The normal distribution

$$p(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$l(x|\mu,\sigma) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

$$-\frac{1}{2}\ln(2\pi) - \ln(\sigma) - \frac{1}{2}(x-\mu)^2\sigma^{-2}$$

$$\partial_\sigma l(x|\mu,\sigma) = -\frac{1}{\sigma} + (x-\mu)^2\sigma^{-3}$$

$$\partial_\sigma^2 l(x|\mu,\sigma) = \sigma^{-2} - 3(x-\mu)^2\sigma^{-4}$$

$$\partial_\mu\partial_\sigma = 2(x-\mu)\sigma^{-3}$$

$$\partial_\mu l(x|\mu,\sigma) = -(x-\mu)\sigma^{-2}$$

$$\partial_\mu^2 l(x|\mu,\sigma) = \sigma^{-2}$$

$$n L_n(x|\theta) = -\frac{n}{2}(\ln 2\pi + 2\ln\sigma) - \frac{1}{2\sigma^2}\sum_i (x_i-\mu)^2$$

$$\partial_\mu L_n(x|\theta) = -\frac{1}{n\sigma^2}\sum_i (x_i-\mu)$$

taking $\partial_\mu L_n = 0 \Rightarrow \sum_i x_i = \sum \mu = n\mu$

$$\Rightarrow \hat{\mu} = \bar{x}$$

$$\partial_\sigma L_n(x|\theta) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum (x_i-\mu)^2$$

taking $\partial_\sigma L_n = 0 \Rightarrow n\sigma^3 = \sigma\sum(x_i-\mu)^2$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_i (x_i-\mu)^2$$

$$I_{ij} = E_{\theta_0} \begin{pmatrix} n\sigma^{-2} & 2n(\bar{x}-\mu)\sigma^{-3} \\ 2n(\bar{x}-\mu)\sigma^{-3} & n\sigma^{-2} - 3n(\bar{x}-\mu)^2\sigma^{-4} \end{pmatrix}$$

$$I_{ij} = \begin{pmatrix} n\sigma_0^{-2} & 0 \\ 0 & n\sigma_0^{-2} \end{pmatrix}$$

Expectations & Variances

$E(\hat{\mu}) = E(\bar{x}) = \mu$

$Var(\hat{\mu}) = Var(\bar{x}) = \frac{1}{n^2} Var(\sum x_i) = \frac{1}{n^2} \sum Var(x_i) = \frac{\sigma_0^2}{n}$

$\hat{\mu} - \mu \sim N(0, \frac{1}{nI_0})$ where $I_0 = \frac{1}{\sigma_0^2}$ by CLT

where $\hat{\mu}$ is maximizer of $L_n(X \mid \mu, \sigma^2)$

$\Rightarrow$ Variance of MLE for large sample size

is asymptotic to $\frac{1}{nI_0}$ (holds in general)

$$E(\hat{\sigma}^2) = E\left[\frac{1}{n}\sum (x_i-\mu)^2\right] = E\left[\frac{1}{n}\sum\left((x_i-\bar{x})^2 + 2(x_i-\bar{x})(\bar{x}-\mu) + (\bar{x}-\mu)^2\right)\right]$$

$$= E\left[\;\ldots \right.$$

$$Var(\hat{\sigma}^2) = Var\left(\frac{1}{n}\sum (x_i-\mu)^2\right) \quad ; \quad \sum (x_i - \bar{x} + \bar{x} - \mu)^2$$

$$= Var\left[\frac{1}{n}\sum\left((x_i-\bar{x})^2 + 2(x_i-\bar{x})(\bar{x}-\mu) + (\bar{x}-\mu)^2\right)\right]$$