As Landau and Lifshitz (1969) described, the motion of $\mathbf{M}$ in the rotating frame can be understood by the use of conservation laws. For the time being let us consider, in the rotating frame, a torque free body with moments of inertia I_1, I_2, I_3. If the spin-vector components are given by $M_i = I_i\Omega_i$ then in momentum space, we can write the conservation of energy and angular momentum as

$$\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} = 2E \tag{0.0.1}$$

$$M_1^2 + M_2^2 + M_3^2 = M^2 \tag{0.0.2}$$

The conservation of energy describes an ellipsoid with semi-axis $\sqrt{2EI_1}, \sqrt{2EI_2}$ and $\sqrt{2EI_3}$. The conservation of angular momentum describes a sphere of radius M. The intersection of the sphere and ellipse at fixed E and M describe the precession of the angular momentum and hence the spin-vector.

0.1 Torque free biaxial body

For a biaxial body free from torques we can parameterise the principle components of the moment of inertia by

$$I_1 = I_2 = I_0, \qquad I_3 = I_0(1+\epsilon_{\mathrm{I}}) \tag{0.1.1}$$

For such a system, the Euler rigid body equations have an exact solution with $\Omega_3 = \text{const.}$ and the other components are given by

$$\Omega_1 = A\cos\left(\epsilon_{\mathrm{I}}\Omega_3 t\right), \tag{0.1.2}$$

$$\Omega_2 = A\sin\left(\epsilon_{\mathrm{I}}\Omega_3 t\right), \tag{0.1.3}$$

where the constant can be written as

$$A = \left(\Omega^2 - \Omega_3^2\right)^{1/2} \tag{0.1.4}$$

$$= \left(\frac{M^2 - M_3^2}{I_0^2}\right)^{1/2}. \tag{0.1.5}$$

We note that Ω^2 is also a constant, since there is no torque. In the rotating body frame, this solution demonstrates that the spin-vector will precess in a circle about the symmetry axis. The circle is precisely the intersection of the ellipsoid and sphere. Since the cone is aligned with the 3 axis, we can define a polar angle θ made by the momentum-vector with the 3 axis. This will be constant during a precessional period and can be calculated as follows:

$$\sin\theta = \frac{M_3}{M} \tag{0.1.6}$$

In the work of Jones and Andersson (2001), the angle θ was referred to as the *wobble angle* meaning the angle of precession.

To calculate the wobble angle in terms of the energy and momentum, from equation (0.0.1) we can rearrange to get

$$M_3^2 = (1+\epsilon_{\mathrm{I}})\left(2EI_0 - M_1^2 - M_2^2\right) \tag{0.1.7}$$

$$= (1+\epsilon_{\mathrm{I}})\left(2EI_0 - \left(M^2 - M_3^2\right)\right) \tag{0.1.8}$$

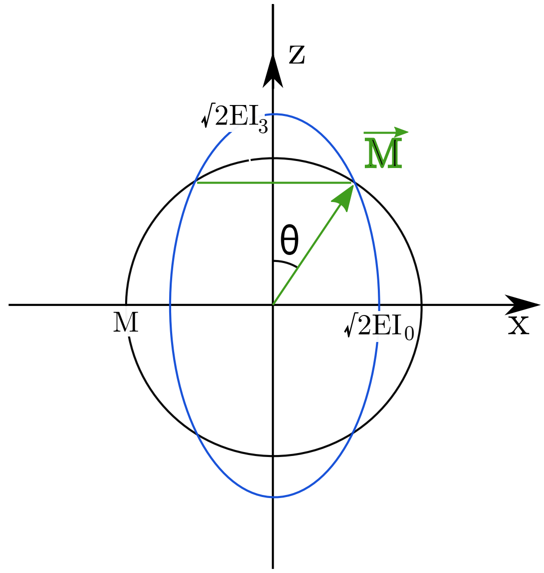

Figure 0.1.1: A slice through the $z - x$ in momentum space showing the sphere described by the conservation of angular momentum and ellisoid due to the conservation of energy. Since both of these conservation laws must be satisfied, this restricts the motion of $\mathbf{M}$ to intersection. For this biaxial body the intersection is a circle about the $\hat{\boldsymbol{z}}$ axis, as a result the momentum vector traces out a cone of half angle θ about this axis.

