0.1 Approximate analytic spin-down modulation

The aim of this work is to calculate an approximate form of the spin-down modulation capable of exhibiting the key features in the appropriate limit. To do this we will first nelect the anomalous torque and work exclusively with the spin-down torque. In component form, this is given by

$$\boldsymbol{T}_{\mathrm{s}} = \begin{pmatrix} -\frac{\omega_{\mathrm{x}}}{2}\left(\cos(2\chi)+1\right) + \frac{\omega_{\mathrm{z}}}{2}\sin(2\chi) \\ -\omega_{\mathrm{y}} \\ \frac{\omega_{\mathrm{x}}}{2}\sin(2\chi) + \frac{\omega_{\mathrm{z}}}{2}\left(\cos(2\chi)-1\right) \end{pmatrix} \tag{0.1.1}$$

Clearly solving this in general as a source in the Euler rigid body equations is difficult. In order to get an approximate solution we will define two assumptions which are acceptable for a small subset of realistic pulsars:

- We assume that the wobble-angle is small, such that $\omega = \omega_{\mathrm{z}} + \eta$ where $\eta \sim \omega_{\mathrm{x}}, \omega_{\mathrm{y}}$ and is small.
- Second, we assume that the magnetic dipole is close to the equator such that $\chi \approx \pi/2$. Expanding the trig. functions about this point we have

$$\sin(2\chi) \approx -2\left(\chi - \pi/2\right) + \mathcal{O}\left((\chi-\pi/2)^2\right) \tag{0.1.2}$$

$$\cos(2\chi) \approx -1 + \mathcal{O}\left((\chi-\pi/2)^2\right) \tag{0.1.3}$$

Under these assumptions, the spin-down torque is:

$$\boldsymbol{T}_{\mathrm{s}} = \begin{pmatrix} 0 \\ 0 \\ -\omega_{\mathrm{z}} \end{pmatrix} + \mathcal{O}(\eta) + \mathcal{O}(\chi - \pi/2) \tag{0.1.4}$$

Considering the three coupled ODEs for the components of $\boldsymbol{\omega}$, unlike in the free-precession case where $\dot{\omega_{\mathrm{z}}} = 0$, we now have:

$$\dot{\omega_{\mathrm{z}}} = -\frac{2R}{3c}\epsilon_{\mathrm{A}}\omega_{\mathrm{z}}^3 \tag{0.1.5}$$

This has a physical solution (where $\omega_{\mathrm{z}} > 0$) given by

$$\omega_{\mathrm{z}}(t) = \left(\frac{4R}{3c}\epsilon_{\mathrm{A}}t + C_0\right)^{-1/2} \tag{0.1.6}$$

Solving for C using the initial condition

$$\omega_{\mathrm{z}}(t) = \left(\frac{4R}{3c}\epsilon_{\mathrm{A}}t + \frac{1}{\omega_{\mathrm{z}}(0)^2}\right)^{-1/2} \tag{0.1.7}$$

We can demonstrate that this approximation holds by comparing with exact numerical solutions as shown in figure **??**. In this plot we see that the full numerical solution both spins-down and undergoes a modulation at the precession frequency. By comparing this with the numerical solution without the anomlous torque, we can understand that the modulations are a result of the anomalous

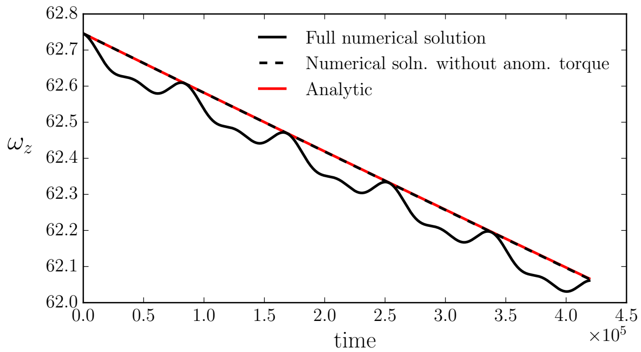

Figure 0.1.1: Comparison of three results for the z component of the spin-vector in the body frame. In solid black is shown the full numerical solution, the dashed black line indicates the numerical solution when the anomalous torque component is neglected, in red is shown the analytic result of equation (0.1.7)

Simulation parameters		
ω_0	$=$	62.8 rad/s
χ	$=$	$88.00°$
a_0	$=$	$3.00°$
τ_P	$\approx$	1.0×10^5
τ_A	$\approx$	3.0×10^5
τ_S	$\approx$	4.0×10^7

Table 0.1.1

torque. Without the anomalous torque the numerical solution agrees well with the analytic solution of equation (0.1.7); it should be noted that deviations are found to exist at the 10^{-3} level.

This result holds for any spin-down strength provided the assumptions listed above are met. However, we can further simplify by working in the weak spin-down limit for which $\epsilon_{\mathrm{A}} \ll 1$. Expanding we have

$$\omega_{\mathrm{z}}(t) \approx \omega_{\mathrm{z}}(0) + \frac{2R}{3c}\epsilon_{\mathrm{A}}\omega_{\mathrm{z}}(0)^3 t + \mathcal{O}(\epsilon_{\mathrm{A}}^2) \tag{0.1.8}$$

Now we refer to Landau and Lifshitz (1969) where, provided suitable initial conditions are used, the ψ euler angle is shown to satisfy:

$$\dot{\psi} = -\epsilon_{\mathrm{I}}\omega_{\mathrm{z}}. \tag{0.1.9}$$

Plugging in the expanded version of ω_{z} and solving we have

$$\psi(t) = -\epsilon_{\mathrm{I}}\omega_{\mathrm{z}}(0)t + \frac{R}{3c}\epsilon_{\mathrm{A}}\epsilon_{\mathrm{I}}\omega_z(0)^3 t^2 + \mathcal{O}(\epsilon_{\mathrm{A}}^2) \tag{0.1.10}$$

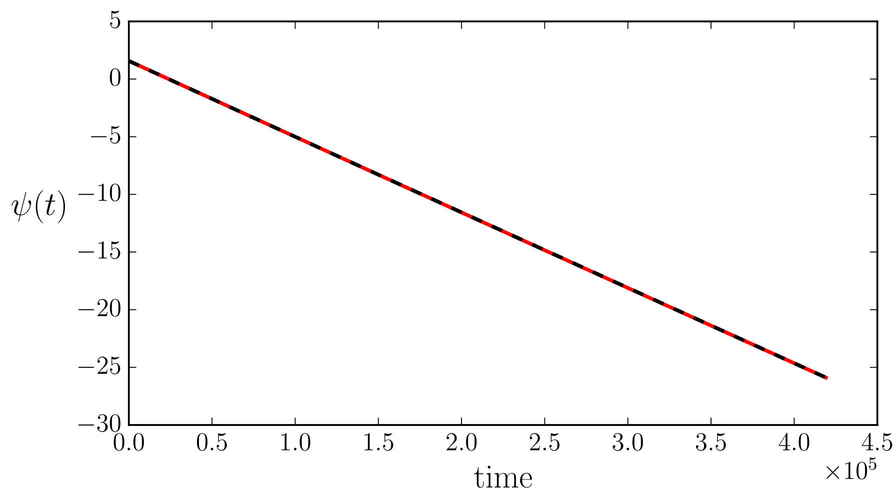

Figure 0.1.2

Bibliography

Landau, L. D. and Lifshitz, E. M. (1969). *Mechanics*. Pergamon press.