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Image Formation and Acquisition (Part 2)

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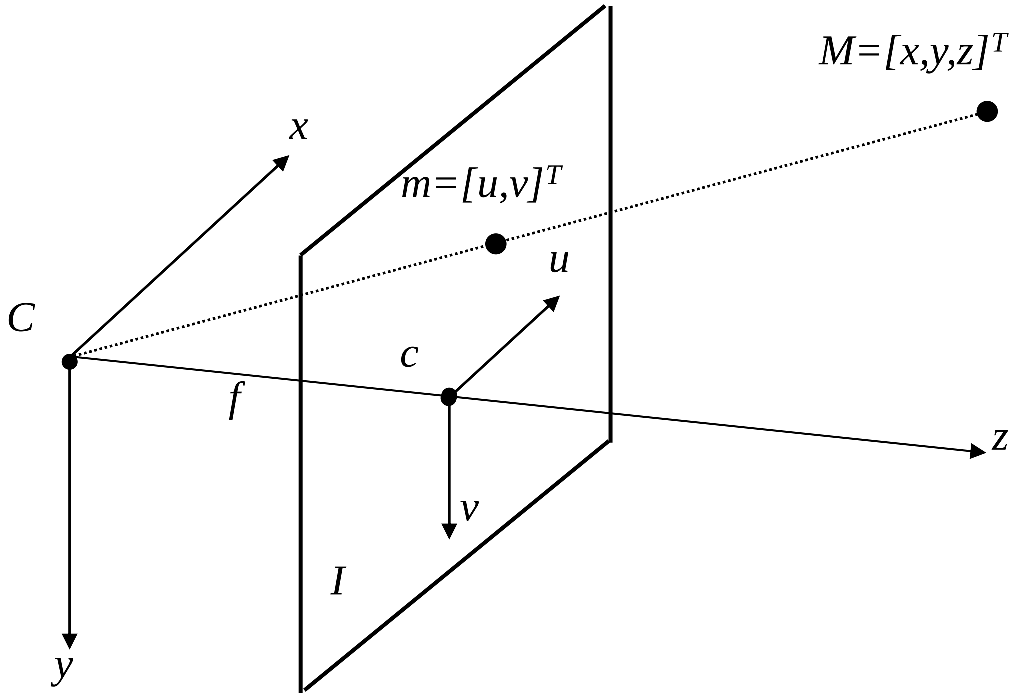
Perspective Projection

- Let us consider
  - A point in the 3D space, $M=[x,y,z]^T$, with coordinates given in the Camera Reference Frame (CRF).
  - Its projection onto the image plane I, denoted as $m=[u,v]^T$
- The **non-linear** equations providing image coordinates as a function of the 3D coordinates in the CRF are as follows:

$$\begin{cases} u = \dfrac{f}{z}x \\ v = \dfrac{f}{z}y \end{cases}$$



Projective Space

- The physical space is a 3D Euclidean Space ($\mathbf{R}^3$) whose points can be represented as 3D vectors in a given reference frame.
  - In this space parallel lines do not intersect, or intersect "at infinity ".
  - Points at infinity cannot be represented in this vector space.
- Let's now append one more coordinate to our Euclidean triples, so that e.g.

$$\begin{pmatrix} x & y & z \end{pmatrix} \text{ becomes } \begin{pmatrix} x & y & z & 1 \end{pmatrix}$$

  and assume that both vectors are correct representations of the same 3D point.
- Moreover, we do not constrain the 4th coordinate to be 1 but instead assume

$$\begin{pmatrix} x & y & z & 1 \end{pmatrix} \equiv \begin{pmatrix} 2x & 2y & 2z & 2 \end{pmatrix} \equiv \begin{pmatrix} kx & ky & kz & k \end{pmatrix} \forall k \neq 0$$

- In this representation a point in space is represented by an **equivalence class** of **quadruples**, wherein equivalent quadruples differ just by a multiplicative factor.
- This is the so-called **homogeneous coordinates** (a.k.a. projective coordinates) representation of the 3D point having Euclidean coordinates (x, y, z). The space associated with the homogeneous coordinates representation is called **Projective Space**, denoted as $\mathbf{P}^3$.
- Extension to Euclidean spaces of any other dimension is straightforward ($\mathbf{R}^n \rightarrow \mathbf{P}^n$)