Expanding and solving for M_3^2/M^2

$$\frac{M_3^2}{M^2} = \frac{1+\epsilon_{\mathrm{I}}}{\epsilon_{\mathrm{I}}}\left(1 - \frac{2EI_0}{M^2}\right) \tag{0.1.9}$$

Then we can write the polar angle as

$$\sin\theta = \left(\frac{1+\epsilon_{\mathrm{I}}}{\epsilon_{\mathrm{I}}}\left(1 - \frac{2EI_0}{M^2}\right)\right)^{1/2} \tag{0.1.10}$$

The term in brackets contains the ratio of the square of smallest semi-axis of the ellisoid and the radius of the sphere. That is

$$\frac{2EI_0}{M^2} = \left(\frac{\sqrt{2EI_0}}{M}\right)^2 \tag{0.1.11}$$

This provides an intuitive way to think about the wobble angle: as the smallest semi-axis of the ellipse approaches the radius of the sphere $\sqrt{2EI_0} \to M$ then θ tends to zero. That is, the circles of intersection between the sphere and ellipse close up around the z axis.

0.2 Torque free triaxial body

Let us now consider the case of a triaxial body with moments of inertia in the body frame satisifying

$$I_1 < I_2 < I_3 \tag{0.2.1}$$

As described by the Landau and Lifshitz (1969) the motion of $\mathbf{M}$ is described by the intersection of the sphere and now triaxial ellipsoid with semiaxis $\sqrt{2EI_1}, \sqrt{2EI_2}$ and $\sqrt{2EI_3}$. These intersection are complicated but a sketch is provided in figure 51 of Landau and Lifshitz (1969).

Solutions to the components of Ω_i in the rotating frame are given by Landau and Lifshitz (1969) in terms of Jacobian elliptic functions. For definiteness the suppose that $M^2 > 2EI_2$ such that the momentum precesses about the $\hat{\boldsymbol{z}}$ axis. If this inequality is reversed then the momentum will precess about the $\hat{\boldsymbol{x}}$ axis. As such as should therefore define the angle θ by replacing 3 with 1 in equation (0.1.6); if this were the case we would swap the 1 and 3 suffixes in the following formulae.

Continuing with precession about the $\hat{\boldsymbol{z}}$ axis, we have that

$$\Omega_3 = \sqrt{\frac{M^2 - 2EI_1}{I_3\,(I_3 - I_1)}}\mathrm{dn}(\tau, k^2). \tag{0.2.2}$$

where $\mathrm{dn}(\tau, k^2)$ is an elliptic function, with the dimensionless time variable τ given by

$$\tau = t\sqrt{(\frac{I_3 - I_2)(M^2 - 2EI_1)}{I_1 I_2 I_3}} \tag{0.2.3}$$

and the elliptic parameter k^2 is bounded by $[0, 1]$ and given by

$$k^2 = \frac{(I_2 - I_1)(2EI_3 - M^2)}{(I_3 - I_2)(M^2 - 2EI_1)} \tag{0.2.4}$$

The polar angle in the triaxial case can then be written

$$\sin\theta = \frac{M_3}{M} \tag{0.2.5}$$

$$= \frac{I_3\Omega_3}{M} \tag{0.2.6}$$

$$= \sqrt{\frac{I_3(1 - 2EI_1/M^2)}{(I_3 - I_1)}}\mathrm{dn}(\tau, k^2) \tag{0.2.7}$$

The coefficients of the elliptic integral is constructed from constants, so only the elliptic function $\mathrm{dn}(\tau, k^2)$ contributes to the time dependance of the polar angle.

The variable k measures the 'triaxiality' of the body: in the limit $k \to 0$ when the body becomes biaxial, $dn(\tau, k) \to 1$. That is, in the biaxial limit Ω_3 tends to a constant such that $\sin\theta$ tends to the constant given in equation(0.1.10).

Aside on the elliptic function

We now briefly review the features of the elliptic function $\mathrm{dn}(\tau, k)$. Firstly we note that there an alternaive convention used by Abramowitz and Stegun (1972) in which $m := k^2$; the elliptic function is then written $\mathrm{dn}(\tau|m)$.

