0.1 Analytic description of the two state switching

The two state switching model is defined by two distinct spindown values and the durations that the system spends at each value. The spindown follows a square wave pattern, numerically integrating this with appropriate initial values yields the frequency and phase. This basic model was investigated by **?** and **?** by fitting and removing a Taylor expansion to the phase to produce the timing residual of such a system. It was found by **?** that this deterministic system must be modified by the introduction of some randomness to match observed residuals.

In this section we will consider only the deterministic system and develop an analytic result matching the numerical result. This analytic result allows us to better understand the simple two-state switching model.

First, let us define two periods labelled by A and B, these refer to the two states which the system switches between. If the system spends time t_A in state A, and t_B in state B then repeats, we can define the total duration of a single cycle and a ratio between the two states as

$$T = t_A + t_B \qquad\qquad R = t_A/t_B. \tag{0.1.1}$$

In each period the spin-down rate is a constant given by $\ddot{f}_X$ where $X \in [A, B]$. Therefore during each period the phase evolution will be described by a Taylor expansion such as (**??**). Let us define the parameters of the Taylor expansion in each period as $\boldsymbol{\lambda}_X = [\phi_X, f_X, \dot{f}_X]$; then we can write a time dependent set of parameters as

$$\boldsymbol{\lambda}(t) = \begin{cases} \boldsymbol{\lambda}_A & t \in A \\ \boldsymbol{\lambda}_B & t \in B \end{cases} \tag{0.1.2}$$

Similar expression can be written for each of the components (e.g. $f(t)$). We also need to define the reference time for each period: we choose to set this halfway through the respective period. The full time-dependant reference time can then be written

$$t_{\mathrm{R}}(t) = \begin{cases} \frac{T}{2}R & t \in A \\ \frac{T}{2}(R+1) & t \in B \end{cases} \tag{0.1.3}$$

Having defined the time-dependant parameters and reference time, the global phase model valid from $[0, T]$ is given by

$$\Phi(t; \boldsymbol{\lambda}(t), t_{\mathrm{R}}(t)) = \phi(t) + 2\pi \left(f(t)(t - t_{\mathrm{R}}(t)) + \frac{\dot{f}(t)}{2} (t - t_{\mathrm{R}}(t))^2 \right) \tag{0.1.4}$$

0.1.1 Calculating the phase residual

The phase residual $\Delta\Phi(t)$ is the difference between the actual phase of the pulsar and a best fit global Taylor expansion. The term global distinguishes a single Taylor expansion over the entire duration from the local templates used to model the phase of model. By *best fit* we refer to the global template which minimises the phase residual. usually this is done using a least-squares method.

First we ignore the best-fit requirement and define only how to analytically calculate the phase residual. Explicitly the phase residual is defined as

$$\Delta\Phi(t) = \Phi(t; \boldsymbol{\lambda}(t), t_{\mathrm{R}}(t)) - \Phi(t; \boldsymbol{\lambda}_G(t), t_{\mathrm{R}}(t)), \tag{0.1.5}$$

The term $\Phi(t;\boldsymbol{\lambda}(t),t_{\mathrm{R}}(t))$ describes the phase evolution in the switched spindown model (eqn. (0.1.4)), while $\Phi(t;\boldsymbol{\lambda}_G(t),t_{\mathrm{R}}(t))$ refers to the phase of the global template.

The global template is defined by a single set of coefficients and corresponding reference time. However, if want to describe the system such that at a given time the model and global template refer to the same reference times $t_{\mathrm{R}}(t)$, we can explicitly break up the global template into segments coinciding with those of the switching model. In this way the global reference time is still defined by a single set of coefficients and reference time, but these are translated to the relevant reference times $t_{\mathrm{R}}(t)$ such that all the individual segments lie on the same global template. This allows the phase residual to be written as

$$\Delta\Phi(t;\Delta\boldsymbol{\lambda}(t),t_{\mathrm{R}}(t)) = \Delta\phi(t) + 2\pi\left(\Delta f(t)(t-t_{\mathrm{R}}(t)) + \frac{\Delta\dot{f}(t)}{2}\left(t-t_{\mathrm{R}}(t)\right)^2\right) \tag{0.1.6}$$

where

$$\Delta\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}(t) - \boldsymbol{\lambda}_G(t) = \begin{cases} \Delta\boldsymbol{\lambda}_A & t\in A \\ \Delta\boldsymbol{\lambda}_B & t\in B \end{cases}. \tag{0.1.7}$$

