A Rose by Any Other Name…

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Abstract

Rhodonea curves, also known as rose curves, have intrigued mathematicians and artists alike since they were first described by Guido Grandi in the 18th century. In the late 20th century Maurer roses, closed polylines derived from rhodonea curves, were introduced. They are notable for the striking patterns they produce from a simple algorithm. Although Maurer roses have often been re-implemented, to date there is little published work on extending the concept since it was first described. In this paper we review previous work, then use that foundation to explore a number of extensions and generalizations of Maurer roses that we use to generate aesthetically pleasing forms.

Rhodonea Curves

Rhodonea curves (or rose curves) were first described by Guido Grandi in 1722 [1], and in modern times are often used in the classroom as an example when introducing polar coordinates. A rhodonea curve is a planar curve defined by the polar equation $r = \cos(k\theta)$. Converting from polar to Cartesian coordinates the equation becomes: $x = \cos(k\theta)\cos(\theta)$, $y = \cos(k\theta)\sin(\theta)$. The name rhodonea comes from the similarity of its shape to a flower with petals, as shown in Figure 1 for different values of k.
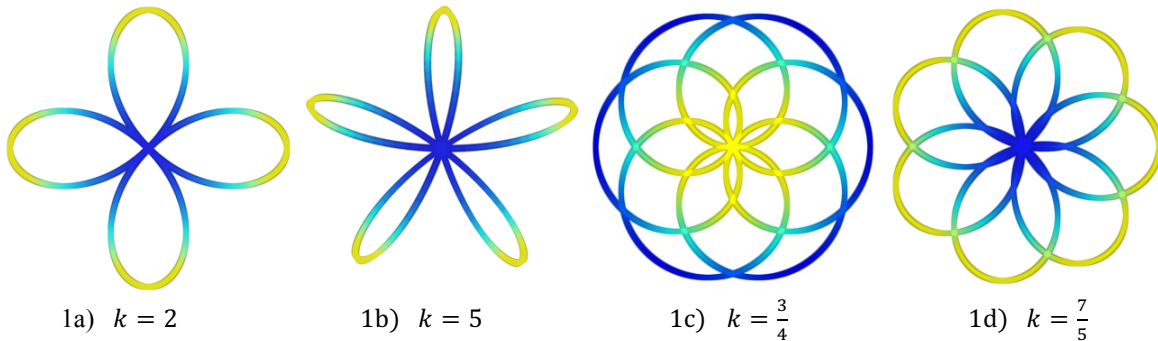

| 1a) $k = 2$ | 1b) $k = 5$ | 1c) $k = \frac{3}{4}$ | 1d) $k = \frac{7}{5}$ |

Figure 1: *Rhodonea curves with different values of k. Shading is based on the speed of a point travelling along the curve with a light (slow) to dark (fast) gradient.*

A number of other interesting properties are also illustrated above. If k is an even integer the curve has $2k$ petals (Fig.1a), and if k is an odd integer it has k petals (Fig.1b). If k is rational and thus can be represented as an irreducible fraction $k = a/b$ then if either a or b is even the curve has *2a* petals and closes (starts repeating) at $\Delta\theta = 2\pi b$ (Fig.1c), otherwise it has a petals and closes at $\Delta\theta = \pi b$ (Fig.1d). If k is irrational the curve never closes, thus there are an infinite number of petals (as $\Delta\theta \rightarrow \infty$) and the curve fills the unit disk (Figure 2). We will refer to even, odd, integer, rational and irrational rhodonea based on the value of k, and we use **R**$\{k\}$ to designate the corresponding rhodonea curve and c[k] for the $\Delta\theta$ at curve closure.
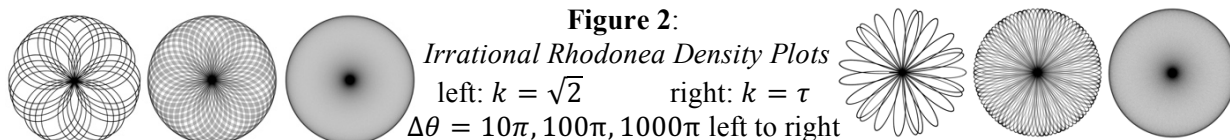

Figure 2:
Irrational Rhodonea Density Plots
left: $k = \sqrt{2}$ right: $k = \tau$
$\Delta\theta = 10\pi, 100\pi, 1000\pi$ left to right

