CH3050 - Assignment № 5

Process Dynamics and Control: Jan-May 2020

Raj Jain | CH17B066 May 4, 2020

Problem 1

Solution code is attached as (**'Q1.m'**)

- Given,

$$G_p(s) = \frac{s^2 + 4s + 8}{s(s+1)(s+3)}; \quad G_{sens}(s) = \frac{1}{s+10}; \quad G_c = K_c$$

Part (a)

- Closed loop characteristic equation is given by

$$1 + G_c G_p G_{sens} = 0 \tag{1}$$

- Wrtiting (1) in the form $1 + \beta L(s) = 0$. We get

$$1 + K_c \left[\frac{s^2 + 4s + 8}{s(s+1)(s+3)(s+10)} \right] = 0 \tag{2}$$

$$\Rightarrow L(s) = \frac{s^2 + 4s + 8}{s(s+1)(s+3)(s+10)}; \quad \beta = K_c \tag{3}$$

Rule 1: Finding poles and zeros of open-loop system $L(s)$

- Solving (1) as $\beta \to 0$, gives us the poles of the open loop system

$$\Rightarrow s(s+1)(s+3)(s+10) = 0 \Rightarrow s = 0, -1, -3, -10 \; \textit{are the poles of OL system}$$

- Solving (1) as $\beta \to \infty$, gives us the zeros of the open loop system

$$\Rightarrow s^2 + 4s + 8 = 0 \Rightarrow s = -2 \pm 2j \; \textit{are the zeros of OL system}$$

Rule 2: Determining number of branches, starting and ending points

- No. of branches = No. of poles of OL system. Hence we have four branches in the root locus diagram
- We have No. of poles($P = 4$) > No. of zeros($Z = 2$), hence the four branches start at four poles of L(s), two of them end at the two zeros and the other two go to infinity

Rule 3: Determining the real axis part of the locus

- A point s_0 on the real axis iff it is to the left of odd number of zeros and poles

$$\Rightarrow s = [-10, -3] \cup [-1, 0] \; \textit{are the parts of locus on the real axis}$$

Rule 4: Determining centroid and asymptote angles

- Asymptote angles are given by

$$\theta_k = \frac{(2k-1)\pi}{P-Z}; \quad k = 1, 2, ..., P-Z$$

$$\Rightarrow \theta_\mathbf{1} = \frac{\pi}{\mathbf{2}}; \quad \theta_\mathbf{2} = \frac{\mathbf{3}\pi}{\mathbf{2}}$$

- Centroid is given by

$$\sigma = \frac{\sum_{i=1}^{P} p_i - \sum_{i=1}^{Z} z_i}{P-Z}$$

$$\Rightarrow \sigma = \frac{(0 + (-1) + (-3) + (-10)) - (-2 + 2j - 2 - 2j)}{4-2} = \mathbf{-5}$$

Rule 5: Determining crossover points, if any

- We substitute $s = j\omega$ into the CE (2), if a solution exists such that $\beta > 0$, then we have a crossover point, otherwise the locus stays in LHP

$$\Rightarrow (-\omega^2 + j\omega)(-\omega^2 + 13j\omega + 30) + \beta(-\omega^2 + 4j\omega + 8) = 0$$

- Equating real and imaginary parts

$$\Rightarrow \omega^4 - \omega^2(43 + \beta) + 8\beta = 0 \qquad and \qquad -14\omega^3 + 30\omega + 4\beta\omega = 0$$

- For the above system of equations, no solution exists such that $\beta > 0$, hence there exists no crossover point

Rule 6: Determining break-in and break-away points

- A break-in or a break-away point exists at the local minima of parameter β, between two zeros or two poles respectively

$$\beta = \frac{-1}{L(s)} = \frac{-(s^4 + 14s^3 + 43s^2 + 30s)}{s^2 + 4s + 8}$$

$$\left.\frac{\mathrm{d}\beta}{\mathrm{d}s}\right|_{s=s^*} = 0$$

$$\Rightarrow \mathbf{s^* = -6.287,\ -0.497\ are\ the\ two\ break-away\ points}$$

