Phys 232
Problem set 2

Problem 1)

$$m\ddot{x} + m\Gamma\dot{x} + kx = 0$$

$$\ddot{x} + \Gamma\dot{x} + \underbrace{\frac{k}{m}}_{\omega_0^2}x = 0$$

Solution to damped oscillations (derived in class) is

$$x(t) = A_0 e^{-\Gamma t/2} \cos(\omega t - \varphi)$$

So the amplitude is $A(t) = A_0 e^{-\Gamma t/2}$

We know the frequency $f = 0.5\,Hz$, and that after one cycle (i.e. when $t = T$ (period) $= \frac{1}{f}$). Amplitude is reduced by a factor of $e^{-1.5}$

$$\Rightarrow \frac{A(T)}{A_0} = e^{-\Gamma T/2} = e^{-1.5}$$

$$\Rightarrow \frac{-\Gamma}{2f} = -1.5$$

$$\Rightarrow \boxed{\Gamma = 1.5\,s^{-1}}$$

We also know that $\omega = 2\pi f = 2\pi \cdot 0.5 = \pi \text{ rad } s^{-1}$

and $\omega = \sqrt{\omega_0^2 - \left(\frac{\Gamma}{2}\right)^2}$

$$\Rightarrow \omega^2 = \omega_0^2 - \left(\frac{\Gamma}{2}\right)^2$$

$$\Rightarrow \omega_0 = \sqrt{\omega^2 + \left(\frac{\Gamma}{2}\right)^2} = \sqrt{\pi^2 + \left(\frac{1.5}{2}\right)^2} \simeq 3.23\,s^{-1}$$

finally, $\omega_0^2 = \frac{k}{m}$

$$\Longrightarrow k = m\omega_0^2 = 5 \times \left(\pi^2 + \left(\frac{1.5}{2}\right)^2\right)$$

$$\Longrightarrow \boxed{k \cong 52.16 \ Nm^{-1}}$$

Problem 2)

$$\ddot{\theta} + \Gamma\dot{\theta} + \frac{g}{\ell}\theta = 0$$

a) $\omega = \sqrt{\omega_0^2 + (\frac{1}{2}\Gamma)^2}$

$= \sqrt{\frac{g}{\ell} - (\frac{1}{2}\Gamma)^2}$

$= \sqrt{\frac{9.8}{0.4} - (\frac{1}{2}\cdot 0.2)^2}$

$$\boxed{\omega \cong 4.949\ s^{-1}}$$

$\omega_0 = \sqrt{\frac{g}{\ell}} = \sqrt{\frac{9.8}{0.4}} \simeq 4.950\ s^{-1}$

ω is only smaller by 0.01 so $\omega \approx \omega_0$.

b) the quality factor is $Q = \frac{\omega_0}{\Gamma} = \frac{4.950\,s^{-1}}{0.2\,s^{-1}} = \underline{\underline{24.75}}$

c) We know from class that the general solution to an underdamped oscillator is

$$\theta(t) = e^{-\Gamma t/2}\left(a\cos(\omega t) + b\sin(\omega t)\right)$$

where a, b are found from initial conditions.

The initial conditions are $\theta(0) = 0$ m

and at $t=0$ $E_k = \frac{1}{2} m \ell^2 \dot{\theta}^2 = 1.5$

$$\Longrightarrow \dot{\theta}(0) = \sqrt{\frac{3}{m\ell^2}} = \sqrt{\frac{3}{3.(0.4)^2}} = 2.5 \text{ m/s}$$

$$\theta(0) = e^{-0}\left(a\cos(0) + b\,\cancel{\sin(0)}\right) = 0$$

$$\Longrightarrow \boxed{a = 0}$$

$$\Longrightarrow \theta(t) = e^{-\Gamma t/2}\, b\sin(\omega t)$$

$$\dot{\theta}(t) = \frac{-\Gamma}{2} e^{-\Gamma t/2}\, b\sin(\omega t) + e^{-\Gamma t/2}\, b\omega\cos(\omega t)$$

