Pulse intensity

Assuming a fixed magnitude of the magnetic dipole, the pulse intensity will depend on the orientation of the magnetic dipole to the observer and the beam geometry. It will be maximal when pointing directly at the observer and presumably fall off as the angle between the two grows. To model this, we take an observers position as $(\Phi_{\rm obs}, \Theta_{\rm obs} = \iota)$ and then assume the beam geometry follows Gaussian profile with a single conal emmision. We will assign a maximum intensity to the beam of I_0, then parameterise the beam width by $\sigma_{\rm beam}$. We can the express the pulse intensity for such a beam geometry as

$$I(\Theta, \Phi, \iota, I_0, \sigma_{\rm beam}) = I_0 \left(\exp\left(-\frac{\Delta d_{\rm F}^2}{2\sigma_{\rm beam}^2}\right) + \exp\left(-\frac{\Delta d_{\rm B} 2^2}{2\sigma_{\rm beam}^2}\right)\right) \tag{0.0.1}$$

Note that there are two beams here: arbtiarily labelled F (front) and B (back) corresponding to the two sides of the magnetic dipole. The Δd quantity measures the central angle between the observers line of sight and each side of the beam. We can calculate these spherical distances from the spherical law of cosines

$$\Delta d_{\rm F} = \cos^{-1}\left[\cos(\Theta)\cos(\iota) + \sin(\Theta)\sin(\iota)\cos(|\Phi - \Phi_{\rm obs}|)\right] \tag{0.0.2}$$

$$\Delta d_{\rm B} = \cos^{-1}\left[\cos(\pi - \Theta)\cos(\iota) + \sin(\pi - \Theta)\sin(\iota)\cos(|\Phi + \pi - \Phi_{\rm obs}|)\right] \tag{0.0.3}$$

$$\tag{0.0.4}$$

At this point we need to take stock of the significance of the two beams. If $\Theta < \pi/2$, then the observer will only observe a single beam, the other will much smaller and go unobserved. If $\Theta \approx \pi/2$, then both pulses could be observed although the ability to do this will depend on the observers sensitivity.

To illustrate the case where $\chi \approx \pi/2$, in figure 0.0.1 we plot the total beam intensity intensity for a torque-free pulsar. This is intended to demonstrate the fast rotation frequency of individual pulses, along with the longer modulation of the precession: the parameters are artificially selected to show both these behaviours on the same timescale. Both sets of pulses can be seen with the second second become larger than the first with $\chi > \pi/2$.

Each observation occurs over a period much shorter than the precession timescale, so changes in the beam will not be observed during a single session, but only between observations. In view of this, we make the assumption that the observer only ever measures the larger of the two beams: we can then generalise our intensity by stating that the polar angle of whichever beam is in the northen hemisphere is

$$\tilde{\Theta} = \frac{\pi}{2} - \left|\frac{\pi}{2} - \Theta\right| \tag{0.0.5}$$

In such a case the intensity can be written

$$I = I_0 \exp\left(-\frac{\left(\cos^{-1}\left[\cos(\tilde{\Theta})\cos(\iota) + \sin(\tilde{\Theta})\sin(\iota)\cos(|\Phi - \Phi_{\rm obs}|)\right]\right)^2}{2\sigma_{\rm beam}^2}\right) \tag{0.0.6}$$

From this general expression for the intensity, we can also calcualte the maximum observed intensity

$$I_{\rm max} = I_0 \exp\left(-\frac{\left(\tilde{\Theta} - \iota\right)^2}{2\sigma_{\rm beam}}\right) \tag{0.0.7}$$

We have included this in figure 0.0.1.

