A Rose is a Rose...

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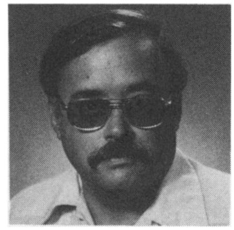


PETER M. MAURER received his B.A. in mathematics in 1969 from St. Benedict's College in Atchison, Kansas. After three years of writing computer programs for the U.S. Army and five more years writing programs for the state government of Iowa, he returned to school at Iowa State University. He received his M.S. in 1979 and his Ph.D. in 1982, both in Computer Science. Since 1982 he has been employed by AT & T Bell Laboratories and AT & T Information Systems. His main professional interest is using mathematics to solve problems in the design of microprocessors.

1. Introduction. The past two decades have seen massive advances in computer graphics. Computer-generated pictures have progressed from simple line drawings [1] to breathtakingly beautiful full-color displays [2]. However, sometimes even a simple line-drawing program can exhibit interesting, and unexpected, mathematical behavior. "The Rose" is an example of such a program. This program has been used as a demo for AT & T's "DMD 5620" terminal [3] and other sophisticated equipment. It gets its name from the polar-coordinate graph of the function $r = \sin(n\theta)$, where n is a positive integer. The graph of this function is an n-petaled rose if n is odd, and a $2n$-petaled rose if n is even, as demonstrated by Figure 1.

2. The basic algorithm. "The Rose" uses the following algorithm, called algorithm-A, to display polygons inscribed in n-petaled and $2n$-petaled roses.

1. Choose integers n, d such that $1 \leqslant n \leqslant 359$ and $1 \leqslant d \leqslant 359$.
2. Set θ equal to zero, and set (*oldx*, *oldy*) to $(0,0)$.
3. Set θ equal to $\theta + \text{d}$. If $\theta \geqslant 360$ replace θ by the remainder obtained when dividing θ by 360. (That is, reduce mod 360.)
4. Compute $n\theta$, reduce it mod 360, convert the result from degrees to radians, and set x equal to the final result.
5. Set r equal to the sin of x.
6. Convert θ from degrees to radians, and set t equal to the result.
7. Convert the point (t, r) from polar to rectangular coordinates to obtain the point (*newx*, *newy*).
8. Draw a line from (*oldx*, *oldy*) to (*newx*, *newy*).
9. If θ is equal to zero then stop, else set (*oldx*, *oldy*) to (*newx*, *newy*) and go back to step 3.

Algorithm-A computes the points $(\theta, \sin(n\theta))$ for $\theta = 0°, d°, 2d°, \ldots,$ and draws lines between each pair of successively computed points. The first computed point and the last computed point always coincide, so the figure drawn is always a closed polygon. The values of n and d can be chosen at random or supplied by the