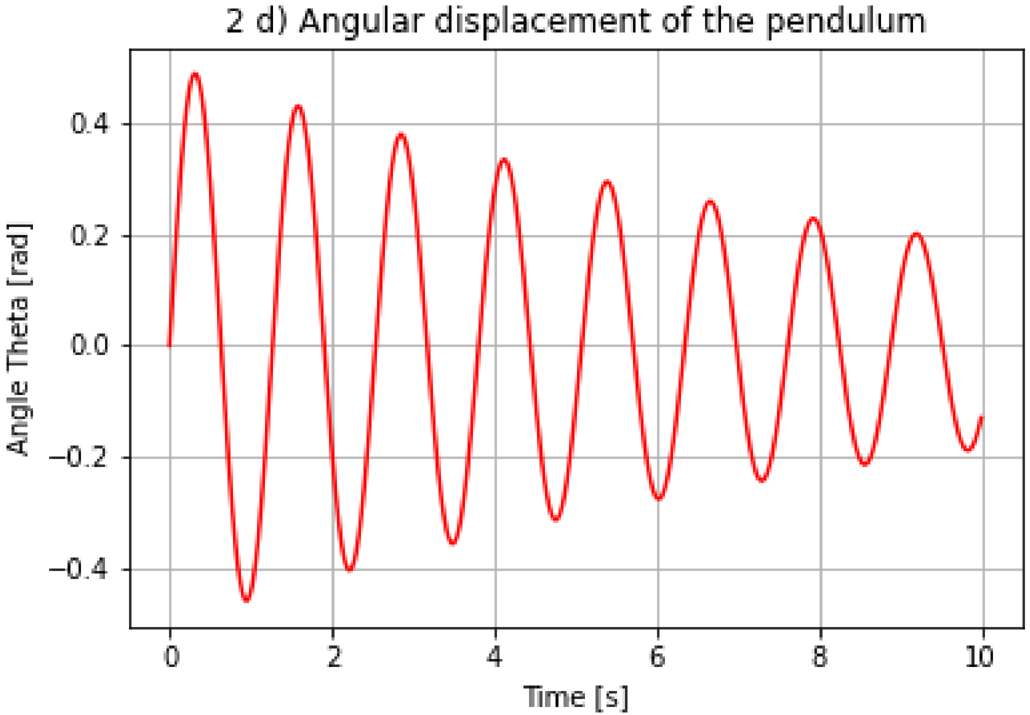
$$\dot{\theta}(0) = \frac{-\Gamma}{2} e^{0} b\,\cancel{\sin(0)} + e^{0} b\omega\cos(0) = 2.5$$

$$\Longrightarrow b = \frac{2.5}{\omega} = \frac{2.5}{4.949} \simeq 0.505$$

So the formula for $\theta(t)$ is

$$\boxed{\theta(t) = 0.505 e^{-0.1t} \sin(4.949t)}$$

d) Plot for $t = 0$ s to $t = 10$ s attached.


2 d) Angular displacement of the pendulum
Angle Theta [rad]
Time [s]
0.4
0.2
0.0
−0.2
−0.4
0
2
4
6
8
10

Problem 3)

a) $$m\ddot{Y} + m\Gamma\dot{Y} + \frac{2TY}{\sqrt{Y^2+L^2}} = 0$$

$$\Rightarrow \ddot{Y} + \Gamma\dot{Y} + \frac{2TY}{m\sqrt{Y^2+L^2}} = 0. \qquad \omega_0 = \sqrt{\frac{2T}{mL}}$$

