Calculation_of_expected_mismatch

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In [1]:
```python
from sympy import init_printing
init_printing()

from sympy import *
import sympy as sp

i, j , k, m, l, N = symbols('i j k m l N', integer=True)
sigmaP0, sigmaF0, sigmaF1 = symbols('\sigma_{\phi} \sigma_{f} \sigma_{\dot{f}}')

T, dT = symbols('T \Delta{T}')
AP0, AF0, AF1 = symbols('A_{\phi} A_{f} A_{\dot{f}}')
```

1 Calculation of E[m]

The aim is to calculate the expected mismatch from a random walk in parameter space in Phase, Frequency and Spindown over N steps. The terms that we require to calculate are those given by taking the expectation of

$$\begin{aligned} m = &\sum_{i=1}^{N}\left(g^{E}_{00}\Delta\phi_i^2 + g^{E}_{11}\Delta f_i^2 + g^{E}_{22}\Delta\dot{f}_i^2 + 2\left(g^{E}_{01}\Delta\phi_i\Delta f_i + g^{E}_{02}\Delta\phi_i\Delta\dot{f}_i + g^{E}_{12}\Delta f_i\Delta\dot{f}_i\right)\right) \\ &+ 2\sum_{i=1}^{N}\left(\sum_{j=1}^{i-1}\left(g^{NE}_{00}\Delta\phi_i\Delta\phi_j + g^{NE}_{11}\Delta f_i f_j + g^{NE}_{22}\Delta\dot{f}_i\Delta\dot{f}_j + g^{NE}_{01}\Delta\phi_i\Delta f_j + g^{NE}_{10}\Delta f_i\Delta\phi_j + g^{NE}_{02}\Delta\phi_i\Delta\dot{f}_j + g^{NE}_{20}\Delta\phi_j\Delta\dot{f}_i + g^{N}_{12}\right.\right. \end{aligned} \tag{1}$$

Note that $j < i$

1.1 Definitions

In order to simplify the calculation we define the dimensionless parameters

$$A_\phi = \sigma_\phi^2 \quad A_f = \sigma_f^2\Delta T^2 \quad A_{\dot{f}} = \sigma_{\dot{f}}^2\Delta T^4,$$

and define some summation properties

$$\left(\sum_{b=1}^{a}X_b\right)^2 = \sum_{b=1}^{a}\left(X_b^2 + \sum_{\substack{c\neq b\\c=1}}^{a}X_bX_c\right) = \sum_{b=1}^{a}\left(X_b^2 + 2\sum_{c=1}^{b-1}X_bX_c\right) \tag{2}$$

$$\sum_{b=1}^{a}\sum_{c=1}^{b}X_c = \sum_{b=1}^{a}(a+1-b)X_b \tag{3}$$

$$\sum_{b=1}^{a}\sum_{c=1}^{b-1} X_c = \sum_{b=1}^{a-1}(a-b)X_b \tag{4}$$

1.2 Defining the departures due to Random walks

We will assume that the spindown, frequency and phase take a random walk. Each step proceeds from the previous value and the step is drawn from a normal distribution, we define

$$\hat{\Delta}\dot{f}_i = \Delta\dot{f}_i - \Delta\dot{f}_{i-1} \sim N(0, \sigma_{\dot{f}}^2)$$

Rearranging this gives an expression for the offset in the i^{th} segment. For the spindown we can look at two terms

$$\Delta\dot{f}_i = \hat{\Delta}\dot{f}_i + \Delta\dot{f}_{i-1}$$
$$\Delta\dot{f}_{i-1} = \hat{\Delta}\dot{f}_{i-1} + \Delta\dot{f}_{i-2}$$

to show that

$$\Delta\dot{f}_i = \sum_{j=1}^{i}\hat{\Delta}\dot{f}_j$$

The same is not true for the frequency and phase. This can be understood by considering only a random walk in the spindown. In such a case the offset in spindown will naturally lead an increase or decrease of the frequency and also the phase. It is possible that the frequency and phase could shift by a corresponding amount such that the net departure in frequency and phase is zero but this is a particularly special case that won't be considered here.

To account for the frequency departure due to a random walk in the spindown we integrate

$$\Delta f(t) = \int_0^t \Delta\dot{f}(t')dt' \tag{5}$$

Because the random walk is discreet and constant in any given segment the integral can be replaced with a summation.

$$\Delta f_i = \sum_{j=1}^{i}\hat{\Delta}f_j + \sum_{j=1}^{i-1}\Delta\dot{f}_j\Delta T \tag{6}$$

$$= \sum_{j=1}^{i}\hat{\Delta}f_j + \sum_{j=1}^{i-1}\sum_{k=1}^{j}\hat{\Delta}\dot{f}_k\Delta T \tag{7}$$

$$= \sum_{j=1}^{i}\hat{\Delta}f_j + \sum_{j=1}^{i-1}(i-j)\hat{\Delta}\dot{f}_j\Delta T \tag{8}$$

Note that we are setting the reference time to the begining of each segment. If this were not the case the replacement of the integral would need an additional term.