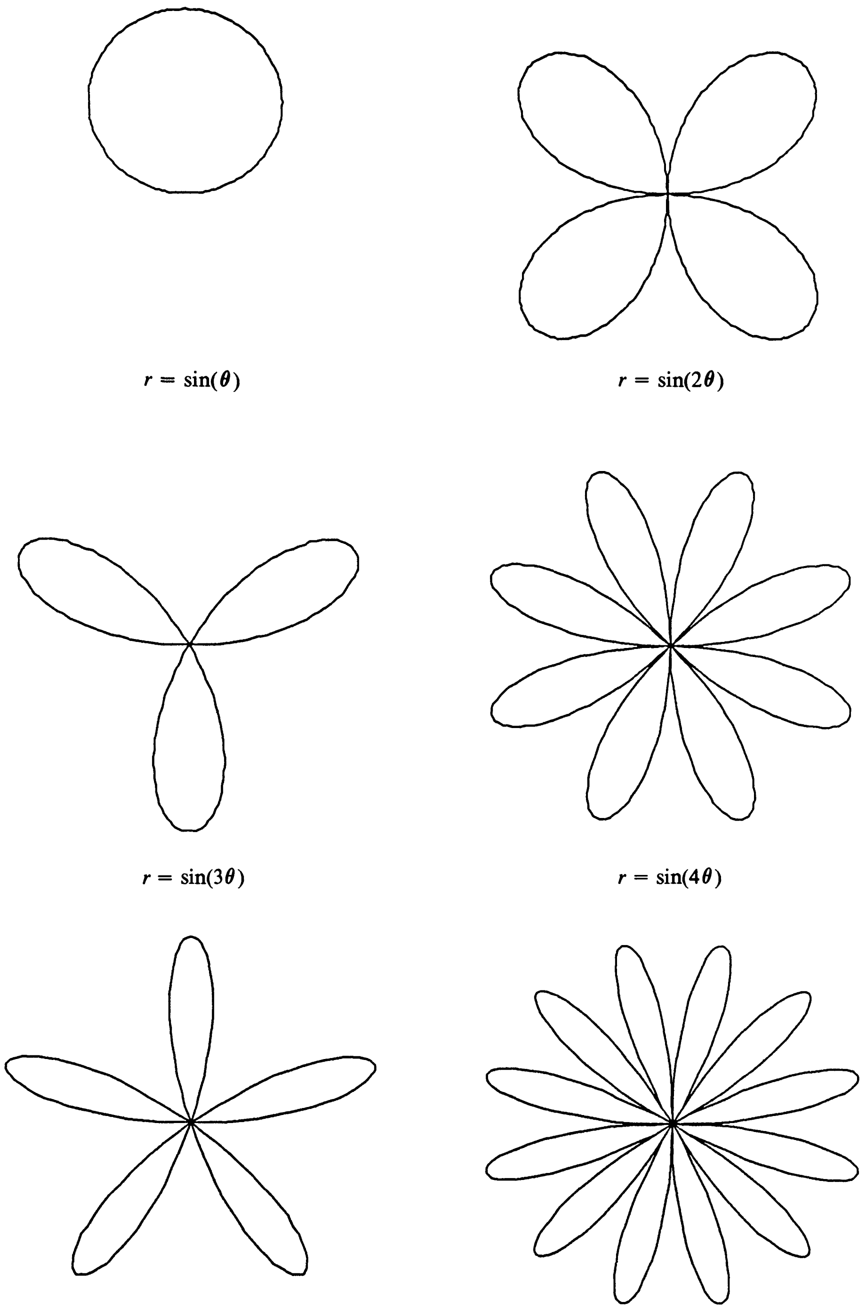
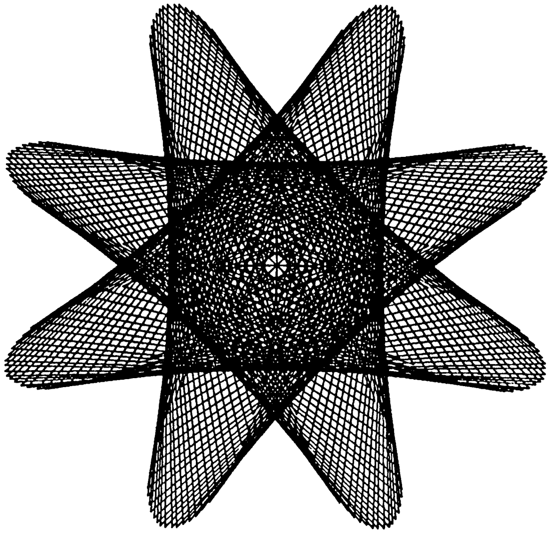
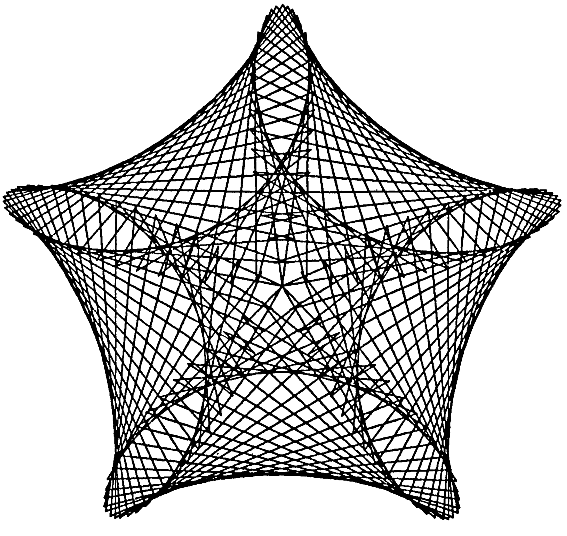


FIG. 1. The graphs of $r = \sin(n\theta)$ for $\theta = 1$–6.



$n = 4,\ d = 43$



$n = 5,\ d = 97$

FIG. 2. "The Rose" drawings for random n and d.

user of the program. Figure 2 gives examples of pictures drawn with randomly-chosen values for n and d.

Unfortunately, not all of the drawings generated by algorithm-A are as beautiful as those shown in Figure 2. Many of the drawings contain only a few lines, and many consist of a single dot. It would be esthetically pleasing to get rid of these degenerate figures but first it is necessary to understand why they occur.

Let G be the additive group of integers mod 360. Examination of steps 3 and 9 of algorithm-A makes it obvious that the number of lines in a drawing is equal to the order of d in G. (Some of the lines may be degenerate with starting and ending points coinciding.) It is a simple matter to show that the order of d in G is equal to $360/k$, where k is the greatest common divisor of d and 360. Now, let H be the subgroup of G generated by d. H has k distinct cosets in G of the form $H, H+1, H+2, \ldots, H+k-1$. A degenerate drawing is produced when the order of H is less than 360. However, when this is the case, a drawing can also be produced for each of the cosets $H+1, H+2, \ldots$. Furthermore, a different drawing will be produced for each distinct coset. The degenerate figures can be eliminated by superimposing the drawings for the cosets $H+1, H+2, \ldots$ over the drawing for H. Algorithm-B does exactly this:

1. Choose integers n, d such that $1 \leqslant n \leqslant 359$ and $1 \leqslant d \leqslant 359$.
2. Set T and c equal to zero.
3. Set θ equal to T. Compute the point $(2\pi\theta/360, \sin(2\pi n\theta/360))$, convert it to rectangular coordinates and set $(oldx, oldy)$ to the result.
4. Set θ equal to $\theta + d$. If $\theta \geqslant 360$ replace θ by the remainder obtained when dividing θ by 360.
5. Compute $n\theta$, reduce it mod 360, convert the result from degrees to radians, and set x equal to the final result.
6. Set r equal to the sin of x.
7. Convert θ from degrees to radians, and set t equal to the result.
8. Convert the point (t, r) from polar to rectangular coordinates to obtain the point $(newx, newy)$.
9. Draw a line from $(oldx, oldy)$ to $(newx, newy)$.
10. Add 1 to c.
11. If θ is equal to T then go to step 12, else set $(oldx, oldy)$ to $(newx, newy)$ and go back to step 4.
12. If $c \geqslant 360$ stop, else add 1 to T and go back to step 3.

