Operations Research (Master's Degree Course)

1. Introduction

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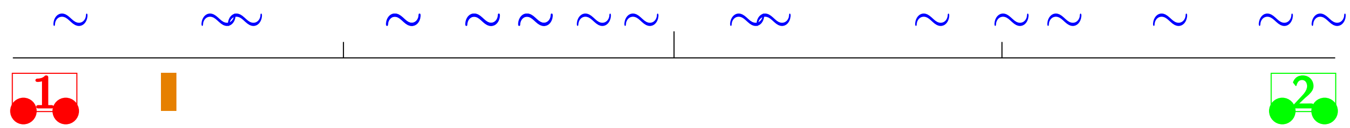
Decision making

Operations Research: application of scientific methods to "decision making problems" which are encountered in any organization in which it is required to manage and coordinate activities and resources so as to obtain the best possible result.

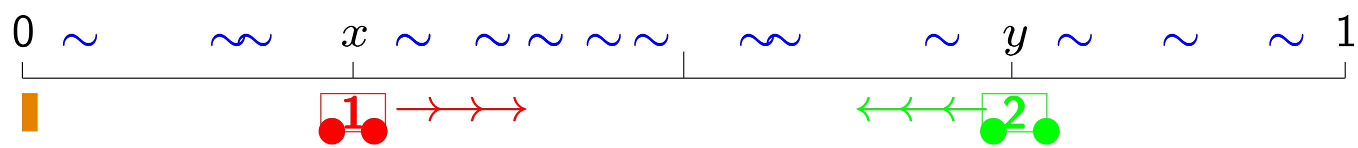
- Decision-making capacity $\leftrightarrow$ intellectual ability.
- According to communication scientists, a normal person makes 70 decisions a day.
  Many more if she has a position of responsibility.
- Organization (private company, public administration, hospitals, army, ...)
  - interacting elements $\Rightarrow$ performance;
  - decisions needed to obtain the best possible performance.
- Traditionally solved through experience, intuition, common sense.
  Starting from the mid-Twentieth century:
- new logical and mathematical methodologies allowing to formulate and solve many decision problems in a scientific way.
- Is it possible to predict the behavior of human beings? Let's see a couple of examples.

Example. Where will two ice cream vendors locate their carts on a beach?

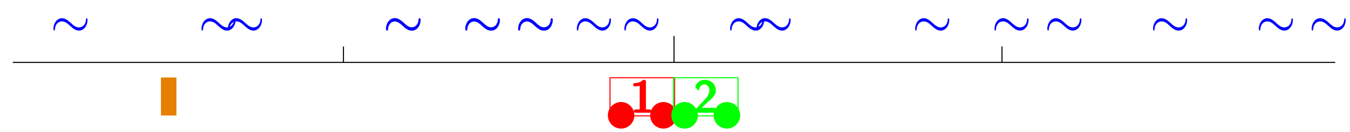
- Beach length: 1 Km , consumers evenly spread, 2 **competing** ice cream vendors:



- Inclination to purchase depends on the distance. **Where will they locate carts?**
  1st possibility:



- **1**: $\frac{1}{2}(x+y)$; **2**: $1 - \frac{1}{2}(x+y)$;
- For both vendors it is **convenient to move towards the center.** **Result:**



- None has advantage from moving: Nash equilibrium. Game theory.
- However, both will lose those consumers that are too far. What about **cooperation**?
  1st possibility: all consumers are closer, both vendors sell more ice creams.
- Is this just a game? **NO:**

  In bipartite political systems, **parties** tendo to position themselves close to the center.

One more example. The prisoners' dilemma.

- Two criminals are arrested. The police do not possess sufficient proof to have them convicted;
- the prisoners are isolated from each other, and cannot communicate;
- the district attorney visit each of them and offer the same deal:
  - if you testify against your partner, and s/he remains silent, you go **free** and s/he gets a **10 year** sentence;
  - if neither confesses, you will both receive a **six-month** sentence for a minor charge;
  - if each testify against the other, each receives a **5-year** sentence.
- **What will they decide?** Each prisoner thinks:

		S/he C	S/he NC
I	C	5 years	free
	NC	10 years	6 months

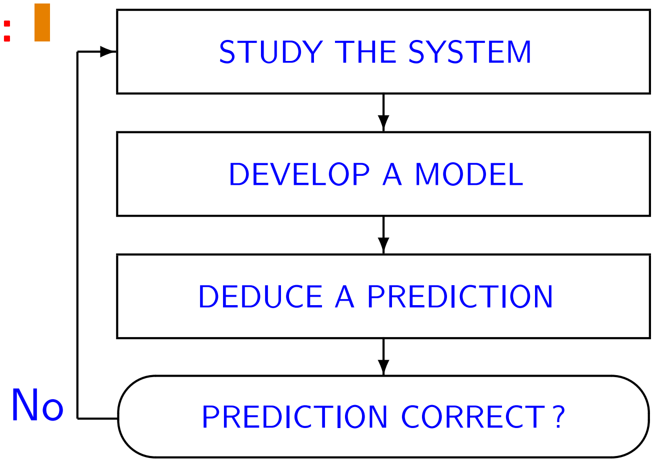
- **Hence I confess.** Result: **both confess** and receive a **5-year** sentence.
- But the **best solution for both** would be not to confess (**6 months**) !
- **If they could communicate?** "We agree that none confesses." But then:

  I have two possibilities: I trust my companion or I don't. If I don't trust ... **Same result**.
- Just a game? **NO:**
- **USA and USSR** in the cold war: should we build more nuclear weapons or disarm?

  Result: both built nuclear weapons!

Scientific method

- Origins of the modern scientific method: Galileo, experiments.
- **Science:** organization of knowledge in a form that allows one to predict the effect of actions.
- Objective: discover rules (**models**) that describe the behavior of a **system**.
  Example: relationship between mass and energy: $E = mc^2$.
- Twentieth century: Abandoning the **inductive method** (use observations to find general laws and theories). Russell and the inductivist turkey ☺.
  Popper and the **deductive method:**



  Example: Atomic models: Thomson (plum pudding)$\rightarrow$Rutherford (planetary model)$\rightarrow$Bohr ...
- Usefulness and reliability of a model:
  - **predictivity**;
  - **falsifiability** (possibility to contradict it by an experiment).