Rule 6: Determining angle of arrival for the complex zeros

- Contribution by pole at origin = $\angle(-2 + 2j - 0) = \tan^{-1}(-1) = 135°$
- Contribution by pole at s=-1 = $\angle(-2 + 2j + 1) = \tan^{-1}(\frac{2}{-1}) = 116.56°$
- Contribution by pole at s=-3 = $\angle(-2 + 2j + 3) = \tan^{-1}(\frac{2}{1}) = 63.44°$
- Contribution by pole at s=-10 = $\angle(-2 + 2j + 10) = \tan^{-1}(\frac{2}{8}) = 14.04°$
- Contribution by zero at s=-2-2j = $\angle(-2 + 2j + 2 + 2j) = \tan^{-1}(\frac{4}{0}) = 90°$
- Hence, the angle of arrival at the complex zero$(-2 + 2j)$ is given by

$$\Rightarrow \phi = \mathbf{180} + (\mathbf{90}) - (\mathbf{135} + \mathbf{116.56} + \mathbf{63.44} + \mathbf{14.04}) = \mathbf{-59.04°}$$

Drawing the root locus

Part (b)

Root locus plot obtained using MATLAB:

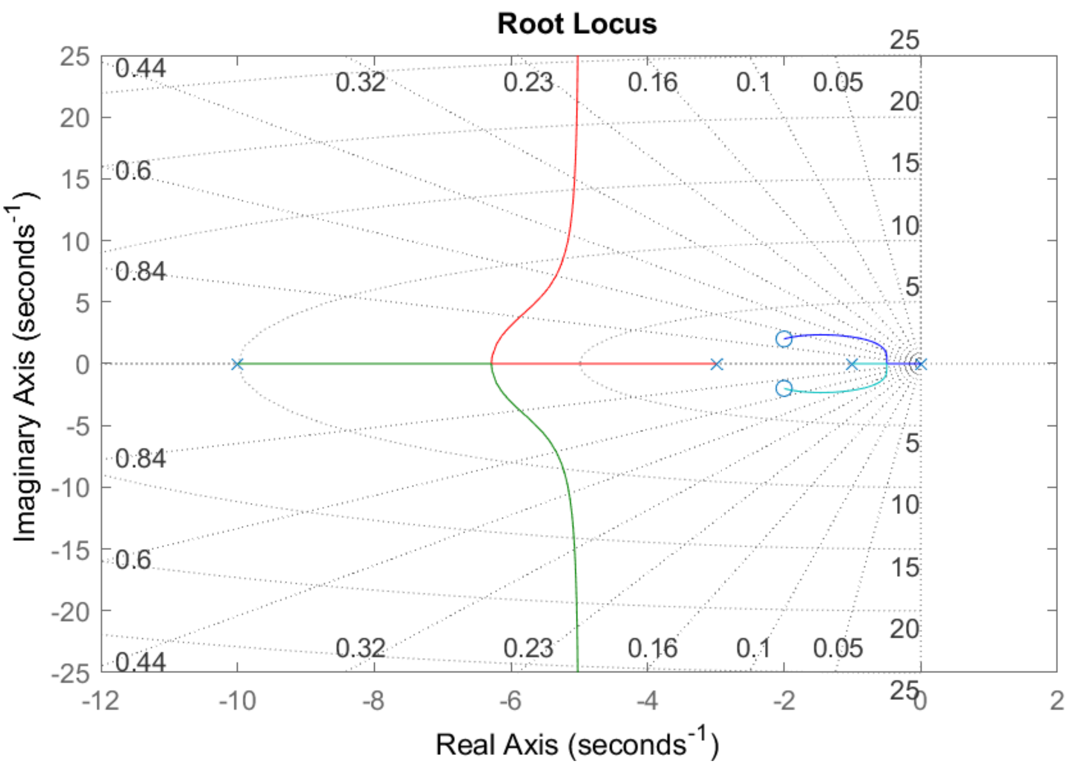

Figure 1: Root Locus plot

Part (c)

- We need to tune K_c such that the damping ratio is 0.4. We use 'rltool' in MATLAB for this, and drag the cursor till it reaches the required damping ratio, and then obtain the corresponding K_c value.

