Modifications for the Differential Evolution algorithm

Vasileios Charilogis, Ioannis G. Tsoulos, Alexandros Tzallas, Evangelos Karvounis

Department of Informatics and Telecommunications, University of Ioannina, Greece

Abstract

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1 Introduction

The location of the global minimum of a continuous and differentiable function $f : S \to R, S \subset R^n$ is formulated as

$$x^* = \mathbf{arg} \min_{x \in S} f(x) \tag{1}$$

where the set S is defined as

$$S = [a_1, b_1] \otimes [a_2, b_2] \otimes \ldots [a_n, b_n]$$

In the recent literature there are a plethora of real world problems that can formulated as global optimization problems such as problems from the physics area [1, 2, 3, 4], chemistry [5, 6, 7], economics [8, 9] etc. There are a variety of proposed methods to handle the global minimum problem such as Adaptive Random Search [10], Completive Evolution [11], Controlled Random Search [12], Simulated Annealing [13, 14, 16], Genetic Algorithms [17, 18], Differential Evolution [21], Particle Swarm Optimization [23], Bee optimization [19, 20], Ant Colony Optimization [21] etc. **The method of Differential Evolution is modified here ....**

After a literature review it was found that differential evolution is used in several areas and many modifications of the original algorithm have been introduced in the recent literature. More specifically, in the research of Zongjun et al [24], genetic and differential calculus algorithms were used to optimize the parameters of two models aimed at estimating evapotranspiration in three regions and it was found that the performance of evolution algorithms was better than the genetic algorithm. Another research focuses on a case study of a cellular neural network aimed at generating fractional classes of neurons. The best solutions provided by differential calculus and the use of accelerated particle swarm optimization (APSO) are presented concretely in the work of Tlelo-Cuautle et al [25]. Another article [26] proposes a regeneration framework based on space search adaptation (ARSA), which can be integrated into different variants of different evolution to address the problems of early convergence and population stability faced by different calculus. Also another interesting variation of the method is the Bernstain Search Differential Evolution Algorithm [27] for optimizing numerical functions. Another interesting algorithm is a new design of differential evolution to solve the travel salesman problem [28].

The differential evolution method was applied also to energy science. Specifically, the article of Liang et al [29] evaluates the parameters of solar photovoltaic models through a self-adjusting differential evolution. Similarly, in the study of Peng et al [30], differential evolution is used for the prediction of electricity prices. Also, differential evolution was also incroporated in neural architecture search [31].

The rest of this article is organized as follows....

2 Modifications

2.1 The base algorithm

2.2 The new termination rule

2.3 The new crossover scheme

3 Experiments

In order to measure the effectiveness of the proposed approach we utilize several benchmark functions from the relevant literature [32, 33].

3.1 Test functions

- **Bf1** function. The function Bohachevsky 1 is given by the equation

$$f(x) = x_1^2 + 2x_2^2 - \frac{3}{10}\cos(3\pi x_1) - \frac{4}{10}\cos(4\pi x_2) + \frac{7}{10}$$

with $x \in [-100, 100]^2$. The value of global minimum is 0.0.

- **Bf2** function. The function Bohachevsky 2 is given by the equation

$$f(x) = x_1^2 + 2x_2^2 - \frac{3}{10}\cos(3\pi x_1)\cos(4\pi x_2) + \frac{3}{10}$$

  with $x \in [-50, 50]^2$. The value of the global minimum is 0.0.

- **Branin** function. The function is defined by $f(x)=\left(x_2-\frac{5.1}{4\pi^2}x_1^2+\frac{5}{\pi}x_1-6\right)^2+10\left(1-\frac{1}{8\pi}\right)\cos(x_1)+10$ with $-5\le x_1\le 10,\ 0\le x_2\le 15$. The value of global minimum is 0.397887.with $x\in[-10,10]^2$. The value of global minimum is -0.352386.

- **CM** function. The Cosine Mixture function is given by the equation

$$f(x)=\sum_{i=1}^{n}x_i^2-\frac{1}{10}\sum_{i=1}^{n}\cos\left(5\pi x_i\right)$$

  with $x\in[-1,1]^n$. The value of the global minimum is -0.4 and in our experiments we have used $n=4$.