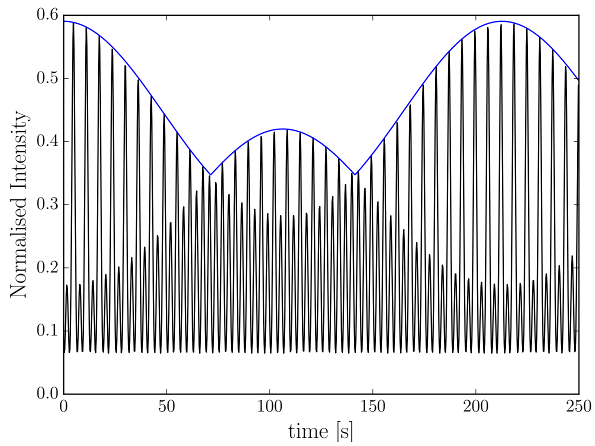

Figure 0.0.1: Amplitude variation using a 2D Gaussian emission. In black is the pulse intensity as given by (0.0.1): this shows each individual pulse at the fast rotation frequency with the slower modulation due to precession. In blue we plot the maximum amplitude as described by Eqn. (0.0.7).

Pulse width

We now need to relate the pulse intensity calculated previously with the quantity measured by pulsar astronomers: the W_{10} or the width at 10% of the observed peak intensity.

Now let us state that Θ varies on the slow precession timescale, while Φ varies on the rapid spin timescale. We are looking to measure the variations with respect to the slow precession timescale. The pulse width is measured by the time spent above some fractional amount $p/100$ of the peak measured amplitude; note the convention is to use say 10% of the peak value, in our notation then p=10. The condition for when the intensity is greater than this fraction is

$$I > I_{\mathrm{max}}\frac{p}{100}. \tag{0.0.8}$$

We can substitute into equations (0.0.1) and (0.0.7) and rearrange. This gives us an expression for when the inequality is satisfied:

$$\cos(|\Phi|) < \frac{\cos\left(\sqrt{(\tilde{\Theta}-\iota)^2 - 2\sigma_{\mathrm{beam}}^2 \ln(\frac{p}{100})}\right) - \cos(\tilde{\Theta})\cos(\iota)}{\sin(\tilde{\Theta})\sin(\iota)} \tag{0.0.9}$$

Since we expect $\tilde{\Theta}$ to vary on a much longer timescale than Φ, over one period in Φ we can treat $\tilde{\Theta}$, and hence the whole right-hand-side as a constant. Lets consider a single rotation with the magnetic dipole starting and ending in the antipodal point to the observers position. Then Φ increase beween $-\pi$ and π during this rotation. The time during which this inequality is true, measures the beam width.

In figure 0.0.2 an illustration is given of this single period showing the constant value on the right hand side of (0.0.9) and the osciallting cosine function. When the cosine is less than this constant this inequality is satisfied.

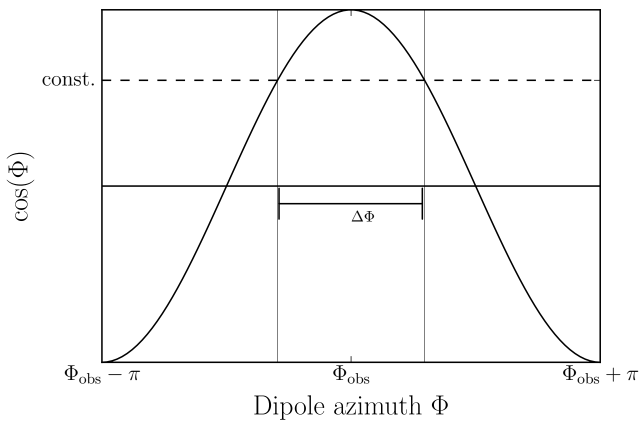

Figure 0.0.2: Illustration of the inequality in equation (0.0.9) the constant value represents the right hand side of this equation. The width $\Delta\Phi$ indicates the angular period during which inequality is satisfied.

We can calculate the beam width measured by observed by first calculating the angular width $\Delta\Phi$ for which the inequality is not satisified:

$$\Delta\Phi = 2\cos^{-1}\left(\frac{\cos\left(\sqrt{(\tilde{\Theta}-\iota)^2 - 2\sigma_{\rm beam}^2\ln(\frac{p}{100})}\right) - \cos(\tilde{\Theta})\cos(\iota)}{\sin(\tilde{\Theta})\sin(\iota)}\right) \tag{0.0.10}$$

Then the angular fraction at which the inequality *is* satisfied is given by $2\pi - \Delta\Phi$. The beam width is measured in the time spent above the fraction f of the peak measured amplitude. So we can convert our angular fraction above into a beam width by