Maurer Roses

Maurer roses were first introduced by Peter Maurer in 1987 [2]. The central idea is to construct a polyline (a connected sequence of straight lines) from a walk along an integer rhodonea curve. Start the walk with any point on the rhodonea curve, with angle parameter $\theta = \theta_0$. Follow the rule that for each successive point in the sequence of the walk, θ is incremented by a fixed amount d to find the next point in the walk. The sequence ends when the walk returns to the original point. Thus the polyline forms a closed polygonal curve, which is the Maurer rose. We use $\mathbf{M}\{k, d\}$ to designate a Maurer rose of incremental angle d, based on the rhodonea curve $\mathbf{R}\{k\}$ (since in this paper we always set $\theta_0=0$, we simplify by omitting this parameter). The first few steps in this process are illustrated in Figure 3a and 3c. Surprising patterns and intriguing envelopes for subsets of line segments can emerge, as shown in Figure 3b and 3d.
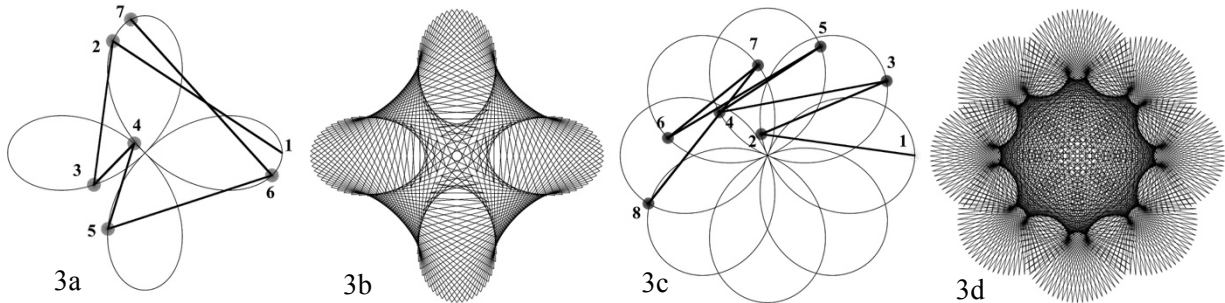

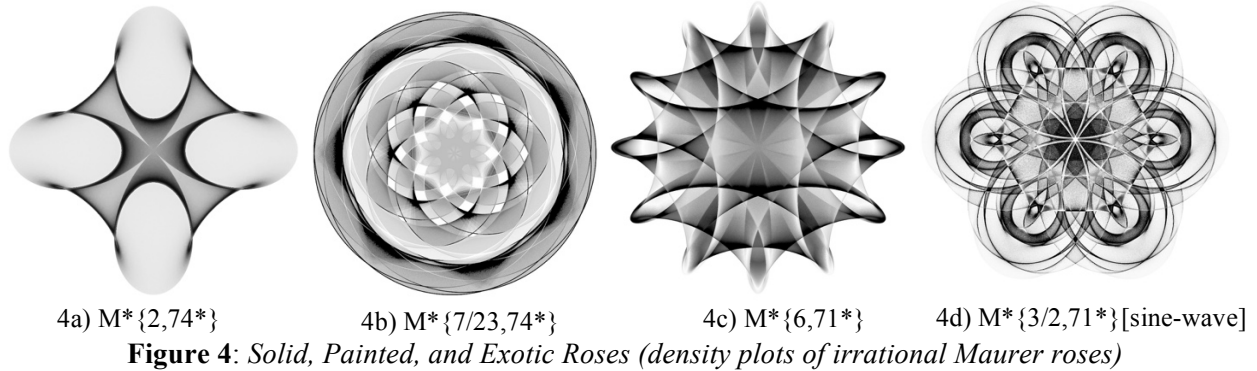**Figure 3**: *Maurer roses and their construction.*
3a) $\mathbf{M}\{2, 74°\}$, *first 6 steps (7 endpoints) of walk.* 3b) $\mathbf{M}\{2, 74°\}$, *walk finished in 180 steps*
3c) $\mathbf{M}\{\frac{4}{3}, 74°\}$, *first 7 steps (8 endpoints) of walk.* 3d) $\mathbf{M}\{\frac{4}{3}, 74°\}$, *walk finished in 540 steps*

