0.1 Physical observables: spin-down rate $\dot{\nu}$

An observable which is often considered in characterising periodic patterns in pulsar signals is the spin-down rate $\dot{\nu}$. In our numerical model, the spin-down rate is the same as the second derivative of the magnetic dipoles azimuthal angle (i.e. $\ddot{\Phi}$).

As we will now show, the spin-down rate is modulated by free precession, but this effect can also be amplified by the electromagnetic torque as first shown by Jones (2004). In order to understand all of the subtle physics which may be present in the spin-down rate, we will now build our intuition by starting with the simplest models and gradually introducing new features.

Calculation of the spin-down rate can be done in two ways: either analytically building on results from Jones (2004), or we can numerically solve the system of equations and measure the spin-down rate (discussed shortly). Using the latter method we include by default all of the interplay between precession and the EM torque. We then compare this 'exact' solution with analytic results which allows us to understand which parts of the system are responsible for the observed phenomena.

In order to measure the spin-down rate from numerical solutions we could numerically differentiate eqn. (**??**) (the angular phase of the magnetic dipole). However, since we are comparing our results with measured values from pulsar astronomers, an alternative is use the method proposed by Lyne et al. (2010): a second order Taylor expansions is fitted to short sections of data of length T and the resulting coefficient $\dot{\nu}$ is recorded the mid-point in time of the section of data. Repeating this process every $\sim T/4$ in a 'sliding-window', we build a picture of how the spindown varies with time. We choose T such that it is a fraction of the precession period over which we expect quantities to be modulated. This is consistent with the observers method where T is chosen in order to resolve the observed modulations, but average out the short-time scale changes.

We now proceed to compare this numerical spin-down rate calculation with analytic predictions gradually introducing new features when required. Since the geometric variations can be quite complex, we consider only $\theta < \chi$ and start with $\theta \approx 3°$ and $\chi = 60°$. This produces simple simple oscialltory behaviour. The solution become more complex if either the inequality is not satisfied or $\chi \approx \pi/2$ so we initially ignore this region and return to it later on.

0.1.1 Spin-down variations due to free-precession

Without an EM torque, the average spin-down rate should be zero. However, if the body undergoes free precession, this will produce periodic variations. These can be calculated exactly by differentiating eqn. (**??**), because the torque is zero we have $\dot{\theta} = 0$ leaving

$$\ddot{\Phi}(t)_{\mathrm{p}} = \ddot{\phi} + \frac{d}{dt}\left(\sin\chi\dot{\psi}\frac{\cos\theta\sin\chi - \sin\psi\sin\theta\cos\chi}{(\sin\theta\cos\chi - \cos\theta\sin\psi\sin\chi)^2 + \cos^2\psi\sin^2\chi}\right) \tag{0.1.1}$$

The resulting expression is unweildy and so we do not report it here. For a freely precessing biaxial body we demonstrated in section **??** that the euler angles are linear functions of time. As such the first term in this expression vanishes and we are left with a complicated trigonometric function of $\psi(t)$ which is given by eqn. (**??**).

It is also useful to calculate the magnitude of variations: this is done by differentiating equation (**??**) twice which gives:

$$|\Delta\dot{\nu}|_{\mathrm{p}} = \frac{\dot{\psi}^2\theta\cot\chi}{2\pi}. \tag{0.1.2}$$

In figure 0.1.1 we plot three curves: the numerical result obtained using Lyne method of calculating the spin-down, the analytic prediction from eqn. (0.1.1) and the magnitude obtained by eqn. (0.1.2)

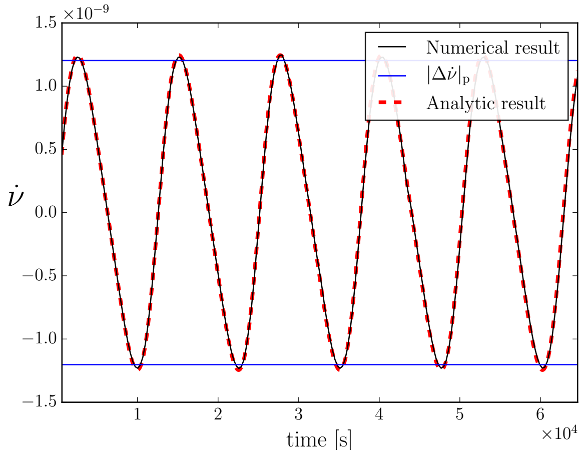

Figure 0.1.1: Modulations in the spin-down rate due to free precesion. The solid black line is the numerical solution. in blue we show the analytic preduction of the magnitude of modulations due to free-precession as given by equation (0.1.2) and the red-dashed line indicates the analytic prediction of eqn /(0.1.1)

Simulation parameters		
ω_0	=	100.0 rad/s
B_0	=	0.0 G
χ	=	60.00°
a_0	=	3.00°
$\tilde{\theta}$	=	3.00°
$\mathcal{A}_{\rm EM}$	=	0.0

Table 0.1.1

The variations seen in figure 0.1.1 are the result of free precession. As there is no torque, the overall spindown is zero. However, due to precession the magnetic dipole performs a slow rotation about the deformation axis. During half the cycle it counter-rotates and in the other half it corotates with the rapid rotation about the angular momentum vector. The result is a symmetric modulation of the spindown about zero with magnitude given by equation (0.1.2).