Point at infinity of a 3D line

Let’s consider the parametric equation of a 3D line:

$$\mathrm{M} = M_0 + \lambda D = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_0 + \lambda a \\ y_0 + \lambda b \\ z_0 + \lambda c \end{bmatrix}$$

and represent the geberic point along the line in projective coordinates:

$$\widetilde{M} = \begin{bmatrix} M \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 + \lambda a \\ y_0 + \lambda b \\ z_0 + \lambda c \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x_0}{\lambda} + a \\ \frac{y_0}{\lambda} + b \\ \frac{z_0}{\lambda} + c \\ \frac{1}{\lambda} \end{bmatrix}$$

by taking the limit with $\lambda \to \infty$ we obtain the projective coordinates of the point at infinity of the given line:

$$\widetilde{M}_\infty = \lim_{\lambda \to \infty} \widetilde{M} = \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix}$$

The projective coordinates of the point at infinity of a 3D line are obtained by taking any Euclidean vector parallel to the line and appending a 0 as fourth coordinate. There exist infinitely many points at infinity in $\mathbf{P}^3$, as many as the directions of the 3D lines.

Points at infinity

- The points of the 3D Projective Space **having the fourth coordinate equal to 0** , e.g. (x, y, z, 0), are the points at infinity of the 3D lines. These points cannot be represented in the 3D Euclidean Space.
- Indeed, to map such points into the Euclidean Space we would divide by the –null- fourth coordinate, so as to get (x/0, y/0, z/0), i.e. infinite coordinates, which is not a valid representation in the Euclidean Space.
- By the homogenous coordinates it is therefore possible to represent and process seamlessly both ordinary points as well as *points at infinity.*
- Point (0, 0, 0, 0) is undefined.
  - indeed, the above point is NOT the origin of the Euclidean Space (0, 0, 0), for such point is represented in homogeneous coordinates as (0, 0, 0, k), $k \neq 0$.
- It can be shown that all points at ininity of $\mathbf{P}^3$ lie on a plane, which is called the **plane at infinity**.

Recap

- Any Euclidean Space $\mathbf{R}^n$ can be extended to a corresponding **Projective Space** $\mathbf{P}^n$ by representing points in homogeneous coordinates.
- The projective representation includes one additional coordinate, referred to here as *k*, wrt the Euclidean representation:
  - $k \neq 0$ denotes point existing in $\mathbf{R}^n$, their coordinates given by x_i / k, $i = 1 \ldots n$
  - $k = 0$ denotes points at infinity (a.k.a. ideal points) in $\mathbf{R}^n$, which do not admit a representation via Euclidean coordinates.
- The Projective Space allows then to represent and process homogeneously (i.e. without introduction of exceptions or special cases) both the ordinary and the ideal points of the Euclidean Space.
- Why are we interested in Projective Spaces ?

Because Perspective Projection is more conveniently dealt with using projective coordinates as it becomes a <u>linear transformation</u> !

Perspective Projection in projective coordinates (1)

- We already know that there exist a non-linear transformation between 3D coordinates and image coordinates:

$$u = \frac{f}{z}x \qquad v = \frac{f}{z}y$$

- Let's now go back to our 3D point *M* (with coordinates expressed in the CRF) and its projection onto the image plane, *m*:

$$\boldsymbol{M} = [x,y,z\,]^T \qquad \boldsymbol{m} = [u,v\,]^T$$

and represent both points in homogenous coordinates (denoted by symbol ~)

$$\tilde{\mathbf{m}} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \qquad \tilde{\mathbf{M}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Perspective Projection in projective coordinates (2)

- In homogeneous coordinates (hence considering the mapping between projective spaces), perspective projection becomes a **linear transformation**:

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f\dfrac{x}{z} \\ f\dfrac{y}{z} \\ 1 \end{bmatrix} = \begin{bmatrix} fx \\ fy \\ z \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- In matrix notation

$$k\tilde{\mathbf{m}} = \tilde{\mathbf{P}}\tilde{\mathbf{M}}$$

often expressed also as below, where $\approx$ means "equal up to an arbitrary scale factor"

$$\tilde{\mathbf{m}} \approx \tilde{\mathbf{P}}\tilde{\mathbf{M}}$$

Vanishing Point of a 3D line

$$\begin{bmatrix} fa \\ fb \\ c \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} fa \\ fb \\ c \end{bmatrix} \rightarrow \begin{bmatrix} f\dfrac{a}{c} \\ f\dfrac{b}{c} \end{bmatrix}$$