Writing the phase residual in such a way removes the requirement to consider the monotonic spindown. Instead, we can focus on the difference between the model and the global template.

0.1.2 Modelling the two-state switching

Now we are ready to start modelling the two state switching. From equation (0.1.2) we can write the spin-down rate as

$$\dot{f}(t) = \begin{cases} \dot{f}_A & t\in A \\ \dot{f}_B & t\in B \end{cases}. \tag{0.1.8}$$

Now we rewrite this expression by defining a time-dependant spin-down offset about the global template:

$$\dot{f}(t) = \dot{f}_G + \Delta\dot{f}(t), \tag{0.1.9}$$

here we assume $\ddot{f}_G = 0$ such that $\dot{f}_G$ is a constant. Then we have implicitly defined the time-dependant offset as

$$\Delta\dot{f}(t) = \begin{cases} \Delta\dot{f}_A & t\in A \\ \Delta\dot{f}_B & t\in B \end{cases}. \tag{0.1.10}$$

with $\Delta\dot{f}_X = \dot{f}_X - \dot{f}_G$. In figure 0.1.1 we illustrate the setup over a single cycle of A and B.

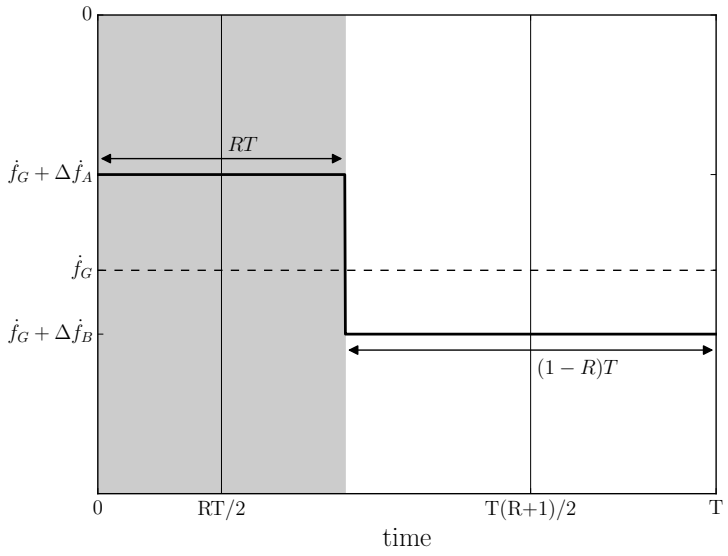

Figure 0.1.1: Illustration of the switched spindown model over one cycle.

0.1.3 Calculating the best fit for the two-state switching

We now need to define what we mean by the 'best fit'. For observational data and the numerical models of **?** and **?**, this resulted from a least-squares fitting procedure. While we could replicate this process analytically, we can also proceed by a more intuitive method.

The phase residual of equation (0.1.6) is given for a single cycle, in reality we want to calculate the best fit over several cycles. Therefore we define the best fit model as the set of coefficients $[\Delta\boldsymbol{\lambda}_A, \Delta\boldsymbol{\lambda}_B]$ which minimises $\Delta\Phi(t)$ over several cycles. The free parameters of the model are the jump in spin-down which occurs between the two segments, and the relative times spent in each segment. This leaves the frequency and phase offsets in each segment and one of the spin-down offsets to be calculated. Note only one spin-down value is required since the other can be set from the jump in spin-down. We can constrain these unknown parameters by setting up 5 simultaneous equations reflecting our choice of best fit. In each case the property is given and the equation resulting from applying equation (0.1.6) is then given.