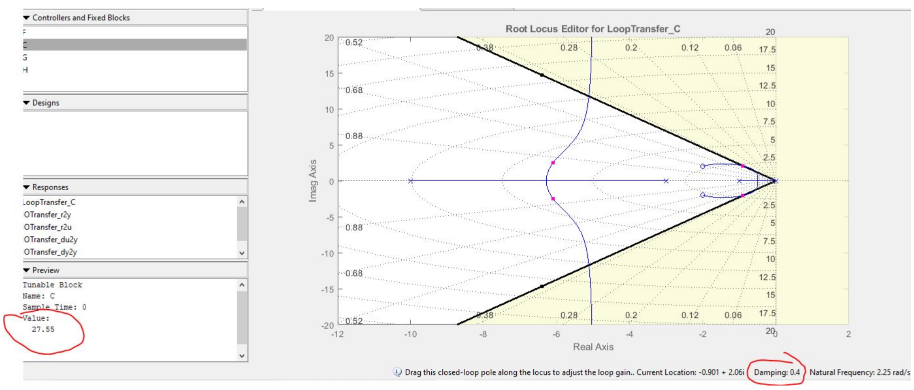

Figure 2: Obtaining Kc such that damping ratio is 0.4 using 'rltool'

- From this we obtain $\mathbf{K_c} = \mathbf{27.55}$

Part (d)

- Given,

$$G_c = K_c + \frac{K_I}{s}$$

- We write closed loop characteristic equation

$$1 + G_c G_p G_{sens} = 0$$

$$1 + K_I \left(\frac{G_{sens} G_p}{s(1 + K_c G_{sens} G_p)} \right) = 0 \Rightarrow L(s) = \frac{G_{sens} G_p}{s(1 + K_c G_{sens} G_p)}$$

- Now we draw the root locus plot using L(s), and obtain ultimate gain, i.e., the ultimate value of K_I