Systems and models

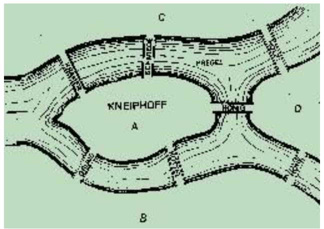
- **System** ($\sigma\acute{\upsilon}\sigma\tau\eta\mu\alpha$ = composite whole) = a whole formed by several interacting parts.
- How can we predict the behavior of a system?
  - *prototypes* (time consuming, costly; sometimes not implementable);
  - **models** (modulus = small measure): scheme corresponding to a theoretical structure, to be experimentally tested.
- **Physical model**=scale prototype of a system (Leonardo da Vinci: mills, engines, war machines);

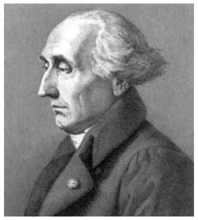
  frequent in the past: nowadays normally only used for aesthetics (e.g., Architecture);
- **Model** = simplification of reality, designed for studying a certain system.
- **Mathematical models** = mathematical and/or logical description of a system.

analytical models:	sets of equations having a closed form solution;
	rarely possible for decision making problems;
numerical models:	**linear programming, graphs, simulation**
	(solution through algorithms and computers);

static models:	systems without time dependency (**linear programming, graphs**);
dynamic models:	behavior of the system over time (**simulation**).

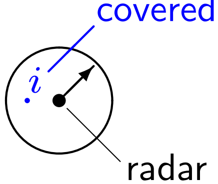
Remote origins

- *The Art of War*, Sun Tzu (circa 5th century BC): 
  military strategy and tactics, with elements of decision theory and game theory;
  Still adopted today in military schools and business strategy studies.
- Leonhard Euler:  

  problem of the seven bridges of Königsberg; invention of **Graph Theory** (1735).
- Joseph-Louis Lagrange (Giuseppe Luigi Lagrangia, b. Turin 1736): 
  constrained optimization, method of Lagrange multipliers.
- Carl Gustav Jacob Jacobi: 
  discovery of the *Hungarian algorithm* (1850) for the assignment problem, re-discovered in 1955.

Modern origins (Western version)

- **UK**, before World War II, circa 1937: invention of the **radar**, lack of experience in the army.
- **Decision making problem**: where to allocate the available radars? Formally:
  - given m radars and n ($n >> m$) potential objectives,
  - given a "value" v_i to each objective i ($i = 1, \ldots, n$),
  - problem: where to locate radars so that $\sum_i$(protected v_i) is a maximum?
  - The solution must exist, but no mathematical method to find it is known.



- P. Blackett (NL 1948): 

  group of mathematicians, physicists, officers: "Research on the operational aspects",

  ($\rightarrow$*Operations Research*$\rightarrow$*Ricerca Operativa* ☹);

  mathematical models: success in radar allocation (Battle of Britain).

  Other mathematical models for **logistic, strategic, and tactical** problems:
  Battle of the Atlantic,
  Pacific War,
  Bombing of Germany.

Modern origins (Eastern version)

- **Leonid Kantorovich** (NL 1975) 

  circa 1935, studies for the Soviet government:
  – optimizing the production in a plywood industry $\Longrightarrow$ *Linear Programming*, LP;

  – *Mathematical methods of organizing and planning production* (in Russian), 1939;

  – **criticisms** from Soviet Academics (capitalist tools?) but support from Stalin.

  – Siege of Leningrad (1941-1944): optimal distance between trucks on iced lake (Ladoga).

Post-War Boom (Fifties and Sixties)

- Decision making problems from the new complexity of production processes:

  – **RAND Project** (Research ANd Development), **John von Neumann** 
  "to help improve policy and decision making through research and analysis";

  – extension of the new techniques and their diffusion (industry, economy, administration, ...);

  – development of theoretical aspects;

  – improvement in computer technology.