Let $\hat{t} = t\omega_0$ and $y = \frac{Y}{L} \Rightarrow Y = yL$

Then $\dot{Y} = \frac{dY}{dt} = \frac{dY}{d\hat{t}}\frac{d\hat{t}}{dt}$ (chain rule)

$$= \frac{d}{d\hat{t}}(yL)\frac{d}{dt}(t\omega_0) = \dot{y}L\omega_0$$

$$\ddot{Y} = \frac{d^2Y}{dt} = \frac{d^2Y}{d\hat{t}^2}\left(\frac{d\hat{t}}{dt}\right)^2 + \frac{dY}{d\hat{t}}\left(\frac{d^2t}{d\hat{t}}\right)$$

$$= \ddot{y}L\omega_0^2 + \dot{y}L(0)$$

$$= \ddot{y}L\omega_0^2$$

Then, plugging $\dot{Y}$ and $\ddot{Y}$ and substituting in the Y's in the EOM, we have

$$\ddot{y}L\omega_0^2 + \Gamma\dot{y}L\omega_0 + \frac{2T}{m}\cdot\frac{Y}{L\sqrt{1+\frac{Y^2}{L^2}}} = 0$$

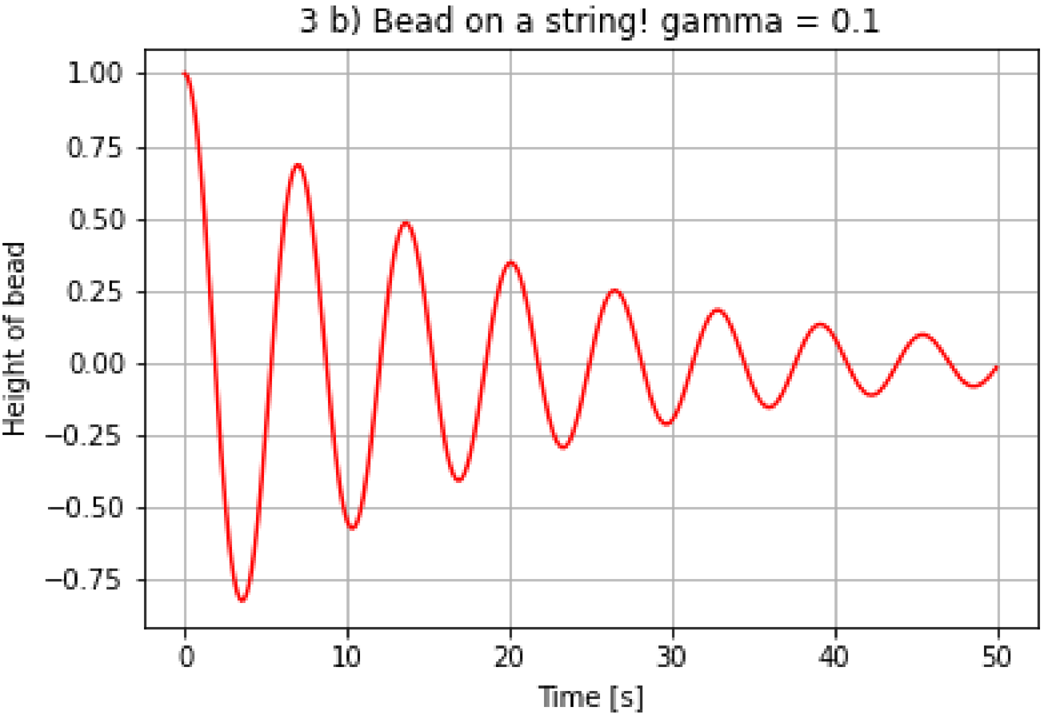
$$\Rightarrow \ddot{y}L\omega_0^2 + \Gamma\dot{y}L\omega_0 + \frac{2T}{m}\cdot\frac{y}{\sqrt{1+y^2}} = 0$$

$$\ddot{y} + \frac{\Gamma \dot{y}}{\omega_0} + \frac{2T}{mL\omega_0^2} \cdot \frac{y}{\sqrt{1+y^2}} = 0$$

$$\ddot{y} + \frac{\Gamma \dot{y}}{\sqrt{\frac{2T}{mL}}} + \frac{2T}{mL} \cdot \frac{mL}{2T} \cdot \frac{y}{\sqrt{1+y^2}} = 0$$

So, $\boxed{\ddot{y} + \gamma \dot{y} + \frac{y}{\sqrt{y^2+1}} = 0}$ where $\gamma \equiv \frac{\Gamma\sqrt{mL}}{\sqrt{2T}}$

b) Plot for $\hat{f}=0$ to $\hat{f}=50$, with $\gamma = 0.1$ and $y(0)=1$, $\dot{y}(0)=0$ attached.