(Euclidean Coordinates)

Perspective Projection Matrix (PPM)

- Matrix $\tilde{\mathbf{P}}$ represents the geometric camera model and is known as **Perspective Projection Matrix (PPM).**

- If we assume distances to be measured in focal lenght units (f = 1), the PPM becomes

$$\tilde{\mathbf{P}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [\mathbf{I} \,|\, \mathbf{0}]$$

- This form is useful to understand the core operation carried out by perspective projection, which is indeed scaling lateral coordinates (i.e. *x*,*y*) according to the distance from the camera (*z*). The actual focal length just introduces an additional, fixed (i.e. independent of *z*) scaling factor of projected coordinates. The above form is usually referred to as ***canonical*** or ***standard*** **PPM**.

A more comprehensive camera model

- To come up with a really useful camera model we need to take into account two additional issues:
  - Image Digitization;
  - The 6 DOF (3D rotation and translation) rigid motion between the Camera Reference Frame (CRF) and the World Reference Frame (WRF).

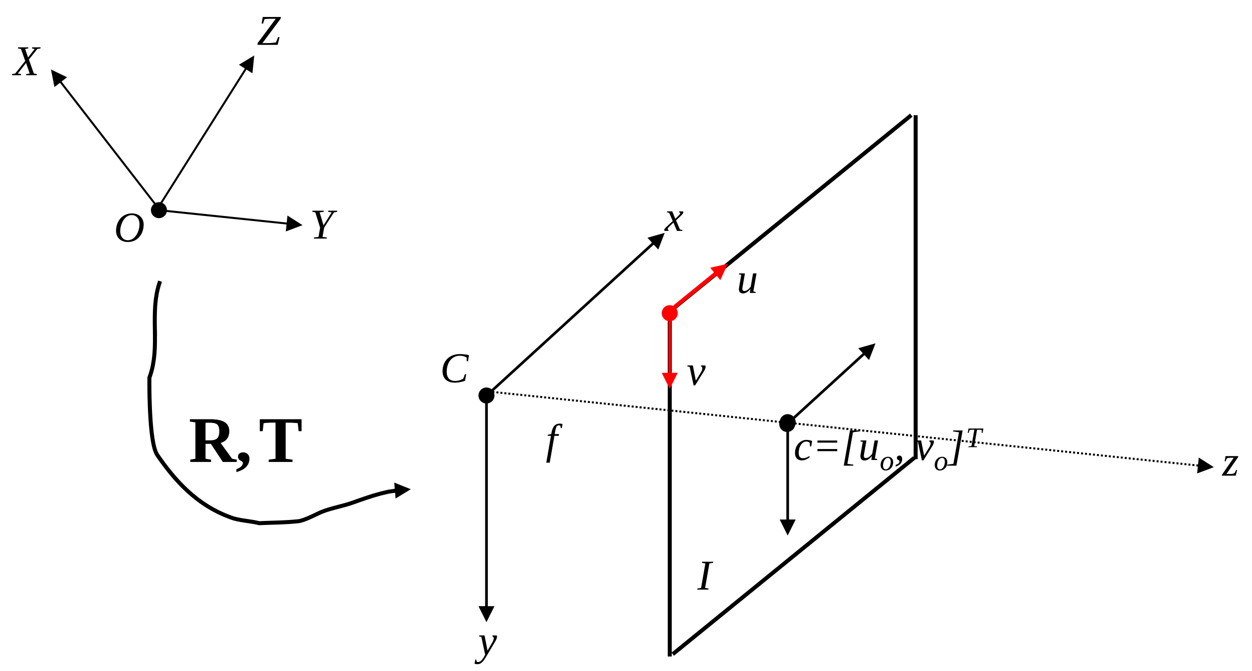
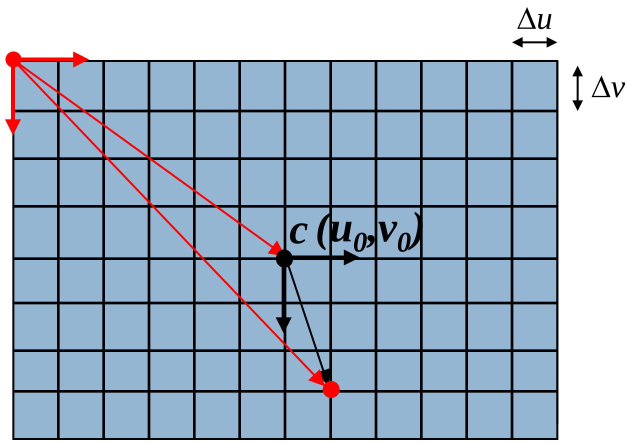