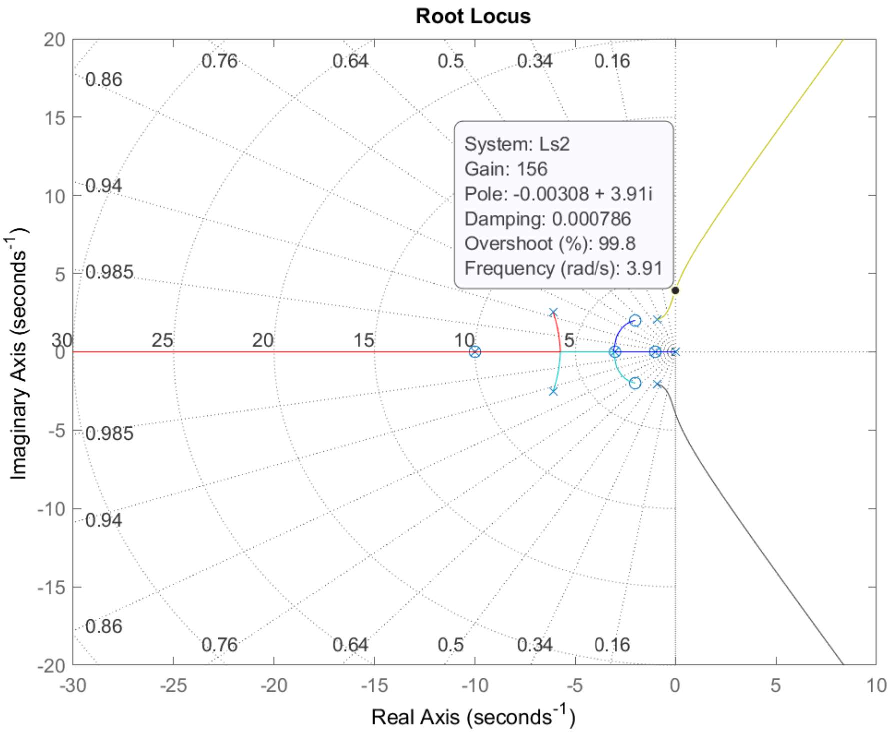

Figure 3: Root Locus plot with design parameter $\beta = K_I$

$$\Rightarrow \mathbf{K_{Iu} = 156}$$

Problem 2

Solution code is attached as (**'Q2.m'**)

- Given,

$$G_p = \frac{2(s+1)}{10s^2 + 7s + 1} e^{-2s}; \qquad G_c = K_c$$

- Loop-Open Transfer function $L_G(s)$

$$L_G(s) = G_p G_c = \frac{2K_c(s+1)}{10s^2 + 7s + 1} e^{-2s}$$

Part (a)

- For a system in feedback controller the ultimate value of controller gain K_{cu} can be written as

$$K_{cu} = K_{c_i} GM_i \tag{4}$$

for some K_{c_i} and corresponding gain margin GM_i

- Now, to determine the value of K_c at which the gain margin is 8.2dB, we will use equation (4) using some arbitrary $K_c(= 1)$ and its corresponding gain margin. And then evaluate K_c corresponding to $GM = 8.2dB$ from there

$$K_{cu} = K_{c_i} GM_i$$

$$\Rightarrow K_{c_1} GM_1 = K_{c_2} GM_2 \tag{5}$$

- Using 'margin' command on MATLAB, we find out the gain margin corresponding to $K_c = 1$

$$For\ K_{c_1} = 1 \Rightarrow GM_1 = 8.89dB = 2.5704$$

- Putting these values in equation (5)

$$\Rightarrow 1 \times 10^{\frac{8.89}{20}} = K_c \times 10^{\frac{8.2}{20}}$$

$$\Rightarrow K_c = 10^{\frac{8.89-8.2}{20}} = 10^{\frac{0.69}{20}} = 1.083$$

$$\mathbf{K_c = 1.083}; \quad \mathbf{GM = 8.2dB}; \quad \mathbf{PM = 69.36^\circ}$$

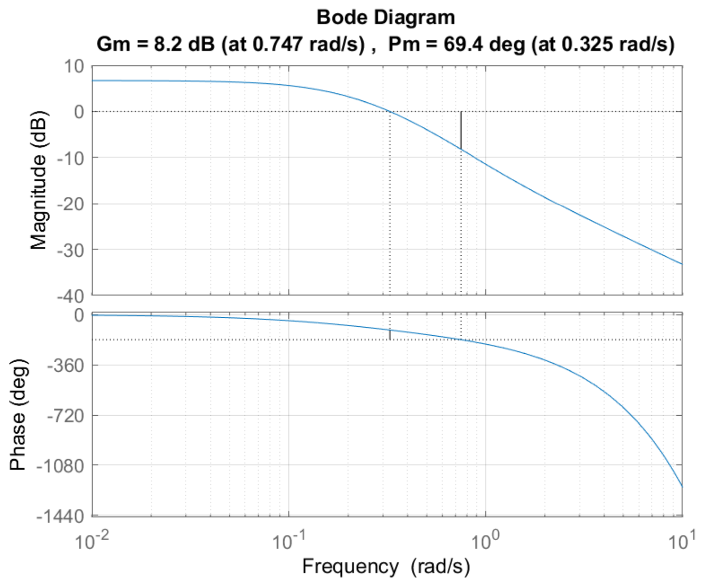

Figure 4: Bode Plot of Loop open system with $K_c = 1.083$

Part (b)

- Now, we have a PI controller of the form

$$G_c = K_c\left(1+\frac{1}{\tau_I s}\right); \qquad K_c = 1.083$$

- We need to find τ_I such that the phase margin is 60°. Since τ_I affects both the gain as well as phase of the system, we need to solve the two equations for gain and phase margin equations and iterate over τ_I to meet the requirements