3 b) Bead on a string! gamma = 0.1
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
Height of bead
0
10
20
30
40
50
Time [s]

c) What is Y at $t = 3.2s$?

$$\hat{t} = \omega_0 t = \sqrt{\frac{2T}{mL}}\, t = \sqrt{\frac{2.50}{0.8 \cdot 3}}\,(3.2) \cong 20.666$$

We can use the numerical solution from b) to find y at $\hat{t} = 20.66$

This gives us $y \cong 0.299$.

We can use $y = \frac{Y}{L}$ to find Y.

$$\Longrightarrow Y = yL = 0.299 \times 0.8m$$

$$\Longrightarrow \boxed{Y = 0.239m}$$

d) $$\ddot{y} + \dot{y}\gamma + \frac{y}{\sqrt{y^2+1}} = 0.$$

for small y: $\sqrt{y^2+1} \sim 1 + \cancel{\frac{y^2}{2}}$ very small ~ 1

So the EOM is

$$\ddot{y} + \gamma\dot{y} + y = 0$$

we know how to solve that, the general solution is

$$y = e^{-\gamma t/2}\left(a\cos(\omega t) + b\sin(\omega t)\right)$$

a, b and determined from initial conditions and these conditions are $y(0) = 1$ and $\dot{y}(0) = 0$.

$$y(0) = e^0 (a\cos(0) + b\cancel{\sin(0)}^{\to 0}) = 1$$

$$a = 1$$

$$y(t) = e^{-\gamma t/2} (\cos(\omega t) + b\sin(\omega t))$$

$$\dot{y}(t) = -\frac{\gamma}{2} e^{-\gamma t/2} \cos(\omega t) - e^{-\gamma t/2} \omega \sin(\omega t)$$

$$- \frac{\gamma}{2} e^{-\gamma t/2} b \sin(\omega t) + e^{-\gamma t/2} b\omega \cos(\omega t)$$

$$\dot{\gamma}(0) = -\frac{\gamma}{2} e^0 \cos(0) - e^0 \cancel{\omega \sin(0)}^{\to 0} - \frac{\gamma}{2} e^0 \cancel{b \sin(0)}^{\to 0}$$

$$+ e^0 b\omega \cos(0) = 0$$

$$-\frac{\gamma}{2} + b\omega = 0$$

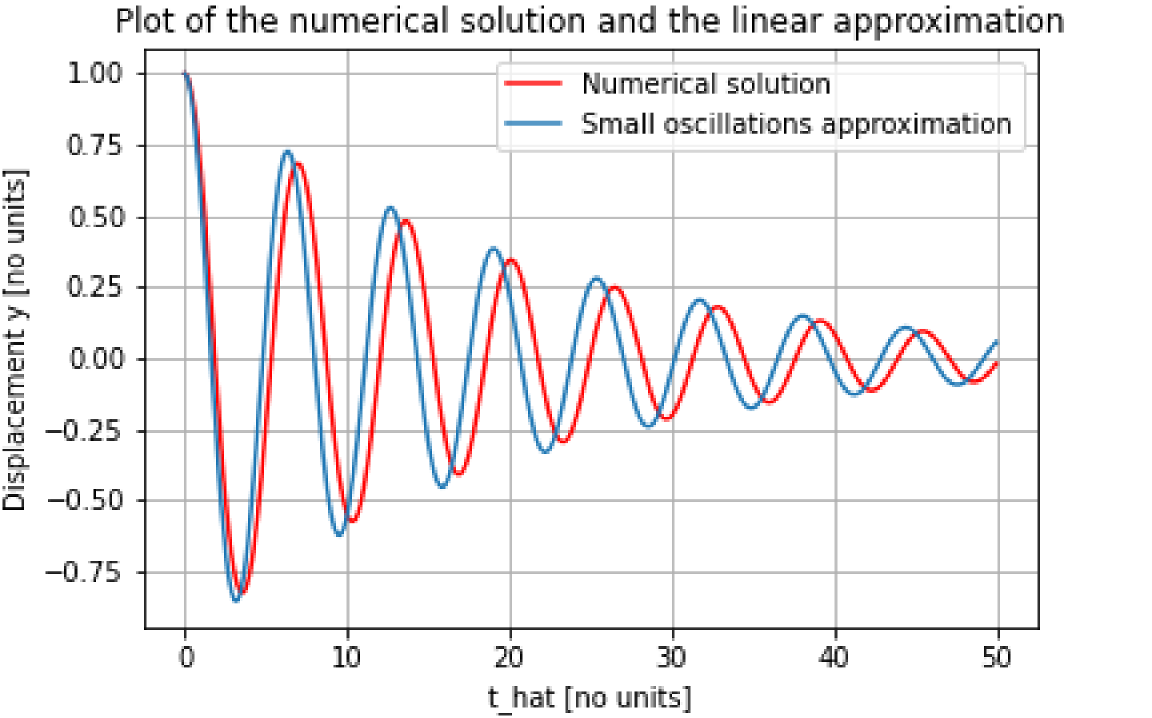
$$\Rightarrow b = \frac{\gamma}{2\omega} = \frac{\gamma}{2\sqrt{1-\left(\frac{\gamma}{2}\right)^2}} = \frac{0.1}{2\sqrt{1-\left(\frac{0.1}{2}\right)^2}}$$

