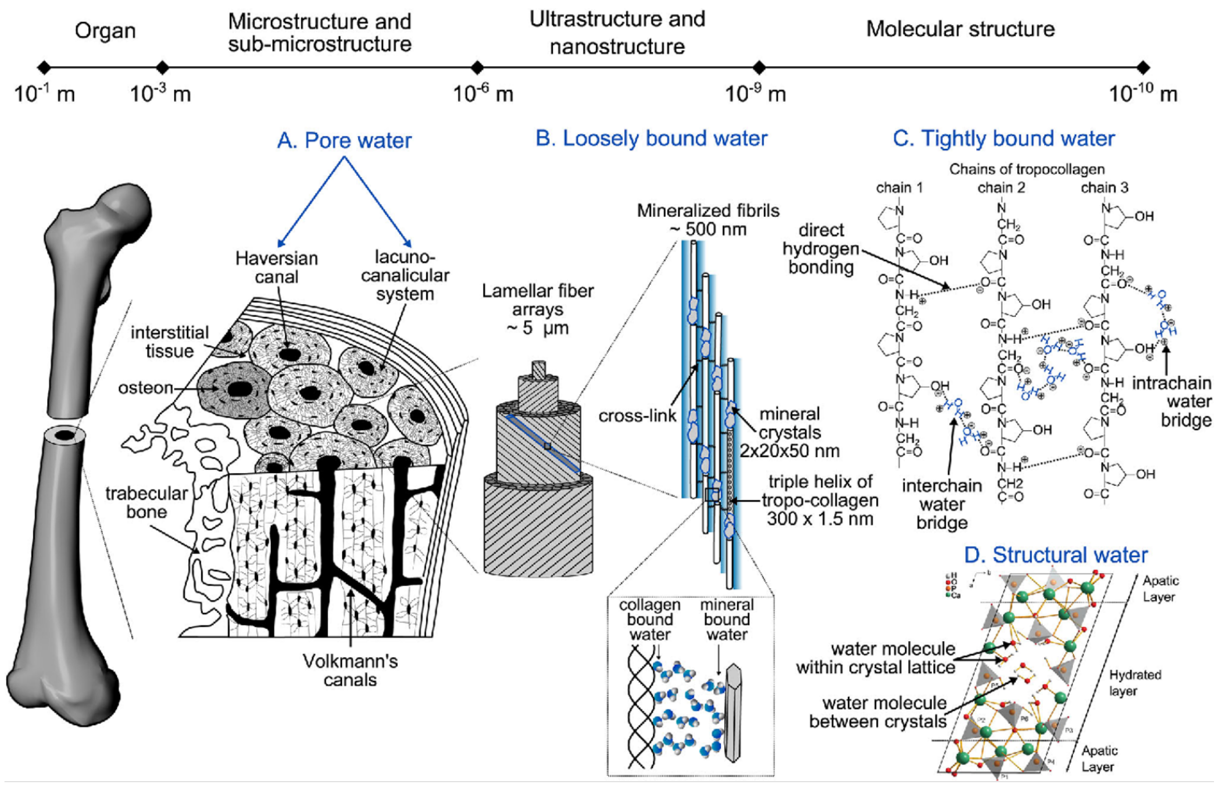

Figure 1: The complex hierarchical architecture of bone spanning of several orders of magnitude from the organ down to the molecular length scale level. Water, in bound and unbound form, play an important role in the formation of hydrogen bonds. Formation and breaking of hydrogen bonds are thought to be time dependent, and as such, this is a compelling mechanism to explain the origin of viscoelasticity in bone.

Viscoelastic materials exhibit both an elastic and a viscous response. Two approaches are available to identify, measure, and estimate the viscoelasticity of such materials: a) the oscillatory method or also known as dynamic mechanical analysis and b) stress relaxation or creep tests.

Experimentally, the oscillatory methods possess an advantage in that they can be used to describe viscoelasticity by estimating complex moduli by analyzing the experimental data (no assumptions made for material behavior or no phenomenological theory needed to analyze the data). On the other hand, stress relaxation or creep tests are analyzed employing a function developed based on a phenomenological model using dashpots and springs in series, in parallel or a combination. In addition, a stress relaxation or creep tests requires, theoretically, an instantaneous step to a constant strain or stress, respectively, but this is not possible, because an instrument has a certain loading speed limitation to how fast it can move to apply a certain strain or stress.

