0.1 Approximate analytic spin-down modulation

The spin-down of a pulsar is obtained by calculating $\ddot{\Phi}$, the second time-derivative of the magnetic dipole phase. The phase itself is constructed from the Euler angles ψ, ϕ and θ which relate the motion in the body frame to the interial frame of an observer. No exact analytic solution for these angles is known when the electromagnetic dipole torque is included; therefore the same is true of the spin-down.

We are interested in calculating the spin-down for pulsar B1828-11. The evidence suggests that it is in a region of parameter space where $\epsilon_{\mathrm{A}} \ll 1$ (or the spin-down timescale is much longer than the precession time scale), and the wobble angle is small. Therefore it is appropriate to model the spin-down with an approximate analytic solution, expanding in both of these term:

$$\ddot{\Phi}(\epsilon_{\mathrm{A}}, \theta)|_{\epsilon_{\mathrm{A}},\theta} = \ddot{\Phi}(0,0) + \frac{\partial \ddot{\Phi}(0,0)}{\partial \epsilon_{\mathrm{A}}}\epsilon_{\mathrm{A}} + \frac{\partial \ddot{\Phi}(0,0)}{\partial \theta}\theta + \mathcal{O}(\epsilon_{\mathrm{A}}^2, \epsilon_{\mathrm{A}}\theta, \theta^2) \tag{0.1.1}$$

The spin-down at the origin of this parameter space is zero so the first term will vanish. We can then identify the other two terms with the corresponding Taylor expansion in one-dimension:

$$\ddot{\Phi}(\epsilon_{\mathrm{A}}, \theta)|_{\epsilon_{\mathrm{A}},\theta} = \ddot{\Phi}|_{\epsilon_{\mathrm{A}}=0} + \ddot{\Phi}|_{\theta=0} + \mathcal{O}(\epsilon_{\mathrm{A}}^2, \epsilon_{\mathrm{A}}\theta, \theta^2) \tag{0.1.2}$$

0.1.1 Small θ limit

The electomagnetic torque produced by a dipole can be written as

$$\ddot{\Phi} = -\frac{2R}{3c}\epsilon_{\mathrm{A}}\dot{\Phi}^3 \sin^2\Theta \tag{0.1.3}$$

where Θ is the polar angle of the dipole with respect to the inertial z axis. In the limit of $\theta \to 0$, we can expand as following

$$\begin{aligned} \cos\Theta &= \sin\theta\sin\psi\sin\chi + \cos\theta\cos\chi & (0.1.4)\\ &\approx \theta\sin\psi\sin\chi + \cos\chi + \mathcal{O}(\theta^2) & (0.1.5)\end{aligned}$$

and then solving for $\sin^2\Theta$ we find

$$\sin^2\Theta \approx \sin^2\chi - 2\theta\sin\psi\sin\chi\cos\chi + \mathcal{O}(\theta^2) \tag{0.1.6}$$

And similarly

$$\begin{aligned} \dot{\Phi} &= \dot{\phi} + \dot{\psi}\sin\chi\frac{\cos\theta\sin\chi - \sin\psi\sin\theta\cos\chi}{(\sin\theta\cos\chi - \cos\theta\sin\psi\sin\chi)^2 + \cos^2\psi\sin^2\chi} & (0.1.7)\\ &\approx \dot{\phi} + \dot{\psi}(1 + \tan\chi\sin\psi\theta) + \mathcal{O}(\theta^2) & (0.1.8)\end{aligned}$$

We can simplify this expression by expanding the following relation about $\theta = 0$:

$$\begin{aligned} \omega_{\mathrm{z}}(t) &= \dot{\phi}\cos\theta + \dot{\psi} & (0.1.9)\\ &\approx \dot{\phi} + \dot{\psi} + \mathcal{O}(\theta^2) & (0.1.10)\end{aligned}$$