In a similiar way the phase is given by the integral of the frequency deviation

$$
\begin{aligned}
\Delta\phi_i &= \sum_{j=1}^{i} \hat{\Delta}\phi_j + 2\pi \left(\sum_{j=1}^{i-1} \Delta f_j \Delta T + \frac{1}{2} \sum_{j=1}^{i-1} \Delta \dot{f}_j \Delta T^2 \right) \\
&= \sum_{j=1}^{i} \hat{\Delta}\phi_j + 2\pi \left(\sum_{j=1}^{i-1} \left(\sum_{k=1}^{j} \hat{\Delta} f_k + \sum_{k=1}^{j-1} (j-k) \hat{\Delta} \dot{f}_k \Delta T \right) \Delta T + \frac{1}{2} \sum_{j=1}^{i-1} \sum_{k=1}^{j} \Delta \dot{f}_k \Delta T^2 \right) \\
&= \sum_{j=1}^{i} \hat{\Delta}\phi_j + 2\pi \left(\sum_{j=1}^{i-1} (i-j) \hat{\Delta} f_j \Delta T + \sum_{j=1}^{i-1} \sum_{k=1}^{j-1} (j-k) \hat{\Delta} \dot{f}_k \Delta T^2 + \frac{1}{2} \sum_{j=1}^{i-1} (i-j) \Delta \dot{f}_j \Delta T^2 \right) \\
&= \sum_{j=1}^{i} \hat{\Delta}\phi_j + 2\pi \left(\sum_{j=1}^{i-1} (i-j) \hat{\Delta} f_j \Delta T + \frac{1}{2} \sum_{j=1}^{i-1} \left((i-j)\,(i-j-1) \right) + (i-j) \right) \hat{\Delta} \dot{f}_j \Delta T^2 \right) \\
&= \sum_{j=1}^{i} \hat{\Delta}\phi_j + 2\pi \left(\sum_{j=1}^{i-1} (i-j) \hat{\Delta} f_j \Delta T + \frac{1}{2} \sum_{j=1}^{i-1} \left((i-j)^2 \right) \hat{\Delta} \dot{f}_j \Delta T^2 \right)
\end{aligned} \tag{9}
$$

1.3 Taking the expectation

We will be taking the expectation of various combinations of these quantities. Since the terms are central normal distributions all cross correlated terms will vanish, and the only non-zero terms are given by

$$
E[\hat{\Delta}\phi_i \hat{\Delta}\phi_j] = \delta_{ij} \sigma_\phi^2, \quad E[\hat{\Delta} f_i \hat{\Delta} f_j] = \delta_{ij} \sigma_f^2, \quad E[\hat{\Delta} \dot{f}_i \hat{\Delta} \dot{f}_j] = \delta_{ij} \sigma_{\dot{f}}^2,
$$

Since there are a significant number of terms to deal with, all obviously cross corelated terms will not be written down. Each term will be calculated separately and then combined at the end

1.3.1 TERM 1: $\Delta\phi_i^2$

$$
\Delta\phi_i^2 = \sum_{k=1}^{i} \sum_{l=1}^{i} \hat{\Delta}\phi_k \hat{\Delta}\phi_l + (2\pi)^2 \left(\sum_{k=1}^{i-1} \sum_{l=1}^{i-1} (i-k)(i-l) \hat{\Delta} f_k \hat{\Delta} f_l \Delta T^2 + \frac{1}{4} \sum_{k=1}^{i-1} \sum_{l=1}^{i-1} (i-k)^2 (i-l)^2 \hat{\Delta} \dot{f}_k \hat{\Delta} \dot{f}_l \Delta T^4 \right) \tag{10}
$$

$$
E\left[\Delta\phi_i^2\right] = \sigma_\phi^2 i + (2\pi)^2 \left[\sum_{k=1}^{i-1} (i-k)^2 \sigma_f^2 \Delta T^2 + \frac{1}{4} \sum_{k=1}^{i-1} (i-k)^4 \sigma_{\dot{f}}^2 \Delta T^4 \right]
$$

In [4]:
```
exp1 = sigmaP0**2 * i + (2*pi)**2 * (
        (sigmaF0 * dT)**2 * summation((i-k)**2, (k, 1, i-1)) +
        (0.25)*(sigmaF1 * dT**2)**2 * summation((i-k)**4, (k, 1, i-1)))

exp1 = simplify(nsimplify(exp1))

TERM1 = simplify(summation(exp1, (i, 1, N)))
TERM1 = TERM1.expand().subs(sigmaF1**2 * dT **4, AF1).subs(sigmaF0**2 * dT**2, AF0).subs(sigmaP(
TERM1 = TERM1.expand().collect([AF0, AF1, AP0])
TERM1
```