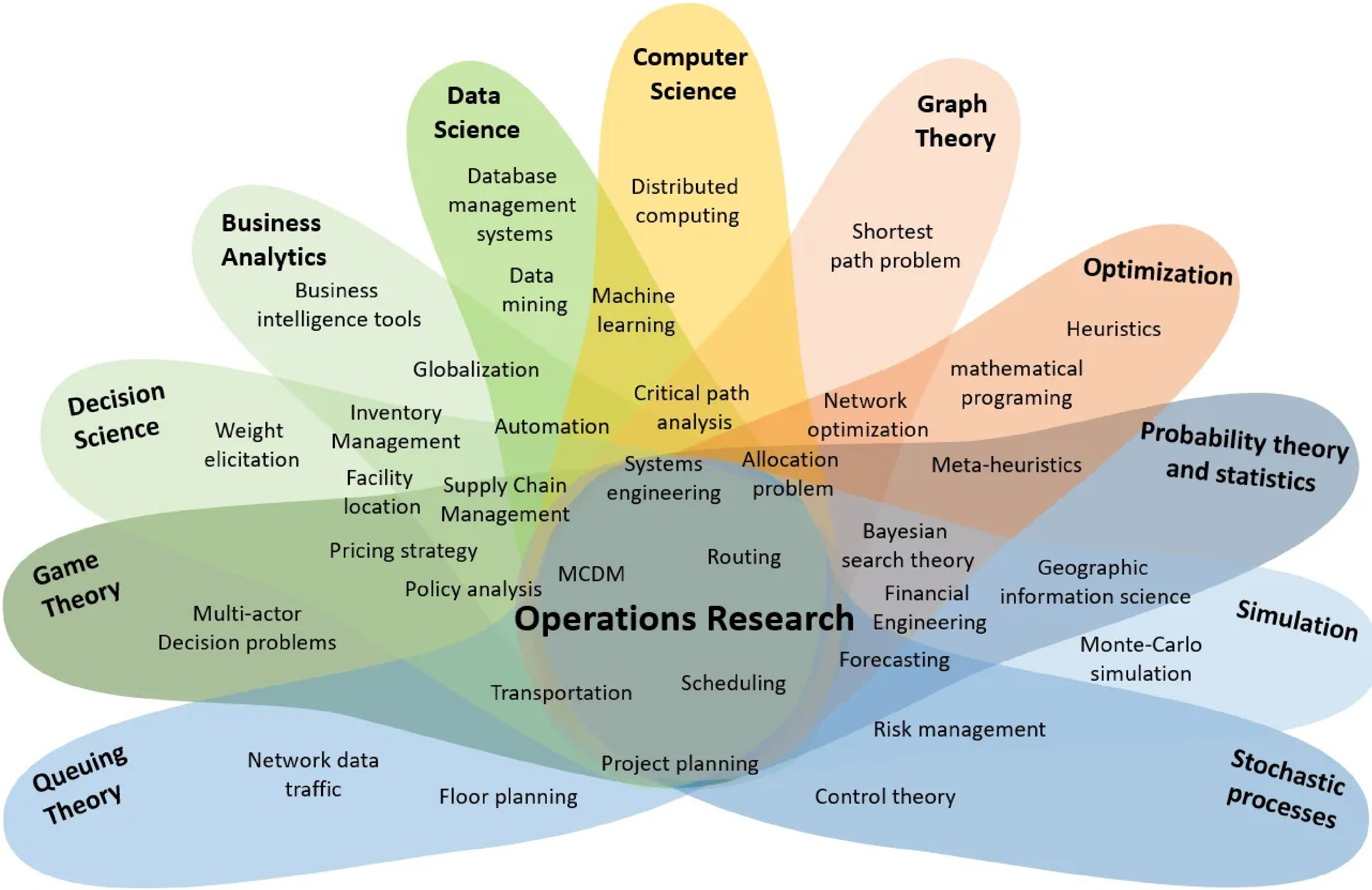
Optimization models for decision making

- **Objective** = apply the **scientific method** to **decision making problems** arising in an organization in which it is required to manage and coordinate activities and resources.
- Various scientific areas (frequently overlapping)
  - **Management science**;
  - **Mathematical programming, Mathematical modeling**;
  - **Optimization and decision sciences**;
  - **Game theory**;
  - **Graph theory**;
  - **Computer science**;
  - **Statistics**;
  - **Algorithm theory**

    and more ... pause
  - (parts of) such sciences merge in

    Operations research (Operational research in UK)

    in which they are used to improve the efficiency of decision making.



Computer Science
Data Science
Graph Theory
Business Analytics
Optimization
Decision Science
Probability theory and statistics
Game Theory
Simulation
Queuing Theory
Stochastic processes
Database management systems
Distributed computing
Shortest path problem
Business intelligence tools
Data mining
Machine learning
Heuristics
mathematical programing
Globalization
Critical path analysis
Network optimization
Inventory Management
Automation
Weight elicitation
Systems engineering
Allocation problem
Meta-heuristics
Facility location
Supply Chain Management
Bayesian search theory
Pricing strategy
Routing
MCDM
Geographic information science
Policy analysis
Financial Engineering
Multi-actor Decision problems
Operations Research
Monte-Carlo simulation
Forecasting
Transportation
Scheduling
Risk management
Network data traffic
Project planning
Floor planning
Control theory

Models and methodology in Operations Research
Queueing problems (Simulation)

- **Optimizing a front office**

- **Step 1: informal problem formulation**
  - It is decided to reduce administrative costs of a front office staff while maintaining an appropriate service quality (vague formulation).

  - Analysis of the problem: different possibilities (precise formulations), e.g.:

    a) minimize the total cost ($\leftrightarrow$ number of workers) needed to ensure that the average queueing time does not exceed three minutes;

    b) ... same ... to ensure that no more than 5% customers queues more than three minutes;

    c) minimize the average queueing time with an annual cost not exceeding €Q;

    d) ...

- **Optimizing a front office (cont'd)**

  Step 2: study the system:

  Collect data and information. For example:

  - λ = **average number of customers per hour**;

    additional issues:

    * does the arrival rate vary according to the time of day?

      according to day of the week? according to season?

  - μ = **average number of customers served in one hour by a front office worker**;

    additional issues:

    * does μ depend on the length of the queue?
    * do customers always join the shortest queue?
    * if a queue becomes shorter, do customers switch from one queue to another?
    * better separate queues or a unique virtual queue?

- **Optimizing a front office (cont'd)**

  Step 3: problem formulation:

  - **given** λ and μ;
  - **given** $s =$ number of workers;
  - **given** ...
  - **determine**: $W =$ **average queueing time**;
  - **determine**: $P =$ **probability that a customer queues more than three minutes**;

  If the system is **simple**, with **stable and well defined** statistical parameters, **analytical mathematical models** are possible.

  For example (**Queueing Theory**): if there is a single counter and arrival and service are Poisson processes (with $\lambda < \mu$), we have $W = \dfrac{\lambda}{\mu(\mu - \lambda)}$

  Usually, in queueing problems the systems are complex, and the parameters are unstable $\Rightarrow$

  logical models (**Simulation**).

- **Optimizing a front office (cont'd)**

  Step 4: test the model:

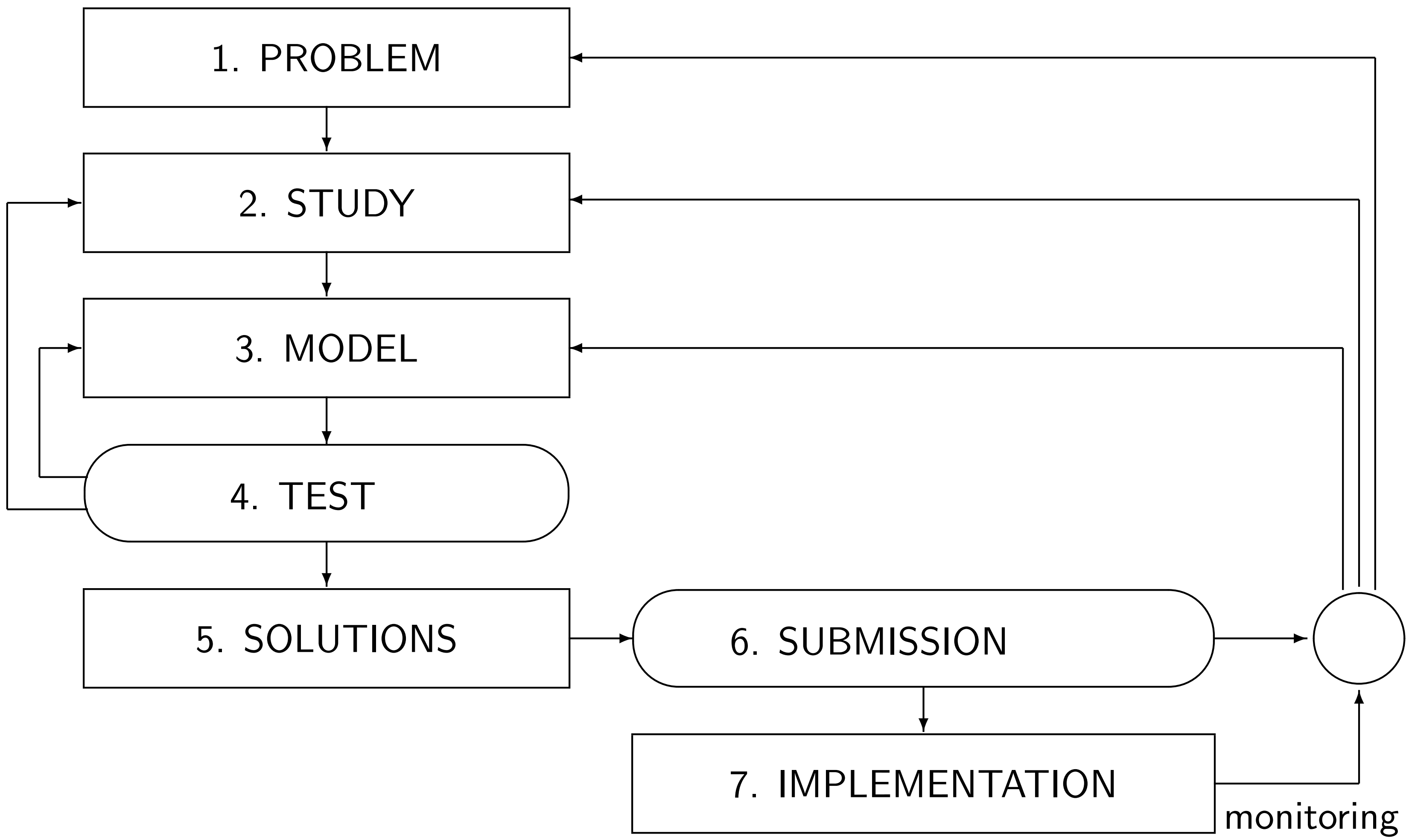
  - **Predictions** using the model (expected values) and comparison with **actual data** (observations, analysis of historical series);
  - in case of consistent gaps $\Rightarrow$ new model, or improved study of the system, ...