Image Digitization



Δu= horizontal pixel size
Δv= vertical pixel size

$$u = \frac{f}{z}x \qquad \rightarrow u = \frac{1}{\Delta u}\frac{f}{z}x = k_u \frac{f}{z}x + u_0$$

$$v = \frac{f}{z}y \qquad \rightarrow v = \frac{1}{\Delta v}\frac{f}{z}y = k_v \frac{f}{z}y + v_0$$

Digitization can be accounted for by including into the projection equations the scaling factors along the two axes due to the quantization associated with the horizontal and vertical pixel size. Morever, we need to model the translation of the image center (aka piercing point, the intersection between the optical axis and the image plane) wrt the origin of the image coordinate system (top-left corner of the image).

Intrinsic Parameter Matrix

Based on the equations in the previous slide, the PPM can be written as

$$\tilde{\mathbf{P}} = \begin{bmatrix} fk_u & 0 & u_0 & 0 \\ 0 & fk_v & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} fk_u & 0 & u_0 \\ 0 & fk_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \mathbf{A}\left[\mathbf{I} \,|\, \mathbf{0}\right]$$

- Matrix **A**, which models the characteristics of the image sensing device, is called **Intrinsic Parameter Matrix** (or **Camera Matrix**).
- Intrinsic parameters can be reduced in number by setting $\alpha_u = fk_u$, $\alpha_v = fk_v$, such quantities representing, respectively, the focal length expressed in horizontal and vertical pixel sizes. The smallest number of intrinsic parameters is thus 4.
- A more general model would include a 5th parameter, known as *skew*, to account for possible non orthogonality between the axis of the image sensor. The *skew* would be A[1,2], but it is usually 0 (= ctg(π/2)) in practice.

Rigid motion between CRF and WRF (1)

- So far we have assumed 3D coordinates to be measured into the CRF, though this is hardly feasible in practice.
- More generally, 3D coordinates are measured into a World Reference Frame (WRF) external to the camera. The WRF will be related to the CRF by:
  - A rotation around the optical centre (e,g. expressed by a 3x3 rotation matrix $\mathbf{R}$)
  - A translation (expressed by a 3x1 translation vector $\mathbf{T}$)
- Therefore, the relation between the coordinates of a point in the two RFs is:

$$\mathbf{W} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \mathbf{M} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \mathbf{M} = \mathbf{R}\mathbf{W} + \mathbf{T}$$

- Which can be rewritten in homogeneous coordinates as follows:

$$\tilde{\mathbf{W}} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}, \tilde{\mathbf{M}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow \tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{0} & 1 \end{bmatrix} \tilde{\mathbf{W}} = \mathbf{G}\tilde{\mathbf{W}}$$

Rigid motion between CRF and WRF (2)

So far So far we have seen how to map a 3D point expressed in the CRF

$$k\tilde{\mathbf{m}} = \mathbf{A}\left[\mathbf{I}\,|\,\mathbf{0}\right]\tilde{\mathbf{M}}$$

- We need now to consider also the rigid motion between the WRF and the CRF :

$$\tilde{\mathbf{M}} = \mathbf{G}\tilde{\mathbf{W}} \longrightarrow k\tilde{\mathbf{m}} = \mathbf{A}\left[\mathbf{I}\,|\,\mathbf{0}\right]\mathbf{G}\tilde{\mathbf{W}}$$

$$k\tilde{\mathbf{m}} = \mathbf{A}\left[\mathbf{I}\,|\,\mathbf{0}\right]\begin{bmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{0} & 1 \end{bmatrix}\tilde{\mathbf{W}}$$

- Accordingly, the general form of the PPM can be expressed as:

$$\tilde{\mathbf{P}} = \mathbf{A}\left[\mathbf{I}\,|\,\mathbf{0}\right]\mathbf{G} \quad \text{or also} \quad \tilde{\mathbf{P}} = \mathbf{A}\left[\mathbf{R}\,|\,\mathbf{T}\right]$$

Extrinsic Parameters

- Matrix $\mathbf{G}$, which encodes the position and orientation of the camera with respect to the WRF, is called **Extrinsic Parameter Matrix**.