Out[4]:

$$
A_{\dot{f}} \left(\frac{\pi^2 N^6}{30} - \frac{\pi^2 N^4}{12} + \frac{\pi^2 N^2}{20} \right) + A_\phi \left(\frac{N^2}{2} + \frac{N}{2} \right) + A_f \left(\frac{\pi^2 N^4}{3} - \frac{\pi^2 N^2}{3} \right)
$$

1.3.2 TERM 2: Δf_i^2

$$\Delta f_i^2 = \sum_{k=1}^{i}\sum_{l=1}^{i}\hat{\Delta} f_k \hat{\Delta} f_l + \sum_{k=1}^{i-1}\sum_{l=1}^{i-1}(i-k)(i-l)\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l \Delta T^2 \tag{11}$$

$$\tag{12}$$

$$E\left[\Delta f_i^2\right] = \sigma_f^2 i + \Delta T^2 \sigma_{\dot{f}}^2 \sum_{k=1}^{i-1}(i-k)^2$$

In [5]:
```
exp2 = i * sigmaF0**2 + (dT * sigmaF1)**2 * summation((i-k)**2, (k, 1, i-1))
TERM2 = summation(exp2, (i, 1, N))
TERM2 = TERM2.expand().subs(sigmaF1**2 * dT **2, AF1/dT**2).subs(sigmaF0**2, AF0/dT**2).collect
TERM2
```

Out[5]:

$$A_{\dot{f}}\left(\frac{N^4}{12\Delta T^2} - \frac{N^2}{12\Delta T^2}\right) + A_f\left(\frac{N^2}{2\Delta T^2} + \frac{N}{2\Delta T^2}\right)$$

1.3.3 TERM 3: $\Delta \dot{f}_i^2$

$$\Delta \dot{f}_i^2 = \sum_{k=1}^{i}\sum_{l=1}^{i}\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l$$

$$E[\Delta \dot{f}_i^2] = \sigma_{\dot{f}}^2 i \tag{13}$$

$$E\left[\sum_{i}^{N}\Delta \dot{f}_i^2\right] = \sigma_{\dot{f}}\sum_{i}^{N} i =$$

In [6]:
```
exp3 = sigmaF1**2 * i
TERM3 = summation(exp3, (i, 1, N))
TERM3 = TERM3.expand().subs(sigmaF1**2, AF1/dT**4).collect([AF0, AF1])
TERM3
```

Out[6]:

$$A_{\dot{f}}\left(\frac{N^2}{2\Delta T^4} + \frac{N}{2\Delta T^4}\right)$$

1.3.4 TERM 4: $\Delta\phi_i\Delta f_i$

$$\begin{aligned}\Delta\phi_i\Delta f_i &= 2\pi\left(\sum_{k=1}^{i-1}(i-k)\hat{\Delta} f_k\Delta T + \frac{1}{2}\sum_{k=1}^{i-1}(i-k)^2\hat{\Delta}\dot{f}_k\Delta T^2\right)\left(\sum_{l=1}^{i}\hat{\Delta} f_l + \sum_{l=1}^{i-1}(i-l)\hat{\Delta}\dot{f}_l\Delta T\right)\\ &= 2\pi\left(\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}(i-k)\hat{\Delta} f_k\hat{\Delta} f_l\Delta T + \frac{1}{2}\sum_{k=1}^{i-1}\sum_{l=1}^{i-1}(i-k)^2(i-l)\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l\Delta T^3\right)\end{aligned} \tag{14}$$

$$E[\Delta\phi_i\Delta f_i] = 2\pi\left(\sum_{k=1}^{i-1}(i-k)\sigma_f^2\Delta T + \frac{1}{2}\sum_{k=1}^{i-1}(i-k)^3\sigma_{\dot{f}}^2\Delta T^3\right)$$

In [7]:
```
exp4 = 2*pi*(sigmaF0**2 * dT * summation((i-k), (k, 1, i-1))
                + 0.5 * sigmaF1**2 * dT**3 * summation((i-k)**3, (k, 1, i-1)))

TERM4 = summation(exp4, (i, 1, N)).nsimplify()
TERM4 = TERM4.expand().subs(dT **3 * sigmaF1**2, AF1/dT).subs(dT * sigmaF0**2, AF0/dT).collect(
TERM4
```

Out[7]:

$$A_{\dot{f}}\left(\frac{\pi N^5}{20\Delta T} - \frac{\pi N^3}{12\Delta T} + \frac{\pi N}{30\Delta T}\right) + A_f\left(\frac{\pi N^3}{3\Delta T} - \frac{\pi N}{3\Delta T}\right)$$

