Following Baez

Recall that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, or

often $\rho = \frac{1}{Z} e^{-\beta H}$. If $\rho^2 = \rho$,

then $\rho \sim$ pure state. Otherwise, $\rho \sim$ mixed state, a statistical mixture of pure states.

In general, $tr(\rho) = \sum_i p_i = 1$.

Thinking more generally, suppose we have fixed various quantities (operators) $X_1, \ldots, X_n$.

Then, we want ρ_* which maximizes

$$S(\rho) = -Tr(\rho \ln \rho) \text{ \& satisfies } \langle X_i \rangle = x_i .$$

Lagrange multipliers:

$$\bar{S}(\rho, \lambda_i) = S(\rho) + \sum_i \lambda_i (\underbrace{\langle X_i \rangle_\rho}_{tr(\rho X_i)} - x_i)$$

$$\frac{\partial \bar{S}}{\partial \rho} = \frac{\partial \bar{S}}{\partial \lambda_i} = 0 \quad ; \quad \frac{\partial}{\partial \rho} = \sum_{ij} \frac{\partial}{\partial \rho_{ij}}$$

Diagonalize ρ $\Rightarrow$ $S = -\sum_k' (\rho_{kk} \ln \rho_{kk})$

$$\Rightarrow \quad -(\ln(\rho_{kk})+1) + \sum_i' \lambda_i \underbrace{\langle X_i \rangle_k}_{\langle X_i\rangle_k = \langle \phi_k | X_i | \phi_k \rangle} = 0$$

$$\rho_{kk} = e^{\sum_i' \lambda_i \langle X_i \rangle_k - 1}$$

Take $X_1 = I$ $\Rightarrow$ $Tr(\rho) = 1$

$$\Rightarrow \quad \rho_{kk} = e^{\lambda_1 - 1 + \sum' \lambda_i \langle X_i \rangle_k}$$

$$= Z^{-1} e^{\sum' \lambda_i \langle X_i \rangle_k}$$

$$\rho = Z^{-1} e^{\sum' \lambda_i \hat{X}_i}$$

since $\rho_{kk} = \langle \phi_k | \rho | \phi_k \rangle = Z^{-1} e^{\sum' \lambda_i \overbrace{\langle \phi_k | \hat{X}_i | \phi_k \rangle}^{\langle X_i \rangle_k}}$

Note: $tr(\rho) = 1 \Rightarrow Z = tr\left(e^{\sum' \lambda_i \hat{X}_i}\right)$

Enter Statistics

commulant generating function

mean $\quad \langle x_i \rangle = -\frac{\partial}{\partial \lambda_i} \ln Z$

variance $\quad \langle x_i^2 \rangle - \langle x_i \rangle^2 = \frac{\partial^2}{\partial \lambda_i^2} \ln Z$

$$= \langle (x_i - \langle x_i \rangle)^2 \rangle$$

covariance :

$$\langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle = \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} \ln Z$$

Since $[\hat{x}_i, \hat{x}_j] \neq 0$ in general.

Define $\quad g_{ij} = \mathrm{Re}\left(\frac{\partial^2}{\partial \lambda_i \partial \lambda_j} \ln Z\right)$

so that $g_{ij} = g_{ji}$ (How to prove)

Geometry

For each $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we

have a Gibbs state ρ s.t. $\langle X_i \rangle = x_i$.

Hoping g_{ij} is positive definite $\Rightarrow$ $g_{ij} \sim$ Riemannian metric

At $T_x\mathbb{R}^n$, ... what?

First, ~~at~~ each point on $\mathbb{R}^n$ we interpret as the mean of a random vector $\vec{X} \in T_x\mathbb{R}^n$ st. $E[\vec{X}] = x^i e_i$ whose distribution is given by ρ. Associate $|\psi_{\vec{x}}\rangle \sim \vec{X}$ & assume non-degeneracy. Then we get $\vec{X}$ with

$$\mathrm{prob}(\vec{X}) = Z^{-1} e^{\lambda^i \langle X_i \rangle_{\psi_{\vec{x}}}}$$