- *No accumulations of phase:* In order for $\Delta\Phi$ to remain bounded and not grow with each cycle we require that the phase offset at the beginning of a cycle be equal to that at the end:

$$\Delta\Phi(0) = \Delta\Phi(T) \tag{0.1.11}$$

$$0 = \Delta\phi_B - \Delta\phi_A + \pi\left(T\left(\Delta f_A R + \Delta f_B(1-R)\right) + \frac{T^2}{4}\left(\Delta\dot{f}_B(1-R)^2 - \Delta\dot{f}_A R^2\right)\right) \tag{0.1.12}$$

- *Smooth phase at the interface:* The interface between the two segments occurs at $t_A = RT$. For the best fit, the phase and hence $\Delta\Phi(t)$ should be smooth at this point

$$\lim_{t\to t_A^-}\Delta\Phi(t) = \lim_{t\to t_A^+}\Delta\Phi(t) \tag{0.1.13}$$

$$0 = \Delta\phi_B - \Delta\phi_A + \pi\left(-T\left(\Delta f_A R + \Delta f_B(1-R)\right) + \frac{T}{4}\left(\Delta\dot{f}_B(1-R)^2 - \Delta\dot{f}_A R^2\right)\right) \quad (0.1.14)$$

In addition, for the best fit the phase offset in each segment must carry opposite signs if the total phase accumulation is to vanish. This must mean that the phase offset at the interface between the two segments is zero such that

$$\lim_{t\to t_A^-}\Delta\Phi(t) = \lim_{t\to t_A^+}\Delta\Phi(t) = 0 \quad (0.1.15)$$

Either of these relation can be used, from the first we have

$$\Rightarrow 0 = \Delta\phi_A + \pi\left(\Delta f_A RT + \Delta\dot{f}_A\frac{R^2T^2}{4}\right) \quad (0.1.16)$$

- *No accumulation of frequency:* In a similar manner to the phase, we expect that

$$\frac{d\Delta\Phi(0)}{dt} = \frac{d\Delta\Phi(T)}{dt} \quad (0.1.17)$$

$$0 = \Delta f_B - \Delta f_A + \frac{T}{2}\left(\Delta\dot{f}_A T + \Delta\dot{f}_B(1-R)\right) \quad (0.1.18)$$

- *Smooth frequency at the interface:*

$$\lim_{t\to t_A^-}\frac{d\Delta\Phi(t)}{dt} = \lim_{t\to t_A^+}\frac{d\Delta\Phi(t)}{dt} \quad (0.1.19)$$

$$0 = \Delta f_B - \Delta f_A - \frac{T}{2}\left(\Delta\dot{f}_A R + \Delta\dot{f}_B(1-R)\right) \quad (0.1.20)$$

Working through this system of equations we end up with the following set of parameters for residual of the best fit

$$\Delta\dot{f}(t) = \begin{cases} \Delta\dot{f}_A & t\in A \\ -\frac{R}{1-R}\Delta\dot{f}_A & t\in B \end{cases} \quad \Delta f(t) = 0 \quad \Delta\phi(t) = \begin{cases} -\frac{\pi}{4}\Delta\dot{f}_A T^2R^2 & t\in A \\ \frac{\pi}{4}\Delta\dot{f}_A T^2(R-R^2) & t\in B \end{cases} \quad (0.1.21)$$

Then the phase residual can be written as

$$\Delta\Phi(t) = \pi\Delta\dot{f}_A\begin{cases} -\left(RT/2\right)^2 + \left(t-(RT/2)\right)^2 & t\in A \\ \left(T/2\right)^2(R-R^2) - \frac{R}{1-R}\left(t-\frac{T}{2}(R+1)\right)^2 & t\in B \end{cases} \quad (0.1.22)$$

We can make this slightly more general by defining the total difference in spindown between two states as $\Delta\dot{f}_T = |\Delta\dot{f}_A - \Delta\dot{f}_B|$ such that the phase residual can be written as

$$\Delta\Phi(t;\Delta\dot{f}_T, t_A, R) = \pi\Delta\dot{f}_T\begin{cases} (1-R)t(t-RT) & t\in A \\ -R(t-T)(t-RT) & t\in B \end{cases} \quad (0.1.23)$$

This gives the best phase residual in a single cycle for t is bound by $[0,T]$ when fitting through many such cycles. Note that this is not the same as the best fit phase residual over a single cycle.