$\mathrm{dn}(\tau, k)$ is a periodic function: in τ the period is $4K$ where K is a complete elliptic integral of the first kind:

$$K(k) = \int_0^{\pi/2} \frac{d\tau}{\sqrt{1 - k^2\sin^2\tau}} \tag{0.2.8}$$

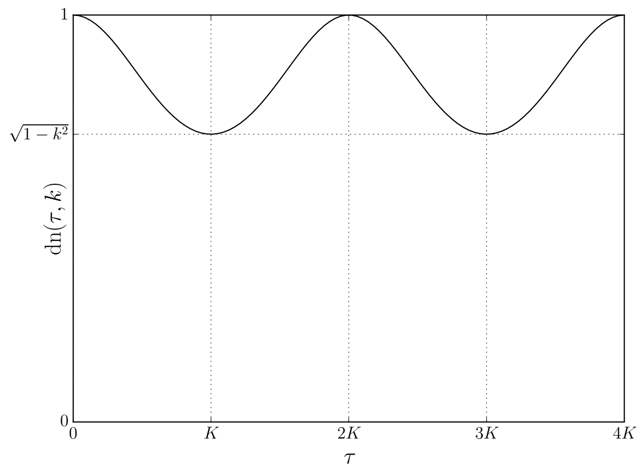

Figure 0.2.1: Illustration of the elliptic function $\mathrm{dn}(\tau, k)$.

In figure 0.2.1 we illustrate $\mathrm{dn}(\tau, k)$ marking some critical values over a single period of $4K$. We can write the periodicity in t by referring to equation (0.2.3) such that the period is:

$$T = 4K\sqrt{\frac{I_1I_2I_3}{(I_3 - I_2)(M^2 - 2EI_1)}} \tag{0.2.9}$$

This period refers to the motion of Ω is in the body frame: after one period the spin-vector returns to its original position. However, the axisymmetric body itself will not return to its original position in the inertial frame as described by Landau and Lifshitz (1969).

Return to the wobble angle

In the body frame then, the polar angle is a periodic function of the precessional phase. It therefore does make sense to define it as a wobble angle since it is not constant. Nevertheless, we would like to define an averaged wobble angle particuarly in the case where the variations from this average are small.

As previously mentiond, the magnitude of the variations is governed by the size of the parameter k. When k is small the variations are small, this limit occurs when

$$|I_2 - I_1| \ll |I_3 - I_2|, \tag{0.2.10}$$

such the the body is close to biaxial. From Abramowitz and Stegun (1972) when $k^2 \ll 1$ we may use the approximation

$$\mathrm{dn}(\tau, k) \approx 1 - \frac{1}{2}k^2 \sin^2 \tau + \mathcal{O}(k^4) \tag{0.2.11}$$

Time averaging over a single period in τ we have

$$\langle \mathrm{dn}(\tau, k)\rangle\Big|_{k\ll 1} \approx \frac{1}{4K}\int_0^{4K} 1 - \frac{1}{2}k^2 \sin^2 \tau d\tau = \frac{1}{4K}\left[\tau - \frac{k^2}{4}(\tau - \sin\tau\cos\tau)\right]_0^{4K} \tag{0.2.12}$$

$$= \left(1 - \frac{k^2}{4}\right) \tag{0.2.13}$$

Therefore we can define the averaged polar angle as

$$\langle \sin\theta \rangle \approx \frac{1}{4K}\int_0^{4K} \sin\theta d\tau \tag{0.2.14}$$

$$= \sqrt{\frac{I_3(1-2EI_1/M^2)}{(I_3-I_1)}}\left(1-\frac{k^2}{4}\right) \tag{0.2.15}$$

We can now simplify by setting the right hand side all under a sqaure root. Then, since we are already neglecting term of $\mathcal{O}(k^4)$, we can expand to give

$$\langle \sin\theta \rangle \approx \left(\frac{I_3}{I_3-I_1}\left(\left(1-\frac{2EI_1}{M^2}\right)-\frac{1}{2}\left(\frac{2EI_3}{M^2}-1\right)\frac{I_2-I_1}{I_3-I_2}\right)\right)^{1/2} \tag{0.2.16}$$

From equation (0.2.10) it is clear the second term is much smaller than the first. Hence in the limit of small triaxiality, the average of $\sin\theta$ tends too the fixed wobble angle in the biaxial body. Therefore provided equation (0.2.10) holds we may still approximate the wobble angle by (0.1.10).

Bibliography

Abramowitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions*.

Jones, D. I. and Andersson, N. (2001). Freely precessing neutron stars: model and observations. *Monthly Notices of the Royal Astronomical Society*, 324(4):811–824.

Landau, L. D. and Lifshitz, E. M. (1969). *Mechanics*. Pergamon press.