0.1.2 Spin-down due to EM torque

Averaged spin-down rate

When we include the EM torque we will have a non-zero average spin-down rate. We can measure this directly from equation (**??**): rearranging for $\dot{\Omega}$ we have

$$\dot{\Omega} = -\frac{B_0^2 R^6 \sin^2\alpha \Omega^3}{6 I_0 c^3}, \tag{0.1.3}$$

writen in terms of the model parameters this is

$$\dot{\nu} = -\frac{1}{3\pi}\frac{R\Omega^3}{c}\sin^2\alpha\epsilon_{\rm A} \tag{0.1.4}$$

In general the spin-down is approximately constant over a given observation time. As a result we estimate the spin-down rate by the initial value:

$$\dot{\nu}_0 \approx -\frac{1}{3\pi}\frac{R\Omega_0^3}{c}\sin^2\alpha\epsilon_{\rm A} \tag{0.1.5}$$

Here α is the angle between the spin-vector and the magnetic dipole. For small wobble angles, the angle $\hat{\theta}$ (see figure **??**) is vanishingly small and so we can approximate $\alpha \approx \chi$ such that the spin-down is

$$\dot{\nu}_0 = -\frac{1}{3\pi}\frac{R\Omega_0^3}{c}\sin^2\chi\epsilon_{\rm A} \tag{0.1.6}$$

This expression will give us the average spin-down under an EM torque; if the star precesses, then the actual spin-down will be modulated about this values due to precession. Before demonstrating how to calculate this modulation we first show that the magnitude of modulation will depend on the EM torque.

Amplification of the spin-down modulation

The magnitude of the spin-down modulation due to precession will depend on the EM amplification factor which we discussed in equation (**??**). For the timing residuals Jones and Andersson (2001) demonstrated that the residuals could be amplified by the EM torque, we now show that a similar effect occurs for the spin-down modulation. Starting with a vacuum point-dipole spin-down torque

$$\ddot{\Phi} = k\dot{\Phi}^3\sin^2\alpha, \tag{0.1.7}$$

then Jones and Andersson (2001) demonstrated that the magnitude of modulations of the spin-down rate is

$$|\Delta\ddot{\Phi}|^{58} \approx -2k\Omega^3\theta\sin\chi\cos\chi \tag{0.1.8}$$

here the superscript refers to the relevant equation in Jones and Andersson (2001). Taking this expression, we now show that it can be rewritten

$$\begin{aligned}
|\Delta\ddot{\Phi}|^{58} &\approx 2k\Omega^3\theta\sin\chi\cos\chi && && (0.1.9)\\
&\approx 2\frac{\ddot{\Phi}}{\sin^2\alpha}\theta\sin\chi\cos\chi && \text{if } \alpha\approx\chi && (0.1.10)\\
&\approx 2\ddot{\Phi}\theta\cot\chi && \tau_{\rm S} = \left|\dot{\Phi}/\ddot{\Phi}\right| && (0.1.11)\\
&\approx 2\frac{\dot{\Phi}}{\tau_{\rm S}}\theta\cot\chi && P/\tau_{\rm P} = \dot{\psi}/\dot{\Phi} && (0.1.12)\\
&\approx 2\dot{\psi}^2\left(\frac{\tau_{\rm P}}{P}\right)\frac{1}{\dot{\psi}}\frac{1}{\tau_{\rm S}}\theta\cot\chi && \dot{\psi} = \frac{2\pi}{\tau_{\rm P}} && (0.1.13)\\
&\approx \frac{1}{\pi}\dot{\psi}^2\left(\frac{\tau_{\rm P}}{P}\right)\left(\frac{\tau_{\rm P}}{\tau_{\rm S}}\right)\theta\cot\chi && && (0.1.14)
\end{aligned}$$

Here we have shown that, just like for the timing residuals, the EM torque can amplify the spin-down modulation by a factor $\mathcal{A}_{\rm EM}$ as defined in equation (**??**).