- As a rotation matrix (3x3=9 entries) has indeed only 3 independent parameters (DOF), which correspond to the rotation angles around the axis of the RF, the total number of extrinsic parameter is 6 (3 translation parameters, 3 rotation parameters).

- Hence, the general form of the PPM can be thought of as encoding the position of the camera wrt the world into $\mathbf{G}$, the perspective projection carried out by a pinhole camera into the canonical PPM $[\,\mathbf{I} \mid \mathbf{0}\,]$ and, finally, the actual characteristics of the camera into $\mathbf{A}$.

P as a Homography

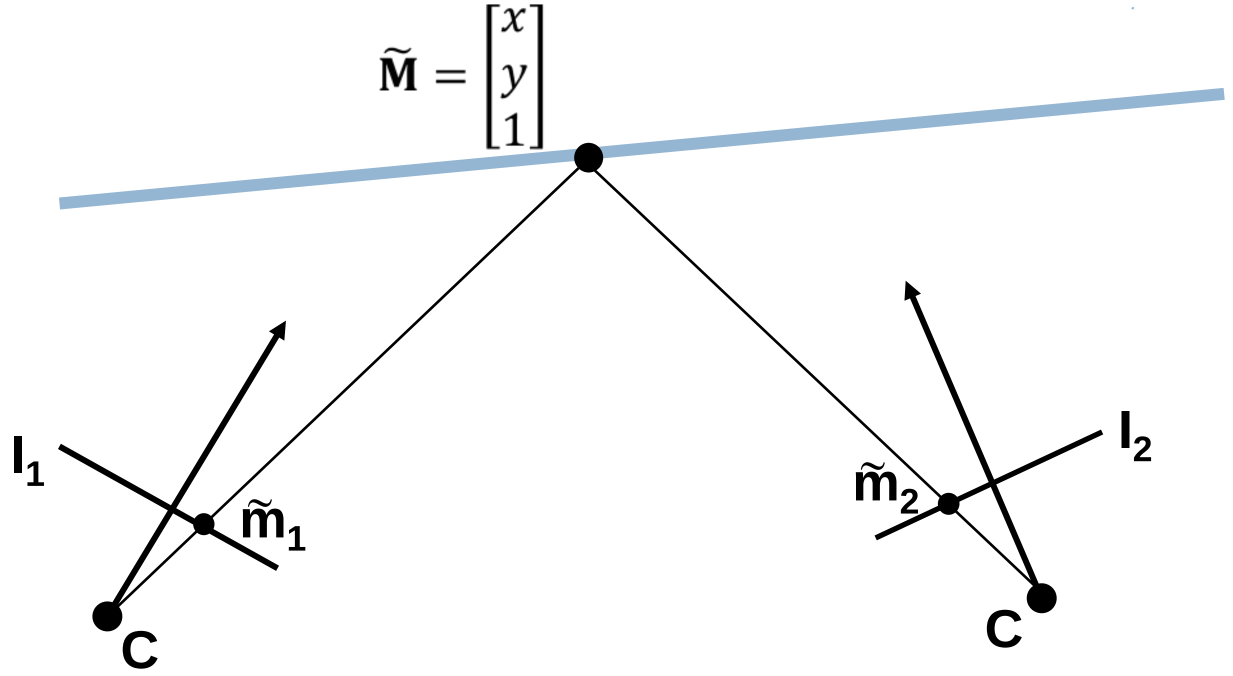
- If the camera is imaging a **planar scene**, we can assume the z-axis of the WRF to be perpendicular to the plane such that all 3D points will have **their z-coordinate equal to 0**. Accordingly, the PPM boils down to a simpler transformation defined by a 3x3 matrix:

$$k\tilde{\mathbf{m}} = \tilde{\mathbf{P}}\,\tilde{\mathbf{W}} = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & p_{1,4} \\ p_{2,1} & p_{2,2} & p_{2,3} & p_{2,4} \\ p_{3,1} & p_{3,2} & p_{3,3} & p_{3,4} \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,4} \\ p_{2,1} & p_{2,2} & p_{2,4} \\ p_{3,1} & p_{3,2} & p_{3,4} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{H}\tilde{\mathbf{M}}$$

- Such a transformation, denoted here as $\mathbf{H}$, is known as **homography** and represents a general linear transformation between projective planes. Above, $\tilde{\boldsymbol{M}}$ represents vector $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$. $\mathbf{H}$ can be thought of as a simplification of $\mathbf{P}$ in case the imaged scene is planar.
- Akin to $\mathbf{P}$, $\mathbf{H}$ is known up to an <u>arbitrary scale factor</u> and thus the independent elements in the 3x3 matrix are just 8.