In figure 0.1.2 we illustrate the critical points of the phase residual during a single cycle. Note that when $R \neq 1/2$ the radius of curvature of the maxima and minima will differ.

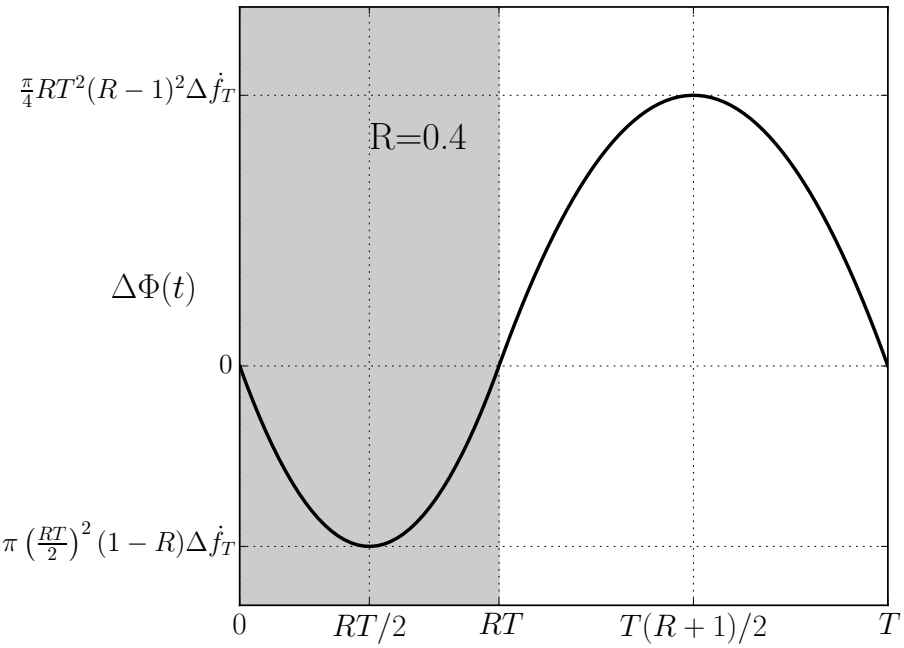

Figure 0.1.2: Phase residual over a single cycle, due to our choice of the best fit this pattern will repeat for each cycle

0.1.4 Extending the model to an arbitrary number of cycles

The phase residual and hence mismatch calculated above consider only fitting to one single cycle of switching. Since we calculated the best fit over several cycles, we know that in any given cycle the residual will be the same, provided that an integer number of cycles is fitted. Therefore we can extend equation (0.1.23) to an any arbitrary number of cycles by defining $\tilde{t} = t \mod T$. The phase residual is then

$$\Delta\Phi(t;\Delta\dot{f}_T,R) = \pi\Delta\dot{f}_T \begin{cases} (1-R)\tilde{t}(\tilde{t}-RT) & t\in A \\ -R(\tilde{t}-T)(\tilde{t}-RT) & t\in B \end{cases} \tag{0.1.24}$$

with the assumption that the initial cycle starts at $t = 0$. Note that this is *not* a fully general model since it assumed we have fitted to an integer number of cycles. If some fraction of a cycle existed at the start and end of the observation then the residual may be skewed by favouring one state or the other. Nevertheless, this method should illustrate the essential features of the residual in the limit that enough full cycles are observed not to favout one or the other.