- **Step 5: selection of the best solution(s):**
  - In general different solutions are possible, because of: different objectives ($\leftarrow$ Step 1), different possibilities (overtime, part-time, ...);
  - selection of the most promising and/or implementable possibilities.

- **Step 6: submission of results:**
  - models and solutions submitted to decision-makers: when they are not approved ($\Leftarrow$ uncorrect definition, incomplete study, missing involvement of decision-makers, ...) $\Rightarrow$ **correction**.

- **Step 7: implementation and monitoring:**
  - In case of changes $\rightarrow$ **revision**.

- **Optimizing a front office (cont'd)**
- **Flow Chart representation:**



Examples of models and methodology in Operations Research

Linear Programming

An example in production planning

- **Step 1: The problem (informal definition)**: Chair production
- The factory makes use of three plants:
  - **Plant # 1**: metal structures;
  - **Plant # 2**: wood structures;
  - **Plant # 3**: plastic parts and assembly.
- The plants are currently underused.
- It is decided to start the production of new products: $\Longrightarrow$ R&D department:
  - **Product 1**: metal;
  - **Product 2**: wood
  - ("competing" on Plant # 3).
- Marketing department: no limit on production.

An example in Production Planning (cont'd)

- **Step 2: System analysis**

- We need to establish production possibilities:

  - percentage of production capacity available at the plants;

  - percentage of production capacity needed by the products for each unit produced per hour; ("fractions" of unit per hour are obviously acceptable).

  - profit per unit for each product.

- **Decision problem:**

how many and which products have to be scheduled per hour,
without violating the production capacities,
so as to obtain the maximum profit?

An example in Production Planning (cont'd)

- **Numerical example**

Plant	Required capacity per unit/hour: Product 1	Product 2	Available capacity
1	1	0	4
2	0	2	10
3	3	2	18
Unit profit	€30	€50	

- **Variables** (their value will solve the problem):
  - x_1 = units of product 1 to be produced per hour;
  - x_2 = units of product 2 to be produced per hour;
- **Objective**: maximize the profit: $\max 30\ x_1 + 50\ x_2$
- **Constraints**: do not violate the available capacities:
  1. $x_1 \leq 4$
  2. $2x_2 \leq 10$
  3. $3x_1 + 2x_2 \leq 18$
  4. production cannot be negative: $x_1 \geq 0,\ x_2 \geq 0$

An example in Production Planning (cont'd)

- **Step 3: mathematical model:**

$$
\begin{array}{rrcrcl}
\max\ z = & 30\,x_1 & + & 50\,x_2 & & \\
 & x_1 & & & \leq & 4 \\
 & & & 2\,x_2 & \leq & 10 \\
 & 3\,x_1 & + & 2\,x_2 & \leq & 18 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$

- It totally replaces the real-world problem.

- All functions are linear:
  Linear Programming (LP) model.

- Linear Programming can model very many real-world optimization problems (planning, production, transportation, personnel assignment, logistics, ...) in very many areas (industry, health service, agriculture, finance, ...)

- **Simplex algorithm**: the most important optimization algorithm; it easily solves LP instances with many thousands of constraints and variables.

Examples of models and methodology in Operations Research

Integer Linear Programming

(Personnel) Assignment Problem

- **Step 1: The problem (informal definition)**:
- A department wants to assign in the best way n duties to n candidates (one each)
- **Step 2: System analysis**
- we know the (presumed) time that each candidate would need for each duty;
- we want to minimize the total time needed to perform all duties.
- **Example:** $n = 2$:

		Duty 1	Duty 2
Candidate	1	20	40
	2	30	25

- Immediate solution: $1 \leftrightarrow 1$, $2 \leftrightarrow 2$; total time = 45.

(Personnel) Assignment Problem (cont'd)

- **Example:** $n = 3$:

		Duty 1	Duty 2	Duty 3
Candidate	1	20	60	30
	2	80	40	90
	3	50	70	80

- By enumeration (6 possible solutions): $1 \leftrightarrow 3$, $2 \leftrightarrow 2$, $3 \leftrightarrow 1$; total time = 120.
- But for **n = 20** $\Rightarrow$ Number of solutions = 20! $\approx 2.4 \cdot 10^{18}$:
- a 3Mhz PC would take **some centuries** to enumerate them.
- And what about using a **supercomputer**?
- Fastest supercomputer (as of November 2023) **Cray HPE Frontier**:

  Speed: 1.194 Exaflop (1 Exaflop = 10^{18} floating point operations per second);

  Processors: $561,664$ EPYC 64C microprocessors.
- A Frontier can indeed enumerate all solutions in less than one minute. **But:**
- for **n = 25** a Frontier needs over **8 years** to enumerate all solutions;
- for **n = 31** a Frontier needs **4.4 billion years** (age of the Universe $= \approx 13.8$ billion years).
- Problems with a **combinatorial structure** cannot be solved **brute force!**
- But we know **algorithms** that solve AP instances with **n = 5 000** in < 1" on a PC.


