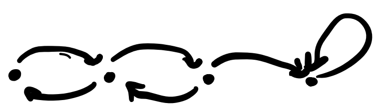
Question:

- Why can't all states be transient
- Why can't the following happen?

$$P_{ij}^{n} > 0 \text{ for some } n>0 \text{ but } P_{ji} = \delta_{ji}$$



1 2 3 4

state 4 is recurrent

others are transient

BUT, this violates

i comm $j \rightarrow j$ comm i

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim P_{\text{in out}}$$

Definition:

- Accessible: j acces. from i if P_{ij}^{n}
- Communicates: $i \leftrightarrow j$ if i & j are accessible from each other
- Absorbing: State i is absorbing if $P_{ij} = \delta_{ij}$

Prop: Communication creates equiv classes. $\left[i \leftrightarrow i, \; i \leftrightarrow j \; \& \; j \leftrightarrow k \rightarrow i \leftrightarrow k \right]$ $\quad i \leftrightarrow j \Leftrightarrow j \leftrightarrow i$

Def: P is irreducible if $\exists!$ class.

Ex: For the above P, there are two classes

$\{1,2,3\}$ & $\{4\}$

Def:

① If $P[\text{ever enter } j \mid \text{start in } i] = \begin{cases} 1, & \text{recurrent} \\ <1, & \text{transient} \end{cases}$

② State i has period d if $P_{ii}^{n} = 0 \;\; \forall n \neq 0 \bmod(d)$

Prop: Reccurence is a class property.

Transience is a class property.

Period is a class property.

Remark: Once the chain enters a class, it can either leave the class, or never leave the class.

Prop: Not all states can be transient.

Proof: Assume all M states are transient.

Then $P_{ii}^n = 0 \ \forall i$ for some sufficiently large n

This implies the system is in none of the M states.

Thus, at least one state must be recurrent.

Prop: For a finite state Markov Chain, irreducibility implies that all states are recurrent.

Proof: $\exists 1$ class. At least one state is recurrent.

Thus the class is recurrent. $\blacksquare$

Def: State i is ____ recurrent if

$$E[\text{time to return to state } i] \sim \begin{cases} < \infty, & \text{positive rec.} \\ \infty, & \text{null rec.} \end{cases}$$

Def: Positive recurrent states are called ergodic

Remark:

1. Pos & null reccurence are class properties.
2. For a finite MC, recurrence $\rightarrow$ pos. recurrence.

Theorem: For an irreducible ergodic MC

1. $\lim_{n\to\infty} P_{ij}^n = \pi_j \quad \forall i \ \& \ j \geq 0$

2. $\sum_j \pi_j = 1$

3. $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j \geq 0$

Note

P_{ij} is one number

Remarks:

under certain conditions

1. $\pi_j \sim$ limiting prop $\iff$ long-run $\underset{\sim}{\text{stationary}}$ time proportions

2. If MC is irreducible, then

   $\exists\, \pi_j$ s.t. $\pi_j = \sum_i \pi_i P_{ij} \iff$ MC is pos. rec.

   - π_j = long-run prop of time spent in state j
   - If MC is also aperiodic, then π_j is also the limiting probability state MC is in state j.

3.

limiting prob	long-run props / stationary
$P^n \approx I\pi$	$\pi_j = \pi_i P^i{}_j$

aperiodicity

④ P_{ij}^n is the $(i,j)^{th}$ component of P^n

⑤ Comparing long time vs. limit
(a) (b)

(a) $$\pi_j = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \text{Indicator}[X_m = j]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \text{Indicator}[X_m = j \mid X_0 = i]$$

$$E[\pi_j] = \pi_j = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P(X_m = j \mid X_0 = i)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P_{ij}^m \quad ; \; \forall i$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P_{ij}^m = \begin{pmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{pmatrix}$$

Convergence in average ~ weak

(b) $$P^n \to \begin{pmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{pmatrix}$$ requires aperiodicity

Stronger Convergence

since if $d > 1$ ∃ an alternating behavior of the limit.

Alternatively, ∃ a subsequence $P_{ii}^{2k+1} = 0 \;\; \forall i$
& hence isn't convergent to π_i. Thus, $\lim_{n\to\infty} P^n$ doesn't converge.

Examples:

① Long-run proportions vs. limiting probabilities (pg 222)

② A null recurrent MC (only possible for infinite MCs?)