- **Camel** function. The function is given by

$$f(x)=4x_1^2-2.1x_1^4+\frac{1}{3}x_1^6+x_1x_2-4x_2^2+4x_2^4,\quad x\in[-5,5]^2$$

  The global minimum has the value of $f\left(x^*\right)=-1.0316$

- **DiffPower** function. The Sum of Different Powers function is defined

$$f(x)=\sum_{i=1}^{n}|x_i|^{i+1}$$

  and the global minimum is is $f\left(x^*\right)=0$. The value $n=10$ was used in the conducted experiments and the associated function is denoted as Diffpower10.

- **Easom** function. The function is given by the equation

$$f(x)=-\cos\left(x_1\right)\cos\left(x_2\right)\exp\left(\left(x_2-\pi\right)^2-\left(x_1-\pi\right)^2\right)$$

  with $x\in[-100,100]^2$. The value of the global minimum is -1.0

- **Exponential** function. The function is given by

$$f(x)=-\exp\left(-0.5\sum_{i=1}^{n}x_i^2\right),\quad -1\le x_i\le 1$$

  The global minimum is located at $x^*=(0,0,...,0)$ with value -1. In our experiments we used this function with $n=8,32$ and the corresponding functions are denoted by the labels EXP8,EXP32.

- **Griewank2** function. The function is given by

$$f(x)=1+\frac{1}{200}\sum_{i=1}^{2}x_i^2-\prod_{i=1}^{2}\frac{\cos(x_i)}{\sqrt{(i)}},\quad x\in[-100,100]^2$$

  The global minimum is located at the $x^*=(0,0,...,0)$ with value 0.

- **Griewank10** function. The function is given by the equation

$$f(x) = \sum_{i=1}^{n} \frac{x_i^2}{4000} - \prod_{i=1}^{n} \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$$

  In our experiments we have used $n = 10$ and the global minimum is 0.0 The function has several local minima in the specified range.

- **Gkls** function. $f(x) = \mathrm{Gkls}(x, n, w)$, is a function with w local minima, described in [34] with $x \in [-1, 1]^n$ and n a positive integer between 2 and 100. The value of the global minimum is -1 and in our experiments we have used $n = 2, 3$ and $w = 50$. The corresponding functions are denoted by the labels GKLS250 and GKLS350.

- **Hansen** function. $f(x) = \sum_{i=1}^{5} i \cos\left[(i-1)x_1 + i\right] \sum_{j=1}^{5} j \cos\left[(j+1)x_2 + j\right]$, $x \in [-10, 10]^2$. The global minimum of the function is -176.541793.

- **Hartman 3** function. The function is given by

$$f(x) = -\sum_{i=1}^{4} c_i \exp\left(-\sum_{j=1}^{3} a_{ij} \left(x_j - p_{ij}\right)^2\right)$$

  with $x \in [0, 1]^3$ and $a = \begin{pmatrix} 3 & 10 & 30 \\ 0.1 & 10 & 35 \\ 3 & 10 & 30 \\ 0.1 & 10 & 35 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ 1.2 \\ 3 \\ 3.2 \end{pmatrix}$ and

$$p = \begin{pmatrix} 0.3689 & 0.117 & 0.2673 \\ 0.4699 & 0.4387 & 0.747 \\ 0.1091 & 0.8732 & 0.5547 \\ 0.03815 & 0.5743 & 0.8828 \end{pmatrix}$$

  The value of global minimum is -3.862782.

- **Hartman 6** function.

$$f(x) = -\sum_{i=1}^{4} c_i \exp\left(-\sum_{j=1}^{6} a_{ij} \left(x_j - p_{ij}\right)^2\right)$$

  with $x \in [0, 1]^6$ and $a = \begin{pmatrix} 10 & 3 & 17 & 3.5 & 1.7 & 8 \\ 0.05 & 10 & 17 & 0.1 & 8 & 14 \\ 3 & 3.5 & 1.7 & 10 & 17 & 8 \\ 17 & 8 & 0.05 & 10 & 0.1 & 14 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ 1.2 \\ 3 \\ 3.2 \end{pmatrix}$

  and

$$p = \begin{pmatrix} 0.1312 & 0.1696 & 0.5569 & 0.0124 & 0.8283 & 0.5886 \\ 0.2329 & 0.4135 & 0.8307 & 0.3736 & 0.1004 & 0.9991 \\ 0.2348 & 0.1451 & 0.3522 & 0.2883 & 0.3047 & 0.6650 \\ 0.4047 & 0.8828 & 0.8732 & 0.5743 & 0.1091 & 0.0381 \end{pmatrix}$$

  the value of global minimum is -3.322368.