1.3.5 TERM 5: $\Delta\phi_i\Delta\dot{f}_i$

$$\begin{aligned}\Delta\phi_i\Delta f_i &= 2\pi\left(\sum_{k=1}^{i-1}(i-k)\hat{\Delta}f_k\Delta T + \frac{1}{2}\sum_{k=1}^{i-1}(i-k)^2\hat{\Delta}\dot{f}_k\Delta T^2\right)\left(\sum_{l=1}^{i}\hat{\Delta}\dot{f}_l\right)\\ &= 2\pi\left(\frac{1}{2}\sum_{k=1}^{i-1}\sum_{l=1}^{i}\left((i-k)^2\right)\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l\Delta T^2\right)\end{aligned} \tag{15}$$

$$E\left[\Delta\phi_i\hat{\Delta}\dot{f}_i\right] = \pi\sum_{k=1}^{i-1}(i-k)^2\sigma_{\dot{f}}^2\Delta T^2$$

In [8]:
```
exp5 = pi * sigmaF1 **2 * dT**2 * summation((i-k)**2, (k, 1, i-1))
TERM5 = summation(exp5, (i, 1, N))
TERM5 = TERM5.expand().subs(sigmaF1**2 * dT **2, AF1/dT**2).subs(sigmaF0**2 * dT**2, AF0).colle
TERM5
```

Out[8]:

$$A_{\dot{f}}\left(\frac{\pi N^4}{12\Delta T^2} - \frac{\pi N^2}{12\Delta T^2}\right)$$

1.3.6 TERM 6: $\Delta f_i\Delta\dot{f}_i$

$$\begin{aligned}\Delta f_i\Delta\dot{f}_i &= \left(\sum_{k=1}^{i}\hat{\Delta}f_k + \sum_{k=1}^{i-1}(i-k)\hat{\Delta}\dot{f}_k\Delta T\right)\left(\sum_{l=1}^{i}\hat{\Delta}\dot{f}_l\right)\\ &= \sum_{k=1}^{i-1}\sum_{l=1}^{i}(i-k)\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l\Delta T\end{aligned} \tag{16}$$

$$E[\Delta f_i\Delta\dot{f}_i] = \sum_{k=1}^{i-1}(i-k)\sigma_{\dot{f}}^2\Delta T$$

In [9]:
```
exp6 = sigmaF1**2 * dT * summation(i-k, (k, 1, i-1))
TERM6 = summation(exp6, (i, 1, N))
TERM6 = TERM6.expand().subs(sigmaF1**2 * dT, AF1/dT**3).collect([AF0, AF1])
TERM6
```

Out[9]:

$$A_{\dot{f}}\left(\frac{N^3}{6\Delta T^3} - \frac{N}{6\Delta T^3}\right)$$

1.3.7 TERM 7: $\Delta\phi_i\Delta\phi_j$

$$\Delta\phi_i\Delta\phi_j = \sum_{k=1}^{i}\sum_{l=1}^{j}\hat{\Delta}\phi_k\hat{\Delta}\phi_l + (2\pi)^2\left(\sum_{k=1}^{i-1}\sum_{l=1}^{j-1}(i-k)(j-l)\hat{\Delta}f_k\hat{\Delta}f_l\Delta T^2 + \frac{1}{4}\sum_{k=1}^{i-1}\sum_{l=1}^{j-1}(i-k)^2(j-l)^2\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l\Delta T^4\right) \tag{17}$$

$$E[\Delta\phi_i\Delta\phi_j] = \sigma_\phi^2 j + (2\pi)^2\left(\sum_{l=1}^{j-1}(i-l)(j-l)\sigma_f^2\Delta T^2 + \frac{1}{4}\sum_{l=1}^{j-1}(i-l)^2(j-l)^2\sigma_{\dot{f}}^2\Delta T^4\right)$$

In [10]:
```
exp7 = sigmaP0**2 * j + (2*pi)**2 * (
        (sigmaF0 * dT)**2 * summation((i-l)*(j-l), (l, 1, j-1)) +
        0.25 * (sigmaF1 * dT**2)**2 * summation((i-l)**2 * (j-l)**2, (l, 1, j-1))
        )
exp7 = nsimplify(exp7)

TERM7 = summation(summation(exp7, (j, 1, i-1)), (i, 1, N))
TERM7 = TERM7.expand().subs(sigmaF1**2 * dT **4, AF1).subs(sigmaF0**2 * dT**2, AF0).subs(sigmal
TERM7
```

Out[10]:

$$A_{\dot{f}}\left(\frac{\pi^2N^7}{126} - \frac{\pi^2N^6}{60} - \frac{7\pi^2}{360}N^5 + \frac{\pi^2N^4}{24} + \frac{\pi^2N^3}{72} - \frac{\pi^2N^2}{40} - \frac{\pi^2N}{420}\right) + A_\phi\left(\frac{N^3}{6} - \frac{N}{6}\right) + A_f\left(\frac{\pi^2N^5}{10} - \frac{\pi^2N^4}{6} - \frac{\pi^2N^3}{6} + \frac{\pi^2N^2}{6}\right.$$