Therefore

$$\dot{\Phi} \approx \omega_{\rm z} + \dot{\psi}\theta \tan\chi \sin\psi \tag{0.1.11}$$
$$\dot{\Phi}^3 \approx \omega_{\rm z}^3 + 3\omega_{\rm z}^2\dot{\psi}\theta \tan\chi \sin\psi + \mathcal{O}(\theta^2) \tag{0.1.12}$$

Putting these two expressions into the dipole spin-down we have

$$\ddot{\Phi}|_{\theta=0} = -\frac{2R}{3c}\epsilon_{\rm A}\left(\omega_{\rm z}^3 \sin^2\chi + \theta\left(3\omega_{\rm z}^2\dot{\psi}\tan\chi\sin\psi\sin^2\chi - 2\omega_{\rm z}^3\sin\psi\sin\chi\cos\chi\right)\right) + \mathcal{O}(\theta^2) \tag{0.1.13}$$

0.1.2 Small $\epsilon_{\rm A}$ limit

At zeroth order (when $\epsilon_{\rm A} =$), we have exact results for the Euler angles and hence the spin-down. We will now derive approximate analytic results for the Euler angles under some conditions, the aim being to derive results for the spin-down.

To do this we will first nelect the anomalous torque and work exclusively with the spin-down torque. In component form, this is given by

$$\boldsymbol{T}_{\rm s} = \begin{pmatrix} -\frac{\omega_{\rm x}}{2}\left(\cos(2\chi)+1\right) + \frac{\omega_{\rm z}}{2}\sin(2\chi) \\ -\omega_{\rm y} \\ \frac{\omega_{\rm x}}{2}\sin(2\chi) + \frac{\omega_{\rm z}}{2}\left(\cos(2\chi)-1\right) \end{pmatrix} \tag{0.1.14}$$

Clearly solving this in general as a source in the Euler rigid body equations is difficult. In order to get an approximate solution we will define two assumptions which are acceptable for a small subset of realistic pulsars:

- We assume that the wobble-angle is small, such that $\omega = \omega_{\rm z} + \eta$ where $\eta \sim \omega_{\rm x}, \omega_{\rm y}$ and is small.
- Second, we assume that the magnetic dipole is close to the equator such that $\chi \approx \pi/2$. Expanding the trig. functions about this point we have

$$\sin(2\chi) \approx -2\left(\chi - \pi/2\right) + \mathcal{O}\left((\chi-\pi/2)^2\right) \tag{0.1.15}$$
$$\cos(2\chi) \approx -1 + \mathcal{O}\left((\chi-\pi/2)^2\right) \tag{0.1.16}$$

Under these assumptions, the spin-down torque is:

$$\boldsymbol{T}_{\rm s} = \begin{pmatrix} 0 \\ 0 \\ -\omega_{\rm z} \end{pmatrix} + \mathcal{O}(\eta) + \mathcal{O}(\chi - \pi/2) \tag{0.1.17}$$

Considering the three coupled ODEs for the components of $\boldsymbol{\omega}$, unlike in the free-precession case where $\dot{\omega_{\rm z}} = 0$, we now have:

$$\dot{\omega_{\rm z}} = -\frac{2R}{3c}\epsilon_{\rm A}\omega_{\rm z}^3 \tag{0.1.18}$$

This has a physical solution (where $\omega_{\rm z} > 0$) given by

$$\omega_{\rm z}(t) = \left(\frac{4R}{3c}\epsilon_{\rm A}t + C_0\right)^{-1/2} \tag{0.1.19}$$

Solving for C using the initial condition

$$\omega_{\mathrm{z}}(t)=\left(\frac{4R}{3c}\epsilon_{\mathrm{A}}t+\frac{1}{\omega_{\mathrm{z}}(0)^2}\right)^{-1/2} \tag{0.1.20}$$

We can demonstrate that this approximation holds by comparing with exact numerical solutions as shown in figure **??**. In this plot we see that the full numerical solution both spins-down and undergoes a modulation at the precession frequency. By comparing this with the numerical solution without the anomlous torque, we can understand that the modulations are a result of the anomalous torque. Without the anomalous torque the numerical solution agrees well with the analytic solution of equation (0.1.20); it should be noted that deviations are found to exist at the 10^{-3} level.

