0.1 Extracted data

In this work I am working to test the hypothesis of Lyne et al. (2010) that they have found compelling evidence for state-switching in at least two pulsars. They contend that the spin-down rate switched discreetly between two distinct states. This is not observed in the corresponding measure of $\dot{\nu}$ (see figure 5 from their paper) because of an averaging process. Instead, they consider a measure of the beam-width W_{10} because this can be measured without averaging.

I have extracted the data for the B1828-11 pulsar from the Lyne et al. (2010) paper and plotted the W_{10} value in figure 0.1.1. In future I intend to apply the same analysis to the other available data sets (3 in total) but begin with just one to develop the methodology.

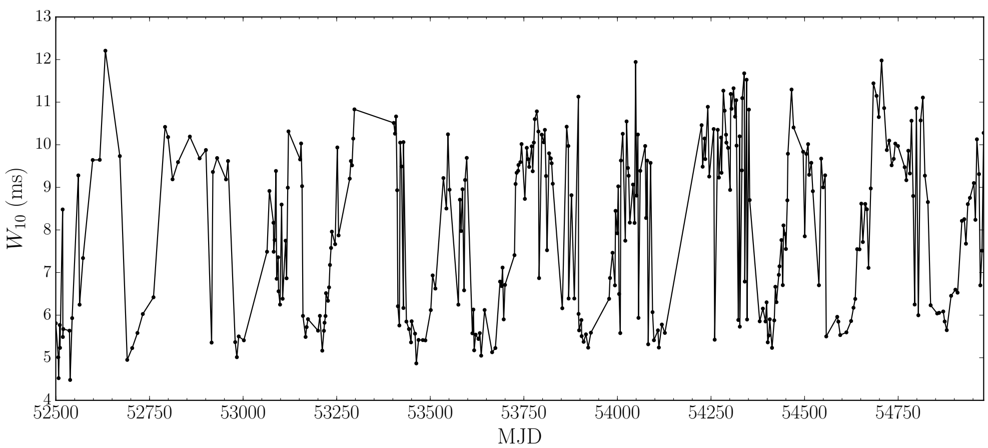

Figure 0.1.1: Raw data extracted from Lyne et al. (2010) on the W_{10} value of B1828-11 which is a measure of the pulse width.

The authors argue that this data is showing the beam width switching discreetly between two values with a period 0.73 yr^{-1}. We intend to apply some data analysis to better understand this statement since there is some doubts about the consistency of the measured values.

0.2 Histogram

Since the claim centers on the fact that the beam width switches it makes sense to begin by binning the measured values in a histogram. In figure 0.2.1 we show such a histogram corresponding to the data in figure 0.1.1

Evidently the histogram illustrates that the beam width follows a bi-modal distribution. The Lyne hypthosis to explain this is:

$\mathcal{H}_{\mathrm{Lyne}}$: The spin-down switches discreetly between two distinct values in a square-wave pattern. This is caused by changes in the magnetosphere which manifest as changes in the beam width. Therefore, the beam width must also follow a square wave pattern.

We would like to question the evidence for this hypothesis, by comparing with some alternatives. We now proceed to analyse competitor models with the aim being to quantify which, if any, of the models best described the observed data.

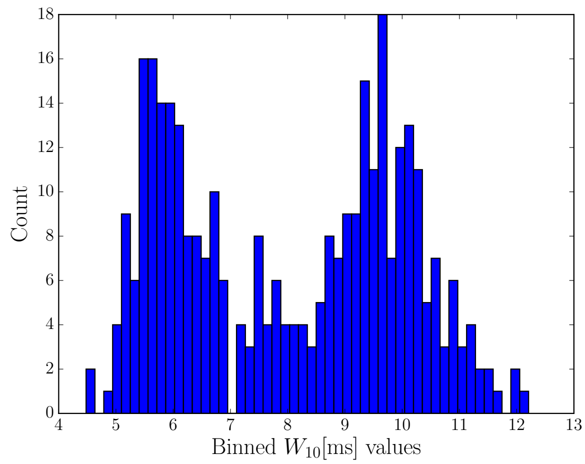

Figure 0.2.1: A histogram of the data presented in figure 0.1.1 binned according the value of W_{10}. This clearly demonstrates a bimodal distribution.

0.3 An alternative hypthesis: A trigonometric function

The primary competitor model to the square wave which we consider will be a simple trig. function. This is motivated by variations in the beam width due to free precession (a model that itself has potential to describe the spin-down variations). Free precession however procudes a complicates oscillatory function that could be the sum of several harmonically related sinusoid. Instead using a single sine-wave captures the idea that the beam width smoothly varies rather than as a square-wave without introducing any additional complexity.

0.3.1 Motivation using simple numerical example

Trig. function without noise

We can begin by demonstrating why the hypothesis that figure 0.1.1 is uniquely described by a square-wave model is open to debate. To do this we take a simply function such as

$$y(x) = y_0 + A\sin(\omega t) \tag{0.3.1}$$

Then writing a simple script which produces data for x and y, we pick some number of random points from the data; this simulated the process of collecting data. We can then 'bin' the y values in a histogram to look at their distribution. This is done in figure 0.3.1 showing both the collected points and their distribution. We note the particular shape of the histogram has two distinct peaks. This can be intuitively understood as the result of the system spending more time near the extrema.