We can now convert this into a pulse width

$$W_p = P\frac{2\pi - \Delta\Phi}{2\pi} \tag{0.0.11}$$

$$= P\left(1 - \frac{1}{\pi}\cos^{-1}\left(\frac{\cos\left(\sqrt{\left(\tilde{\Theta}-\iota\right)^2 - 2\sigma_{\rm beam}^2\ln(\frac{p}{100})}\right) - \cos(\tilde{\Theta})\cos(\iota)}{\sin(\tilde{\Theta})\sin(\iota)}\right)\right) \tag{0.0.12}$$

where P is the spin period which we have then written in terms of the spin frequency and p is the percentage of beam width. the beam width will vary with both the changes in spin-frequency, and with $\tilde{\Theta}$.

In the following figures we demonstrate typical beamwidths alongside the Θ behaviour which drives changes in the widths.

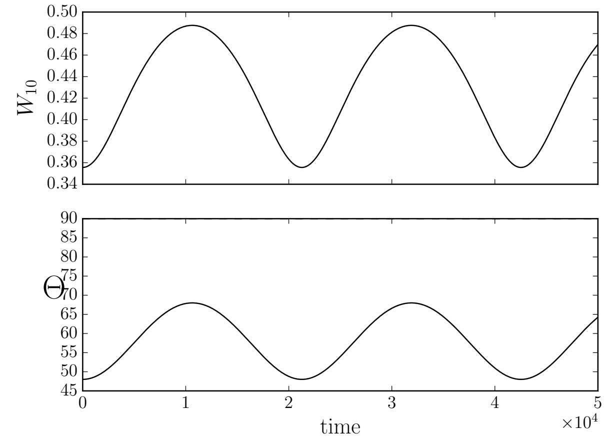

Figure 0.0.3: The beamwidth and polar angle Θ of the brightest beam for a pulsar with $\Theta < \pi/2$

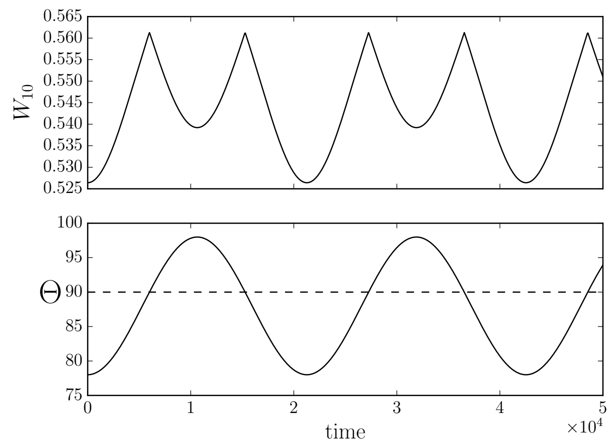

Figure 0.0.4: The beamwidth and polar angle Θ of the brightest beam for a pulsar with $\Theta \approx \pi/2$

Limits on σ_{beam}

So far we have ignored the physical implications of σ_{beam}, we will now discuss its limits which will help us to gain some intuition about. Inherited from the our Gaussian beam structure, σ_{beam} provides a measure of the beam width, as such we know $\sigma_{\text{beam}} > 0$ in order for us to observe it, but the upper limit is somewhat more subtle. The observed pulses arrive in a 'pulse-train' as Φ increases in Eqn. (0.0.6). In order then to measure the p^{rm} percentage of the beamwidth, we need

$$I_{\min} < \frac{p}{100} I_{\max} \tag{0.0.13}$$

where, for the generalise pulse intensity

$$I_{\max} = I_0 \exp\left(-\frac{(\tilde{\Theta} - \iota)^2}{2\sigma_{\text{beam}}^2}\right) \tag{0.0.14}$$

$$I_{\min} = I_0 \exp\left(-\frac{(\tilde{\Theta} + \iota)^2}{2\sigma_{\text{beam}}^2}\right) \tag{0.0.15}$$

Substituting and rearranging for σ_{beam} we find that

$$\sigma_{\text{beam}}^2 < \frac{2\tilde{\Theta}\iota}{\log(100/P)} \tag{0.0.16}$$