$$b = 0.05$$

So $y(t)$ for small oscillations is

$$y(t) = e^{-0.1t/2} \left(\cos(0.99t) + 0.05\sin(0.99t)\right)$$

We can plot this alongside the numerical solution. Attached.


Plot of the numerical solution and the linear approximation
Numerical solution
Small oscillations approximation
Displacement y [no units]
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
0
10
20
30
40
50
t_hat [no units]

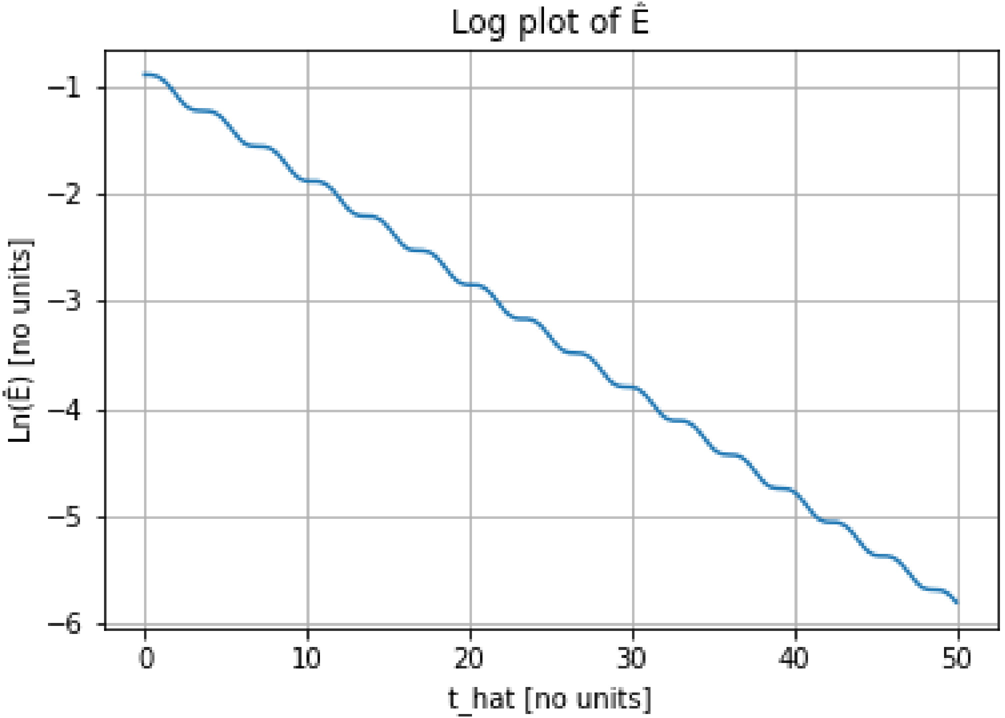
It seems like the two solutions start in phase, but start to separate more and more as time advances. The blue curve (numerical solution) oscillates more rapidly than the real one (linear approximation). Both have the same values for all parameters, so it seems like non-linearity changes the angular frequency ω of vibrations (ω increases in this case).