A. Dynamic mechanical analysis

Let us assume the sample is exposed to an oscillatory force of the form:

$$F=F_0 \sin(\omega t) \tag{1}$$

where F_0 is the maximum applied force, ω is the frequency of the oscillating force (here in radians per time) and t is the time (Figure 2).

As a response to that force, the material will be displaced by a displacement of the form:

$$d=d_0 \sin(\omega t-\delta) \tag{2}$$

where d_0 is the maximum displacement reached by the sample.

Applied forces and displacements are normalized based on the sample geometry to write the stress, σ and σ_0, and strain, ε and ε_0, as follows:

$$\sigma=\sigma_0 \sin(\omega t) \tag{3}$$

$$\varepsilon=\varepsilon_0 \sin(\omega t-\delta) \tag{4}$$

If the material behaves linearly elastic then the phase shift, δ, will be zero, and force and displacement will be *in-phase*. But if the material behaves linearly viscoelastic then the phase shift will be $\delta \in (0, \frac{\pi}{2})$.

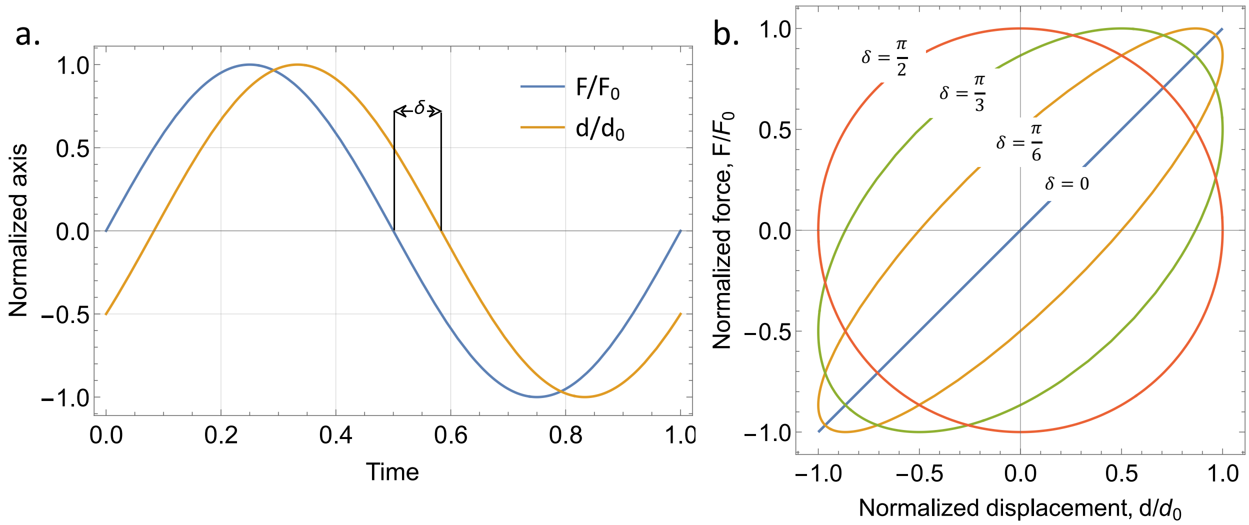

Figure 2: a. Stress and strain signals in the time domain. b. The phase shift between the applied force, or stress, and the response of displacement, or strain. The blue straight line corresponds to zero phase shift and to a purely elastic response, while the extreme case of the red circle, corresponds to a purely viscous response, resulting to a π/2 phase shift.

Example 2.1: Show that δ=0 for a purely elastic response and that δ=π/2 for a purely viscous response.