This histogram result provides an insight at an underlying probability distribution. Namely, if we measure values from an oscillating system, what is the probability of getting some particular value? We analytically calculate this underlying distribution in section 0.3.2.

Trig. function with noise

Cleary to fit with the observed data, we must also include a noise component to avoid having a hard cutoff at $y = y_0 \pm A$. This can be done by assuming the noise is normally distributed such

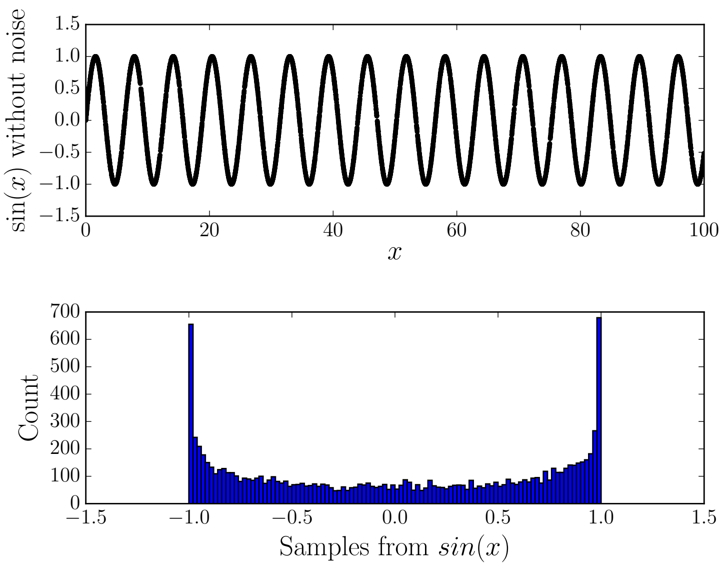

Figure 0.3.1: In the top panel is the collected data for eqn. (0.3.1) with $A = 1$, $y_0 = 0$. In the lower panel is the binned data on the collected values of y in a histogram.

that

$$y(x) = y_0 + A\sin(\omega t) + \mathcal{N}(0, \sigma^2) \tag{0.3.2}$$

We can again simulate this numerically and the result for a particular realisation is given in figure 0.3.1.

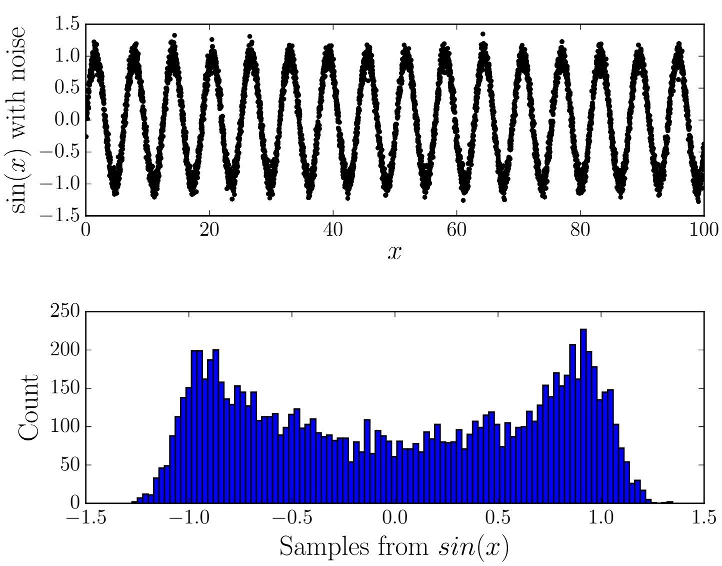

Figure 0.3.2: In the top panel is the collected data for eqn. (0.3.2) with $A = 1$, $y_0 = 0$ and $\sigma = 0.1$. In the lower panel is the binned data on the collected values of y in a histogram.

The addition of noise notably broadens the histogram but can, if the noise is sufficiently weak, retain the bimodal structure.

0.3.2 Analytic calculation

Trig. function without noise

As we have demonstrated, the histogram of values from randomly selecting points on a sine wave is bi-modal. This suggests their is an underlying probability distribution which we will now calculate. That is, we can think of making a measurement y_i on the system, then the distribution from which this is drawn is the underlying probability distribuition $f(y)$ which we wish to calculate. This can be calculated from the inverse-CDF method, however a simpler and more intuitive approach follows.

Let us assume that we choose a point x_i at which to measure the system, and that the x_i are uniformly distributed on $[0, X]$ where $X = \frac{2\pi}{\omega}$ is the period of the sine wave. Note that while we are considering picking points over only a single period, this is the same as pick points over an arbitary number of periods since the sine-wave is a repetitive oscillation.