Figure 3 gives drawings for various cosets with $n = 3$ and $d = 72$. Figure 4 demonstrates the difference between algorithm-A and algorithm-B for $n = 4$ and $d = 120$.

With this modification, it is possible to study the evolution of a drawing for a fixed n, as d ranges from 1–360, without worrying about degenerate drawings. Figures 5a and 5b illustrate this evolution for $n = 2$. First the apparently smooth line of the drawing becomes wider and more "lacy" until the space between the loops disappears. At the same time a squarish figure appears in the center.

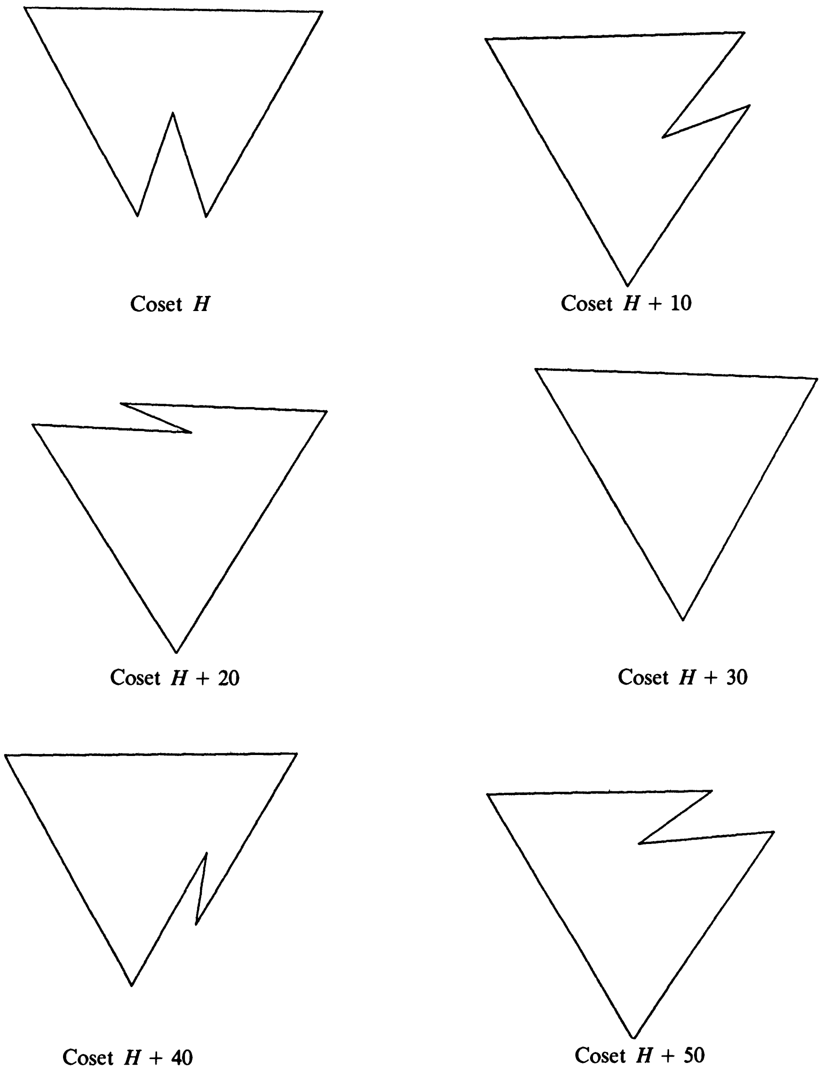
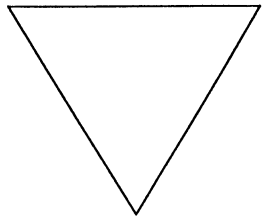


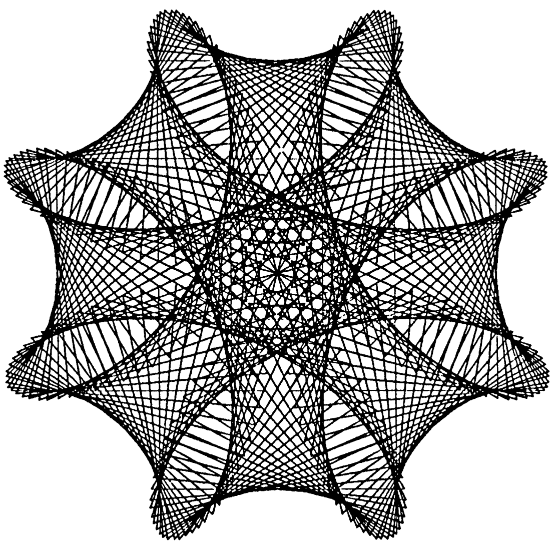
FIG. 3. Coset drawings for $n = 3$, $d = 72$.