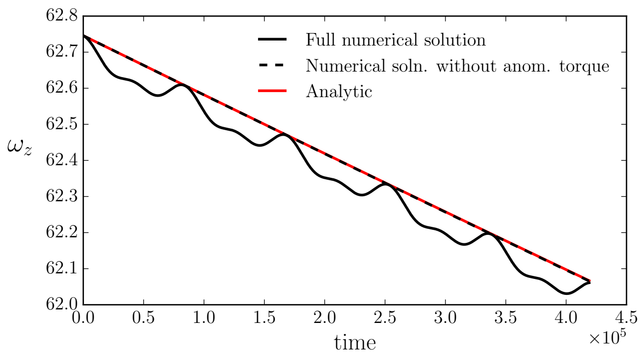

Figure 0.1.1: Comparison of three results for the z component of the spin-vector in the body frame. In solid black is shown the full numerical solution, the dashed black line indicates the numerical solution when the anomalous torque component is neglected, in red is shown the analytic result of equation (0.1.20)

Simulation parameters		
ω_0	$=$	62.8 rad/s
χ	$=$	$88.00°$
a_0	$=$	$3.00°$
τ_P	$\approx$	1.0×10^5
τ_A	$\approx$	3.0×10^5
τ_S	$\approx$	4.0×10^7

Table 0.1.1

This result holds for any spin-down strength provided the assumptions listed above are met. However, we can further simplify by working in the weak spin-down limit for which $\epsilon_{\mathrm{A}} \ll 1$. Expanding we have

$$\omega_{\mathrm{z}}(t)\approx\omega_{\mathrm{z}}(0)-\frac{2R}{3c}\epsilon_{\mathrm{A}}\omega_{\mathrm{z}}(0)^3t+\mathcal{O}(\epsilon_{\mathrm{A}}^2) \tag{0.1.21}$$

Now we refer to Landau and Lifshitz (1969) where, provided suitable initial conditions are used, the ψ euler angle is shown to satisfy:

$$\dot{\psi}=-\epsilon_{\mathrm{I}}\omega_{\mathrm{z}}. \tag{0.1.22}$$

Plugging in the expanded version of ω_{z} and solving we have

$$\psi(t)=-\epsilon_{\mathrm{I}}\omega_{\mathrm{z}}(0)t+\frac{R}{3c}\epsilon_{\mathrm{A}}\epsilon_{\mathrm{I}}\omega_z(0)^3t^2+\pi/2+\mathcal{O}(\epsilon_{\mathrm{A}}^2) \tag{0.1.23}$$

Finally from Landau and Lifshitz (1969) again we have

$$\dot{\phi}=\omega_{\mathrm{z}}-\dot{\psi}=\omega_{\mathrm{z}}(1+\epsilon_{\mathrm{I}}).+\mathcal{O}(\theta^2) \tag{0.1.24}$$