FRONTIER
OAK RIDGE
National Laboratory
U.S. DEPARTMENT OF
ENERGY
Hewlett Packard
Enterprise
AMD

(Personnel) Assignment Problem (cont'd)

- **Step 3: mathematical model:**
- Data matrix: $T = (t_{ij})$ = time needed by candidate i to perform duty j.
- **Variables:** $x_{ij} = \begin{cases} 1 & \text{if candidate } i \text{ performs duty } j \\ 0 & \text{otherwise} \end{cases}$
- **Model:**

$$
\begin{array}{lll}
\min & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}x_{ij} & \text{total time needed} \\
& \displaystyle\sum_{i=1}^{n} x_{ij} = 1 \ \text{ for } j = 1,\dots,n & \text{one candidate per duty} \\
& \displaystyle\sum_{j=1}^{n} x_{ij} = 1 \ \text{ for } i = 1,\dots,n & \text{one duty per candidate} \\
& x_{ij} \in \{0,1\} \ \text{ for } i,j = 1,\dots,n &
\end{array}
$$

- linear functions, integer variables: **Integer Linear Programming (ILP)**.
- **ILP** can model many real-world optimization problems: telecommunications networks, facility location, allocation problems, supply chain management, capital budgeting, ...
- ILP is much more difficult to solve than LP.
- Algorithms based on iterative execution of the simplex algorithm for LP.

(Personnel) Assignment Problem (cont'd)

- **Use of mathematical models to handle problem variants**

  Variant: n duties but $m < n$ candidates

  assign all duties (at least one per candidate).

- **Example:** $n = 4$, $m = 3$:

		Duty 1	2	3	4
Candidate	1	20	60	30	20
	2	80	40	30	10
	3	50	70	80	20

- **Variables:** $x_{ij} = \begin{cases} 1 & \text{if candidate } i \text{ performs duty } j \\ 0 & \text{otherwise} \end{cases}$

- **Model:**

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} x_{ij} \qquad \text{total time needed}$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad \text{for } j = 1, \ldots, n \qquad \text{one candidate per duty}$$

$$\sum_{j=1}^{n} x_{ij} \geq 1 \quad \text{for } i = 1, \ldots, m \qquad \text{at least one duty per candidate}$$

$$x_{ij} \in \{0, 1\} \quad \text{for } i = 1, \ldots, m \text{ and } j = 1, \ldots, n$$

Examples of models and methodology in Operations Research

Scheduling problem

- **Step 1: The problem (informal definition)**:

  Find the optimal scheduling sequence for processing jobs on machines.

- **Step 2: System analysis**

- 2 machines and 3 kinds of job;

- each job must be processed on both machines, in any order;

- each machine processes one job at a time;

- **processing times:**

		Job 1	Job 2	Job 3
Machine	1	30	50	30
	2	40	70	20

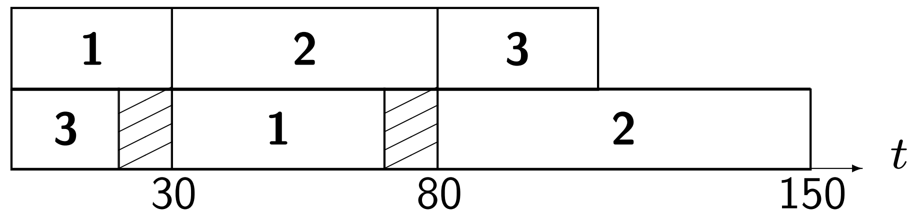
- **objective:** complete all processing in minimum time

  $\Rightarrow$ cyclic optimal schedule.

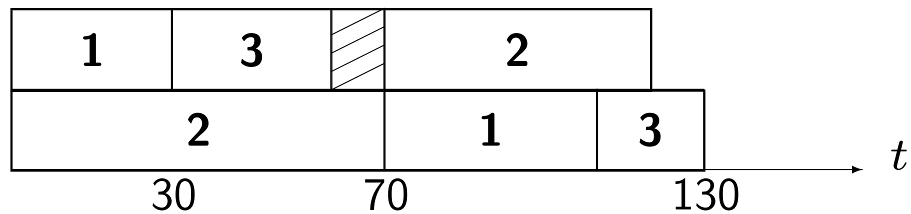
Scheduling problem (cont'd)

		Job 1	Job 2	Job 3
Machine	1	30	50	30
	2	40	70	20

Feasible solution: machine 1 / machine 2



Optimal solution: machine 1 / machine 2



- **Step 3: mathematical model:** the problem can be modeled as a (complicated) ILP.

- In general: m machines; n kinds of job; processing
  - in any order
  - in prefixed order
  - with precedence constraints
  - . . .

- Many other **applications**: courses–teachers, boats–garaging piers, . . .