Eventually the squarish figure grows until it overwhelms the entire figure leaving holes for the original petals. Then the petals become more "hairy" looking and the squarish shape begins to degenerate into filigree between the petals, until the shape disappears entirely leaving only the hairy petals. This form of evolution takes place (at different rates) for all figures with small n.

3. Very large *n*. All of the examples given so far have used fairly small values of n, even though step 1 of both algorithms allows n to range from 1 to 359. As n



$n = 4,\ d = 120$ (Algorithm-A)



$n = 4,\ d = 120$ (Algorithm-B)

FIG. 4. The difference between algorithms A and B.

becomes very large (say, greater than 60) the structure of the underlying rose disappears, but other puzzling phenomena begin to occur. Figure 6 gives examples of some of these phenomena.

Consider the drawing for $n = 181$, $d = 2$. This looks suspiciously like two copies of the drawing for $n = 1, d = 1$, rotated 180 degrees from one another. Further experimentation with the program will show that the drawing for $n = 121$, $d = 3$ is three circles whose centers are 120 degrees apart, and the drawing for $n = 91$, $d = 4$ is four circles whose centers are 90 degrees apart. Furthermore, the drawing for $n = 183$, $d = 2$ resembles two copies of the drawing for $n = 3$, $d = 1$, rotated 180 degrees from each other.

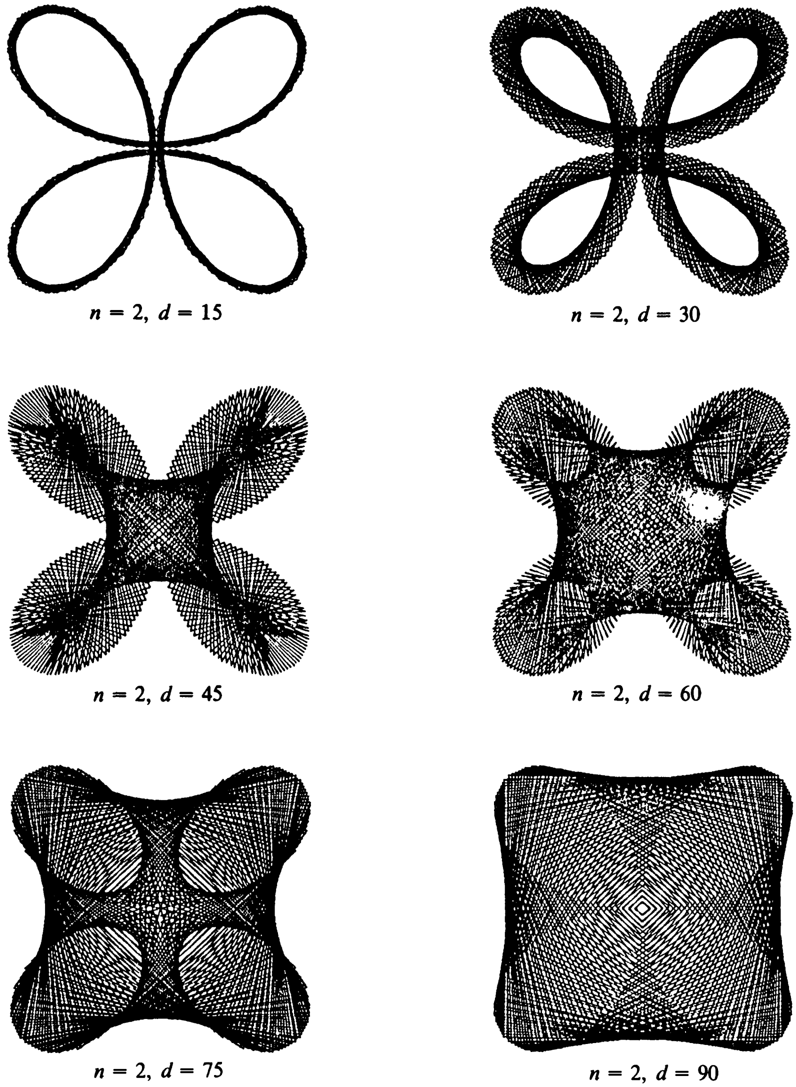
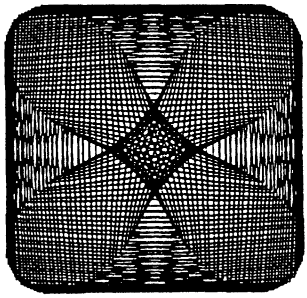


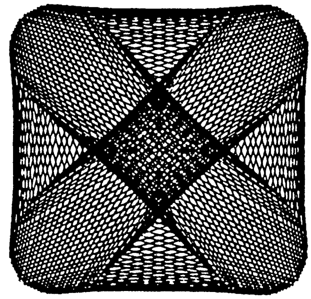
FIG. 5a. The evolution of $n = 2$ for $d = 1$–90.

These resemblances are not superficial as the following theorem shows. We will call this theorem the *zero* + theorem because it involves adding an integer to a zero-divisor in the ring of integers mod 360. The proof is a simple calculation and is omitted.