A. Considering the constitutive equation of the purely elastic response:

$$\sigma(t) = E\varepsilon(t) \Rightarrow \sigma_0 \sin(\omega t) = E\varepsilon_0 \sin(\omega t - \delta)\,, which\ is\ only\ true\ when\ \delta = 0$$

B. Considering the constitutive equation of the purely viscous response (with μ the viscosity of the dashpot):

$$\sigma(t) = \mu \frac{d\varepsilon(t)}{dt} \Rightarrow \sigma_0 \sin(\omega t) = \mu\varepsilon_0\omega\, cos(\omega t - \delta) \Rightarrow \sigma_0 \sin(\omega t)$$

$$= \mu\varepsilon_0\omega[\cos(\delta) cos(\omega t) + \sin(\delta)\sin(\omega t)], which\ is\ true\ only\ when\ \delta = \frac{\pi}{2}$$

$$Reminder:\ cos(\omega t + \varphi) = \cos(\varphi) cos(\omega t) - \sin(\varphi)\sin(\omega t)$$

$$for\ \varphi = -\delta, i.e. \cos(\omega t - \delta),\ we\ can\ then\ write$$

$$cos(\omega t - \delta) = \cos(-\delta) cos(\omega t) - \sin(-\delta)\sin(\omega t) =$$

$$= \cos(\delta) cos(\omega t) + \sin(\delta)\sin(\omega t) \overset{\delta=\frac{\pi}{2}}{\Longrightarrow} cos\left(\omega t - \frac{\pi}{2}\right) = \sin(\omega t)$$

Four parameters are defined and can be calculated from experimental data:

Storage modulus, E' $\qquad E' = \frac{\sigma_0}{\varepsilon_0} cos\delta$

Loss modulus, E" $\quad E" = \frac{\sigma_0}{\varepsilon_0} sin\delta$

Complex modulus, $E^* = E' + iE"$ $\quad E^* = E' + iE" = \frac{\sigma_0}{\varepsilon_0} e^{i\delta} = sqrt(E'^2 + E"^2)$

Loss tangent, $tan\delta$ $\quad tan\delta = \frac{E"}{E'}$

B. Stress relaxation or creep tests

Data analysis in both stress relaxation and creep tests require the formulation of an empirical function derived from a phenomenological model. These functions merely describe the mathematical relationship between the variables of the model and does not explain as to why they interact the way they do. Once an empirical function has been derived, regression analysis is employed to fit the experimental data with that function, having the constants of the model (spring constants, viscosity and/or relaxation time) as fitting parameters.

The basic principle here is to define a first order linear ordinary differential equation with the dependent variables of stress and strain, and constants the elastic modulus and viscosity. To this end, and for any mechanical model, equilibrium and kinematic equations for the system, and constitutive equations for the elements are used. Solving the first order differential equation (either stress control, i.e., stress constant, or strain control, i.e., strain constant) a function, either for stress or strain dependent on time is derived. This function is then used to fit experimental data and find the elastic modulus, viscosity, and relaxation time.

In stress relaxation tests, a strain is instantly applied to a certain value, ε_0, and is kept constant while recording the response of stress over time. In creep tests an instant stress, σ_0, is applied while the response of strain is recorded over that period of holding time. It is very important for such tests that the strain or stress are applied instantly, but experimentally this is not possible hence we need to use the fastest possible loading or displacement speed that our instrument can perform.

If the loading or displacement speeds are not fast enough, the dashpot will have time to deform and thus contribute to the response of stress before the relaxation or creep tests starts and hence the estimations of elastic modulus and viscosity will not be accurate.

B.1 Maxwell and Kelvin-Voigt material

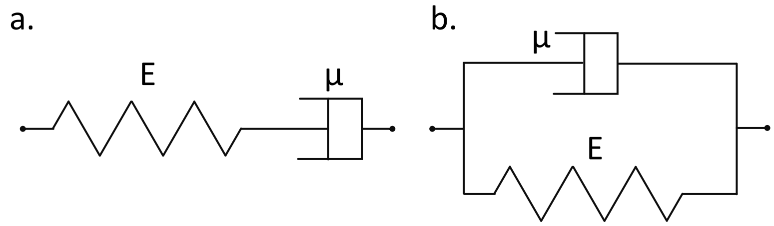

Figure 3: Sketches of the a. Maxwell material the b. Kelvin-Voigt material.

B.1.1 Maxwell material

The Maxwell material is described by a spring, with elastic modulus E, and a dashpot with viscosity μ in series (Figure 3a). The equilibrium equations for the system gives:

$$\sigma = \sigma_s = \sigma_d \tag{2.1}$$

where σ is the stress experienced by the spring and dashpot, σ_s the stress of the spring and σ_d the stress of the dashpot.

