Pulse intensity

Assuming a fixed magnitude of the magnetic dipole, the pulse intensity will depend on the orientation of the magnetic dipole to the observer and the beam geometry. It will be maximal when pointing directly at the observer and presumably fall off as the angle between the two grows. To model this, we take an observers position as $(\Phi_{\mathrm{obs}} = 0, \Theta_{\mathrm{obs}})$ and then assume the beam geometry follows Gaussian profile with a single conal emmision. We can the express the pulse intensity for such a beam geometry as

$$I(\Theta, \Phi, \Theta_{\mathrm{obs}}, I_0, \sigma_B) = I_0 \exp\left(-\frac{\Delta d^2}{2\sigma_B^2}\right) \tag{0.0.1}$$

where Θ and Φ are the usual beam coordinates, I_0 and σ_B are shape parameters of the beam itself, and finally σd is the central angle between the observers line of sight and the beam. We can calulate Δd exactly from the spherical law of cosines

$$\Delta\sigma = \cos^{-1}\left[\sin(\Theta)\sin(\Theta_{\mathrm{obs}}) + \cos(\Theta)\cos(\Theta_{\mathrm{obs}})\cos(|\Phi|)\right] \tag{0.0.2}$$

In figure 0.0.1 an illustration is given of variations in intensity for a torque-free pulsar with some arbitary paramaters. This is intended to demonstrate the fast rotation frequency of individual pulses, along with the longer modulation of the precession: the parameters are artificially selected to show both these behaviours on the same timescale.

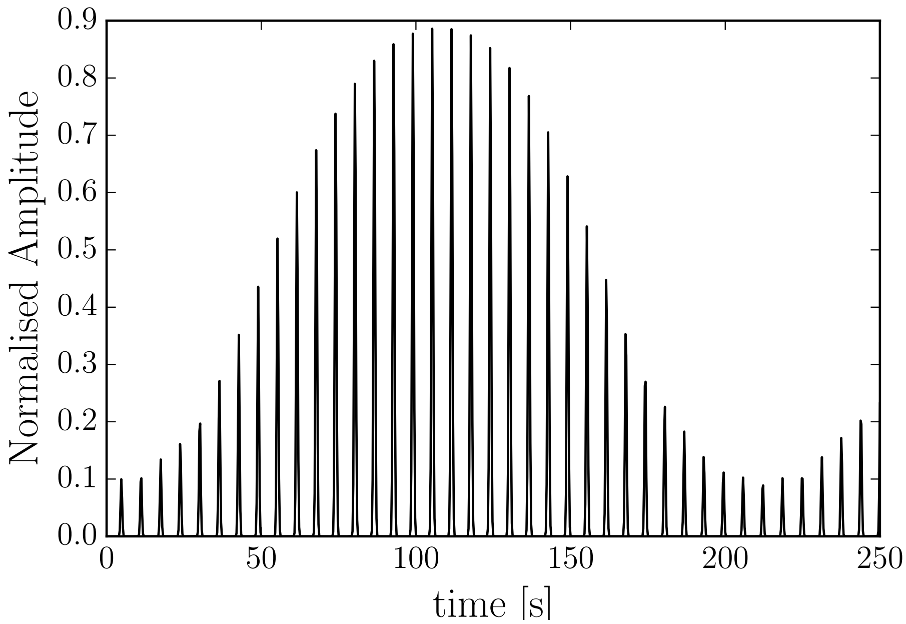

Figure 0.0.1: Amplitude variation using a 2D Gaussian emission.