In figure **??** we have plotted the analytic prediction for the phase residual over several cycles. This is compared with a numerical result obtained in the manor of **?**: starting by defining a time dependant spin-down, we integrate twice to get the phase, then fit and subtract a timing model to get the residual.

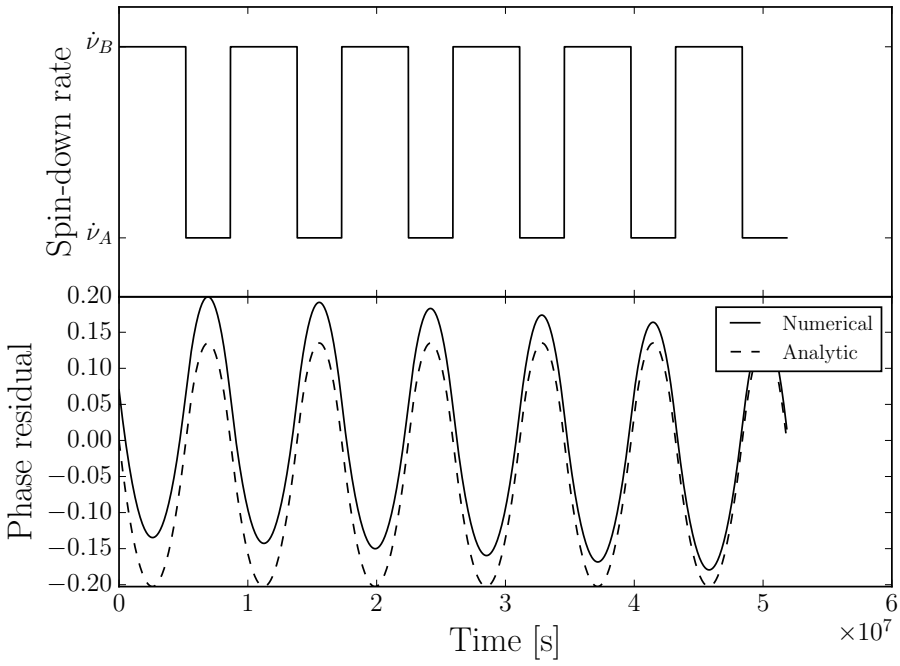

Figure 0.1.3

0.1.5 Verifying the model

Having an analytic expression for the phase residual means observational results could be tested to see if they agree with this model. Ideally we would like to apply this to the results in **?**. This could help quantify how well the observed timing residuals can be explained by a two-state switching model compared to a random walk model. Two methods are presented which could easily be applied to data.

Magnitude of peaks

Firstly if we divide the absolute values of the phase residuals at the extrema, which occur at the reference times in this parameterisation, then we have

$$\left| \frac{\Delta\Phi\left(RT/2\right)}{\Delta\Phi\left(T(R+1)/2\right)} \right| = \frac{\frac{\pi}{4}\Delta\dot{f}_T T^2(R-R^2)}{\frac{\pi}{4}\Delta\dot{f}_T T^2(R-1)^2} = \frac{R}{R-1} \tag{0.1.25}$$

This is exactly the ratio of the durations of the two segments.

0.1.6 Radii of curvature

? found that 'the residuals are generally asymmetric in that the radii of curvature of local maxima and minima are often consistently different'. The radii of curvature for a curve $y(x)$ is given by

$$r(x) = \frac{\left(1 + y'(x)\right)^{3/2}}{y''(x)} \tag{0.1.26}$$

For equation (0.1.22) the radii of curvature at the extrema are then

$$r\left(RT/2\right) = \frac{-R}{2\pi\Delta\dot{f}_T(R-1)} \qquad\qquad r\left(T(R+1)/2\right) = \frac{-1}{2\pi\Delta\dot{f}_T} \tag{0.1.27}$$

For the observation that the radii of curvature to be different in this model we then require $R \neq 1/2$. This can also be measured from the timing residual.