The original publication describing Maurer roses [2] showed that for any division of 2π into integer z equal sections, and any d an integer multiple of $2\pi/z$, if d and z are relatively prime then the Maurer rose curve will close at step z of the walk. Thus the Maurer rose will be composed of z lines. If d and z are integer but not relatively prime, then the Maurer rose is degenerate and closes at $z/\gcd(d{:}z)$, where $\gcd(d{:}z)$ is the greatest common divisor of d and z. Typically z is set to be the number of standard degrees in a circle, $z=360$. So for example, for $\mathbf{M}\{2, 74°\}$ in Figure 3b, the walk completes in $360/\gcd(74{:}360) = 180$ steps. We have extended this work to cover fractional roses, which were not considered in [2]. We do this by replacing 2π in the above walk length calculation with the c[k] for the associated $\mathbf{R}\{k\}$ fractional rhodonea curve to close. Thus to work in standard degrees we define $z =(c[k]*360)/2\pi$. So for $\mathbf{M}\{\frac{4}{3}, 74°\}$ in Figure 3c, we have $\mathbf{R}\{\frac{4}{3}\}$, and $c\left[\frac{4}{3}\right]=6\pi$, so $z=1080$, thus the walk completes in $1080/\gcd(74{:}1080) = 540$ steps. When not using standard degrees, we include z in the Maurer rose definition, which becomes $\mathbf{M}\{k,d,z\}$.

Solid Roses (Irrational Maurer Roses), Painted Roses, Exotic Roses

Looking again at Figure 2 we can see that although an irrational rhodonea curve fills the unit disk, it does not fill evenly. As $\Delta\theta \to \infty$, the density of points from the curve is greater both at the center of the disk and at the edges. This observation motivated us to explore the idea of irrational Maurer roses. We add a small irrational term to the Maurer rose incremental angle d to get irrational $d*$ (usually we use d*=d+0.01e), and thus create an irrational Maurer rose $\mathbf{M}*\{k, d*\}$. Like irrational rhodonea the irrational Maurer rose polyline will also never close, so we simply choose a large $\Delta\theta$ at which to end the polyline. Examples of irrational Maurer roses are shown in Figure 4 as density plots. They possess a combination of properties we find appealing. Namely that a Maurer rose $\mathbf{M}$ of interest can be easily converted to an irrational Maurer rose $\mathbf{M}*$ whose overall form strongly resembles the original (compare Figure 4a to its rational counterpart in Figure 3b), but that can be used in density plot form as an artistic element in settings that call for something different than the dense line art of rational Maurer roses.

446

We have also explored applying color and grayscale mappings to these density plots based on various mathematical properties of **M*** to create "painted roses". For example in Fig 4b and 4d, each line segment of the walk is shaded based on the length of the segment. And in Fig 4c each line segment is shaded based on the angle of the line (relative to the x-axis). Furthermore we have experimented with "exotic roses", where each polyline is replaced with a different shape. In Fig 4d for example sine waves are used in place of polylines.

| 4a) M*{2,74*} | 4b) M*{7/23,74*} | 4c) M*{6,71*} | 4d) M*{3/2,71*}[sine-wave] |

Figure 4: *Solid, Painted, and Exotic Roses (density plots of irrational Maurer roses)*

P-Curves (Generalized Maurer Polylines)

We can generalize the strategy for constructing a Maurer rose from a rhodonea curve to apply to any $\mathbb{R}^n$ curve parameterized by a single changing variable t. For any $\mathbb{R}^n$ curve **C** defined parametrically by $\mathbf{f}(t)$ we define a polyline curve **P** as the **C**-based analog of the **R**-based Maurer rose **M**. The sequence of endpoints for the polyline walk are determined by three parameters: n, the number of lines; t_0 the starting t for the walk; and d the increment of t between points. Thus for all points $\mathbf{p}_0\ldots\mathbf{p}_{n+1}$ in the walk, $\mathbf{p}_m = \mathbf{f}(t_0+md)$ and we designate the polyline by $\mathbf{P_N}\{\mathbf{C}, t_o, d, n\}$. For curves that close it can be more elegant to reintroduce z as the number of divisions of the curve up till closure, from which n can be calculated, and for this paper we can simplify by setting $t_0=0$, giving another form for closed curves: $\mathbf{P_Z}\{\mathbf{C}, d, z\}$. And the Maurer rose $\mathbf{M}\{k,d\}$ can be redefined as $\mathbf{P_Z}\{\mathbf{R}\{k\}, d, z\}$. Figure 5 shows examples of **P** curves constructed from various base curves: 5a) circle, 5b) Fay butterfly [3], 5c) Gielis super-rose [4], 5d) Farris "mystery" curve [5].