Homographies (1)

Any two images of a planar scene are related by a homography:



$$\tilde{\mathbf{m}}_1 = \mathbf{H}_1 \tilde{\mathbf{M}} \qquad \tilde{\mathbf{m}}_2 = \mathbf{H}_2 \tilde{\mathbf{M}}$$

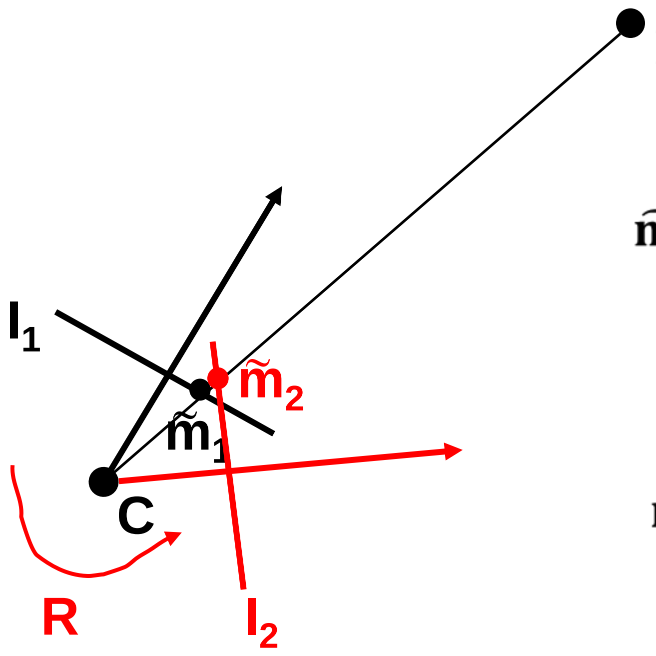
$$\tilde{\mathbf{m}}_1 = \underbrace{\mathbf{H}_1 \mathbf{H}_2^{-1}}_{\mathbf{H}_{21}} \tilde{\mathbf{m}}_2 \qquad \tilde{\mathbf{m}}_2 = \underbrace{\mathbf{H}_2 \mathbf{H}_1^{-1}}_{\mathbf{H}_{12}} \tilde{\mathbf{m}}_1$$

Homography

These homographies link pixels in the first image with pixels in the second image

Homographies (2)

Any two images taken by a camera rotating about the optical center are related by a homography:



$$\tilde{\mathbf{M}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M} \\ 1 \end{bmatrix}$$

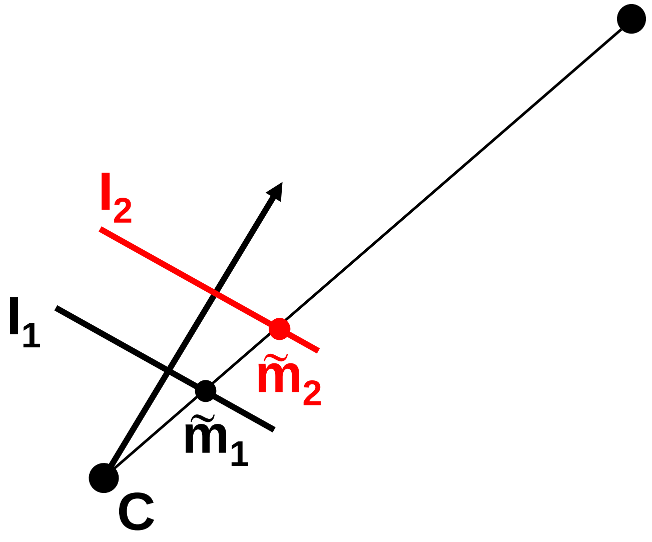
$$\tilde{\mathbf{m}}_1 = \mathbf{A}[\mathbf{I}\ \mathbf{0}]\tilde{\mathbf{M}} = \mathbf{AM} \qquad \tilde{\mathbf{m}}_2 = \mathbf{A}[\mathbf{R}\ \mathbf{0}]\tilde{\mathbf{M}} = \mathbf{ARM}$$

$$\tilde{\mathbf{m}}_1 = \underbrace{\mathbf{AR}^{-1}\mathbf{A}^{-1}}_{\mathbf{H}_{21}}\tilde{\mathbf{m}}_2 \qquad \tilde{\mathbf{m}}_2 = \underbrace{\mathbf{ARA}^{-1}}_{\mathbf{H}_{12}}\tilde{\mathbf{m}}_1$$