1.3.8 TERM 8: $\Delta f_i\Delta f_j$

$$\Delta f_i\Delta f_j = \sum_{k=1}^{i}\sum_{l=1}^{j}\hat{\Delta}f_k\hat{\Delta}f_l + \sum_{k=1}^{i-1}\sum_{l=1}^{j-1}(i-k)(j-l)\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l\Delta T^2 \tag{18}$$

$$\tag{19}$$

$$E\left[\Delta f_i\Delta f_j\right] = \sigma_f^2 j + \Delta T^2\sigma_{\dot{f}}^2\sum_{l=1}^{j-1}(i-l)(j-l)$$

In [11]:
```
exp8 = sigmaF0**2 * j + (dT * sigmaF1)**2 * summation((i-l) * (j-l), (l, 1, j-1))
TERM8 = summation(summation(exp8, (j, 1, i-1)), (i, 1, N))
TERM8 = TERM8.expand().subs(sigmaF1**2 * dT **2, AF1/dT**2).subs(sigmaF0**2, AF0/dT**2).collec
TERM8
```

Out[11]:

$$A_{\dot{f}}\left(\frac{N^5}{40\Delta T^2} - \frac{N^4}{24\Delta T^2} - \frac{N^3}{24\Delta T^2} + \frac{N^2}{24\Delta T^2} + \frac{N}{60\Delta T^2}\right) + A_f\left(\frac{N^3}{6\Delta T^2} - \frac{N}{6\Delta T^2}\right)$$

1.3.9 TERM 9: $\Delta\dot{f}_i\Delta\dot{f}_j$

$$\Delta\dot{f}_i\Delta\dot{f}_j = \sum_{k=1}^{i}\sum_{l=1}^{j}\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l$$

$$E\left[\Delta\dot{f}_i\Delta\dot{f}_j\right] = \sigma_{\dot{f}}^2 j \tag{20}$$

```
In [12]: exp9 = sigmaF1**2 * j
         TERM9 = summation(summation(exp9, (j, 1, i-1)), (i, 1, N))
         TERM9 = TERM9.expand().subs(sigmaF1**2, AF1/dT**4).collect([AF0, AF1])
         TERM9
```

Out[12]:

$$A_{\dot{f}}\left(\frac{N^3}{6\Delta T^4} - \frac{N}{6\Delta T^4}\right)$$

1.3.10 TERM 10: $\Delta\phi_i\Delta f_j$

$$\begin{aligned}\Delta\phi_i\Delta f_j &= 2\pi\left(\sum_{k=1}^{i-1}(i-k)\hat{\Delta}f_k\Delta T + \frac{1}{2}\sum_{k=1}^{i-1}(i-k)^2\hat{\Delta}\dot{f}_k\Delta T^2\right)\left(\sum_{l=1}^{j}\hat{\Delta}f_l + \sum_{l=1}^{j-1}(j-l)\hat{\Delta}\dot{f}_l\Delta T\right)\\ &= 2\pi\left(\sum_{k=1}^{i-1}\sum_{l=1}^{j}(i-k)\hat{\Delta}f_k\hat{\Delta}f_l\Delta T + \frac{1}{2}\sum_{k=1}^{i-1}\sum_{l=1}^{j-1}(i-k)^2(j-l)\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l\Delta T^3\right) \qquad (21)\\ E[\Delta\phi_i\Delta f_j] &= 2\pi\left(\sum_{l=1}^{j}(i-l)\sigma_f^2\Delta T + \frac{1}{2}\sum_{l=1}^{j-1}(i-l)^2(j-l)\sigma_{\dot{f}}^2\Delta T^3\right)\end{aligned}$$

```
In [13]: exp10 = 2*pi * (
                  sigmaF0**2 * dT * summation((i-l), (l, 1, j)) +
                  0.5 * sigmaF1**2 * dT**3 * summation((i-l)**2 * (j-l), (l, 1, j-1))
                    )

         TERM10 = nsimplify(summation(summation(exp10, (j, 1, i-1)), (i, 1, N)))
         TERM10 = TERM10.expand().subs(sigmaF1**2 * dT **3, AF1/dT).subs(sigmaF0**2 * dT, AF0/dT).colle
         TERM10
```

Out[13]:

$$A_{\dot{f}}\left(\frac{\pi N^6}{60\Delta T} - \frac{\pi N^5}{40\Delta T} - \frac{\pi N^4}{24\Delta T} + \frac{\pi N^3}{24\Delta T} + \frac{\pi N^2}{40\Delta T} - \frac{\pi N}{60\Delta T}\right) + A_f\left(\frac{\pi N^4}{6\Delta T} - \frac{\pi N^2}{6\Delta T}\right)$$