- Instead I have used to 'margin' command in MATLAB and iterated over different values of τ_I such that PM=60°. (Code for the same is attached in file **'Q2.m'**)

Listing 1: Code to find τ_I.

```
s=tf('s');
Gp=tf([2,2],[10,7,1],'iodelay',2); %Process TF
Kc=1.083; %Kc obtained from part (a)
Pm0=60; %Target phase margin
for Ki=0:0.0001:0.2 %iterating for different values of KI=1/tauI
    [Gm2,Pm2,wcg2,wcp2]=margin(series(Kc*(1+Ki/s),Gp));
    if abs(Pm2-Pm0)<0.01
        break
    end
end
tau=1/Ki %tauI
Gm2=mag2db(Gm2) %Gain Margin
Pm2 %Phase Margin
```

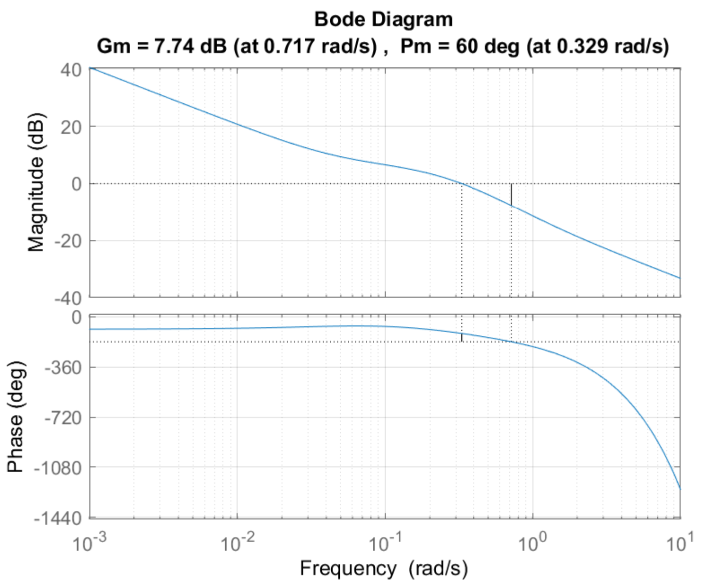

Figure 5: Bode Plot of Loop open system with $\tau_I = 20.367$

$$\tau_\mathbf{I} = \mathbf{20.367}; \quad \mathbf{PM = 60°}; \quad \mathbf{GM = 7.74dB}$$

Part (c)