Homography

Homographies (3)

Any two images taken by different cameras (i.e. different A) in a fixed pose (i.e. same CRF) are related by a homography:



$$\tilde{\mathbf{M}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M} \\ 1 \end{bmatrix}$$

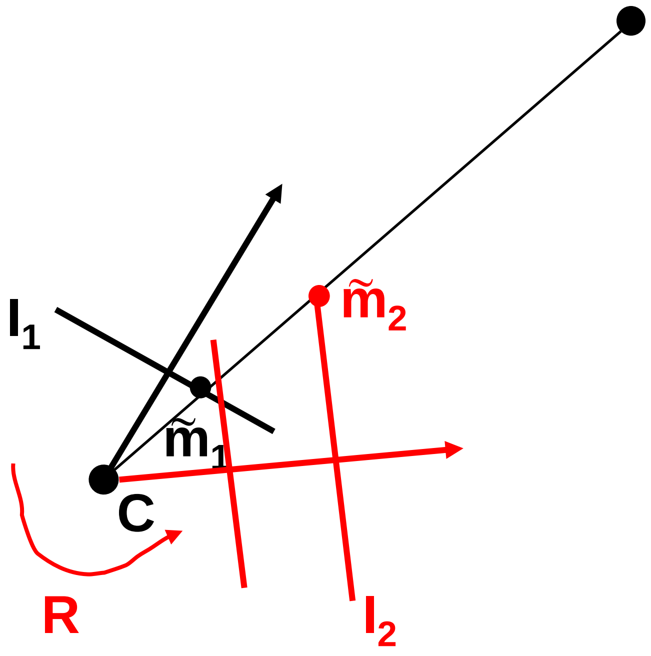
$$\tilde{\mathbf{m}}_1 = \mathbf{A}_1[\mathbf{I}\ \mathbf{0}]\tilde{\mathbf{M}} = \mathbf{A}_1\mathbf{M} \qquad \tilde{\mathbf{m}}_2 = \mathbf{A}_2[\mathbf{I}\ \mathbf{0}]\tilde{\mathbf{M}} = \mathbf{A}_2\mathbf{M}$$

$$\tilde{\mathbf{m}}_1 = \underbrace{\mathbf{A}_1\mathbf{A}_2^{-1}}_{\mathbf{H}_{21}}\tilde{\mathbf{m}}_2 \qquad \tilde{\mathbf{m}}_2 = \underbrace{\mathbf{A}_2\mathbf{A}_1^{-1}}_{\mathbf{H}_{12}}\tilde{\mathbf{m}}_1$$

Homography

Homographies (4)

Should we both rotate the camera about the optical center and change the intrinsic we still end up again with images related by a homography:



$$\tilde{\mathbf{M}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M} \\ 1 \end{bmatrix}$$

$$\tilde{\mathbf{m}}_1 = \mathbf{A}_1[\mathbf{I}\ \mathbf{0}]\tilde{\mathbf{M}} = \mathbf{A}_1\mathbf{M} \qquad \tilde{\mathbf{m}}_2 = \mathbf{A}_2[\mathbf{R}\ \mathbf{0}]\tilde{\mathbf{M}} = \mathbf{A}_2\mathbf{R}\mathbf{M}$$

$$\tilde{\mathbf{m}}_1 = \underbrace{\mathbf{A}_1\mathbf{R}^{-1}\mathbf{A}_2^{-1}}_{\mathbf{H}_{21}}\tilde{\mathbf{m}}_2 \qquad \tilde{\mathbf{m}}_2 = \underbrace{\mathbf{A}_2\mathbf{R}\mathbf{A}_1^{-1}}_{\mathbf{H}_{12}}\tilde{\mathbf{m}}_1$$

Homography

Lens Distortion

However, to explain observed images we often need to model also the effects due to the optical distortion induced by lenses, which indeed renders the pure pinhole not accurate enough a model in many applications. Lens distortion is modelled through additional parameters that do not alter the form of the PPM.