Geometric dominated spin-down modulation

When the $\mathcal{A}_{\rm EM} \ll 1$, the geometric variations in the spin-down dominate. We can predict the spin-down then by allowing $\ddot{\phi}$ in equation (0.1.1) to be non-zero and given exactly by $\ddot{\phi} = 2\pi\dot{\nu}_0$. Then under an EM torque and in the low $\mathcal{A}_{\rm EM}$ the analytic prediction for the spin-down modulation is given by

$$\ddot{\Phi}(t) = 2\pi\dot{\nu}_0 + \frac{d}{dt}\left(\sin\chi\dot{\psi}\frac{\cos\theta\sin\chi - \sin\psi\sin\theta\cos\chi}{(\sin\theta\cos\chi - \cos\theta\sin\psi\sin\chi)^2 + \cos^2\psi\sin^2\chi}\right) \tag{0.1.15}$$

In figure 0.1.2 we plot a typical spin-down calculation with $\mathcal{A}_{\rm EM} \approx 0.4$. Here we see the non-zero average spin-down (given by eqn. (0.1.6)) due to EM torque modulated by the geometric variations of free precession. The black (numerical results) and dashed red lines (analytic prediction) show good agreement.

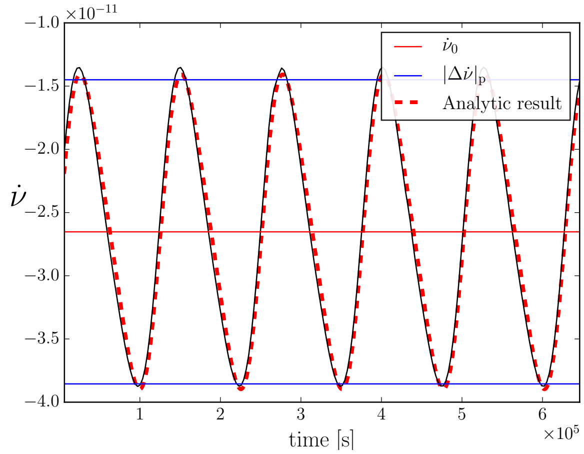

Figure 0.1.2: Modulation in the spin-down rate due to precession. The solid black line indicates the numerical solution including a torque; the solid red line indicates the approximate average spin-down rate due to the EM toque as calculated from equation (0.1.6); the blue lines indicate the magnitude of modulation about the average spin-down due to precession; finally the red-dashed line is the prediction of equation (0.1.15)

Simulation parameters		
ω_0	$=$	10.0 rad/s
B_0	$=$	1.897×10^{14} G
χ	$=$	$60.00°$
a_0	$=$	$3.00°$
$\tilde{\theta}$	$=$	$3.05°$
$\mathcal{A}_{\rm EM}$	$=$	0.42

Table 0.1.2

"F",

EM torque amplification of the spin-down modulation

When $\mathcal{A}_{\rm EM}$ is greater than unity the spin-down modulations are amplified by the torque. We can generate an analytic prediction for this by by inserting eqn. (??) and (??) into the vacuum point-dipole spin-down torque:

$$\ddot{\Phi} = k\dot{\Phi}\sin^2\Theta \tag{0.1.16}$$

$$= -\frac{2R}{3c}\epsilon_{\rm A}\left(\dot{\phi} + \frac{\dot{\psi}\sin(\chi)(\cos\theta\sin\chi - \sin\psi\sin\theta\cos\chi)}{(\sin\theta\cos\chi - \cos\theta\sin\psi\sin\chi)^2 + \cos^2\psi\sin^2\chi}\right)^3 \sin^2\left(\cos^{-1}\left(\sin\theta\sin\psi\sin\chi + \cos\theta\cos\chi\right)\right) \tag{0.1.17}$$

$$= -\frac{2R}{3c}\epsilon_{\rm A}\left(\dot{\phi} + \frac{\dot{\psi}\sin(\chi)(\cos\theta\sin\chi - \sin\psi\sin\theta\cos\chi)}{(\sin\theta\cos\chi - \cos\theta\sin\psi\sin\chi)^2 + \cos^2\psi\sin^2\chi}\right)^3 \left(\sin\theta\sin\psi\sin\chi + \cos\theta\cos\chi\right) \tag{0.1.18}$$

where we are implicitly assuming $\theta = \text{const.}$, this is not true when we have any EM torque. However, provided the variations are small we can to assume it as a first aproximation. Finally we assume that the Euler angles are approximately the same as in the torque free case, that is:

$$\phi(t) = \omega_0 t + \phi_0 \qquad\qquad \psi(t) = -\epsilon_{\rm I}\omega_0 t + \pi/2 \tag{0.1.19}$$

$$\ddot{\Phi} = -\frac{2R}{3c}\omega_0^3\epsilon_{\rm A}\left(1 - \epsilon_{\rm I}\frac{\sin(\chi)(\cos\theta\sin\chi - \sin\psi\sin\theta\cos\chi)}{(\sin\theta\cos\chi - \cos\theta\sin\psi\sin\chi)^2 + \cos^2\psi\sin^2\chi}\right)^3(\sin\theta\sin\psi\sin\chi + \cos\theta\cos\chi) \tag{0.1.20}$$

In figure 0.1.3 we plot the results of a simulation for which this amplifcation factor is greater than unity. This shows that the spindown variations are given by 0.1.14 rather than (0.1.2).