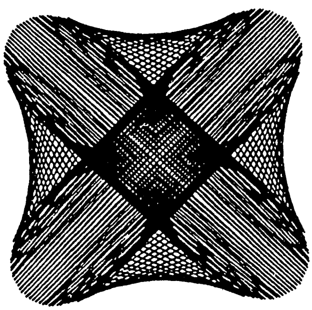
THE ZERO + THEOREM. *Let* $\sin(\theta)$ *be evaluated for* θ *in degrees and let the points* $(\theta, \sin(\theta))$ *represent the angle and radius of points in the usual polar coordinate system.*



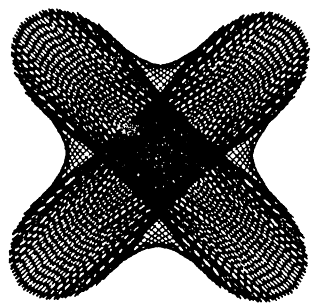
$n = 2,\ d = 105$



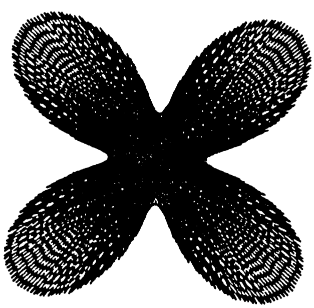
$n = 2,\ d = 120$



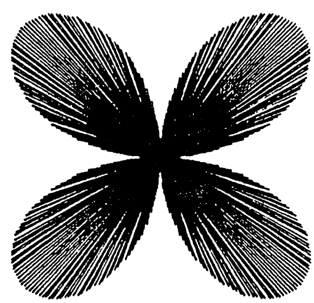
$n = 2,\ d = 135$



$n = 2,\ d = 150$



$n = 2,\ d = 165$

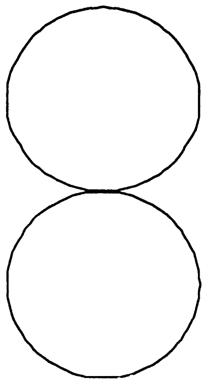


$n = 2,\ d = 180$

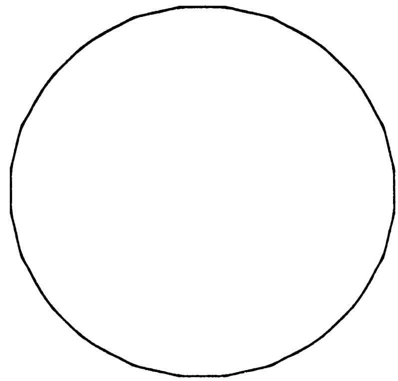
FIG. 5b. The evolution of $n = 2$ for $d = 90$–180.

Let θ, k, n, and m be integers and let $nm = 360$. If θ is of the form $mj + i$ where i and j are integers, then the point $(\theta, \sin((n + k)\theta))$ lies on the curve defined by $(\theta, \sin(k\theta))$ rotated ni degrees clockwise about the origin.

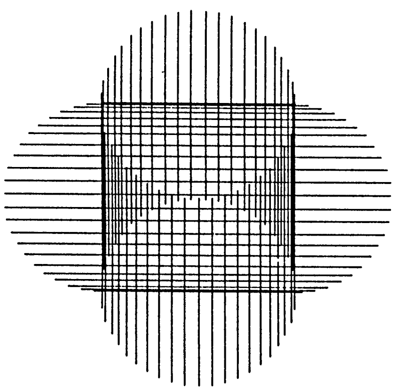
Note that the *zero +* theorem assumes that figures are drawn using a d that divides 360. This implies that each copy of the "small n" figure is generated by a distinct coset of the subgroup H of G generated by d. It turns out that the points



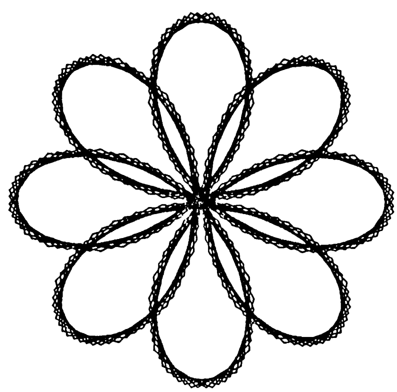
$n = 181,\ d = 2$



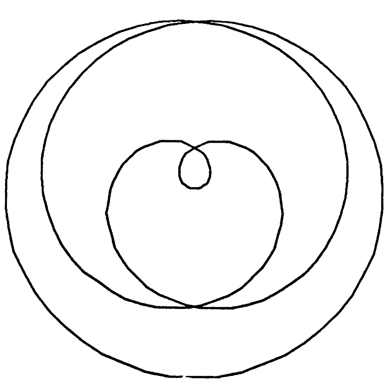
$n = 90,\ d = 4$



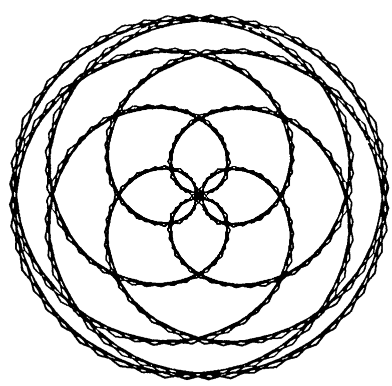
$n = 91,\ d = 90$



$n = 92,\ d = 16$



$n = 103,\ d = 7$



$n = 206,\ d = 28$

FIG. 6. Drawings with very large n.

