# Fish, Bathtubs and Analogue Black Holes

Michael Hogg

School of Mathematics,
University of Southampton.

Southampton, 18th Febuary 2013

# Outline

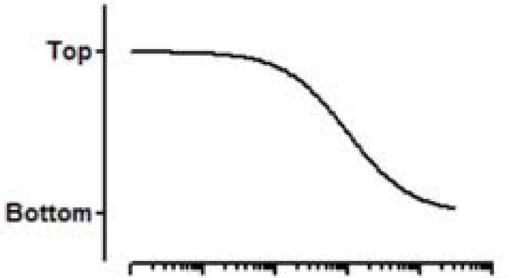
# Meet Timmy and .....

# His Home

Source:http:

//www.astroscu.unam.mx/neutrones/dany.html

# Timmy's Travels and...




# His New Home

Source:http:

//www.astroscu.unam.mx/neutrones/dany.html

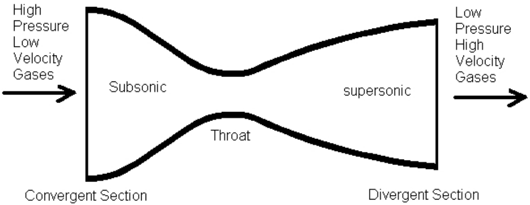
# How to Replace Timmy

- Speed of as fish swimming is not really a suitable standard for an analogue of a black hole.
- Two main possibilities for a measurable quantity that we might use to define a maximum speed in some analogue spacetime. (The analogue of the speed of light in relativity.)
- Speed of surface waves on a fluid.
- Speed of pressure waves (speed of sound) within the body of a fluid.

# Surface Waves

- Within the ridge, the water flows outward too quickly to allow surface waves to pass inside. This is the analogue of a white hole, the time reversed black hole.

# Pressure Waves and the Laval Nozzle




1. The onset of supersonic flow occurs somewhere within the throat. To the right of this, Sound can only flow from left to right.

# How Useful is the Laval Nozzle.

- In terms of spacetime geometry, this is at best a simple one-d model of a Schwarzschild black hole, with the event horizon as the sole key feature.
- In a perfect world, perturbations on the acoustic event horizon causes phonons to be ejected in an analogue for Hawking radiation.
- In the real world, the energy of this radiation is dependent on the size of the nozzle, in a manner similar to the evaporation of black holes happening more quickly with smaller bodies.
- Make the hole too small and the edge effects become significant.
- Although Schwarzschild is a solution to the Einstein equations. Do non-rotating black holes exist?

# Bathtub?

So how about this?

A bit like

Bathtub as an analogue for a Kerr blackhole?

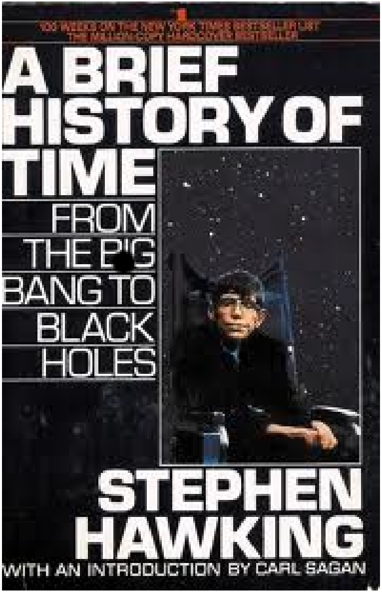
# Superfluid Vortices

- Superfluid vortices display the same features as our bath plug, but are controllable.
- Can only reproduce 2-d + 1 spacetime
- Can we model the equatorial region only?

# A Quick Aside - What is a Metric?

- In general a metric is a mathematical abstraction of distance.
- Put simply, the metric defines some scalar quantity which is invariant for all observers within the geometry or metric space under consideration.

# Sleep Time?




- "Someone told me that each equation I included in the book would halve the sales" Stephen Hawking.

# The Kerr Metric

$$
\begin{aligned}
ds^2 = &-\left(1-\frac{2mr}{\rho^2}\right)dt^2 - \frac{4mar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 \\
&+\rho^2 + \left(r^2 + a^2 + \frac{2ma^2 r\sin^2\theta}{\rho^2}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
a &\equiv J/m \\
\rho^2 &\equiv r^2 + a^2\cos^2\theta \\
\Delta &\equiv r^2 - 2mr + a^2
\end{aligned}
$$

# Features of the Kerr Black Hole I

- Asymptotically flat. For large $r$, the metric approaches that of Special Relativity.
- The metric is both stationary and axi-symmetric.
- In the limit $J = a = 0$, it becomes the Schwarzchild Metric.
- There are two horizons at

$$r = m \pm \sqrt{m^2 - a^2} \tag{1}$$

# Features of the Kerr Black Hole II

- There exists a surface where the purely temporal component of the metric changes sign. Within this surface, but beyond the first horizon, an observer cannot remain at rest. They may, however, still escape the black hole.
- The metric displays a ring singularity. We are hoping to model only up to the event horizon, so we can not hope to observe this.