Step-Response of the resulting closed loop system

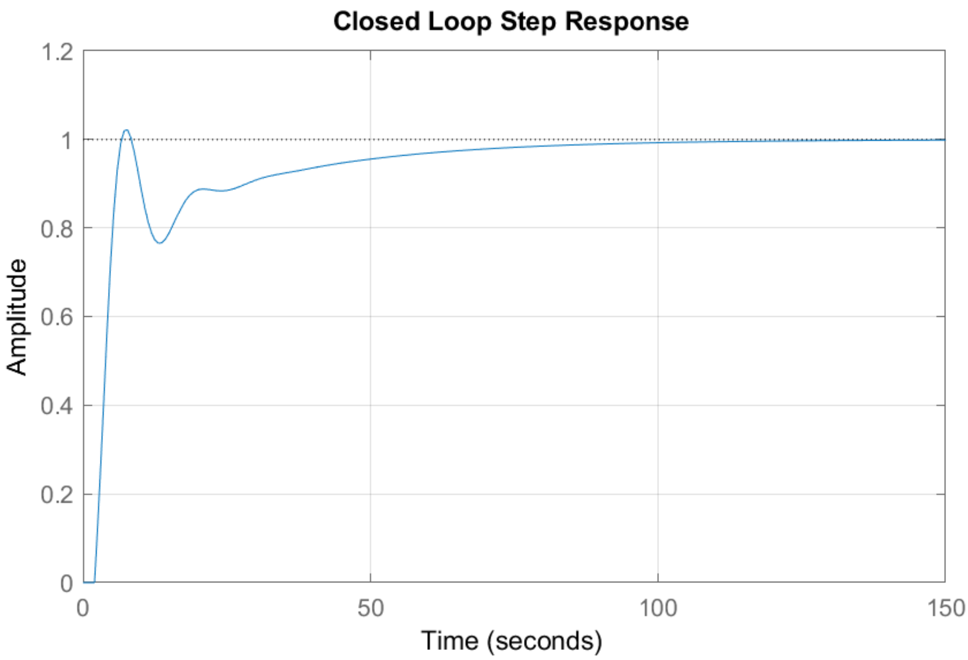

Figure 6: Step-Response of the resulting closed loop system

Part (d)

- Sensitivity function is given by

$$S = \frac{1}{1 + G_c G_p}$$

- Bode's integral states that

$$\int_0^\infty log|G(j\omega)|d\omega = \pi \Sigma Re(p_k) - \frac{\pi}{2} \lim_{s \to \infty} sL(s)$$

$$\int_0^\infty log|G(j\omega)|d\omega = 0 - 0 = 0$$

- We approximate the LHS integral using trapezoidal integration in MATLAB

Listing 2: To calculate the integral.

```
Gc2=Kc*(1+1/tau/s);
S=1/(1+Gc2*Gp);
w=linspace(0.001,1000,10000000);
[mag,phase,l]=bode(S,w);
trapz(l, log(abs(mag(:))))
```

- The integral is $0.0057 (\approx 0)$. Hence, the **bode's sensitivity integral holds**

Problem 3

Solution code is attached as (**'Q3.m'**)

- Given,

$$G_p = \frac{2}{s^2 + 3s - 10} e^{-s}$$

Part (a)

- Ignoring the delay, we need to design a P-controller $G_{c1} = K_{c1}$ such that the resulting closed loop system has a pole at $s = -1$
- Closed loop characteristic equation is given by

$$1 + G_{c1} G_p = 0$$

$$1 + \frac{2K_{c1}}{s^2 + 3s - 10} = 0 \tag{6}$$

- $s = -1$ satisfies the equation (6)

$$1 + \frac{2K_{c1}}{(-1)^2 + 3(-1) - 10} = 0$$

$$\Rightarrow \mathbf{G_{c1} = K_{c1} = 6}$$

Part (b)

- We need to design a P-controller $G_{c2} = K_{c2}$ such that the gain margin is 1.5dB

$$GM = \frac{1}{|L_G(j\omega_c)|} \tag{7}$$

$$L_G = G_{c2} G_p; \quad \omega_c = \text{Phase crossover frequency}$$

- Gain of a P-controller doesn't affect the phase of L_G and hence doesn't affect the phase crossover frequency
- Using Nyquist plot, we determine the phase crossover frequency ω_c

$$\Rightarrow \omega_c = 0$$

$$|L_G(j\omega)| = \left| \frac{2K_{c2}}{(j\omega)^2 + 3(j\omega) - 10} \right|$$

$$\Rightarrow |L_G(j\omega)| = \frac{2K_{c2}}{\sqrt{\omega^4 + 29\omega^2 + 100}}$$

$$\Rightarrow |L_G(j\omega_c)| = \frac{2K_{c2}}{\sqrt{100}} = \frac{K_{c2}}{5}$$

$$\Rightarrow GM = \frac{1}{|L_G(j\omega_c)|} = \frac{5}{K_{c2}} \tag{8}$$

- From equation 8 and given GM=1.5dB

$$\Rightarrow 10^{1.5/20} = \frac{5}{K_{c2}} \Rightarrow K_{c2} = \frac{5}{10^{1.5/20}} = 4.207$$