e) $$\tilde{E} = \frac{\dot{y}^2}{2} + \sqrt{1+y^2} - 1$$

for a linear approximation, we expect the energy to decrease as $e^{-\gamma t}$. Since the amplitude goes as $y \sim e^{-\gamma t/2}$, energy goes as $y^2 \sim (e^{-\gamma t/2})^2 \sim e^{-\gamma t}$.

So the exponential decay will be $e^{-0.1t}$ for energy, in the linear approximation regime.

However, since we know $\dot{y}(t)$ and $y(t)$ from our numerical solution in b), we can make a log plot of $\tilde{E}(t)$. The plot is attached.


Log plot of Ê
Ln(Ê) [no units]
t_hat [no units]
−1
−2
−3
−4
−5
−6
0
10
20
30
40
50

It almost looks like a linear plot with slope ~ -0.14 which makes sense, because our linear approximations yielded $e^{-0.1t}$, which has a slope of -0.1 when we take the log. Of course there are bumbs in the log plot of $\hat{E}$, but that's probably due to non-linearity.

Python code for assignment 2

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
from scipy.optimize import curve_fit


# Question 2, d)

def theta(t):
    return 0.505*np.exp(-0.1*t)*np.sin(4.949*t)

a = np.linspace(0, 10, 1000)
position = theta(a)

plt.plot(a, position, 'r')
plt.xlabel('Time [s]')
plt.ylabel('Angle Theta [rad]')
plt.grid()
plt.title('2 d) Angular displacement of the pendulum')
plt.savefig('Q2d)')
plt.show()


# Question 3, b)

# Solving the ODE
def eom(t, state):
    y, v = state

    return [v, -gamma*v -y/np.sqrt(y**2+1)]

gamma = 0.1
sol = solve_ivp(eom, [0, 50], [1,0], dense_output= True)

t = np.linspace(0, 50, 400)

#plotting
plt.plot(t, sol.sol(t)[0], "r")
plt.xlabel("Time [s]")
plt.ylabel('Height of bead')
plt.title("3 b) Bead on a string! gamma = 0.1")
plt.grid()
plt.savefig('Q3b)')
plt.show()

sol.sol(20.66) # ([ 0.29861052, -0.16729394])


# Question 3, d)

# y(t) for the linear approximation
def linear(t):
    return np.exp(-0.05*t)*(np.cos(0.99*t) + 0.05*np.sin(0.99*t))
```

```
t = np.linspace(0, 50, 400)

# Plotting
plt.plot(t, sol.sol(t)[0], "r", label = "Numerical solution")
plt.plot(t, linear(t), label = "Small oscillations approximation")
plt.grid()
plt.legend()
plt.title('Plot of the numerical solution and the linear approximation')
plt.xlabel("t_hat [no units]")
plt.ylabel('Displacement y [no units]')
plt.savefig('Q3 d)')
plt.show()


# Question 3, e)

# Energy as a function of time
def energy(a):
    return (sol.sol(a)[1] ** 2)/2 + np.sqrt(1 + sol.sol(a)[0] ** 2) - 1

# Linear fit on the energy just to make sure the value is close to -0.1
def linear(x, a, b):
    return a*x+ b

popt, pcov = curve_fit(linear, t, np.log(energy(t)))

# popt = [-0.099387   -0.85041396]
# The slope of the linear fit is -0.099387, which is almost -0.1, as expected for a linear approximation

# Plotting
plt.plot(t, np.log(energy(t)),)
plt.grid()
plt.xlabel("t_hat [no units]")
plt.ylabel('Ln(ÃŠ) [no units]')
plt.title("Log plot of ÃŠ")
plt.savefig("Q3 e)")
plt.show()
```