1.3.11 TERM 11: $\Delta f_i\Delta\phi_j$

$$\begin{aligned}\Delta\phi_j\Delta f_i &= 2\pi\left(\sum_{k=1}^{j-1}(j-k)\hat{\Delta}f_k\Delta T + \frac{1}{2}\sum_{k=1}^{j-1}(j-k)^2\hat{\Delta}\dot{f}_k\Delta T^2\right)\left(\sum_{l=1}^{i}\hat{\Delta}f_l + \sum_{l=1}^{i-1}(i-l)\hat{\Delta}\dot{f}_l\Delta T\right)\\ &= 2\pi\left(\sum_{k=1}^{j-1}\sum_{l=1}^{i}(j-k)\hat{\Delta}f_k\hat{\Delta}f_l\Delta T + \frac{1}{2}\sum_{k=1}^{j-1}\sum_{l=1}^{i-1}(j-k)^2(i-l)\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l\Delta T^3\right) \qquad (22)\\ E[\Delta\phi_j\Delta f_i] &= 2\pi\left(\sum_{l=1}^{j-1}(j-l)\sigma_f^2\Delta T + \frac{1}{2}\sum_{l=1}^{j-1}(i-l)(j-l)^2\sigma_{\dot{f}}^2\Delta T^3\right)\end{aligned}$$

```
In [14]: exp11 = 2*pi * (
                  sigmaF0**2 * dT * summation((j-l), (l, 1, j-1)) +
                  0.5 * sigmaF1**2 * dT**3 * summation((i-l) * (j-l)**2, (l, 1, j-1))
                    )

         TERM11 = nsimplify(summation(summation(exp11, (j, 1, i-1)), (i, 1, N)))
         TERM11 = TERM11.expand().subs(sigmaF1**2 * dT **3, AF1/dT).subs(sigmaF0**2 * dT, AF0/dT).colle
         TERM11
```

Out[14]:

$$A_{\dot{f}}\left(\frac{\pi N^6}{90\Delta T}-\frac{\pi N^5}{40\Delta T}-\frac{\pi N^4}{72\Delta T}+\frac{\pi N^3}{24\Delta T}+\frac{\pi N^2}{360\Delta T}-\frac{\pi N}{60\Delta T}\right)+A_f\left(\frac{\pi N^4}{12\Delta T}-\frac{\pi N^3}{6\Delta T}-\frac{\pi N^2}{12\Delta T}+\frac{\pi N}{6\Delta T}\right)$$

1.3.12 TERM 12: $\Delta\phi_i\Delta\dot{f}_j$

$$\begin{aligned}\Delta\phi_i\Delta f_j &= 2\pi\left(\sum_{k=1}^{i-1}(i-k)\hat{\Delta}f_k\Delta T+\frac{1}{2}\sum_{k=1}^{i-1}(i-k)^2\hat{\Delta}\dot{f}_k\Delta T^2\right)\left(\sum_{l=1}^{j}\hat{\Delta}\dot{f}_l\right)\\ &= 2\pi\left(\frac{1}{2}\sum_{k=1}^{i-1}\sum_{l=1}^{j}(i-k)^2\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l\Delta T^2\right)\end{aligned}\tag{23}$$

$$E\left[\Delta\phi_i\hat{\Delta}\dot{f}_j\right]=\pi\sum_{k=1}^{j}(i-k)^2\sigma_{\dot{f}}^2\Delta T^2$$

In [15]:
```
exp12 = pi * sigmaF1 **2 * dT**2 * summation((i-k)**2, (k, 1, j))
TERM12 = summation(summation(exp12, (j, 1, i-1)), (i, 1, N))
TERM12 = TERM12.expand().subs(sigmaF1**2 * dT **2, AF1/dT**2).subs(sigmaF0**2 * dT**2, AF0).co
TERM12
```

Out[15]:

$$A_{\dot{f}}\left(\frac{\pi N^5}{20\Delta T^2}-\frac{\pi N^3}{12\Delta T^2}+\frac{\pi N}{30\Delta T^2}\right)$$

1.3.13 TERM 13: $\Delta\phi_i\Delta\dot{f}_j$

$$\Delta\phi_j\Delta f_i = 2\pi\left(\sum_{k=1}^{j-1}(j-k)\hat{\Delta}f_k\Delta T+\frac{1}{2}\sum_{k=1}^{j-1}(j-k)^2\hat{\Delta}\dot{f}_k\Delta T^2\right)\left(\sum_{l=1}^{i}\hat{\Delta}\dot{f}_l\right)\tag{24}$$

$$= 2\pi\left(\frac{1}{2}\sum_{k=1}^{j-1}\sum_{l=1}^{i}(j-k)^2\hat{\Delta}\dot{f}_k\hat{\Delta}\dot{f}_l\Delta T^2\right)\tag{25}$$

$$E\left[\Delta\phi_i\hat{\Delta}\dot{f}_i\right]=\pi\sum_{k=1}^{j-1}(j-k)^2\sigma_{\dot{f}}^2\Delta T^2$$

In [16]:
```
exp13 = pi * sigmaF1 **2 * dT**2 * summation((j-k)**2, (k, 1, j-1))
TERM13 = summation(summation(exp13, (j, 1, i-1)), (i, 1, N))
TERM13 = TERM13.expand().subs(sigmaF1**2 * dT **2, AF1/dT**2).subs(sigmaF0**2 * dT**2, AF0).co
TERM13
```