Pulse width

We now need to relate the pulse intensity calculated previously with the quantity measured by pulsar astronomers: the W_{10} or the width at 10% of the peak intensity. The subtlely here is that the peak intensity is the measured value and not the peak of the beam (which in our model is simply I_0). We will label this measured maximum as $I_{\max}$ and we can see that it must occur when Δd is at a minimum. From the functional form of Eqn. (0.0.2) we find the minimum occurs when $\Phi = 0$ in which cases we can simplify

$$\Delta d = \Theta - \Theta_{\mathrm{obs}} \tag{0.0.3}$$

and so

$$I_{\max} = I_0 \exp\left(-\frac{(\Theta - \Theta_{\mathrm{obs}})^2}{2\sigma_B^2}\right) \tag{0.0.4}$$

Now let us state that Θ varies on the slow precession timescale, while Φ varies on the rapid spin timescale. We are looking to measure the variations with respect to the slow precession timescale. The pulse width is measured by the time spent above some fractional amount $p/100$ of the peak measured amplitude; note the convention is to use say 10% of the peak value, in our notation then p=10. The condition for when the intensity is greater than this fraction is

$$I(\Phi, \Theta, \Theta_{\mathrm{obs}}, \Phi_{\mathrm{obs}}, \sigma_B) > I_{\max}\frac{p}{100}. \tag{0.0.5}$$

We can substitute into equations (0.0.1) and (0.0.4) and rearrange. This gives us an expression for when the inequality is satisfied:

$$\cos(\Phi) < \frac{\cos\left(\sqrt{\Theta - \Theta_{\mathrm{obs}} + 2\sigma_B^2 \ln(\frac{p}{100})}\right) - \sin(\Theta)\sin(\Theta_{\mathrm{obs}})}{\cos(\Theta)\cos(\Theta_{\mathrm{obs}})} \tag{0.0.6}$$

Since we expect Θ to vary on a much longer timescale than Φ, over one period in Φ we can treat Θ, and hence the whole right-hand-side as a constant. Lets consider a single rotation with the magnetic dipole starting and ending in the antipodal point to the observers position. Then Φ increase beween $-\pi$ and π during this rotation. The time during which this inequality is true, measures the beam width.

In figure 0.0.2 an illustration is given of this single period showing the constant value on the right hand side of (0.0.6) and the osciallting cosine function. When the cosine is less than this constant this inequality is satisfied.

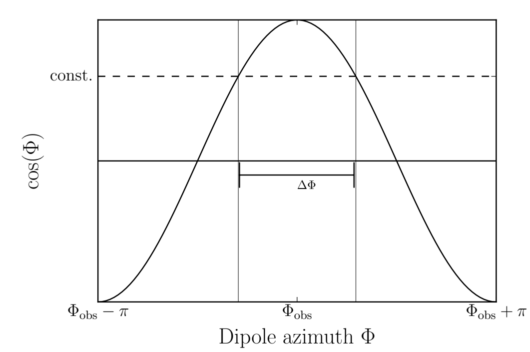

Figure 0.0.2: Illustration of the inequality in equation (0.0.6) the constant value represents the right hand side of this equation. The width $\Delta\Phi$ indicates the angular period during which inequality is satisfied.

We can calculate the beam width measured by observed by first calculating the angular width $\Delta\Phi$ for which the inequality is not satisified:

$$\Delta\Phi = 2\cos^{-1}\left(\frac{\cos\left(\sqrt{\Theta - \Theta_{\text{obs}} + 2\sigma_B^2 \ln(\frac{p}{100})}\right) - \sin(\Theta)\sin(\Theta_{\text{obs}})}{\cos(\Theta)\cos(\Theta_{\text{obs}})}\right) \tag{0.0.7}$$

Then the angular fraction at which the inequality *is* satisfied is given by $2\pi - \Delta\Phi$. The beam width is measured in the time spent above the fraction f of the peak measured amplitude. So we can convert our angular fraction above into a beam width by

We can now convert this into a pulse width

$$W_p = T\frac{2\pi - \Delta\Phi}{2\pi} \tag{0.0.8}$$

$$= T\left(1 - \frac{1}{\pi}\cos^{-1}\left(\frac{\cos\left(\sqrt{\Theta - \Theta_{\text{obs}} + 2\sigma_B^2 \ln(\frac{100}{p})}\right) - \sin(\Theta)\sin(\Theta_{\text{obs}})}{\cos(\Theta)\cos(\Theta_{\text{obs}})}\right)\right) \tag{0.0.9}$$

where T is the spin period which we have then written in terms of the spin frequency and p is the percentage of beam width. the beam width will vary with both the changes in spin-frequency, and with Θ.