Problems on Graphs

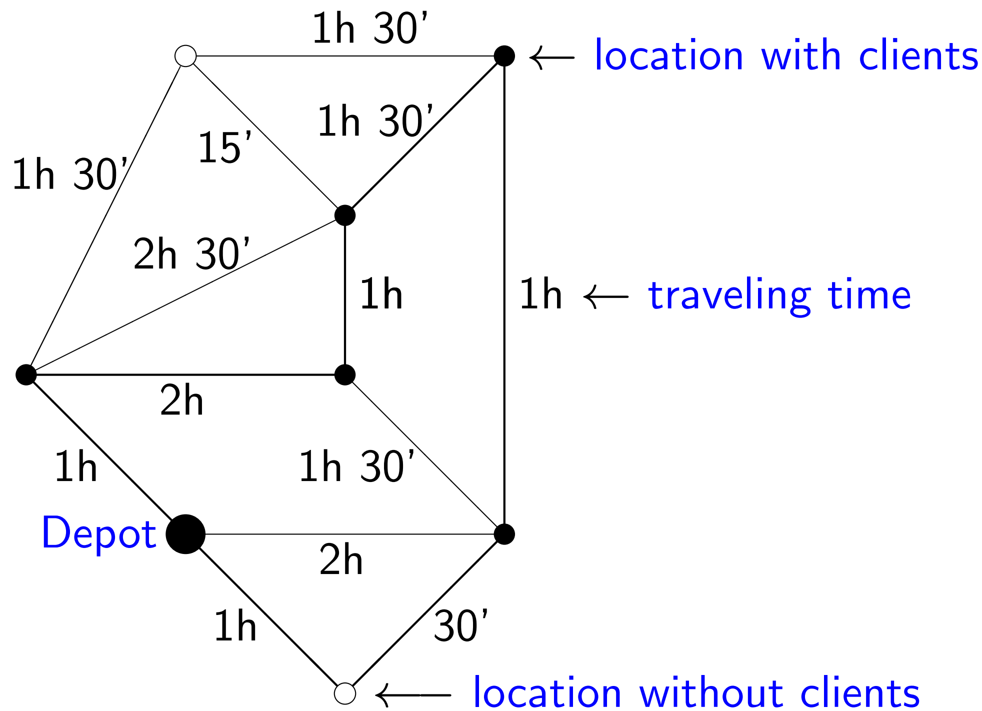
Optimal tours

- **Step 1: The problem (informal definition)**:
- A factory wants to determine the best way to deliver products from a depot to clients.
- **Step 2: System analysis**
- we know the road map;
- for each road we know the presumed traveling time;
- **objective:** find a tour that starts at the depot, visits all clients and returns to the depot, in minimum total time.

Optimal tours (cont'd)

- **Step 3: Graph Theory model**:

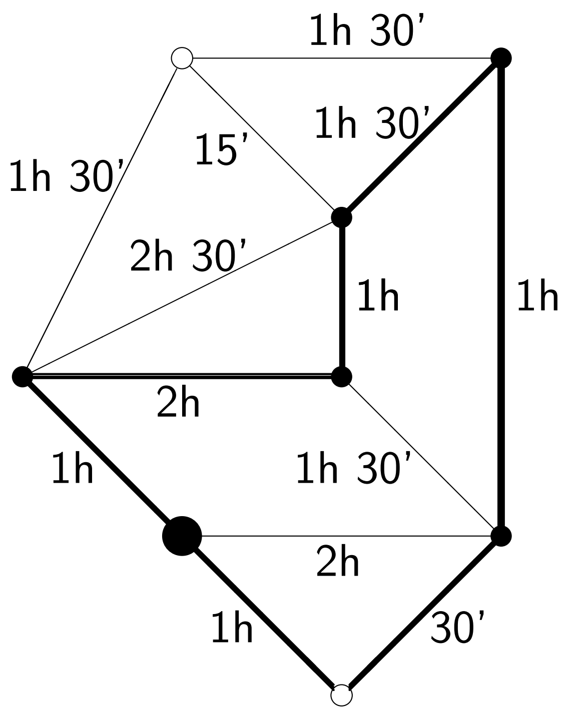
Graph (Network)



- The "locations" are called **vertices**.
- Instead of traveling times one could use costs (fuel + tolls), or other measures.
- Graph theory models can also be expressed through mathematical models.

Optimal tours (cont'd)

- Solution:



- Many additional constraints are possible: clients' opening hours, vehicles capacity, drivers' working hours, ...

- Freight transportation takes 10-25% of the final cost for consumer goods: optimization methods can produce 5-20% saving of the total transportation costs.

- This problem also finds application in many different areas.

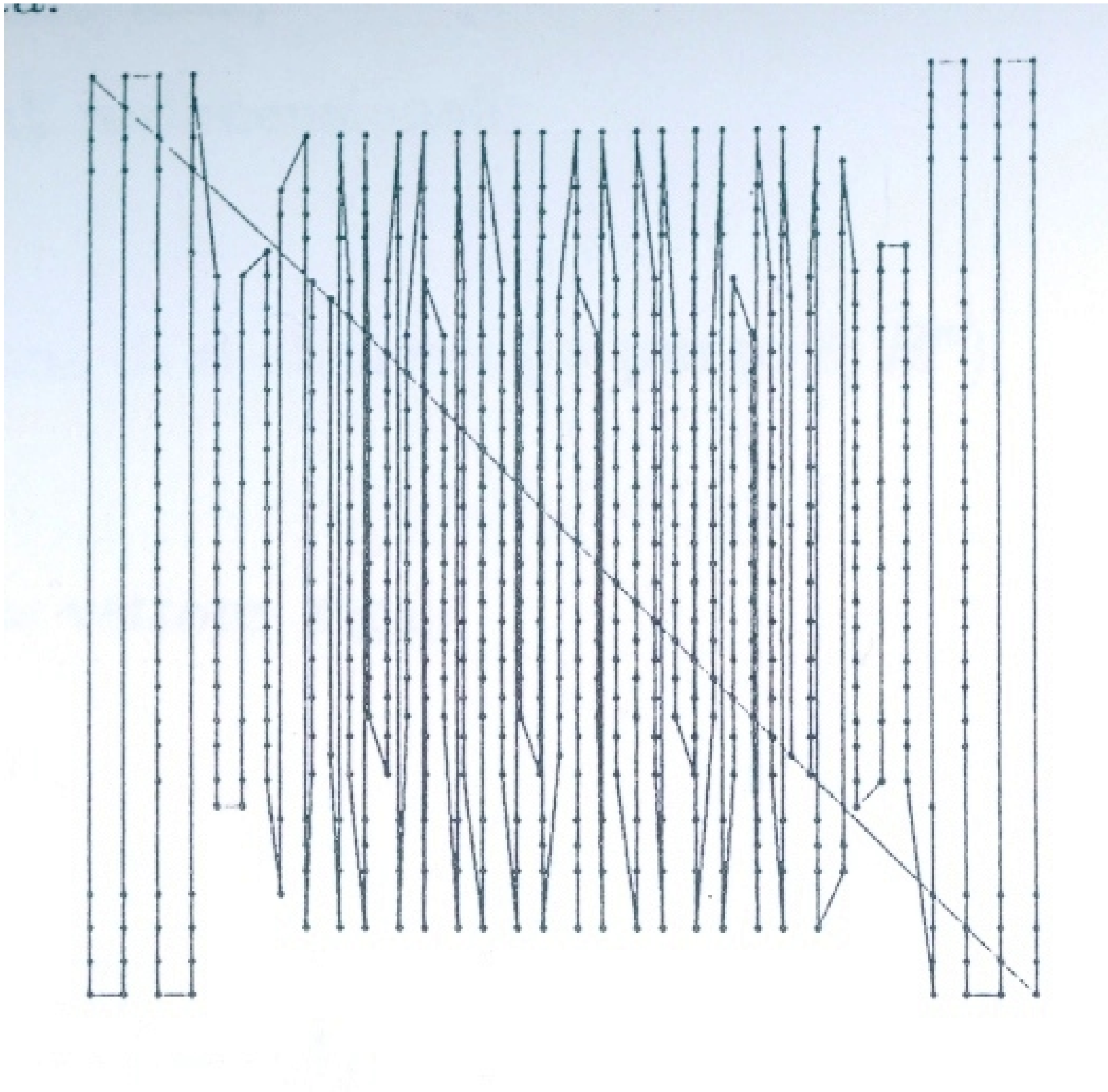


Figura 1: **Application of optimal tours:** Perforation of ceramic support for multi-chip modules (644 holes): **Engineering solution**.