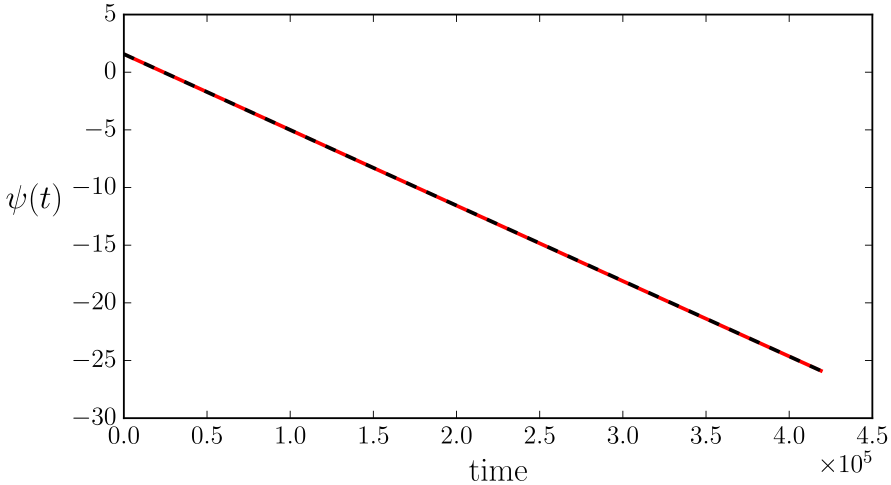

Figure 0.1.2

Substituting for ω_{z} and differentiating we have

$$\ddot{\phi} = -(1+\epsilon_{\mathrm{I}})\left(\frac{2R}{3c}\epsilon_{\mathrm{A}}\omega_{\mathrm{z}}(0)^3\right) + \mathcal{O}(\epsilon_{\mathrm{A}}^2) \tag{0.1.25}$$

Plugging terms into the spin-down

From **?**, the instantaneous EM frequency is given by

$$\dot{\Phi} = \dot{\phi} + \dot{\psi}\sin\chi\frac{\cos\theta\sin\chi - \sin\psi\sin\theta\cos\chi}{(\sin\theta\cos\chi - \cos\theta\sin\psi\sin\chi)^2 + \cos^2\psi\sin^2\chi}. \tag{0.1.26}$$

To simplify the calcualtion, we define a function as such

$$f(\theta,\chi,\psi) = \sin\chi\frac{\cos\theta\sin\chi - \sin\psi\sin\theta\cos\chi}{(\sin\theta\cos\chi - \cos\theta\sin\psi\sin\chi)^2 + \cos^2\psi\sin^2\chi}. \tag{0.1.27}$$

Then differentiating the frequency, we get the spin-down:

$$\ddot{\Phi} = \ddot{\phi} + \ddot{\psi}f(\theta,\chi,\psi) + \dot{\psi}\frac{d}{dt}f(\theta,\chi,\psi) \tag{0.1.28}$$

Since the only time dependence in f is that of ψ, we may simplify

$$\ddot{\Phi} = \ddot{\phi} + \ddot{\psi}f(\theta,\chi,\psi) + \dot{\psi}^2\frac{d}{d\psi}f(\theta,\chi,\psi). \tag{0.1.29}$$

Since we know that $\epsilon_{\mathrm{I}} \ll 1$, we can neglect terms of order $\epsilon_{\mathrm{I}}\epsilon_{\mathrm{A}}$. The coefficients then are

$$\ddot{\psi} = \frac{2R}{3c}\epsilon_{\mathrm{A}}\epsilon_{\mathrm{I}}\omega_{\mathrm{z}}(0)^3 + \mathcal{O}(\epsilon_{\mathrm{A}}^2) \tag{0.1.30}$$

$$\dot{\psi}^2 = (\epsilon_{\mathrm{I}}\omega_{\mathrm{z}}(0))^2 + \frac{4R}{3c}\epsilon_{\mathrm{A}}\epsilon_{\mathrm{I}}^2\omega_{\mathrm{z}}(0)^4 t + \mathcal{O}(\epsilon_{\mathrm{A}}^2) \tag{0.1.31}$$

$$\ddot{\phi} = -(1+\epsilon_{\mathrm{I}})\left(\frac{2R}{3c}\epsilon_{\mathrm{A}}\omega_{\mathrm{z}}(0)^3\right) + \mathcal{O}(\epsilon_{\mathrm{A}}^2) \tag{0.1.32}$$

Bibliography

Landau, L. D. and Lifshitz, E. M. (1969). *Mechanics*. Pergamon press.