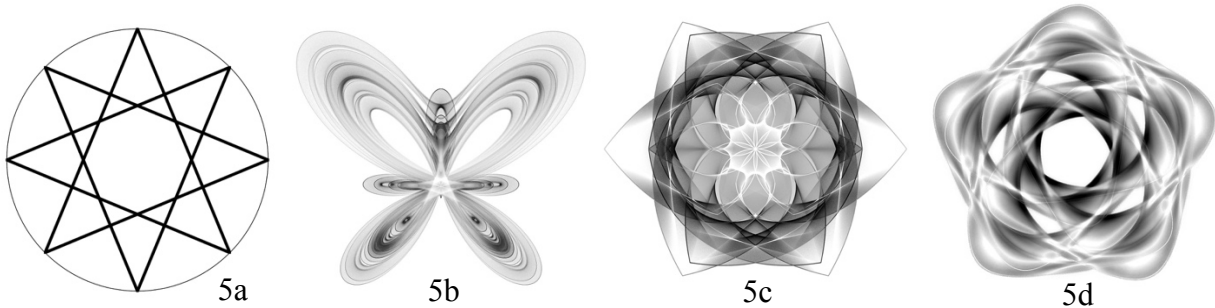| 5a | 5b | 5c | 5d |

Figure 5. *P-curves constructed from various base parametric curves*

Figure 5a is simple but illuminating, the curve $\mathbf{P_Z}\{\mathbf{C}, 3, 8\}$ where **C** (also shown) is the unit circle. However this curve is also exactly the octagram, the star polygon designated with Schläfli symbol {8/3}. Thus we see that in the case of the circle, for integer d and z, $\mathbf{P_Z}\{\mathbf{C}, d, z\}$ is the same as the star polygon $\{p/q\}$ where $p=z$ and $q=d$. Furthermore the degenerate $\mathbf{P_Z}$ (where d, z are not relatively prime) are the same as star figures (also known as "improper" star polygons) [6]. So by generalizing Maurer roses we see that they are closely related to star polygons. This is promising, as there is a considerable body of existing work applying ideas from abstract algebra to star polygons, and in future publications we plan to explore applying some of these ideas to Maurer roses and the more generalized **P**-curves.

In our research into generalized **P**-curves we have found a number of publications where similar techniques are applied to specific curves. For example, in [7] Sharp uses an approach similar to Maurer roses but applied to Rigge envelope rosettes. However we are not aware of any previous publication that considers the generalized case in detail.

P-Curves as Curve Approximation and Interpolation

Another way to think of the curve $\mathbf{P_N}\{\mathbf{C}, t_o, d, n\}$ is as an approximation of **C** based on the sequence of $n+1$ vertices from **C** that the walk is constructed from, where the polylines represent piecewise linear interpolation between each pair of successive points. With this in mind we have experimented with replacing the polylines with other forms of approximating a curve from the walk vertices, including interpolation with cubic Hermite splines using exact tangents, and circle splines [8], with intriguing results. The first four steps of the walks using these methods are shown in Figure 6a. Completed curves are shown in Figure 6b and 6c (compare to the polyline version in Figure 3b).
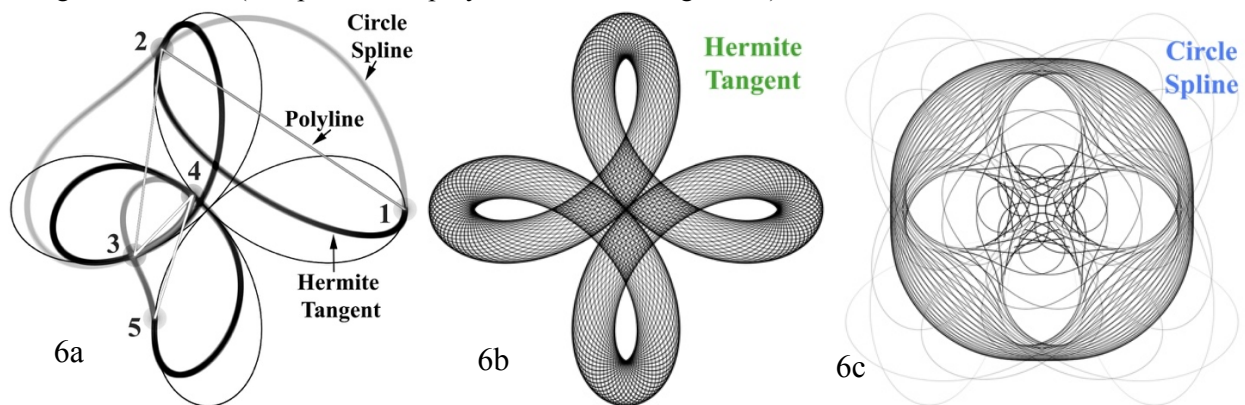

Figure 6. *Replacing **P**-curve polylines with spline curve approximations for Maurer rose **M**{2,74°}*

Source code for our work is available on GitHub as a plugin for JWildfire [9], a freely available open source application for creating algorithmic art, and will soon be integrated into the standard JWildfire application. Additional information and examples are available at http://genomancer.org.

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