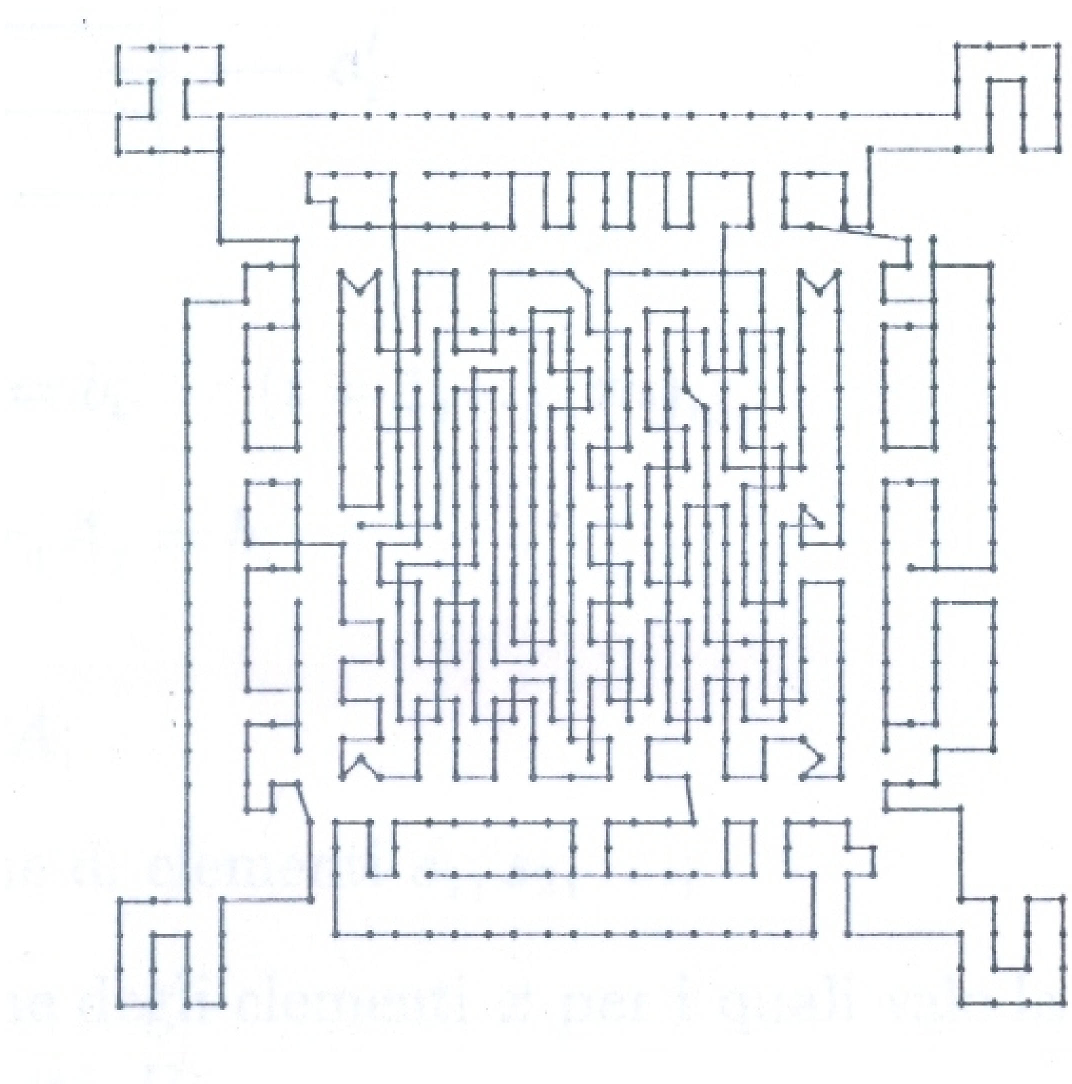
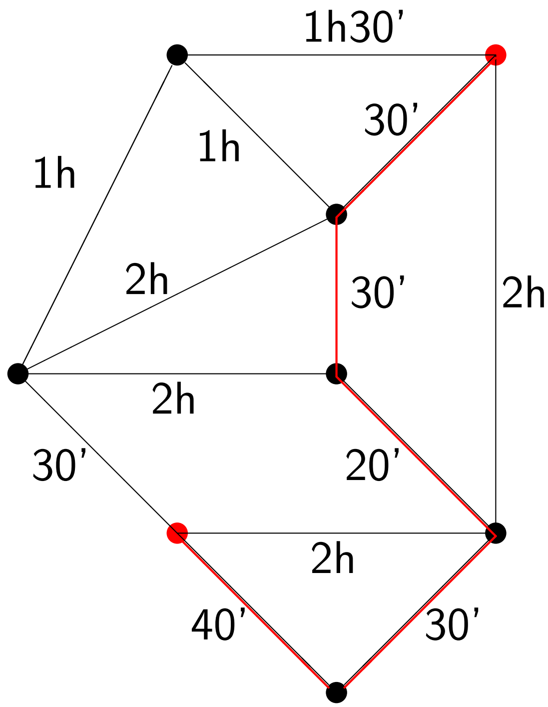
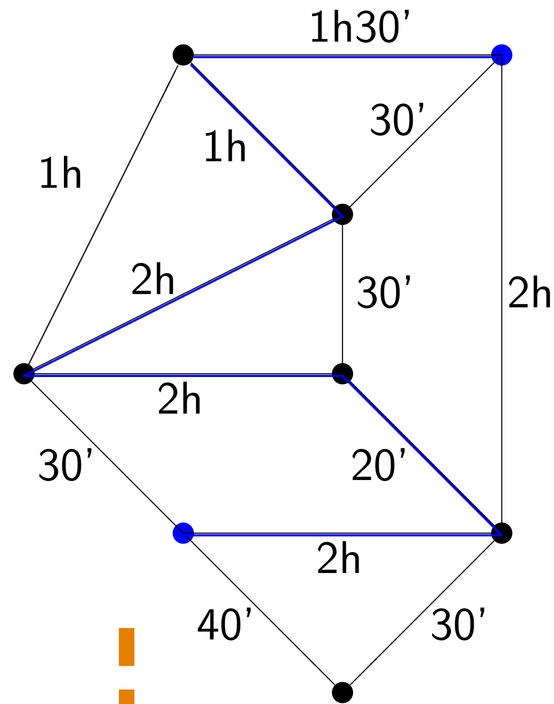


Figura 2: **Optimized solution:** Improvement: $\approx$ 15% on total processing time; $\approx$ 30% on total traveling time.