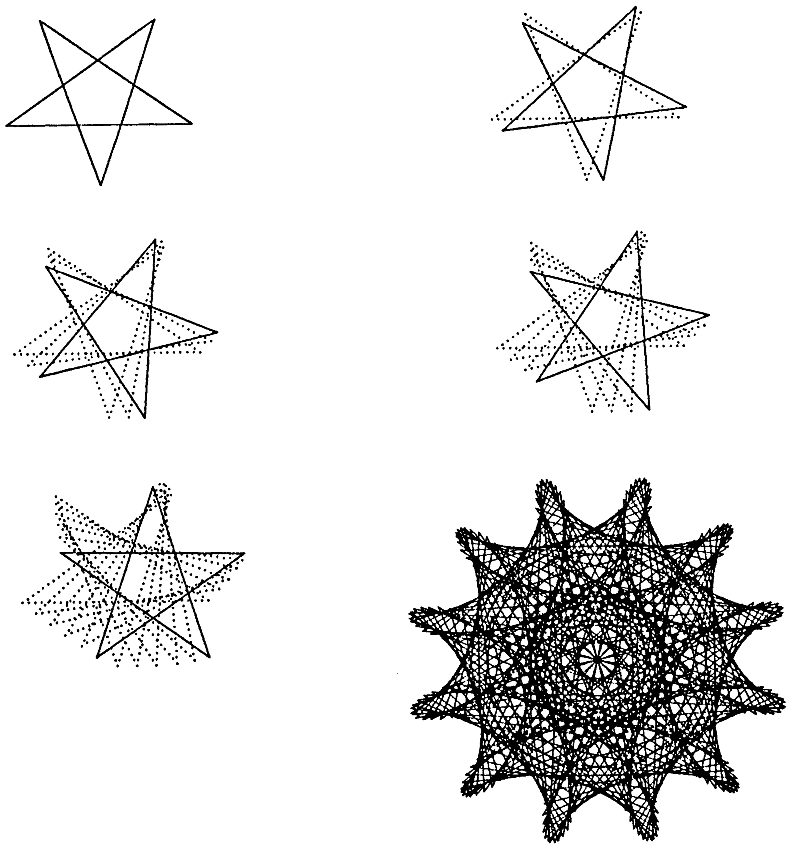


FIG. 7. The rotating pentangle.

computed from each successive coset are ni degrees further along the graph of the "small n" figure as well as being rotated ni degrees about the origin. For certain values of n and d, this produces an amazing visual effect when the figure is drawn. For example when the polygon for $n = 5 + 1 = 6$ and $d = 72$ is drawn, the observer sees a rotating pentangle which eventually produces a figure which looks nothing like a pentangle (see Figure 7).

The *zero* + theorem can be used to explain the appearance of the drawings for $n = 181$, $d = 2$ (two copies of $r = \sin(\theta)$); $n = 91$, $d = 90$ (four underlying circles joined by horizontal and vertical lines); and $n = 92$, $d = 16$ (four "evolved" copies of $r = \sin(2\theta)$, that overlap in pairs).

The *zero* + theorem can be extended to negative offsets from zero-divisors by observing that the graph of $r = \sin((-n)\theta)$ is identical to the graph of $r = \sin(n\theta)$ rotated 180 degrees about the origin. (Proof: $\sin(n\theta + 180) + \sin(n\theta)\cos(180) + \cos(n\theta)\sin(180) = -\sin(n\theta) = \sin(-n\theta)$.) It is a consequence of the *zero* + theorem that the figures for $n \geqslant 360$ are identical to those for $0 \leqslant n \leqslant 359$. Therefore, selecting n in this range provides the richest possible set of drawings.

Now consider the drawing for $n = 90$, $d = 4$. This drawing consists of two coinciding circles of radius 1 (as opposed to the radius .5 circle generated by $n = 1$, $d = 1$) and two coinciding dots in the center (the dots may not be visible in Figure 6). Similar drawings are generated for $n = 120$, $d = 3$ and for $n = 72$, $d = 5$. The following theorem explains the appearance of these drawings. We will call this theorem the *zero* theorem, because it concerns the zero-divisors of the ring of integers mod 360.

THE ZERO THEOREM. *Let* $\sin(\theta)$ *be evaluated for* θ *in degrees, and let* θ *and n be integers and let n divide* 360. *Then all points of the form* $(\theta, \sin(n\theta))$ *lie on* $m = 360/n$ *concentric circles centered on the origin* (*some of the circles may be of equal radius, and some may be of radius zero*).

Proof. Let m be the integer such that $nm = 360$. Consider all points θ of the form $mk + a$ with a constant. Then $\sin(n\theta) = \sin(n(mk + a)) = \sin(nmk + na) = \sin(360k + na) = \sin(360k)\cos(na) + \cos(360k)\sin(na) = \sin(na)$. Since both n and a are constant, so is $\sin(na)$. As a ranges from zero to $m - 1$, m distinct sets of values are produced. The graph of $r = k$ is a circle of radius $|k|$ about the origin. □

For $n = 90$, there are four distinct sets of values which produce the curves $r = \sin(0)$, $r = \sin(90)$, $r = \sin(180)$, and $r = \sin(270)$, which are circles of radius 0, 1, 0, and 1, respectively. When the polygon for $n = 90$ and $d = 4$ is drawn, one can see the second big circle being plotted, since the plotting points don't coincide. The dot in the center is, obviously, the plot of the zero-radius circles.