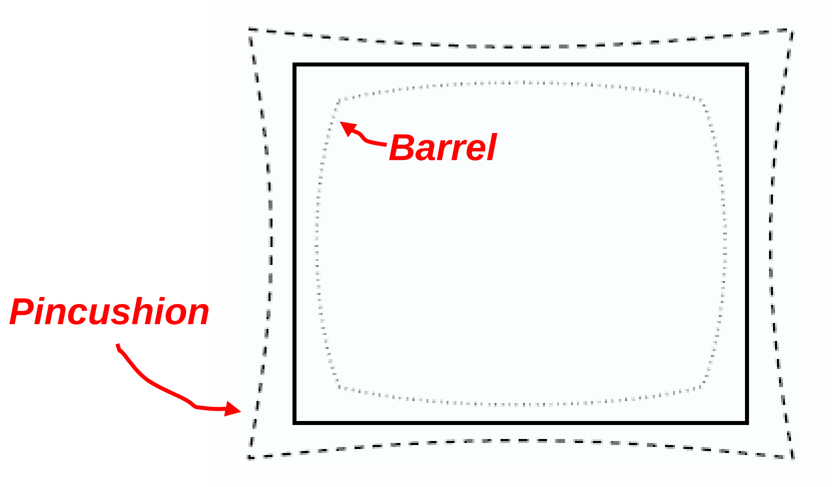
Modelling Lens Distortion

- The PPM is based on the pinhole camera model. However, real lenses introduce distortions wrt to the pure pinhole model, this being true especially for cheap and/or short focal length lenses. The most significant deviation from the ideal pinhole model is known as **radial distortion** (lens "curvature"). Second order effects are induced by **tangential distortion** ("misalignement" of optical components and/or defects).
- We adopt a model whereby lens distortion is modelled through a non-linear transformation which maps ***undistorted*** **continuous** image coordinates into ***distorted*** **continuous** image coordinates:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = L(r)\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} d\tilde{x} \\ d\tilde{y} \end{pmatrix}$$

where r is the distance from the distortion centre $(\tilde{x}_c, \tilde{y}_c)$:

$$r = \sqrt{(\tilde{x}-\tilde{x}_c)^2 + (\tilde{y}-\tilde{y}_c)^2}$$



Lens Distortion Parameters

- The radial distortion function $L(r)$ is defined for positive r only and such as $L(0) = 1$. This non-linear function is typically approximated by its Taylor series (up to a certain approximation order)

$$L(r) = 1 + k_1 r^2 + k_2 r^4 + k_3 r^6 + \ldots$$

- The tangential distortion vector is instead approximated as follows

$$\begin{pmatrix} d\tilde{x} \\ d\tilde{y} \end{pmatrix} = \begin{pmatrix} 2p_1\tilde{x}\tilde{y} + p_2\left(r^2 + 2\tilde{x}^2\right) \\ p_1\left(r^2 + 2\tilde{y}^2\right) + 2p_2\tilde{x}\tilde{y} \end{pmatrix}$$

- The radial distortion coefficients $k_1, k_2, \ldots, k_n$, together with the distortion centre $(\tilde{x}_c, \tilde{y}_c)$ and the two tangential distortion coefficients p_1 and p_2 form the set of the lens distortion parameters, which extends the set of parameters required to build a useful and realistic camera model. Typically, for the sake of simplicity, the distortions centre is taken to coincide with the image centre (i.e. the piercing point).

Lens distortion in the image formation flow

- Lens distortion is modelled as a non-linear mapping taking place after canonical perspective projection onto the image plane. Afterwards, the intrinsic parameter matrix applies an affine transformation which maps continuous image coordinates into pixel coordinates.
- Accordingly, the image formation flow can be summarized as follows:
  1. Transformation of 3D points from the WRF to the CRF, according to extrinsic parameters:

$$\mathbf{M} = \mathbf{R}\mathbf{W} + \mathbf{T}$$

  2. Canonical perspective projection (i.e. scaling by the third coordinate):

$$\tilde{x} = x / z \quad , \quad \tilde{y} = y / z$$

  3. Non-linear mapping due to lens distortion:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = L(r) \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} d\tilde{x} \\ d\tilde{y} \end{pmatrix}$$

  4. Mapping from continuous image coordinates to pixels coordinates according to the intrinsic parameters:

$$\tilde{\mathbf{m}} = \mathbf{A}\,(x'\ y'\ 1)^T$$