③ An MC w/ π that never attains it

④ Something showing why aperiodic matters

- All finite irr. MC are pos. rec

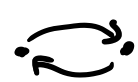
~ want pos. rec. but periodic

Example 3: Irreduc., pos recur, but periodic

Take states $\{i\}_{i=1}^{N}$ & periodic BCs

$P_{ij} = \delta_{i,j+1} \Rightarrow$ period $= N$ & irreducible

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & 0 & & \vdots \\ 0 & & & \ddots \; 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$



Take $N = 2$; $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- each ~~state~~ has period $d = 2$.

$$\det(P-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

$$\Rightarrow \lambda_\pm = \pm 1 \; ; \; e_+ = \tfrac{1}{2}(1,1) \; , \; e_- = \tfrac{1}{2}(1,-1)$$

left eigenvectors!

$\boxed{u^T = u^{-1}}$

$$\Rightarrow \exists\, U \in O(2,\mathbb{R}) \text{ s.t. } D = \mathrm{diag}(1,-1) = U^T P U$$

$$P^n = (UDU^T)^n = UD^nU^T \quad \text{since} \quad UU^T = 1$$

$$P = UDU^T \quad \text{where} \quad U^T = \begin{pmatrix} e_+ \\ e_- \end{pmatrix} ; \; U = \begin{pmatrix} e_+ & e_- \end{pmatrix}$$

$$Pe_+ = UDU^Te_+ = UDe_1 = \lambda_+ Ue_1 = \lambda_+ e_+$$

$$P^n = U\,\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ (-1)^n & (-1)^{n+1} \end{pmatrix} = \tfrac{1}{4}\begin{pmatrix} 1+(-1)^n & 1+(-1)^{n+1} \\ 1+(-1)^{n+1} & 1+(-1)^{n+2} \end{pmatrix}$$

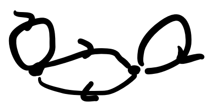
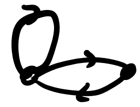
$$\Rightarrow P^n = \tfrac{1}{4}\begin{pmatrix} 1+(-1)^n & 1+(-1)^{n+1} \\ 1+(-1)^{n+1} & 1+(-1)^n \end{pmatrix} = \begin{cases} \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & n \text{ even} \\ \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & n \text{ odd} \end{cases}$$

but P^n has no limiting prob

but $e_+P = e_+ \Rightarrow P$ has longtime/stationary prob!

* P^n has π_{stat} but no π_{limit}. $Q = \tfrac{1}{2}(P+P^2) = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is aperiodic ...

Example 3 cont.:

 or 

1 1
2 0

$\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ or $\begin{pmatrix}1/2 & 1/2\\ 1 & 0\end{pmatrix}$

P $\qquad$ Q

- period is a class function. Since periods of state 1 are $d=1$, the two chains are both aperiodic.

$$C(2P) = (1-\lambda)^2 - 1$$
$$= \lambda^2 - 2\lambda + 1 - 1$$
$$= \lambda(\lambda - 2) = 0$$

$$\lambda_{2P} = 2, 0$$

$$\lambda_P = 1, 0$$

$$C(2P) = -(1-\lambda)\lambda - 2 = 0$$
$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda_{2P} = 2, -1$$

$$\lambda_P = 1, -1/2$$

$$(a \;\; b)\begin{pmatrix}-1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix} = 0$$

$a = b \Rightarrow \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$

$$(a \;\; b)\begin{pmatrix}-1/2 & 1/2\\ 1 & -1\end{pmatrix} = 0$$

$a = 2b$

$\Rightarrow \frac{1}{3}\begin{pmatrix}2\\1\end{pmatrix}$

* Both chains have the dff stationary distributions! *

$P^{1000} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $Q^{1000} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

$\Rightarrow \pi = \frac{1}{2}(1,1)$ $\qquad$ $\pi = \frac{1}{3}(2 \ 1)$

<u>Claim</u>: $P_{ii} > 0$ $\forall$ at least one i is enough (for irrep MCs) to yield aperiodicity. (Proof: period is a class property)

<u>Ending Remark</u>:

While irreducibility implies the MC P_{ij} converges <u>in average</u> to π_j $\forall i$, this is less desirable than having

P_{ij} converge in the limit, independent of i

Thus, ergodicity is **VERY NICE**!

It says we get π_j regardless of X_0,

when $X_0 = i \longrightarrow P_{ij}^n \to \pi_j \quad \forall i$.