Lastly, consider the two most puzzling drawings in Figure 6, namely, those for $n = 103$, $d = 7$ and $n = 206$, $d = 28$. Experimentation with the parameters $n = 103$, $d = 7$ will show that even a slight change in n or d will make the spiral disappear. Furthermore, there are only a few combinations of n and d that give rise to spirals in the first place. It turns out that the curve for $n = 103$, and $d = 7$ coincides exactly with the graph of $r = \sin(\theta/7)$ plotted in 7-degree increments from 0 to 2520 degrees. The critical factor is that the product of 103 and 7 is equivalent to 1 mod 360. The following theorem, which we will call the *unity* theorem, explains this phenomenon.

THE UNITY THEOREM. *Let* $\sin(\theta)$ *be evaluated for* θ *in degrees, and let n and m be integers such that* $nm \equiv 1 \bmod 360$. *Let* α *be an arbitrary integer. Then the points* $(\alpha, \sin(n\alpha))$ *all lie on the graph of the equation* $r = \sin(\theta/m)$. *Furthermore, if* $r = \sin(n\alpha)$ *is evaluated in m degree increments, an approximation to the graph of* $r = \sin(\theta/m)$ *will be produced.*

Proof. Given an integer $\alpha \sin(n\alpha) = \sin((m/m)n\alpha) = \sin(nm\alpha/m)$. Since $nm\alpha \equiv \alpha \bmod 360$, the points $(\alpha, \sin(n\alpha))$ and $(nm\alpha, \sin(nm\alpha/m))$ coincide. To show that $0, m, 2m, 3m, \ldots$ produce successive points along the curve, observe that $\sin(nim) = \sin(i)$, $i = 0, 1, 2, 3, \ldots$. □

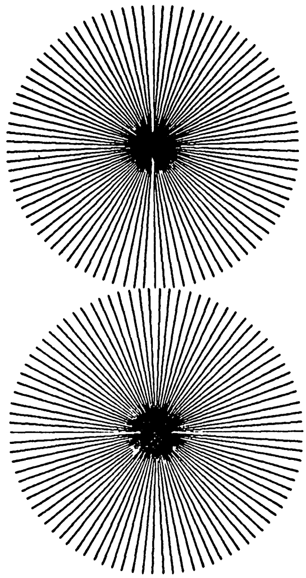
The *unity* theorem has the following corollary: Let m and n be two integers, $1 \leqslant m \leqslant 359$, $1 \leqslant n \leqslant 359$, such that n has a multiplicative inverse mod 360. Then it is possible to produce an approximation to the graph of $r = \sin(m\theta/n)$ using algorithm-B and an appropriate selection of the parameters n and d. The final drawing of Figure 6, with $n = 206$ and $d = 28$ is an "evolved" graph of $r = \sin(2\theta/7)$.

4. Dividing the circle into an arbitrary number of parts. Although dividing the circle into 360 equal parts is a time-honored tradition, there is no reason why some other number of subdivisions cannot be used. In fact, sometimes a small change in the number of circle subdivisions can make a profound difference in the drawings generated for a given n and d. The following algorithm, called "algorithm-C" allows the circle to be divided into z parts, where z is an arbitrary positive integer.

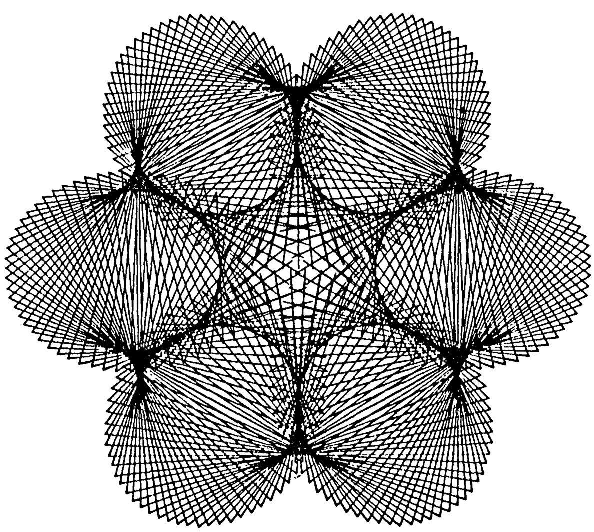
1. Choose integers z, n, d such that $1 \leqslant n < z$ and $1 \leqslant d < z$.
2. Set T and c equal to zero.
3. Set θ equal to T. Compute the point $(2\pi\theta/z, \sin(2\pi n\theta/z))$, convert it to rectangular coordinates and set (*oldx*, *oldy*) to the result.
4. Set θ equal to $\theta + d$. If $\theta \geqslant z$ replace θ by the remainder obtained when dividing θ by z.
5. Compute $n\theta$, reduce it mod z, multiply by $2\pi/z$, and set x equal to the final result.
6. Set r equal to the sin of x.
7. Set t equal to $2\pi\theta/z$.
8. Convert the point (t, r) from polar to rectangular coordinates to obtain the point (*newx*, *newy*).
9. Draw a line from (*oldx*, *oldy*) to (*newx*, *newy*).
10. Add 1 to c.
11. If θ is equal to T then go to step 12, else set (*oldx*, *oldy*) to (*newx*, *newy*) and go back to step 4.
12. If $c \geqslant z$ stop, else add 1 to T and go back to step 3.