Out[16]:

$$A_{\dot{f}}\left(\frac{\pi N^5}{60\Delta T^2}-\frac{\pi N^4}{24\Delta T^2}+\frac{\pi N^2}{24\Delta T^2}-\frac{\pi N}{60\Delta T^2}\right)$$

1.3.14 TERM 14: $\Delta f_i \Delta \dot{f}_j$

$$\begin{aligned}
\Delta f_i \Delta \dot{f}_j &= \left(\sum_{k=1}^{i} \hat{\Delta} f_k + \sum_{k=1}^{i-1} (i-k) \hat{\Delta} \dot{f}_k \Delta T \right) \left(\sum_{l=1}^{j} \hat{\Delta} \dot{f}_l \right) \\
&= \sum_{k=1}^{i-1} \sum_{l=1}^{j} (i-k) \hat{\Delta} \dot{f}_k \hat{\Delta} \dot{f}_l \Delta T
\end{aligned} \tag{26}$$

$$E[\Delta f_i \Delta \dot{f}_i] = \sigma_{\dot{f}}^2 \Delta T \sum_{k=1}^{j} (i-k)$$

In [17]:
```
exp14 = sigmaF1 **2 * dT * summation(i-k, (k, 1, j))
TERM14 = summation(summation(exp14, (j, 1, i-1)), (i, 1, N))
TERM14 = TERM14.expand().subs(sigmaF1**2 * dT, AF1/dT**3).collect([AF0, AF1])
TERM14
```

Out[17]:

$$A_{\dot{f}} \left(\frac{N^4}{12\Delta T^3} - \frac{N^2}{12\Delta T^3} \right)$$

1.3.15 TERM 15: $\Delta f_j \Delta \dot{f}_i$

$$\begin{aligned}
\Delta f_j \Delta \dot{f}_i &= \left(\sum_{k=1}^{j} \hat{\Delta} f_k + \sum_{k=1}^{j-1} (j-k) \hat{\Delta} \dot{f}_k \Delta T \right) \left(\sum_{l=1}^{i} \hat{\Delta} \dot{f}_l \right) \\
&= \sum_{k=1}^{j-1} \sum_{l=1}^{i} (j-k) \hat{\Delta} \dot{f}_k \hat{\Delta} \dot{f}_l \Delta T
\end{aligned} \tag{27}$$

$$E[\Delta f_j \Delta \dot{f}_i] = \sigma_{\dot{f}}^2 \Delta T \sum_{k=1}^{j-1} (j-k)$$

In [18]:
```
exp15 = sigmaF1 **2 * dT * summation(j-k, (k, 1, j-1))
TERM15 = summation(summation(exp15, (j, 1, i-1)), (i, 1, N))
TERM15 = TERM15.expand().subs(sigmaF1**2 * dT, AF1/dT**3).collect([AF0, AF1])
TERM15
```

Out[18]:

$$A_{\dot{f}} \left(\frac{N^4}{24\Delta T^3} - \frac{N^3}{12\Delta T^3} - \frac{N^2}{24\Delta T^3} + \frac{N}{12\Delta T^3} \right)$$

2 Coefficients

We now write out the terms of the metric, note the $g^{E}_{\alpha\beta}$ refer to the metric $g_{\alpha\beta ij}$ with $i = j$ and $g^{NE}_{\alpha\beta}$ when $i \neq j$. To simplify things the symbolic maths program will be given the metric in terms of $N = T/\Delta T$.

$$g_{\alpha\beta ij} = \begin{bmatrix} \delta_{ij} N^{-1} - N^{-2} & \pi \Delta T \left(\delta_{ij} N^{-1} - N^{-2} \right) & \frac{\Delta T^2 \pi}{3} \left(\delta_{ij} N^{-1} - N^{-2} \right) \\ \pi \Delta T \left(\delta_{ij} N^{-1} - N^{-2} \right) & \pi^2 \Delta T^2 \left(\delta_{ij} \frac{4N^{-1}}{3} - N^{-2} \right) & \pi^2 \Delta T^3 \left(\delta_{ij} \frac{N^{-1}}{2} - \frac{N^{-2}}{3} \right) \\ \frac{\Delta T^2 \pi}{3} \left(\delta_{ij} N^{-1} - N^{-2} \right) & \pi^2 \Delta T^3 \left(\delta_{ij} \frac{N^{-1}}{2} - \frac{N^{-2}}{3} \right) & \pi^2 \Delta T^4 \left(\frac{N^{-1}}{5} - \frac{N^{-2}}{9} \right) \end{bmatrix}^{\alpha,\beta} \tag{28}$$