$$\mathbf{G_{c2} = K_{c2} = 4.207}$$

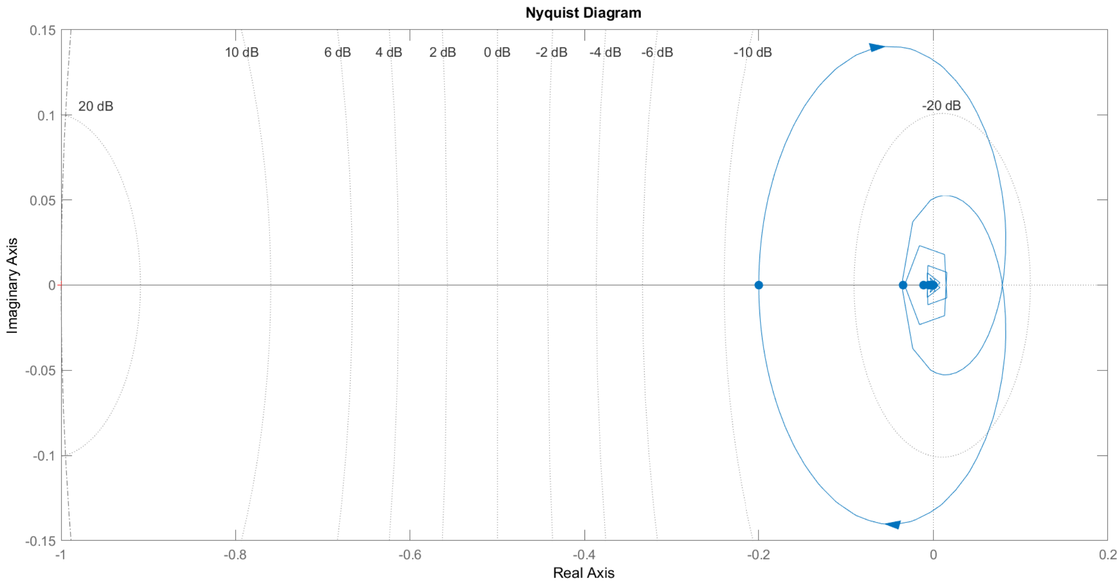

Figure 7: Nyquist plot of L_G for $K_{c2} = 1$

Part (c)

- Step response of the closed loop system with G_{c1} obtained in part (a), and ignoring the process delay (Figure: 8)

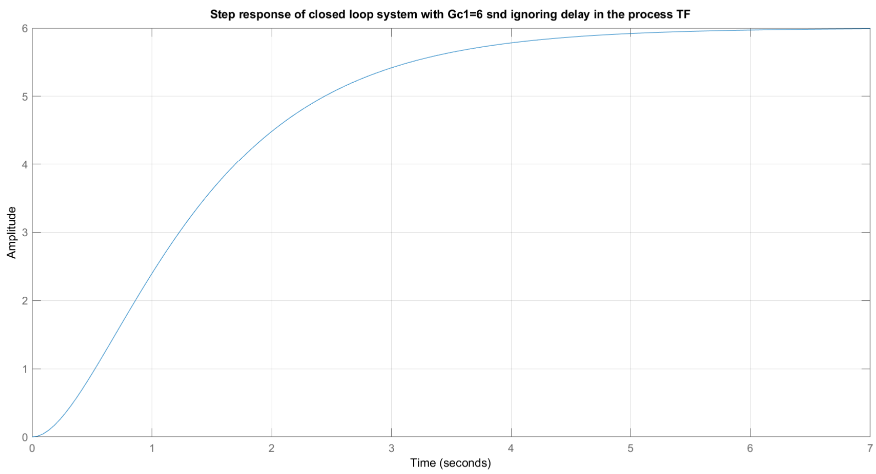

Figure 8: Step response part (a)

- Step response of the closed loop system with G_{c2} obtained in part (b) (Figure: 9)

- Pade's approximation of the process delay:

$$G_p = \frac{2}{s^2 + 3s - 10} e^{-s} = \frac{2}{s^2 + 3s - 10} \frac{1 - \frac{s}{2} + \frac{s^2}{12}}{1 + \frac{s}{2} + \frac{s^2}{12}}$$