Figure 8 demonstrates the effect of changing the number of circle-subdivisions from 360 to 359. Since 359 is prime, there are no analogs of the *zero* and *zero* + theorems, but the *unity* theorem still applies. Figure 9 gives some examples of drawings created with 359 circle subdivisions. None of these drawings could have been created with 360 subdivisions.

5. Conclusion. Because they are static, the drawings presented in this article cannot do justice to "The Rose" program. Readers with access to high-speed computer graphics equipment are encouraged to implement their own versions of "The Rose" and view the construction of the drawings first-hand. Many of the drawings exhibit apparent motion as they are being drawn, and in some cases, such as the rotating pentangle described above, the visual effect is quite stunning.

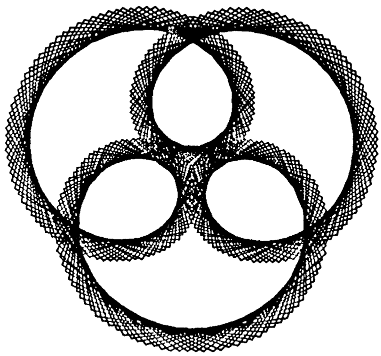


$n = 181,\ d = 90,\ z = 360$

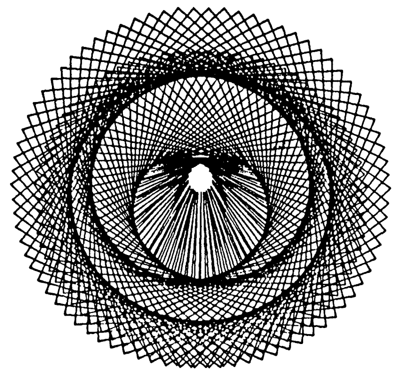


$n = 181,\ d = 90,\ z = 359$

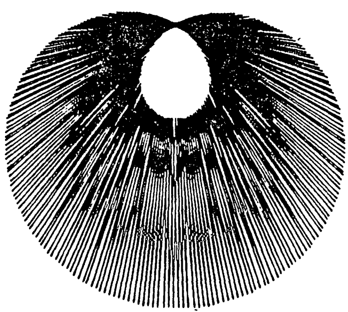
FIG. 8. The effect of changing circle subdivisions.



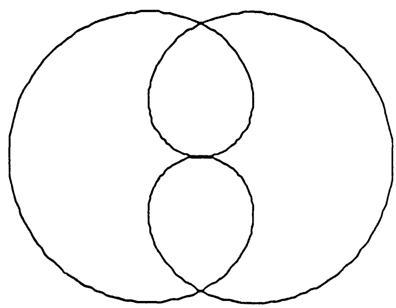
$n = 216,\ d = 50$



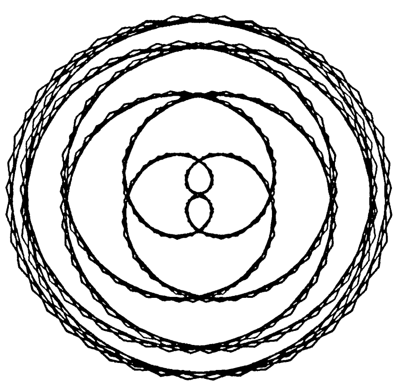
$n = 40,\ d = 89$



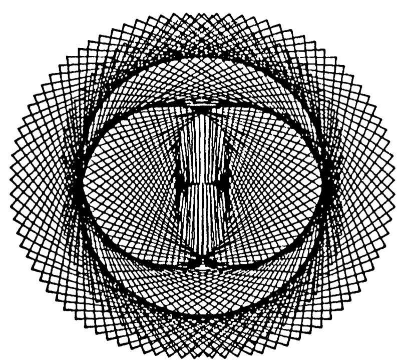
$n = 120,\ d = 92$



$n = 180,\ d = 2$



$n = 45,\ d = 32$



$n = 90,\ d = 90$

FIG. 9. Drawings created with $z = 359$.

There are a number of interesting, and probably not too difficult, problems that remain to be solved. Among them are:

1. What are the rules that govern the shape of the individual cosets drawn by algorithm-B?
2. Many of the drawings have apparent curves that are generated by intersections of lines. What are the parametric equations for these curves? Are any of these curves well known? How do the parametric equations evolve as the figure evolves?
3. Algorithms A and B both draw illustrations of the additive group of integers mod 360 and its subgroups. What about the multiplicative group and its subgroups?
4. When n is odd, each line drawn by algorithms A and B is drawn twice. Can any use be made of this?
5. In this article, considerable use was made of the fact that $\sin(n\theta)$ is periodic in $360°$. In fact, the periods of these functions are usually much smaller than $360°$. Is this important?
6. What about sums and products of sin and cos functions?

There are undoubtedly many other interesting questions that could be asked. The reader is encouraged to try his hand at discovering them.

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2. N. Greene, Frame from Inside a Quark, Computer Graphics, 18 (July 1984), Front Cover.
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The Mathematician's Dictionary

What mathematical concept can be defined as:

Its value on any bowlfull is the length of the longest floating noodle?

(See page 702)