The kinematic equation is the total strain is the sum of the strains of each element.

$$\varepsilon = \varepsilon_s + \varepsilon_d \tag{2.2}$$

The constitutive equations are:

$$\sigma_s = E\varepsilon_s = \sigma \tag{2.3}$$

and

$$\sigma_d = \mu \frac{d\varepsilon_d}{d\tau} \xRightarrow{\dot{\varepsilon}_d = \frac{d\varepsilon_d}{dt}} \sigma_d = \mu\dot{\varepsilon}_d = \sigma \tag{2.4}$$

Differentiating eq. 2.2 and replacing the strain rates of the spring and dashpot from eq. 2.3 and 2.4 gives the following:

$$\dot{\sigma} + \frac{E}{\mu}\sigma = E\dot{\varepsilon} \tag{2.5.1}$$

or after rearranging the equation

$$\sigma + \frac{\mu}{E}\dot{\sigma} = \mu\dot{\varepsilon} \;\; or \; \sigma + p_1\dot{\sigma} = q_1\dot{\varepsilon} \tag{2.5.2}$$

Reminder: Solution of a linear first order ordinary differential equation
Let the differential equation: $\dot{y}(x) + p(x)y(x) = q(x)$ The solution is: $y(x) = \frac{\int u(x)q(x)dx + C}{u(x)}$

Where $u(x) = e^{\int p(x)dx}$

In the case of Eq. 2.4, $p(x) = \frac{E}{\mu}$ and $q(x) = 0$.

- Solution for stress relaxation tests

In the case of Eq. 2.4, $p(x) = \frac{E}{\mu}$, $q(x) = E\dot{\varepsilon} = 0$ and with the initial condition that $\sigma(t = 0) = \varepsilon_0 E$.

The solution of Eq. 2.4 is:

$$\sigma(t) = \frac{C\varepsilon_0 E}{e^{\left(\frac{E}{\mu}\right)t}} = \varepsilon_0 E\ e^{-\left(\frac{E}{\mu}\right)t}$$

$$\sigma(t) = \varepsilon_0 E\ e^{-\left(\frac{E}{\mu}\right)t} \tag{2.6.1}$$

Or with defining the relaxation time: τ=μ/E then 2.6 becomes:

$$\sigma(t) = \varepsilon_0 E\ e^{-t/\tau} \tag{2.6.2}$$

Where the relaxation modulus is:

$$\mathrm{E}(t) = E\ e^{-t/\tau} \tag{2.7}$$

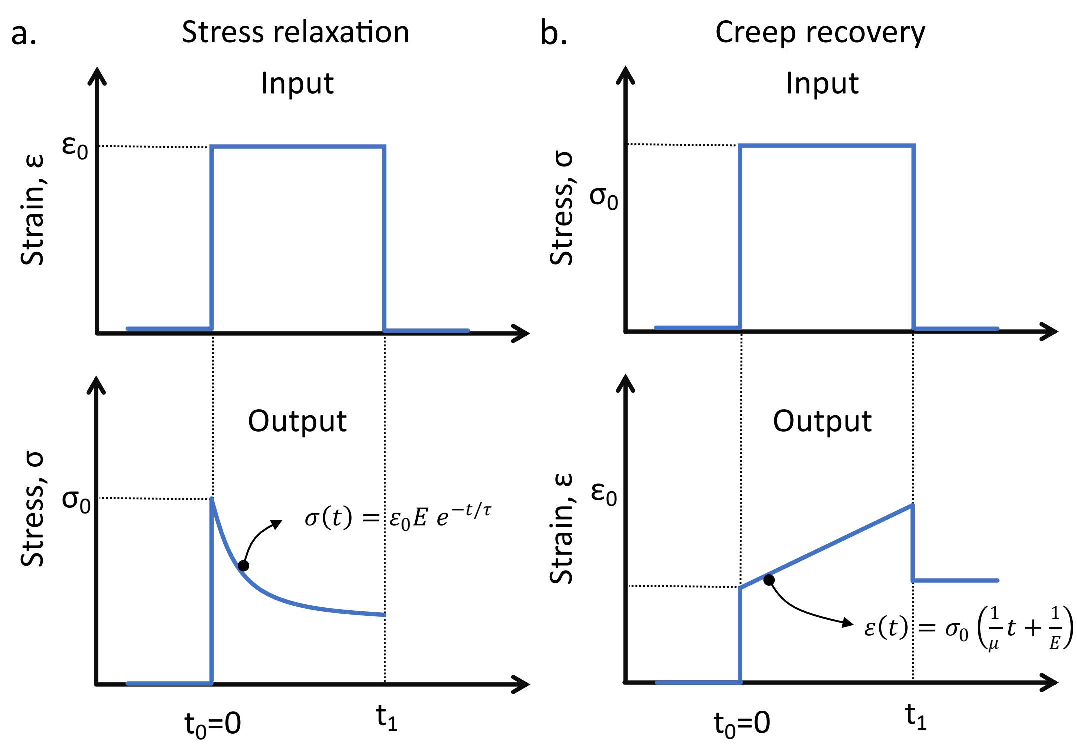

Figure 4: a. Stress relaxation and b. creep recovery of a Maxwell material.

- Solution for creep recovery tests

In creep recovery the stress input is applied instantly and kept constant at σ_0 while the strain increases. The increase of the strain in the Maxwell element is given by the solution of eq. 2.5.1:

$$\dot{\sigma} + \frac{E}{\mu}\sigma = E\dot{\varepsilon} \Rightarrow$$

$$\frac{1}{E}\frac{d\sigma_0}{dt} + \frac{1}{E}\frac{E}{\mu}\sigma_0 = \frac{d\varepsilon(t)}{dt} \Rightarrow$$

$$\frac{1}{\mu}\sigma_0 dt = d\varepsilon(t) \Rightarrow$$

$$\int d\varepsilon(t) = \int \left(\frac{1}{\mu}\sigma_0\right) dt \Rightarrow$$

$$\varepsilon(t) = \frac{1}{\mu}\sigma_0 t + C \tag{2.8}$$

where the constant C is found from the initial condition at t=t_0=0:

$$\varepsilon(t=0) = \frac{\sigma_0}{E}$$

which from eq. 2.8 we have that

$$\varepsilon(t=0) = \frac{\sigma_0}{E} = \frac{1}{\mu}\sigma_0 0 + C \Rightarrow C = \frac{\sigma_0}{E}$$

$$\varepsilon(t=0) = \frac{\sigma_0}{E} = \frac{1}{\mu}\sigma_0 0 + C \Rightarrow$$

$$C = \frac{\sigma_0}{E} \tag{2.9}$$

and therefore eq.2.8 becomes

$$\varepsilon(t) = \sigma_0\left(\frac{1}{\mu}t + \frac{1}{E}\right) \tag{2.10}$$

(see the plot in Figure 4b)

or

$$\varepsilon(t) = \sigma_0 D(t) \tag{2.11}$$

where

$$D(t) = \left(\frac{1}{\mu}t + \frac{1}{E}\right) \tag{2.12}$$

is the creep compliance.

B.1.2 Kelvin-Voigt material

Example 2.2: Given the Kelvin-Voigt material, a spring and a dashpot in parallel as shown in figure 3b:

a. *Describe the constitutive equations (same as for the Maxwell material)*
b. *Write the kinematic and equilibrium equations*
c. *And, based on a and b, show that the differential equation that describes the Kelvin-Voigt material is* $\sigma = E\varepsilon + \mu\dot{\varepsilon}$ *or* $\sigma = q_0\varepsilon + q_1\dot{\varepsilon}$
d. *Find the solutions*
    1. *For creep and*
    2. *For stress relaxation*

B.2 Generalized Maxwell model (Maxwell–Wiechert model)

One can formulate a mathematical model of more than one Maxwell (or Kelvin-Voigt) elements in parallel. Such models better describe quantitatively real materials, while the simple linear Maxwell and Kelvin-Voigt materials are mostly useful for conceptual analysis.