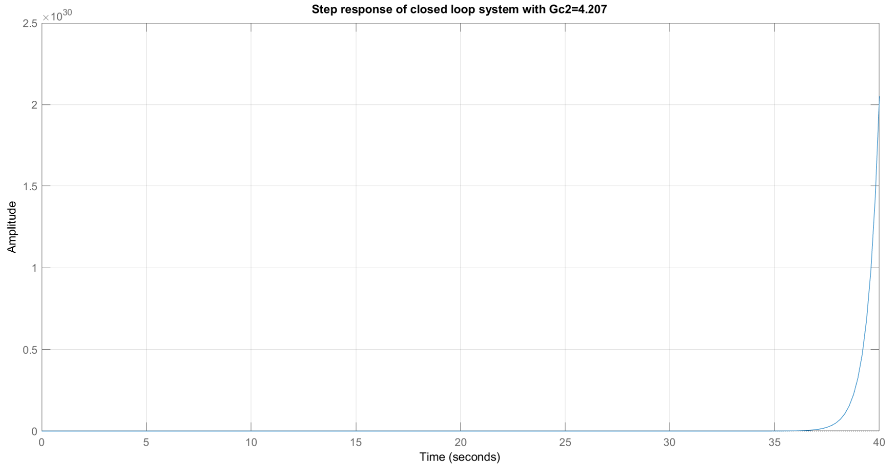

Figure 9: Step response part (b)

- $s = -1$ satisfies the closed loop CE

$$1 + L_G(-1) = 0 \Rightarrow 1 + K_{c3}\frac{2}{(-1)^2 + 3(-1) - 10}\frac{1 - \frac{(-1)}{2} + \frac{(-1)^2}{12}}{1 + \frac{(-1)}{2} + \frac{(-1)^2}{12}} = 0$$

$$\Rightarrow 1 + K_{c3}\frac{2}{(-12)}\frac{\frac{19}{12}}{\frac{7}{12}} = 0$$

$$\Rightarrow 1 + K_{c3}\frac{(-19)}{42} = 0$$

$$\Rightarrow K_{c3} = \frac{42}{19} = 2.21$$

- Drawing the step response of CL system with pade's approximation in figure 10

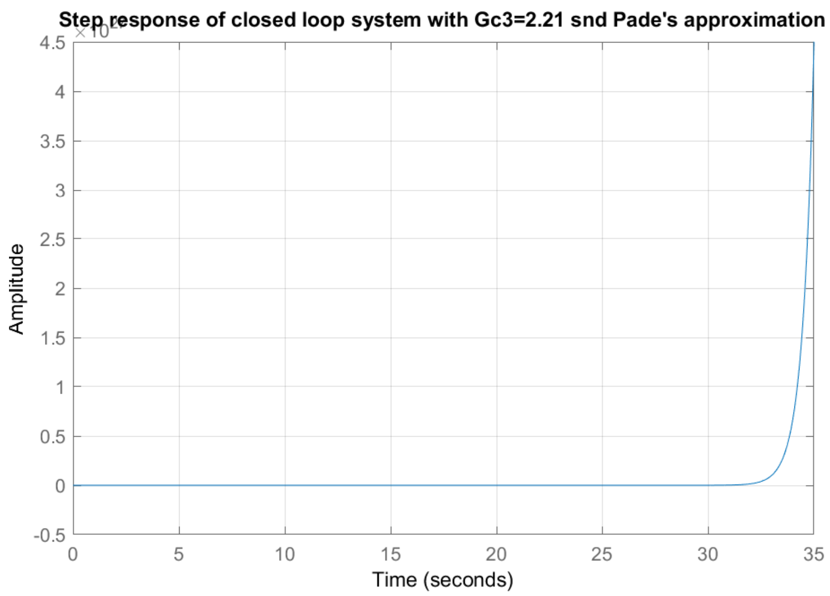

Figure 10: Step response of CL system with pade's approximation

- Step response of closed loop is becoming unstable by Pade's approximation

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