# The Equatorial Region of the Kerr Metric

- Here $\theta = \frac{\pi}{2}$
  $\Rightarrow d\theta = 0, \qquad \sin^2\theta = 1, \qquad \cos^2\theta = 0.$
- So the metric simplifies to
  $ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{4ma}{r} d\phi dt + \frac{r^2}{\Delta} dr^2 + \left(r^2 + a^2 + \frac{2ma^2}{r}\right) d\phi^2$
- This retains all of the features of the full metric, but in the (2+1) dimensions of the vortex

# Dynamical Equations for Fluid Flow

- We consider the velocity, $\boldsymbol{v}$, of a barotropic, inviscid fluid, which we define as the gradient of some scalar field, $\psi$. Hence
  $\nabla \times \boldsymbol{v} = 0 \qquad \boldsymbol{v} = \nabla\psi$.
- We the continuity equation,
  $\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v})$
- The Euler equation,
  $\rho\left(\partial_t \boldsymbol{v} + [\boldsymbol{v} \cdot \nabla]\, \boldsymbol{v}\right) = -\nabla p$
- and the barotropic equation of state
  $p = p\left(\rho\right)$

# Linearised Fluctuations in the Background Field

- Assume we have exact solutions of the dynamical equations, $\rho_0\left(t, \boldsymbol{x}\right)$, $p_0\left(t, \boldsymbol{x}\right)$ and $\psi\left(t, \boldsymbol{x}\right)$
- Consider small linearised fluctuations of this field, e.g. density $\rho\left(t, \boldsymbol{x}\right) = \rho_0\left(t, \boldsymbol{x}\right) + \epsilon\rho_1\left(t, \boldsymbol{x}\right) + \mathcal{O}\left(\epsilon^2\right)$
- Combining with dynamical equations gives a system of partial differential equations which give, after a lot of algebra, the full (3+1) metric tensor for our scalar field

# The Metric Tensor for The Analogue Spacetime

- The full metric tensor for our fluid system, assuming the first sound speed to be analogous to the speed of light is given by

$$g^{\mu\nu}(t, \boldsymbol{x}) = \frac{1}{\rho_0 c} \begin{pmatrix} -1 & -v_0^1 & -v_0^2 & -v_0^3 \\ -v_0^1 & c^2 - \left(v_0^1\right)^2 & -v_0^1 v_0^2 & -v_0^1 v_0^3 \\ -v_0^2 & -v_0^2 v_0^1 & c^2 - \left(v_0^2\right)^2 & -v_0^2 V_0^3 \\ -v_0^3 & -v_0^3 v_0^1 & -v_0^3 v_0^2 & c^2 - \left(v_0^3\right)^2 \end{pmatrix}$$

with
$g = [\det(g^{\mu\nu})] = \rho_0^4 / c^2$

# The General Scalar Field for the Fluid

So our final scalar field is
$\nabla\psi \equiv \frac{1}{\sqrt{-g}}\delta_\mu\left(\sqrt{-g}\,[g^{\mu\nu}\delta_\nu\psi]\right) = 0$

- This is identical to the massless scalar field of (3+1) Lorentzian geometry
- We only have a (2+1) model.
- What background should we choose for our analogue?

# A (2+1) Dimensional flow with a Sink

- We chose co-ordinates $(t, r, \phi)$
- For the continuity equation we state
  $\rho v^r \propto 1/r$
- Locally irrotational flow dictates
  $v^\phi \propto 1/r$
- Conservation of angular momentum also gives
  $\rho v^\phi \propto 1/r$
- This tells us that density is constant.
- So we can describe the velocity by
  $\boldsymbol{v} = \left(-A\boldsymbol{r} + B\boldsymbol{\phi}\right)/r$

# The Final Metric for the Fluid Vortex

- We substitute this velocity in the metric tensor
- We also make the following co-ordinate transformation, which is analogous to the transformation from Schwarzschild to Boyer-Lindquist co-ordinates
- $dt \rightarrow d\tilde{t} + \left(\frac{Ar}{r^2c^2 - A^2}\right) dr$ and $d\phi \rightarrow \tilde{+}\phi + \left(\frac{BA}{r\left[r^2c^2 - A^2\right]}\right) dr$
- This gives the metric for the vortex
- $ds^2 = -\left(1 - \frac{A^2+B^2}{c^2r^2}\right) c^2 d\tilde{t}^2 + \left(1 - \frac{A^2}{c^2r^2}\right)^{-1} dr^2 - 2Bd\tilde{\phi}d\tilde{t} + r^2 d\tilde{\phi}^2$
- On inspection, this looks a lot like the (2+1) Kerr metric. But does it display the same feature/

# Features of the Vortex Metric

- As $r \to \infty$, the metric approaches that of a non moving fluid. So it is asymptotically flat.
- The metric is independent of $t$, stationary, and of $\phi$, axisymmetric. Both features of the Kerr metric.
- $B = 0$ is analogous to $J = a = 0$
- When $r = (A/c)$, $v^r > c$. So we have an event horizon.
- There exists a region outside of this event horizon, where $|\boldsymbol{v}| > c$ but $v^r < c$. An ergosphere. This occurs when $\sqrt{A^2 + B^2} > rc > |A|$.
- On the whole, displays most of the key features.
- Also displays quasi normal modes and Hawking radiation. Not going to be covered today.

Thank-you.
Questions or Lunch?