In [19]:
```
Ninv = 1/N

gE00 = (Ninv) - (Ninv)**2
gE01 = pi * dT * (Ninv - Ninv**2)
gE02 = nsimplify(pi * dT **2 * (Ninv - Ninv**2) / 3.0)
gE10 = gE01
gE11 = nsimplify((pi * dT) **2 * ((4.0/3.0) * Ninv - Ninv**2))
gE12 = nsimplify(pi**2 * dT **3 * (0.5 * Ninv - Ninv**2 / 3.0))
gE20 = gE02
gE21 = gE12
gE22 = nsimplify(pi**2 * dT **4 * (Ninv/5.0 - (Ninv**2)/9.0))

gNE00 = - (Ninv)**2
gNE01 = - pi * dT * (Ninv**2)
gNE02 = - nsimplify(pi * dT **2 * (Ninv**2) / 3.0)
gNE10 = gNE01
gNE11 = - nsimplify((pi * dT) **2 * (Ninv**2))
gNE12 = - nsimplify(pi**2 * dT **3 * (Ninv**2 / 3.0))
gNE20 = gNE02
gNE21 = gNE12
gNE22 = - nsimplify(pi**2 * dT **4 * ((Ninv**2)/9.0))
```

3 Combining terms

In [20]:
```
# Add the terms up
TERM1_TERM7 = gE00 * TERM1 + 2 * gNE00 * TERM7
TERM1_TERM7 = TERM1_TERM7.expand().collect([AF0, AF1])

TERM2_TERM8 = gE11 * TERM2 + 2 * gNE11 * TERM8
TERM2_TERM8 = TERM2_TERM8.expand().collect([AF0, AF1])

TERM3_TERM9 = gE22* TERM3 +  2 * gNE22 * TERM9
TERM3_TERM9 = TERM3_TERM9.expand().collect([AF0, AF1])

TERM4_TERM10_TERM11 = 2 * gE01 * TERM4 + 2 * (gNE01 * TERM10 + gNE10 * TERM11)
TERM4_TERM10_TERM11 = TERM4_TERM10_TERM11.expand().collect([AF0, AF1])

TERM5_TERM12_TERM13 = 2 * gE02 * TERM5 + 2 * (gNE02 * TERM12 + gNE20 * TERM13)
TERM5_TERM12_TERM13 = TERM5_TERM12_TERM13.expand().collect([AF0, AF1]).nsimplify()

TERM6_TERM14_TERM15 = 2 * gE12 * TERM6 + 2* (gNE12 * TERM14 + gNE21 * TERM15)
TERM6_TERM14_TERM15 = TERM6_TERM14_TERM15.expand().collect([AF0, AF1])
```

In [21]:
```
# Combine as the expectation
exp_m = (TERM1_TERM7 + TERM2_TERM8 + TERM3_TERM9 +
         TERM4_TERM10_TERM11 + TERM5_TERM12_TERM13 + TERM6_TERM14_TERM15
         )

term = nsimplify(simplify(exp_m))
term = term.expand().collect([AF0, AF1, AP0])
term
```

Out[21]:

$$A_{\dot{f}} \left(\frac{11\pi^{2}}{630} N^{5} + \frac{2\pi^{2}}{45} N^{4} + \frac{\pi^{2} N^{3}}{36} + \frac{\pi^{2} N}{540} - \frac{\pi^{2}}{378 N}\right) + A_{\phi} \left(\frac{N}{6} - \frac{1}{6 N}\right) + A_{f} \left(\frac{2\pi^{2}}{15} N^{3} + \frac{\pi^{2} N^{2}}{6} + \frac{\pi^{2}}{30 N}\right)$$

3.1 Get python version

In [22]:
```
print_python(nsimplify(term.coeff(AP0)))
```

```
N = Symbol('N')
e = N/6 - 1/(6*N)
```

In [23]:
```
print_python(simplify(term.coeff(AF0)))
```

```
N = Symbol('N')
e = pi**2*(N**3*(4*N + 5) + 1)/(30*N)
```

In [24]:
```
print_python(simplify(term.coeff(AF1)))
```

```
N = Symbol('N')
e = pi**2*(N**2*(66*N**4 + 168*N**3 + 105*N**2 + 7) - 10)/(3780*N)
```

3.2 Getting Latex version

In [25]:
```
simple_form = factor_terms(AP0 * term.coeff(AP0)) + factor_terms(AF0 * term.coeff(AF0)) + AF1 *
simple_form
```

Out[25]:

$$\frac{\pi^{2} A_{\dot{f}}}{3780} \left(66 N^{5} + 168 N^{4} + 105 N^{3} + 7 N - \frac{10}{N}\right) + \frac{A_{\phi}}{6} \left(N - \frac{1}{N}\right) + \frac{\pi^{2} A_{f}}{30} \left(4 N^{3} + 5 N^{2} + \frac{1}{N}\right)$$

In [26]:
```
print latex(simple_form, mode="equation")
```

```
\begin{equation}\frac{\pi^{2} A_{{\dot{f}}}}{3780} \left(66 N^{5} + 168 N^{4} + 105 N^{3} + 7 N - \frac{1
```

In [26]: