Operations Research (Master's Degree Course)

4. The Simplex Algorithm

Silvano Martello

DEI "Guglielmo Marconi", Università di Bologna, Italy

The Simplex algorithm

- **Summarizing the main points seen so far:**

$$
\begin{array}{llcll}
\text{(LP)} & \min c'x & & & \\
 & Ax & = & b & \qquad A(m \times n),\ m < n \\
 & x & \geq & 0 &
\end{array}
$$

  - $F = \{x \in R^n : Ax = b,\ x \geq 0\} \quad \leftrightarrow \quad$ polytope P.
  - Assumptions: 1. A is of rank m;

    2. $F \neq \emptyset$;

    3. F bounded in direction in which $c'x$ decreases.
  - Base: $\mathcal{B} = \{A_{\beta(1)}, \ \ldots \ , A_{\beta(m)}\} \ \leftrightarrow B = [A_{\beta(i)}]$ (det$(B) \neq 0$);

    basic solution: $\left.\begin{array}{ll} (x_j)=0 & (A_j \notin \mathcal{B}) \\ (x_j)=B^{-1}b & (A_j \in \mathcal{B}) \end{array}\right\}$; if $\in F \Rightarrow BFS \leftrightarrow$ vertex of P.
  - $\exists$ optimal BFS.
  - Degenerate BFS $\Leftrightarrow$ more than $n - m$ zeroes $\Leftrightarrow$ corresponds to more than one base.

- Two BFSs, corresponding to $\mathcal{B}'$ and $\mathcal{B}''$, are called **adjacent** if $\exists\ j, k$ s.t.
$$\mathcal{B}'' = (\mathcal{B}' \setminus \{A_j\}) \cup \{A_k\}.$$

- **Simplex algorithm** (**G.B. Dantzig**, 1947): start with a BFS, and iteratively replace the current BFS with an adjacent BFS having no greater cost, until an optimal BFS is found.

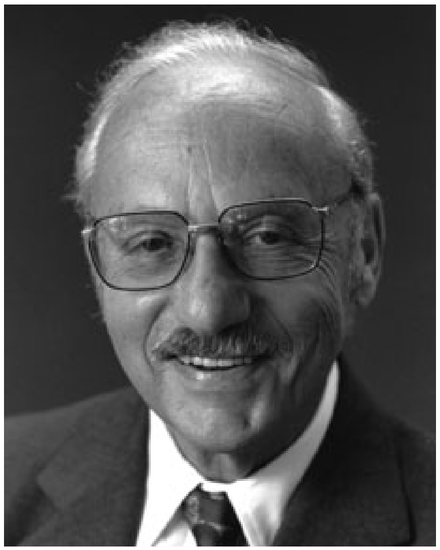
The Simplex algorithm

The **Simplex Algorithm** has been selected as one of the

Top Ten Algorithms of the 20th Century

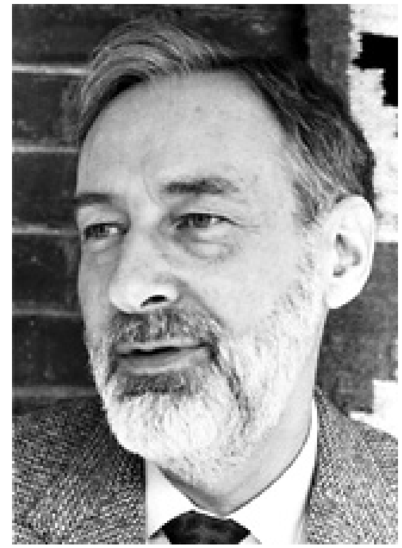
(American Institute of Physics and the IEEE Computer Society):

1. 1946: The Metropolis Algorithm for Monte Carlo.
2. 1947: Simplex Method for Linear Programming.
3. 1950: Krylov Subspace Iteration Method.
4. 1951: The Decompositional Approach to Matrix Computations.
5. 1957: The Fortran Optimizing Compiler.
6. 1959: QR Algorithm for Computing Eigenvalues.
7. 1962: Quicksort Algorithms for Sorting.
8. 1965: Fast Fourier Transform.
9. 1977: Integer Relation Detection.
10. 1987: Fast Multipole Method.



George B. Dantzig

Leonid Kantorovich



Tjalling Koopmans

In 1975 Kantorovich and Koopmans were awarded the Nobel Prize in Economics for their studies on LP and its application to the solution of industrial and transportation problems.

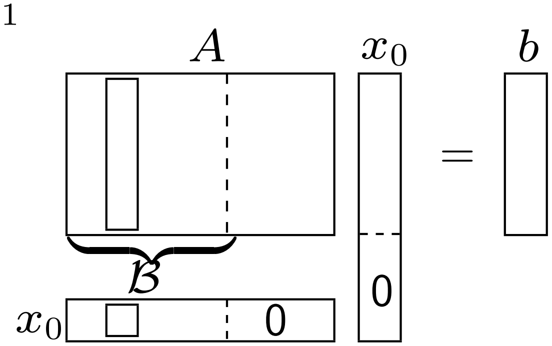
Koopmas wrote to Kantorovich suggesting that they both refuse the prize, because Dantzig had not been included.

Moving from BFS to BFS

- Given a base $\mathcal{B} = \{A_{\beta(i)};\ i = 1, \dots, m\} \rightarrow$ BFS x_0 with basic components y_{i0}:

$$x_0 = (y_{10}, y_{20}, \dots, y_{m0}, 0, \dots, 0)$$

- $Ax_0 = b \iff \sum_{i=1}^{m} a_{k,\beta(i)} y_{i0} = b_k\ \forall\ k \iff \sum_{i=1}^{m} y_{i0} A_{\beta(i)} = b \qquad (\alpha)$



- $A_j \in \mathcal{B}$ linearly independent $\Rightarrow \forall\ A_j \notin \mathcal{B}\ \ \exists\ y_{ij} : \ \sum_{i=1}^{m} y_{ij} A_{\beta(i)} = A_j \qquad (\beta)$
- $(\alpha) - \vartheta \cdot (\beta) \quad (\vartheta$ scalar $> 0)$:

$$\sum_{i=1}^{m} (y_{i0} - \vartheta y_{ij}) A_{\beta(i)} + \vartheta A_j = b\ \forall\ A_j \notin \mathcal{B}$$

- if x_0 is not degenerate then $y_{i0} > 0\ \forall\ i \Rightarrow$
  increasing ϑ from 0, we move to new feasible solutions with $m+1$ positive components;
- max ϑ value: the one for which the first component $(y_{i0} - \vartheta y_{ij})$ becomes 0 ($\Rightarrow$ new base!):

$$\vartheta_{\max} = \min_{i: y_{ij} > 0} \frac{y_{i0}}{y_{ij}}.$$

Moving from BFS to BFS (cont'd)

- **Example:** $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}.$

- $\mathcal{B} = \{A_1, A_3, A_6, A_7\} \Rightarrow \beta(1) = 1, \beta(2) = 3, \beta(3) = 6, \beta(4) = 7$

$$x_0 = (\underset{y_{10}}{2}, 0, \underset{y_{20}}{2}, 0, 0, \underset{y_{30}}{1}, \underset{y_{40}}{4})$$

- Let j=5: then

$$\begin{aligned} A_5 &= y_{15} A_1 + y_{25} A_3 + y_{35} A_6 + y_{45} A_7 = \\ &= 1 \cdot A_1 - 1 \cdot A_3 + 1 \cdot A_6 + 1 \cdot A_7 \end{aligned}$$

$$(2-\vartheta)A_1 + (2+\vartheta)A_3 + (1-\vartheta)A_6 + (4-\vartheta)A_7 + \vartheta A_5 = b$$

- Family of points: $(2-\vartheta,\ 0\ ,\ 2+\vartheta,\ 0\ ,\vartheta,\ 1-\vartheta,\ 4-\vartheta)$. $\vartheta_{\max} = \min\{\frac{2}{1}, \frac{1}{1}, \frac{4}{1}\} = 1$
- Increasing ϑ from 0 to 1 we move from x_0 to $\widetilde{x}_0 = (1, 0, 3, 0, 1, 0, 3)$, corresponding to the new base

$$\widetilde{\mathcal{B}} = \{A_1, A_3, A_5, A_7\}.$$

Moving from BFS to BFS (cont'd)

- **Special cases:**

  1) x_0 **is degenerate** $\Leftrightarrow \; \exists \; y_{i'0} = 0$; then

  if $y_{i'j} > 0 \quad \Rightarrow \quad \vartheta_{\max} = 0$

  $\Leftrightarrow$ **we move to a new basis, but the solution does not change.**

  (Terminology: j enters the base *at zero level*).

  - What happens on the polytope?
    normally: we move from a vertex to a different vertex, "walking" along an edge;

    if x_0 **is degenerate**: we can stay on the same vertex.

  2) $y_{ij} \leq 0$ for $i = 1, \ldots, m \Rightarrow \vartheta$ can indefinitely increase

  if we are moving to a lower cost basis $\Rightarrow$ F **is unbounded: Assumption 3 is violated.**

- **What's left?** We still need to
  - prove that the new solution is a BFS;
  - find an easy way to have the y_{ij} values available;
  - decide how to select A_j.

Moving from BFS to BFS (cont'd)

- **Theorem** *Given a BFS $x_0 = (y_{10}, \dots, y_{m0}, 0, \dots, 0)$ with base $\mathcal{B} = \{A_{\beta(i)};\ i = 1, \dots, m\}$, let $j : A_j \notin \mathcal{B}$. Then the new feasible solution $\widetilde{x}_0$ defined as:*

$$\vartheta_{\max} = \min_{i: y_{ij} > 0} \frac{y_{i0}}{y_{ij}} = \frac{y_{\ell 0}}{y_{\ell j}}, \quad \widetilde{y}_{i0} = \begin{cases} y_{i0} - \vartheta_{\max} y_{ij} & \text{if} \quad i \neq \ell \\ \vartheta_{\max} & \text{if} \quad i = \ell \end{cases}$$

is a BFS with base $\widetilde{\mathcal{B}}$ given by: $\widetilde{\beta}(i) = \begin{cases} \beta(i) & \text{if } i \neq \ell \\ j & \text{if } i = \ell \end{cases}$

(**Terminology:** $y_{\ell j}$ is called the **pivot**; A_j **enters** the base, $A_{\beta(\ell)}$ **leaves** the base.)

Proof We already proved that $\widetilde{x}_0$ is feasible. Let's show that the columns of $\widetilde{\mathcal{B}}$ are linearly independent. Suppose by absurd that $\exists\ d_i \neq 0 \ : \ \sum_{i=1}^{m} d_i A_{\widetilde{\beta}(i)} = 0$. Then

$$d_\ell A_j + \sum_{\substack{i=1 \\ i \neq \ell}}^{m} d_i A_{\widetilde{\beta}(i)} = d_\ell \sum_{i=1}^{m} y_{ij} A_{\beta(i)} + \sum_{\substack{i=1 \\ i \neq \ell}}^{m} d_i A_{\widetilde{\beta}(i)} = \sum_{\substack{i=1 \\ i \neq \ell}}^{m} (d_\ell y_{ij} + d_i) A_{\beta(i)} + d_\ell y_{\ell j} A_{\beta(\ell)} = 0$$

combination of the original base $\Rightarrow$ all coefficients must be 0 $\Rightarrow$
$d_\ell y_{\ell j} = 0 \Rightarrow d_\ell = 0$ ($y_{\ell j} \neq 0$ by hypothesis) $\Rightarrow d_i = 0\ \forall i$, a contradiction. $\square$

- **Corollary** If operation $\vartheta_{\max} = \min_{i: y_{ij} > 0} \frac{y_{i0}}{y_{ij}}$ has a tie then the new BFS is degenerate.

Proof $\vartheta_{\max} = \frac{y_{\ell 0}}{y_{\ell j}} = \frac{y_{k0}}{y_{kj}} \Rightarrow \widetilde{y}_{k0}$ becomes 0. $\square$

The tableau

- **Tableau** (initial definition): $m \times (n+1)$ matrix $b|A$,

$$\begin{array}{c|c} b & \quad A \quad \\ \hline 0 & 1 \quad \dots \quad n \end{array}$$

- **Example:** $\begin{cases} x_1 + x_2 + x_3 = 2 \\ x_1 + 3x_2 + x_4 = 3 \\ x_1 + 4x_2 + x_3 + x_5 = 4 \end{cases}$ $\mapsto$

	x_1	x_2	x_3	x_4	x_5
2	1	1	1	0	0
3	1	3	0	1	0
4	1	4	1	0	1

- **Elementary row operations**: 1) multiply a row for a nonzero constant;
  2) sum a multiple of a row to another row.
- Given $\mathcal{B}$, elementary row operations to obtain I in the columns of $\mathcal{B}$:
- **Example:** $\mathcal{B} = \{A_3, A_4, A_5\}$, subtract row 1 from row 3 $\Rightarrow$ **equivalent system**

y_{i0}	x_1	x_2	x_3	x_4	x_5
2	1	1	1	0	0
3	1	3	0	1	0
2	0	3	0	0	1

$\underbrace{x_1\ x_2}_{[y_{ij}]}$ $\underbrace{x_3\ x_4\ x_5}_{\mathcal{B}}$

All information we need is in the tableau!

The tableau (cont'd)

- **Changing base in the tableau:** **Example:** $j = 2$: $\vartheta_{max} = \min\left\{\frac{2}{1}, \frac{3}{3}, \frac{2}{3}\right\} = \frac{2}{3} = \frac{y_{30}}{y_{32}}$.

  We need to obtain vector (0, 0, 1) in column 2:

$$
\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
2 & 1 & 1 & 1 & 0 & 0 \\
3 & 1 & 3 & 0 & 1 & 0 \\
2 & 0 & 3 & 0 & 0 & 1
\end{array}
\longrightarrow
\begin{array}{c|ccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
\frac{4}{3} & 1 & 0 & 1 & 0 & -\frac{1}{3} \\
1 & 1 & 0 & 0 & 1 & -1 \\
\frac{2}{3} & 0 & 1 & 0 & 0 & \frac{1}{3}
\end{array}
$$

  row 3 := (row 3)/3;

  row 1 := (row 1) − (new row 3);

  row 2 := (row 2) − 3·(new row 3);

- **General methodology** (elementary row operations):
  1. divide the pivoting row by the pivot $y_{\ell j}$;
  2. **for each** other row i **do** subtract y_{ij}·(new pivoting row) from row i, i.e.,

- *Given the* **tableau** $[y_{ij}]$ $(i = 1, \dots, m;\ j = 0, \dots, n)$ *with* **base** $\beta(i)$, *given the* **pivot** $y_{\ell j}$, *the* **new tableau** $[\widetilde{y}_{ij}]$ *is produced by*

  1. $\widetilde{y}_{\ell q} = \dfrac{y_{\ell q}}{y_{\ell j}}$ $(q = 0, \dots, n)$;

  2. $\widetilde{y}_{iq} = y_{iq} - \widetilde{y}_{\ell q} y_{ij}$ $(i = 1, \dots, m,\ i \neq \ell;\ q = 0, \dots, n)$,

  and the **new base** *is:* $\widetilde{\beta}(i) = \begin{cases} \beta(i) & i \neq \ell \\ j & i = \ell \end{cases}$.

The tableau (cont'd)

- **Pivoting consequences on the solution value:**
- BFS x_0 with base $\mathcal{B}$: current solution value: $z_0 = \sum_{i=1}^{m} y_{i0} c_{\beta(i)}$.
- Pivoting which enters A_j into the base: $\left(\sum_{i=1}^{m} (y_{i0} - \vartheta y_{ij}) A_{\beta(i)} + \vartheta A_j = b \right)$.
- If 1 unit of x_j enters the base ($\Leftrightarrow \vartheta = 1$), the cost becomes:

$$\widetilde{z}_0 = \sum_{i=1}^{m} (y_{i0} - y_{ij}) c_{\beta(i)} + 1 \cdot c_j = z_0 - \sum_{i=1}^{m} y_{ij} c_{\beta(i)} + c_j$$

- i.e., for each unit of x_j which enters the base, the cost changes by $c_j - \sum_{i=1}^{m} y_{ij} c_{\beta(i)}$.
- By defining $z_j = \sum_{i=1}^{m} y_{ij} c_{\beta(i)}$, we call $\overline{c}_j = c_j - z_j$ the **relative cost of column j.**
- **Hence:**
  1. **Only columns A_j for which $\overline{c}_j < 0$ are profitable**;
  2. the solution value changes by $\vartheta_{\max} \overline{c}_j = \vartheta_{\max}(c_j - z_j)$.

The tableau (cont'd)

- **Getting the relative costs from the tableau:**
- **cost equation:**

$$0 = c_1 x_1 + \cdots + c_n x_n - z$$

- in the 0-th row of the tableau, by considering $(-z)$ as a new variable:

$$
\begin{array}{|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{\dots \qquad x_n} & \multicolumn{1}{c}{(-z)} \\
\hline
0 & c_1 & \dots \qquad c_n & 1 \\
\hline
y_{i0} & [y_{ij}] & \underbrace{\begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{matrix}}_{\mathcal{B}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\
\hline
\end{array}
$$

- Subtract each row i, multiplied by $c_{\beta(i)}$, to row 0:

$$c_j - \sum_{i=1}^{m} y_{ij} c_{\beta(i)}$$

$$\downarrow$$

$$0 - \sum_{i=1}^{m} y_{i0} c_{\beta(i)} \rightarrow \begin{array}{|c|c|ccc|c|} \hline -z_0 & \overline{c}_j & 0 & \dots & 0 & 1 \\ \hline & & 1 & & \mathbf{0} & 0 \\ y_{i0} & [y_{ij}] & & 1 & & 0 \\ & & \mathbf{0} & & 1 & 0 \\ \hline \end{array}$$

- In the pivoting operation we also execute $\widetilde{y}_{0q} = y_{0q} - \widetilde{y}_{\ell q} y_{0j} \quad (q = 0, \dots, n)$.
- The last column gives no information $\Rightarrow$ it is usually not shown.

Optimality criterion

Theorem *If $\overline{c}_j \geq 0 \ \forall \ j$ then the current solution x_0 is optimal.*

Proof Row 0 equation: $z_{\text{opt}} = z_0 + \sum_{j=1}^{n} \overline{c}_j x_j = z_0 + \sum_{A_j \notin \mathcal{B}} \overline{c}_j x_j$, i.e.,

$$\text{(optimal solution value)} = \text{(current solution value)} + \sum_{A_j \notin \mathcal{B}} \overline{c}_j x_j .$$

Since we must have $x_j \geq 0 \ \forall j$, if $\overline{c}_j \geq 0 \ \forall j$ the current solution value cannot decrease. $\square$

The tableau (cont'd)

Example:

$$
\begin{array}{lccccccccccc}
\min z = - & x_1 & - & 2x_2 & & & & & & & & \\
& x_1 & + & x_2 & + & x_3 & & & & & = & 2 \\
& x_1 & + & 3x_2 & & & + & x_4 & & & = & 3 \\
& & & 3x_2 & & & & & + & x_5 & = & 2 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0,
\end{array}
$$

Initial tableau:

		x_1	x_2	x_3	x_4	x_5
$-z =$	0	-1	-2	0	0	0
$x_3 =$	2	1	1	1	0	0
$x_4 =$	3	1	3	0	1	0
$x_5 =$	2	0	3	0	0	1

Choose A_1 (between A_1 and A_2) to enter the base. Pivoting:

The tableau (cont'd)

		x_1	x_2	x_3	x_4	x_5
$-z =$	2	0	-1	1	0	0
$x_1 =$	2	1	1	1	0	0
$x_4 =$	1	0	2	-1	1	0
$x_5 =$	2	0	3	0	0	1

The solution is not optimal: choose A_2 to enter the base. Pivoting:

		x_1	x_2	x_3	x_4	x_5
$-z =$	$\frac{5}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
$x_1 =$	$\frac{3}{2}$	1	0	$\frac{3}{2}$	$-\frac{1}{2}$	0
$x_2 =$	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	0
$x_5 =$	$\frac{1}{2}$	0	0	$\frac{3}{2}$	$-\frac{3}{2}$	1

Optimal solution: $x' = (\frac{3}{2}, \frac{1}{2}, 0, 0, \frac{1}{2})$, of value $z = -\frac{5}{2}$.

Simplex algorithm (1st version)

Let us assume that: 1) we have an initial BFS with I in the basic columns;
2) no degenerate BFS is encountered.

procedure SIMPLEX:
begin
$optimal := unbounded :=$ false;
while $optimal =$ false **and** $unbounded =$ false **do**
if $\bar{c}_j \geq 0 \ \forall j$ **then** $optimal :=$ true
else
begin
choose any j such that $\bar{c}_j < 0$;
if $y_{ij} \leq 0 \ \forall i > 0$ **then** $unbounded :=$ true (**comment:** Assumption 3 is violated)
else $\vartheta_{\max} := \min_{i: y_{ij} > 0} \frac{y_{i0}}{y_{ij}} = \frac{y_{\ell 0}}{y_{\ell j}}$, and perform a pivoting on $y_{\ell j}$
end
end.

Convergence: in absence of degeneration, z_0 decreases at each iteration $\Rightarrow$
each BFS is different from the previous ones, hence the algorithm converges.

We still need: a policy to decide which column must enter the base;
a method to obtain an initial BFS.

Can degenerate bases produce cycling?

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z =$	3	$-\frac{3}{4}$	20	$-\frac{1}{2}$	6	0	0	0
$x_5 =$	0	$\frac{1}{4}$	-8	-1	9	1	0	0
$x_6 =$	0	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Assume we use as pivoting rule:
1) the variable with "most negative" $\overline{c}_j$ enters the base;
2) in case of tie the variable with lowest index leaves the base.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z =$	3	0	-4	$-\frac{7}{2}$	33	3	0	0
$x_1 =$	0	1	-32	-4	36	4	0	0
$x_6 =$	0	0	4	$\frac{3}{2}$	-15	-2	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Can degenerate bases produce cycling? (cont'd)

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z =$	3	0	0	-2	18	1	1	0
$x_1 =$	0	1	0	8	-84	-12	8	0
$x_2 =$	0	0	1	$\frac{3}{8}$	$-\frac{15}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	0
$x_7 =$	1	0	0	1	0	0	0	1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z =$	3	$\frac{1}{4}$	0	0	-3	-2	3	0
$x_3 =$	0	$\frac{1}{8}$	0	1	$-\frac{21}{2}$	$-\frac{3}{2}$	1	0
$x_2 =$	0	$-\frac{3}{64}$	1	0	$\frac{3}{16}$	$\frac{1}{16}$	$-\frac{1}{8}$	0
$x_7 =$	1	$-\frac{1}{8}$	0	0	$\frac{21}{2}$	$\frac{3}{2}$	-1	1

Can degenerate bases produce cycling? (cont'd)

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z =$	3	$-\frac{1}{2}$	16	0	0	-1	1	0
$x_3 =$	0	$-\frac{5}{2}$	56	1	0	2	-6	0
$x_4 =$	0	$-\frac{1}{4}$	$\frac{16}{3}$	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0
$x_7 =$	1	$\frac{5}{2}$	-56	0	0	-2	6	1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z =$	3	$-\frac{7}{4}$	44	$\frac{1}{2}$	0	0	-2	0
$x_5 =$	0	$-\frac{5}{4}$	28	$\frac{1}{2}$	0	1	-3	0
$x_4 =$	0	$\frac{1}{6}$	-4	$-\frac{1}{6}$	1	0	$\frac{1}{3}$	0
$x_7 =$	1	0	0	1	0	0	0	1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z =$	3	$-\frac{3}{4}$	20	$-\frac{1}{2}$	6	0	0	0
$x_5 =$	0	$\frac{1}{4}$	-8	-1	9	1	0	0
$x_6 =$	0	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0	1	0
$x_7 =$	1	0	0	1	0	0	0	1

Loop!

Pivoting rules

- **Two decisions:**

  1) which column (among those with $\overline{c}_j < 0$) must enter the base;

  2) what to do in case of tie among rows.

- **Selecting the column:**
  - column A_j with most negative $\overline{c}_j$ (**Dantzig rule**, still the most efficient one);
  - column A_j which produces the maximum cost decrease (computationally much heavier);
  - Bland's rule . . .

- **How to avoid cycling**
  - resolve ties in a random way: probability 1 of escaping from loops;
  - **Bland's rule**:
    - · the column A_j with minimum index j (among those with $\overline{c}_j < 0$) enters the base;
    - · in case of tie, the column A_j with minimum index j leaves the base.
    - · It can be proved that no cycling can occur, **but**
    - · **convergence is much slower** $\Rightarrow$ only used to escape loops.

- In the cycling example the first four pivots have been selected according to Bland's rule. Continuing with Bland's rule:

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z =$	3	$-\frac{1}{2}$	16	0	0	-1	1	0
$x_3 =$	0	$-\frac{5}{2}$	56	1	0	2	-6	0
$x_4 =$	0	$-\frac{1}{4}$	$\frac{16}{3}$	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0
$x_7 =$	1	$\frac{5}{2}$	-56	0	0	-2	6	1

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z =$	$\frac{16}{5}$	0	$\frac{24}{5}$	0	0	$-\frac{7}{5}$	$\frac{11}{5}$	$\frac{1}{5}$
$x_3 =$	1	0	0	1	0	0	0	1
$x_4 =$	$\frac{1}{10}$	0	$-\frac{4}{15}$	0	1	$\frac{2}{15}$	$-\frac{1}{15}$	$\frac{1}{10}$
$x_1 =$	$\frac{2}{5}$	1	$-\frac{112}{5}$	0	0	$-\frac{4}{5}$	$\frac{12}{5}$	$\frac{2}{5}$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7
$-z =$	$\frac{17}{4}$	0	2	0	$\frac{21}{2}$	0	$\frac{3}{2}$	$\frac{5}{4}$
$x_3 =$	1	0	0	1	0	0	0	1
$x_5 =$	$\frac{3}{4}$	0	-2	0	$\frac{15}{2}$	1	$-\frac{1}{2}$	$\frac{3}{4}$
$x_1 =$	1	1	-24	0	6	0	2	1

Optimal!

Obtaining an initial BFS

Two-phase method (Phase 2 = SIMPLEX):

1. If $\exists\ b_i < 0$ then multiply the corresponding equation by -1 ($\Rightarrow\ b \geq 0$);

2. Add m artificial variables $x_i^a \geq 0$ (sum x_i^a to each equation i):

$$
\begin{array}{c|ccc|ccc|}
 & x_1^a & \dots & x_m^a & x_1 & \dots & x_n \\
\hline
 & 1 & & \mathbf{0} & & & \\
b & & 1 & & & A & \\
 & \mathbf{0} & & 1 & & & \\
\hline
\end{array}
$$

$\Rightarrow\ \exists$ BFS $x_i^a = b_i$ $(i = 1, \dots, m)$.

3. Use SIMPLEX to minimize an artificial objective function: $\zeta = \sum_{i=1}^{m} x_i^a$.

Three cases may occur:

3.a $\zeta = 0$ and no x_i^a is basic: we have a BFS for the original problem $\Rightarrow$ Phase 2;

3.b $\zeta > 0$: it is impossible to satisfy the constraints without artificial variables $\Rightarrow \nexists$ feasible solution, **Assumption 2 is violated**;

3.c $\zeta = 0$ but some x_i^a is basic (at zero level): additional operations are needed . . . but first let's see some examples.

Obtaining an initial BFS (cont'd)

Example:

$$\begin{array}{rrrrrrrrrrrrrrr}
\min z = & x_1 & & & + & x_3 & & & & & & & & & \\
& x_1 & + & 2x_2 & & & \leq & 5 & \rightarrow & x_1 & + & 2x_2 & + & s_1 & = & 5 \\
& & & x_2 & + & 2x_3 & = & 6 & & & & & & s_1 & \geq & 0 \\
& x_1 & , & x_2 & , & x_3 & \geq & 0 & & & & & & & &
\end{array}$$

Bland's rule: **Phase 1:**

		x_1^a	x_2^a	x_1	x_2	x_3	s_1
	0	1	1	0	0	0	0
	5	1	0	1	2	0	1
	6	0	1	0	1	2	0

		x_1^a	x_2^a	x_1	x_2	x_3	s_1
$-\zeta$	-11	0	0	-1	-3	-2	-1
x_1^a	5	1	0	1	2	0	1
x_2^a	6	0	1	0	1	2	0

Obtaining an initial BFS (cont'd)

		x_1^a	x_2^a	x_1	x_2	x_3	s_1
$-\zeta$	-6	1	0	0	-1	-2	0
x_1	5	1	0	1	2	0	1
x_2^a	6	0	1	0	1	2	0

		x_1^a	x_2^a	x_1	x_2	x_3	s_1
$-\zeta$	$-\frac{7}{2}$	$\frac{3}{2}$	0	$\frac{1}{2}$	0	-2	$\frac{1}{2}$
x_2	$\frac{5}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$
x_2^a	$\frac{7}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0	2	$-\frac{1}{2}$

		x_1^a	x_2^a	x_1	x_2	x_3	s_1
$-\zeta$	0	1	1	0	0	0	0
x_2	$\frac{5}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$
x_3	$\frac{7}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	1	$-\frac{1}{4}$

$\leftarrow$ typical structure

Obtaining an initial BFS (cont'd)

Phase 2:

		x_1^a	x_2^a	x_1	x_2	x_3	s_1
	0	0	0	1	0	1	0
	$\frac{5}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$
	$\frac{7}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	1	$-\frac{1}{4}$

		x_1^a	x_2^a	x_1	x_2	x_3	s_1
$-z$	$-\frac{7}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{5}{4}$	0	0	$\frac{1}{4}$
x_2	$\frac{5}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$
x_3	$\frac{7}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	1	$-\frac{1}{4}$

Optimal solution: $z = \frac{7}{4}$; $x_1 = 0$, $x_2 = \frac{5}{2}$, $x_3 = \frac{7}{4}$.

Observations:

1) In Phase 2 the x_i^a columns could be eliminated (but we will see cases where they can be useful); if maintained, only relative costs of non-artificial variables must be considered for pivoting.

2) Columns x_1^a and s_1 are identical $\Leftrightarrow$ x_1^a could have been avoided:

In general: if $\exists$ columns of I then we only introduce the necessary artificial variables.

Ex: $\min z = x_1 + x_3$

$$\begin{array}{rcl} x_1 + 2x_2 & \le & -5 \\ x_2 + 2x_3 & = & 6 \\ x_1, x_2, x_3 & \ge & 0 \end{array}$$

$x_1 + 2x_2 \le -5 \rightarrow x_1 + 2x_2 + s_1 = -5$

$\downarrow$

$-x_1 - 2x_2 - s_1 = 5$

$s_1 \ge 0$

		x_1^a	x_2^a	x_1	x_2	x_3	s_1
	0	1	1	0	0	0	0
	5	1	0	-1	-2	0	-1
	6	0	1	0	1	2	0

		x_1^a	x_2^a	x_1	x_2	x_3	s_1
$-\zeta$	-11	0	0	1	1	-2	1
x_1^a	5	1	0	-1	-2	0	-1
x_2^a	6	0	1	0	1	2	0

		x_1^a	x_2^a	x_1	x_2	x_3	s_1
$-\zeta$	-5	0	1	1	2	0	1
x_1^a	5	1	0	-1	-2	0	-1
x_3	3	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0

Optimal solution with $\zeta > 0$

$\Rightarrow \nexists$ solution

Obtaining an initial BFS (cont'd)

Ex:
$$\begin{array}{rcrcrcrcr} \min z = & x_1 & + & x_2 & + & 10x_3 & & & \\ & & & x_2 & + & 4x_3 & = & 2 & \\ & 2x_1 & - & x_2 & + & 6x_3 & = & -2 & \rightarrow \; -2x_1 + x_2 - 6x_3 = 2 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0 & \end{array}$$

Phase 1:

	x_1^a	x_2^a	x_1	x_2	x_3
0	1	1	0	0	0
2	1	0	0	1	4
2	0	1	-2	1	-6

		x_1^a	x_2^a	x_1	x_2	x_3
$-\zeta$	-4	0	0	2	-2	2
x_1^a	2	1	0	0	1	4
x_2^a	2	0	1	-2	1	-6

		x_1^a	x_2^a	x_1	x_2	x_3
$-\zeta$	0	2	0	2	0	10
x_2	2	1	0	0	1	4
x_2^a	0	-1	1	-2	0	-10

$\zeta = 0$, but x_2^a in base (at 0 level):

special pivoting with any $x_{2j} \neq 0$

(even if < 0, even if $\overline{c}_j \geq 0 \Leftarrow \vartheta_{\max} = 0$)

		x_1^a	x_2^a	x_1	x_2	x_3
$-\zeta$	0	1	1	0	0	0
x_2	2	1	0	0	1	4
x_1	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	5

we can continue with **Phase 2**

	x_1^a	x_2^a	x_1	x_2	x_3
0	0	0	1	1	10
2	1	0	0	1	4
0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	5

		x_1^a	x_2^a	x_1	x_2	x_3
$-z$	-2	$-\frac{3}{2}$	$\frac{1}{2}$	0	0	1
x_2	2	1	0	0	1	4
x_1	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	5

Opt. sol. $z = 2$; $x_1 = 0, x_2 = 2, x_3 = 0$

Obtaining an initial BFS (cont'd)

Resuming the end of **Phase 1**:

3.a $\zeta = 0$ and no x_i^a is basic: $\Rightarrow$ BFS $\Rightarrow$ Phase 2;

3.b $\zeta > 0$: no solution (**Assumption 2 violated**);

3.c $\zeta = 0$ but $\exists\ x_i^a$ is basic (at zero level) in row i:

- perform a pivoting with any $y_{ij} \neq 0$ (even if < 0, even if $\overline{c}_j \geq 0$: the solution will not change) corresponding to a non-artificial variable
- **if** we can drive all artificial variables out of the basis **then** **Phase 2**;
- **else** $\exists\ x_i^a$ in base s.t. $y_{ij} = 0\ \forall$ non-artificial variables

  $\Rightarrow$ row $0\ 0 \ldots 0$ obtained through elementary row operations

  $\Rightarrow$ A is not of full rank m, **Assumption 1 is violated**. Hence

  $\forall\ \ x_i^a$ in base such that $y_{ij} = 0\ \forall$ non-artificial variable **do** eliminate row i

Simplex algorithm (complete version)

procedure TWO_PHASE:
begin
  $impossible := redundant :=$ false;
  for each $b_i < 0$ **do** multiply the ith equation by -1;
  for i:=1 **to** m **do** add the term x_i^a to the ith equation;
  insert the objective function $\zeta = \sum_{i=1}^{m} x_i^a$ in row 0;
  for $i := 1$ **to** m **do** row 0 := row 0 $-$ row i;
  call SIMPLEX;
  if ζ^* (= solution value) > 0 **then** $impossible :=$ true (**comment:** Assumption 2 violated)
  else
    begin
      for each artificial variable x_i^a in base **do**
        if $\exists\ y_{ij} \neq 0\ :\ x_j$ is non-artificial **then** perform a pivoting on y_{ij}
        else begin
            $redundant :=$ true (**comment:** Assumption 1 violated);
            eliminate row i, and set $m := m - 1$
          end;
      insert the original objective function in row 0;
      for $i := 1$ **to** m **do** row 0 := row 0 $-$ $c_{\beta(i)} \cdot$ (row i);
      call SIMPLEX
    end
end.

Geometric view of the simplex algorithm

Example: $\min z = 2x_2 + x_4 + 5x_7$

$$\begin{array}{rcrcrcrcrcrcrcl}
x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & & & = & 4 \\
x_1 & & & & & & & + & x_5 & & & & & = & 2 \\
 & & & & x_3 & & & & & + & x_6 & & & = & 3 \\
 & & 3x_2 & + & x_3 & & & & & & & + & x_7 & = & 6 \\
x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & , & x_7 & \geq & 0
\end{array}$$

Bland's rule. Phase 1 not needed.

	0	0	2	0	1	0	0	5	
$-z$	-34	-1	-14	-6	0	0	0	0	
x_4	4	1	1	1	1	0	0	0	
x_5	2	1	0	0	0	1	0	0	**1**
x_6	3	0	0	1	0	0	1	0	
x_7	6	0	3	1	0	0	0	1	

Geometric view of the simplex algorithm (cont'd)

$-z$	-32	0	-14	-6	0	1	0	0
x_4	2	0	1	1	1	-1	0	0
x_1	2	1	0	0	0	1	0	0
x_6	3	0	0	1	0	0	1	0
x_7	6	0	3	1	0	0	0	1

2

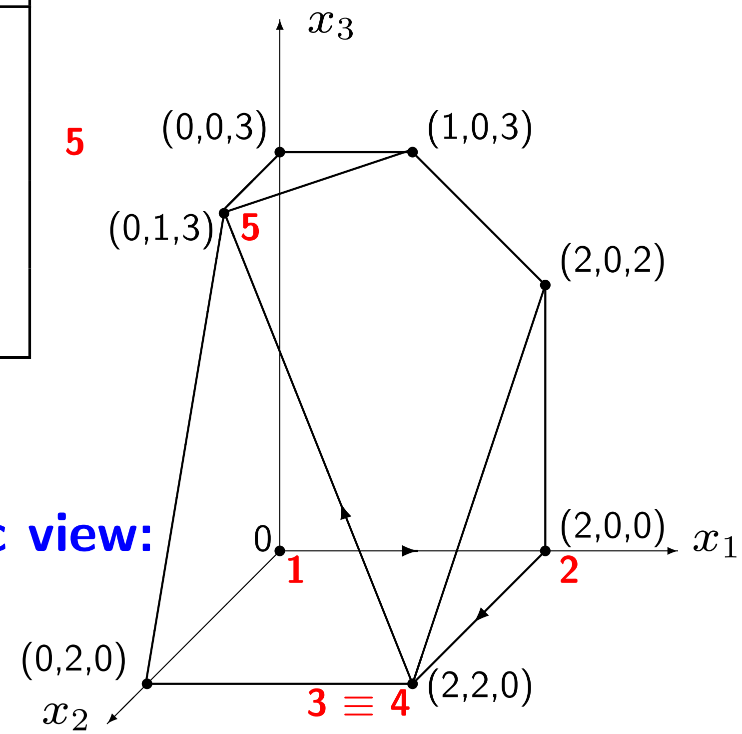
$-z$	-4	0	0	8	14	-13	0	0
x_2	2	0	1	1	1	-1	0	0
x_1	2	1	0	0	0	1	0	0
x_6	3	0	0	1	0	0	1	0
x_7	0	0	0	-2	-3	3	0	1

3

$-z$	-4	0	0	$-\frac{2}{3}$	1	0	0	$\frac{13}{3}$	
x_2	2	0	1	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$	
x_1	2	1	0	$\frac{2}{3}$	1	0	0	$-\frac{1}{3}$	**4**
x_6	3	0	0	1	0	0	1	0	
x_5	0	0	0	$-\frac{2}{3}$	-1	1	0	$\frac{1}{3}$	

$-z$	-2	1	0	0	2	0	0	4	
x_2	1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	
x_3	3	$\frac{3}{2}$	0	1	$\frac{3}{2}$	0	0	$-\frac{1}{2}$	**5**
x_6	0	$-\frac{3}{2}$	0	0	$-\frac{3}{2}$	0	1	$\frac{1}{2}$	
x_5	2	1	0	0	0	1	0	0	

Geometric view:



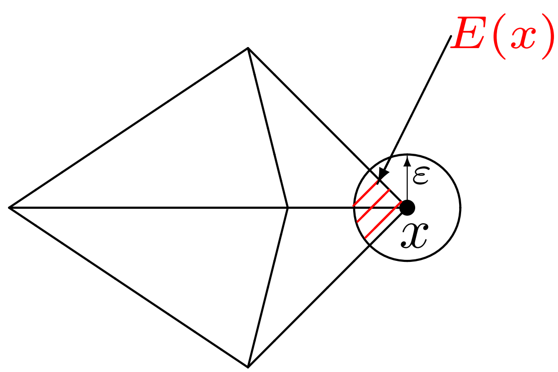
Geometric view of the simplex algorithm (cont'd)

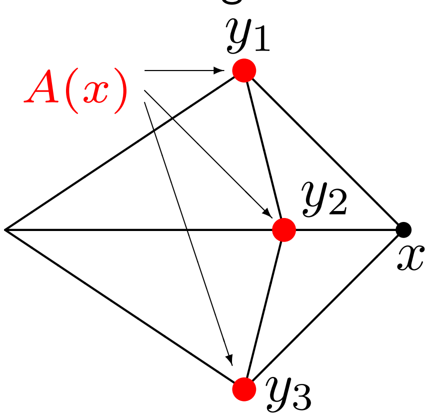
Two vertices of a polytope are called **adjacent** if they are connected by an edge.

Theorem *Given the polytope defined by the constraints of an* LP, *a line segment* $[\overline{x}, \overline{y}] \in P$ *is an edge if and only if the corresponding vectors x, y are adjacent BFSs of the* LP.

Proof omitted.

An LP is a convex programming problem $\Rightarrow$ the Euclidean neighborhood $E(x)$ is exact:





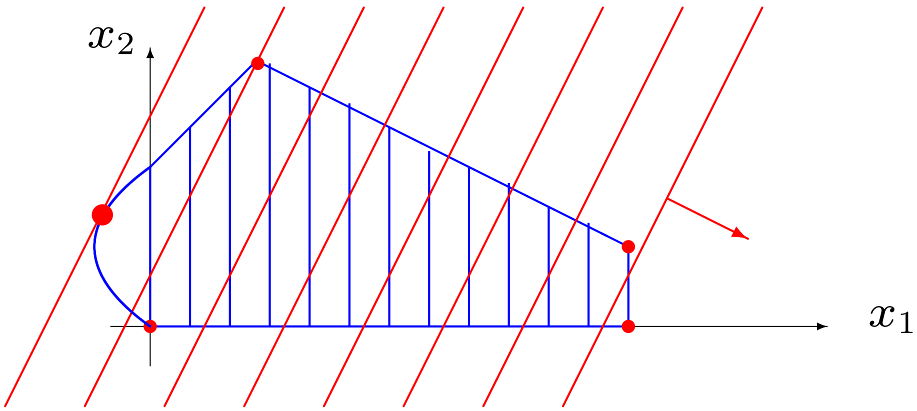
The simplex algorithm proves the existence of a more important neighborhood:
$A(x) = \{y \in P \; : \; y$ is a vertex adjacent to $x\}$

The Optimality criterion ensures that A is exact for LP.

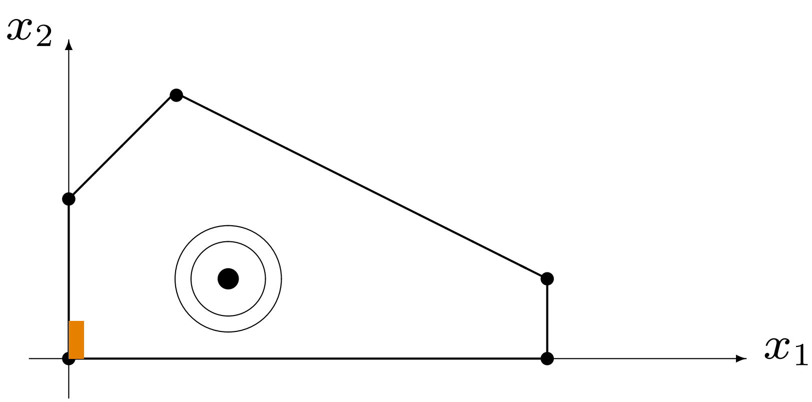
$A(x)$ contains "few" vertices (at most $n - m$ BFSs are adjacent to the current one), and can be searched very fast $\Rightarrow$ "good" algorithm.

Why is it difficult to solve Non-Linear Programming?

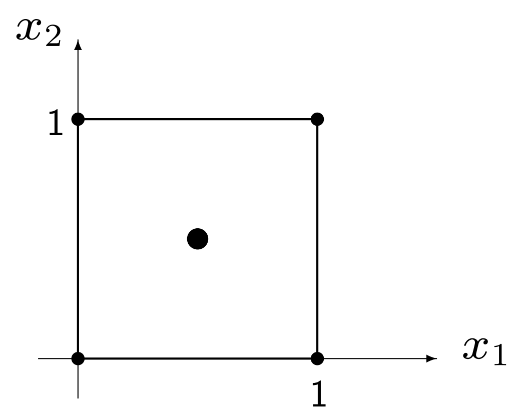
- If constraints are non-linear:



- If the objective function is non-linear:



Ex: $\min \varphi(x) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2$



- In both cases the optimal solution must be searched among an **infinite number of points**.