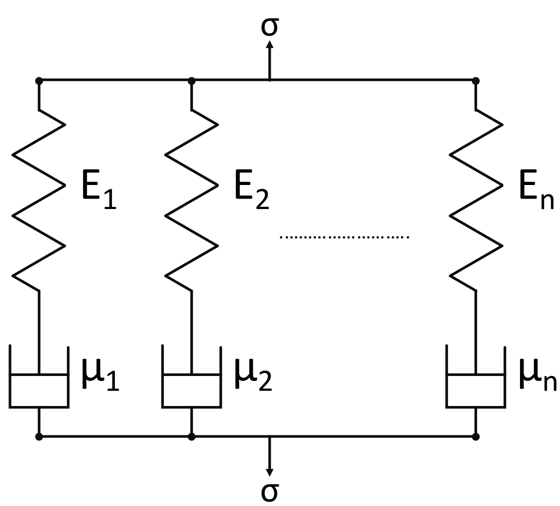

Figure 4: Maxwell elements in parallel.

If *n* parallel Maxwell elements, the differential equation is:

$$\sigma + p_1 D^1\sigma + p_2 D^2\sigma + \cdots + p_n D^n\sigma = q_1 D^1\varepsilon + q_2 D^2\varepsilon + \cdots + q_n D^n\varepsilon, with\ D^i = \frac{\partial^i}{\partial x^i} \tag{2.13}$$

The solution of the Generalized Maxwell model for a given strain input ε(t) can be found by superposition of *n* first order differential equation solutions or by solution of the n[th] order differential equation. The n first order differential equations are of the form of 2.5.1 or 2.5.2,

$$\sigma + \frac{\mu_i}{E_i} D\sigma_i = \mu_i D\varepsilon_i(t) \tag{2.14}$$

where *i* ranges from 1 to n.

Similar to the simple Maxwell model, the kinematic constrain provides that the strain in each Maxwell element, $\varepsilon_i(t)$, is the same as the global strain $\varepsilon(t)$.

The equilibrium constrain is that the global stress equals the sum of the individual stresses, $\sigma(t) = \sum_{i=0}^{n} \sigma_i(t)$. For the condition of stress relaxation, $\varepsilon(t) = \varepsilon_0$ (for a defined time interval), the solution of these linear differential equations is found to be:

$$\sigma(t) = \varepsilon_0 \sum_{i=1}^{n} E_i \, e^{t/\tau_i} \tag{2.15}$$

where $\tau_i = \mu_i / E_i$, the i^{th} relaxation time and:

$$E(t) = \sum_{i=1}^{n} E_i \, e^{t/\tau_i} \tag{2.16}$$

the relaxation modulus.

It is very common to use a free spring in parallel to the Maxwell elements (known as the Wiechert model), so to limit the fluid-like behavior of the Maxwell elements in series.

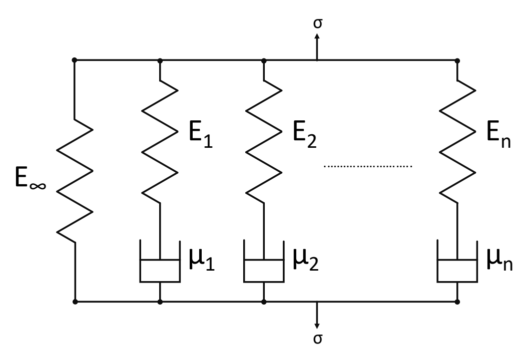

Figure 5: Wiechert model.

The solution for stress relaxation and the relaxation modulus are:

$$\sigma(t) = \varepsilon_0 \left(E_\infty + \sum_{i=1}^{n} E_i \, e^{t/\tau_i} \right) \tag{2.17}$$

$$E(t) = E_\infty + \sum_{i=1}^{n} E_i \, e^{t/\tau_i} \tag{2.18}$$

Where E_∞ is the equilibrium modulus.

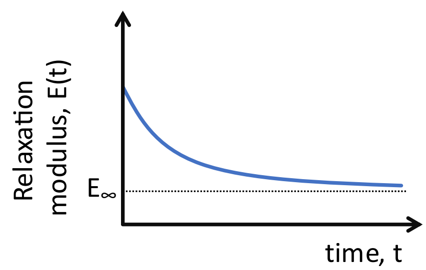

Figure 5: Relaxation modulus

Read more about viscoelasticity

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3. Marques, Severino PC, and Guillermo J. Creus. "Rheological models: integral and differential representations." Computational Viscoelasticity. Springer, Berlin, Heidelberg, 2012. 11-21.