GVAR - long hand

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1 In long-hand

We consider a model with three variables: real GDP (x_{1it}), inflation (x_{2it}), and real interest rate (x_{3it}), for 1 country (call it country i), with a lag of one period. Equation 1 represents the relationship between these variables and the corresponding coefficients (a_{1it}, a_{2it}, a_{3it}).

$$\begin{pmatrix} x_{1it} \\ x_{2it} \\ x_{3it} \end{pmatrix} = \begin{pmatrix} a_{1i0} \\ a_{2i0} \\ a_{3i0} \end{pmatrix} + \begin{pmatrix} a_{1i1} \\ a_{2i1} \\ a_{3i1} \end{pmatrix} \text{t} + \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{pmatrix} \begin{pmatrix} x_{1it-1} \\ x_{2it-1} \\ x_{3it-1} \end{pmatrix} + \begin{pmatrix} \Lambda & & \\ & \Lambda & \\ & & \Lambda \end{pmatrix} \begin{pmatrix} x^*_{1it1} \\ x^*_{2it1} \\ x^*_{3it} \end{pmatrix} + ... + \begin{pmatrix} \epsilon_{1it} \\ \epsilon_{2it} \\ \epsilon_{3it} \end{pmatrix}$$

We now move on to the weight, the foreign variable as the article calls it. Considering we have countries i, j, and l (i.e., N = 3).

$$\begin{pmatrix} x^*_{1it} \\ x^*_{2it} \\ x^*_{3it} \end{pmatrix} = \begin{pmatrix} \sum_{j \neq i}^N \omega_{1ij} x_{1jt} = \omega_{1ij} x_{1jt} + \omega_{1il} x_{1lt} \\ \sum_{j \neq i}^N \omega_{2ij} x_{2jt} = \omega_{2ij} x_{2jt} + \omega_{2il} x_{2lt} \\ \sum_{j \neq i}^N \omega_{3ij} x_{3jt} = \omega_{3ij} x_{3jt} + \omega_{3il} x_{3lt} \end{pmatrix}$$

Here, x_{1jt} and x_{1lt} is the real GDP of country j and l respectively. ω_{ij} and ω_{il} are the weights, here they represent the percentage of trade flowing from country j to country i and from country l to country i respectively. In the set-up of input-output modelling, this would represent the a version of Leontief matrix we saw in the PowerPoint deck.

Next, we take $\begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{pmatrix} \begin{pmatrix} x_{1it-1} \\ x_{2it-1} \\ x_{3it-1} \end{pmatrix}$ to the left hand side, and also define $z_{it} = (x'_{it} x'^*_{it})'$.

We therefore have:

$$\begin{pmatrix} x_{1it} \\ x_{2it} \\ x_{3it} \end{pmatrix} - \begin{pmatrix} \Lambda & & \\ & \Lambda & \\ & & \Lambda \end{pmatrix} \begin{pmatrix} x^*_{1it1} \\ x^*_{2it1} \\ x^*_{3it} \end{pmatrix} = \begin{pmatrix} a_{1i0} \\ a_{2i0} \\ a_{3i0} \end{pmatrix} + \begin{pmatrix} a_{1i1} \\ a_{2i1} \\ a_{3i1} \end{pmatrix} \text{t} + \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{pmatrix} \begin{pmatrix} x_{1it-1} \\ x_{2it-1} \\ x_{3it-1} \end{pmatrix} + ... + \begin{pmatrix} \epsilon_{1it} \\ \epsilon_{2it} \\ \epsilon_{3it} \end{pmatrix}$$

To express in $z_{it} = (x'_{it} x'^*_{it})'$ as defined above, and continuing with three variables: real GDP (x_{1it}), inflation (x_{2it}), and real interest rate (x_{3it}), for country i, we have :

$$\begin{pmatrix} x_{1it} \\ x_{2it} \\ x_{3it} \end{pmatrix} - \begin{pmatrix} \Lambda & & \\ & \Lambda & \\ & & \Lambda \end{pmatrix} \begin{pmatrix} x^*_{1it1} \\ x^*_{2it1} \\ x^*_{3it} \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & -\Lambda_{11} & -\Lambda_{12} & -\Lambda_{13} \\ 0 & I & 0 & -\Lambda_{21} & -\Lambda_{22} & -\Lambda_{23} \\ 0 & 0 & I & -\Lambda_{31} & -\Lambda_{32} & -\Lambda_{33} \end{pmatrix} \begin{pmatrix} x_{1it} \\ x_{2it} \\ x_{3it} \\ x^*_{1it} \\ x^*_{2it} \\ x^*_{3it} \end{pmatrix} = \mathrm{A}_i z_{it}$$

In the paper, they define as $A_i z_{it}$. z_{it} is further decomposed into $W_i x_t$

Continuing the example that we have 3 variables and 3 countries (i.e. N = 3, which are i, j, and l). We will decompose the following:

$$\underset{(6\cdot 1)}{z_{it}} = \underset{(6\cdot 9)}{W_i}\ \underset{(9\cdot 1)}{x_t}$$

with long-hand, into this:

$$\begin{pmatrix} x_{1it} \\ x_{2it} \\ x_{3it} \\ x^*_{1it} \\ x^*_{2it} \\ x^*_{3it} \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{1ij} & 0 & 0 & \omega_{1il} & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_{2ij} & 0 & 0 & \omega_{2il} & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_{3ij} & 0 & 0 & \omega_{3il} \end{pmatrix} \begin{pmatrix} x_{1it} \\ x_{2it} \\ x_{3it} \\ x_{1jt} \\ x_{2jt} \\ x_{3jt} \\ x_{1lt} \\ x_{2lt} \\ x_{3lt} \end{pmatrix}$$

Okay, we got this far, we now combine $A_i z_{it}$ and $z_{it} = W_i x_t$ together, and get a one big monster matrix $A_i W_i x_t$, will lead us to the form: $A_i W_i x_t = a_{i0} + a_{i1} t + B_i Z_{it-1} + \epsilon_{it}$ (i.e., equation 5 and 6 in the reading), where $A_i W_i x_t$ is:

$$\begin{pmatrix} I & 0 & 0 & -\Lambda_{11} & -\Lambda_{12} & -\Lambda_{13} \\ 0 & I & 0 & -\Lambda_{21} & -\Lambda_{22} & -\Lambda_{23} \\ 0 & 0 & I & -\Lambda_{31} & -\Lambda_{32} & -\Lambda_{33} \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{1ij} & 0 & 0 & \omega_{1il} & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_{2ij} & 0 & 0 & \omega_{2il} & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_{3ij} & 0 & 0 & \omega_{3il} \end{pmatrix} \begin{pmatrix} x_{1it} \\ x_{2it} \\ x_{3it} \\ x_{1jt} \\ x_{2jt} \\ x_{3jt} \\ x_{1lt} \\ x_{2lt} \\ x_{3lt} \end{pmatrix}$$

, and, a_{i0} is simply $\begin{pmatrix} a_{1i0} \\ a_{2i0} \\ a_{3i0} \\ a_{1j0} \\ a_{2j0} \\ a_{3j0} \\ a_{1l0} \\ a_{2l0} \\ a_{3l0} \end{pmatrix}$