Other problems on graphs

- Find the **shortest path** between **two vertices** (colored dots):



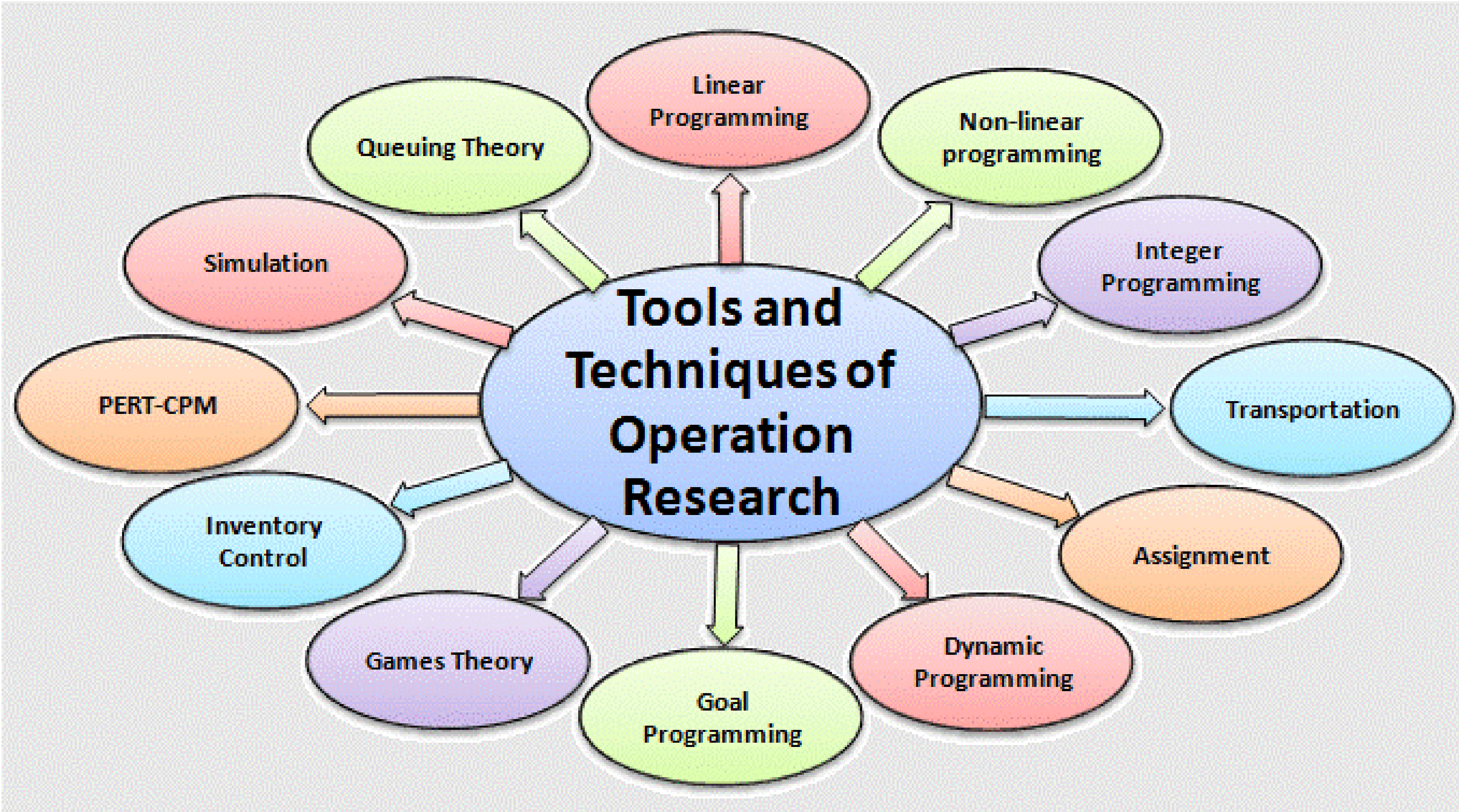


- Find the **longest path** between **two vertices** (without passing twice by the same vertex).
- Hundreds of other important problems can be modeled through graphs.
- Graph theory models can also be expressed through mathematical models.
- Graph theory models are not only useful for transportation and distribution problems but also for important problems in telecommunications, computational biology, economy, ...
- Usually solved through specialized algorithms, but the use of ILP algorithms is also frequent.

Operations Research has many other applications

LP, ILP and **graphs** can model a tremendous number of applications:

- **Location of facilities** (factories, plants, supermarkets, hospitals, fire stations, ...)
- **Telecommunications** (network planning, computer networks, ...)
- **Problems on graphs** (optimal tours, freight transportation, delivery systems, ...)
- **Project management** (CPM (*Critical Path Method*), PERT (*Program Evaluation and Review Technique*), ...)
- **Scheduling** (personnel scheduling, scheduling of operations on automated machines, ...)
- **Cutting and packing** (cutting from rolls or plates, container loading, packaging, ...)
- **Military operations** (strategy, tactics, logistics, unmanned aerial vehicles (UAVs), ...)
- **Supply chain management ...**


Linear Programming
Queuing Theory
Non-linear programming
Simulation
Integer Programming
Tools and Techniques of Operation Research
PERT-CPM
Transportation
Inventory Control
Assignment
Games Theory
Goal Programming
Dynamic Programming

- Most widely used optimization tools in industry applications, according to a research by **Institute for Operations Research and the Management Sciences (INFORMS)**
  - linear programming and integer linear programming;
  - methods based on graphs (in particular, PERT-CPM);
  - simulation.
- **Course(s) syllabus:**
  - **Operations Research M**
    - optimization and mathematical programming (linear and non linear);
    - linear programming and the simplex algorithm;
    - duality theory, dual simplex algorithm and integer linear programming;
    - complexity theory;
    - discrete simulation.
  - **Network Optimization M**
    - introduction to graph theory;
    - shortest trees, shortest paths, maximum flows, PERT-CPM method;
    - routing problems;
    - relaxations;
    - approximation algorithms and metaheuristics.