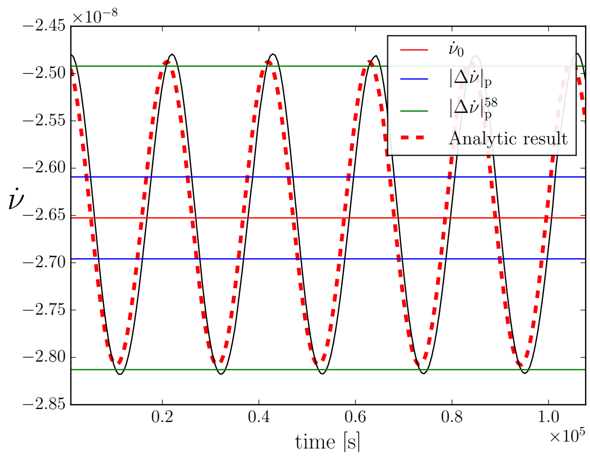

Figure 0.1.3: EM amplification of the modulation in the spin-down rate due to precession. The solid black line indicates the numerical solution including a torque; the solid red line indicates the approximate average spin-down rate due to the EM toque as calculated from equation (0.1.6); the blue region indicates the modulation about the average spin-down due to precession from equation (0.1.2) while the green region indicates the modulation about as calculated from equation (0.1.14).

Simulation parameters		
ω_0	=	100.0 rad/s
B_0	=	1.897×10^{14} G
χ	=	60.00°
a_0	=	3.00°
$\tilde{\theta}$	=	3.09°
$\mathcal{A}_{\rm EM}$	=	12.0

Table 0.1.3

0.1.3 Double peaked spin-down values

When the magnetic dipole can range over the equatorial line (in the inertial frame), the spin-down variations can become doubly peaked. This happens when $\chi + \theta \sim \pi/2$. The magnitude of variations is still tied to the size of the EM amplification as such we treat the two cases separately again.

Geometric dominated spin-down modulation

With $\mathcal{A}_{\mathrm{EM}} \ll 1$, we set $\chi = 88.6°$ and plot the spin-down modulations in figure 0.1.4. This shows the exact numerical result along with the free precession prediction of eqn. (0.1.15) and the EM torque prediction of (0.1.20). The EM torque prediction shows no visible modulation which we expect since the geometric free precession features dominate in this regime. The free precession result agrees with good precision with the exact numerical result.

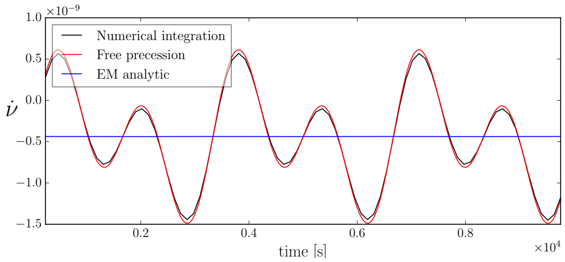

Figure 0.1.4

Simulation parameters		
ω_0	=	62.8 rad/s
B_0	=	4.243×10^{13} G
χ	=	88.60°
a_0	=	2.00°
$\tilde{\theta}$	=	2.00°
$\mathcal{A}_{\mathrm{EM}}$	=	0.49×10^{-2}

Table 0.1.4

Amplification of the spin-down modulation

We now consider the case when $\mathcal{A}_{\mathrm{EM}} \gg 1$. In figure 0.1.5 we show the exact numerical result along with the free precession prediction of eqn. (0.1.15) and the EM torque prediction of (0.1.20). This illustrates that the EM torque has amplified the size of modulations (that is the free precession result is inconsistent with the numerical result). The EM analytic result gets the correct magnitude but there is clearly differences in the structure, this is due to the assumptions made on the Euler angles.

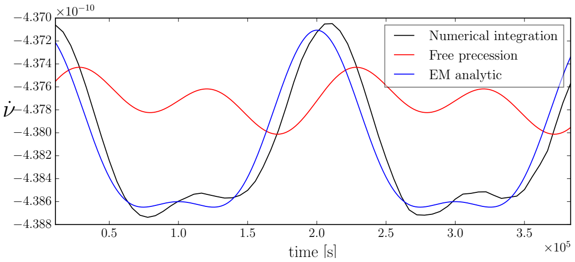

Figure 0.1.5

Simulation parameters		
ω_0	=	62.8 rad/s
B_0	=	4.243×10^{13} G
χ	=	88.60°
a_0	=	2.00°
$\tilde{\theta}$	=	2.00°
$\mathcal{A}_{\mathrm{EM}}$	=	18.0

Table 0.1.5

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