Since the x_i are uniformly distributed, the probability density function for y must satisfy

$$f(y)dy = \frac{dx}{X}. \tag{0.3.3}$$

Inserting the definition for the period and rearranging

$$f(y) = \frac{dt}{dy}\frac{\omega}{2\pi}. \tag{0.3.4}$$

Inverting the relation between y and t and differentiating we end up with a normalised probability distribution

$$f(y) = \frac{1}{2\pi A}\left(1 - \left(\frac{y - y_0}{A}\right)^2\right)^{-1/2} \tag{0.3.5}$$

We plot the distribution in figure 0.3.3 and can be qualitatively compared with the histogram in 0.3.1 to verify that this is the underlying probability distribution.

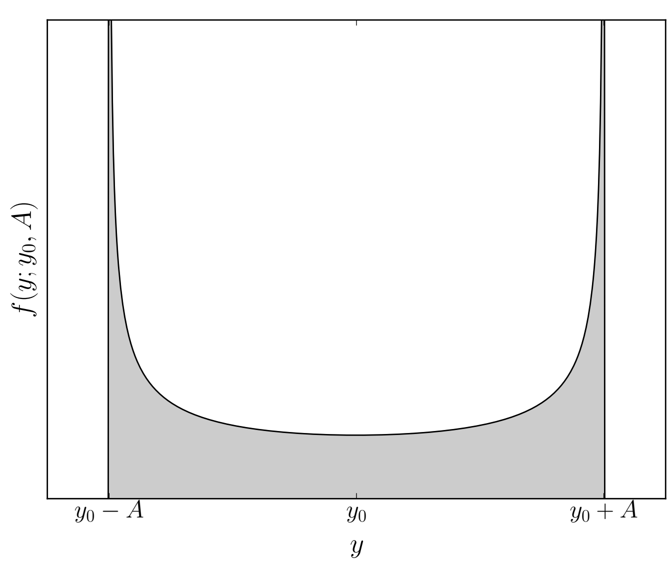

Figure 0.3.3: Plot of the probability density function in eqn. (0.3.5)

Trig. function with noise

Adding noise to each measurement means that our measured value y_i^M is given by

$$y_i^M = y_i + \epsilon_i \tag{0.3.6}$$

where y_i is the deterministic value with probability distribution given by (0.3.5) and ϵ_i is a random variable representing the error. This could be either an intrinsic error which depends in a complex way on the properties of the y_i or a measurement error. In the latter case the distribution will be independant; this simplifies the problem and so we consider only this case for now. Let the measurement error be normally distributed such that

$$\epsilon_i \sim N(0, \sigma^2) \tag{0.3.7}$$

then the probability density function is given by

$$g(\epsilon_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\epsilon^2/2\sigma^2\} \tag{0.3.8}$$

If the random variables y_i and ϵ_i are independant, then the probability distribution of the measured value (their sum) is given by the convolution of their probability density functions:

$$\begin{aligned} P(y^M) = (f * g)(y) &= \int_{-\infty}^{\infty} f(\tilde{y}) g(y - \tilde{y}) d\tilde{y} && (0.3.9) \\ &= \frac{1}{2\pi A\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp\{-(y-\tilde{y})^2/2\sigma^2\}}{\sqrt{(1-((\tilde{y}-y_0)/A)^2}} d\tilde{y} && (0.3.10) \end{aligned}$$

This integration is not easy to analytically integrate. Nevertheless, we can calculate any particular realisation by numerical integration. In figure 0.3.4 we plot the numerical integration for some particular values of A, y_0 and σ.

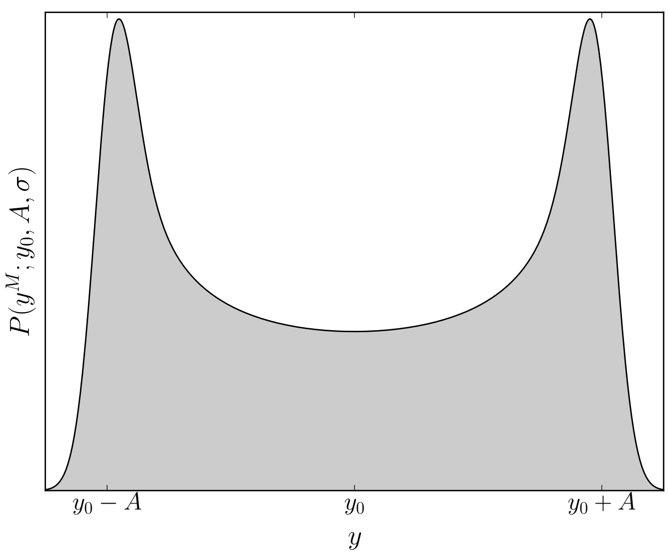

Figure 0.3.4: A numerical calculation of the probability distribution resulting from the convolution of the noise process and the probability for picking values from a trig. function.

Bibliography

Lyne, A., Hobbs, G., Kramer, M., Stairs, I., and Stappers, B. (2010). Switched Magnetospheric Regulation of Pulsar Spin-Down. *Science*, 329:408–.