- **Potential** function. The molecular conformation corresponding to the global minimum of the energy of N atoms interacting via the Lennard-Jones potential[35] is used as a test case here. The function to be minimized is given by:

$$V_{LJ}(r) = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right] \tag{2}$$

  In the current experiments three different cases were studied: $N = 5,\ 10,\ 20$

- **Rastrigin** function. The function is given by

$$f(x) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2), \quad x \in [-1,1]^2$$

  The global minimum is located at $x^* = (0,0)$ with value -2.0.

- ***Rosenbrock** function.*
  This function is given by

$$f(x) = \sum_{i=1}^{n-1}\left(100\left(x_{i+1} - x_i^2\right)^2 + \left(x_i - 1\right)^2\right), \quad -30 \le x_i \le 30.$$

  The global minimum is located at the $x^* = (0,0,...,0)$ with $f(x^*) = 0$. In our experiments we used this function with $n = 2$.

- **Shekel 7** function.

$$f(x) = -\sum_{i=1}^{7}\frac{1}{(x-a_i)(x-a_i)^T + c_i}$$

with $x \in [0,10]^4$ and $a = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 \\ 3 & 7 & 3 & 7 \\ 2 & 9 & 2 & 9 \\ 5 & 3 & 5 & 3 \end{pmatrix}$, $c = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \\ 0.6 \\ 0.3 \end{pmatrix}$. The value of global minimum is -10.342378.

- **Shekel 5** function.

$$f(x) = -\sum_{i=1}^{5}\frac{1}{(x-a_i)(x-a_i)^T + c_i}$$

with $x \in [0,10]^4$ and $a = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 \\ 3 & 7 & 3 & 7 \end{pmatrix}$, $c = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \end{pmatrix}$. The value of global minimum is -10.107749.

- **Shekel 10** function.

$$f(x) = -\sum_{i=1}^{10} \frac{1}{(x - a_i)(x - a_i)^T + c_i}$$

with $x \in [0, 10]^4$ and $a = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 \\ 3 & 7 & 3 & 7 \\ 2 & 9 & 2 & 9 \\ 5 & 5 & 3 & 3 \\ 8 & 1 & 8 & 1 \\ 6 & 2 & 6 & 2 \\ 7 & 3.6 & 7 & 3.6 \end{pmatrix}$, $c = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \\ 0.6 \\ 0.3 \\ 0.7 \\ 0.5 \\ 0.6 \end{pmatrix}$. The value of global minimum is -10.536410.

- **Sinusoidal** function. The function is given by

$$f(x) = -\left(2.5 \prod_{i=1}^{n} \sin(x_i - z) + \prod_{i=1}^{n} \sin(5(x_i - z))\right), \quad 0 \le x_i \le \pi.$$

The global minimum is located at $x^* = (2.09435, 2.09435, ..., 2.09435)$ with $f(x^*) = -3.5$. In our experiments we used $n = 8, 32$ and $z = \frac{\pi}{6}$ and the corresponding functions are denoted by the labels SINU8 and SINU32 respectively.

- **Test2N** function. This function is given by the equation

$$f(x) = \frac{1}{2} \sum_{i=1}^{n} x_i^4 - 16x_i^2 + 5x_i, \quad x_i \in [-5, 5].$$

The function has 2^n in the specified range and in our experiments we used $n = 4, 5, 6, 7$. The corresponding values of global minimum is -156.664663 for $n = 4$, -195.830829 for $n = 5$, -234.996994 for $n = 6$ and -274.163160 for $n = 7$.

- **Test30N** function. This function is given by

$$f(x) = \frac{1}{10} \sin^2(3\pi x_1) \sum_{i=2}^{n-1} \left((x_i - 1)^2 \left(1 + \sin^2(3\pi x_{i+1})\right)\right) + (x_n - 1)^2 \left(1 + \sin^2(2\pi x_n)\right)$$

with $x \in [-10, 10]$. The function has 30^n local minima in the specified range and we used $n = 3, 4$ in our experiments. The value of global minimum for this function is 0.0

Table 1: Experiments with the termination rule of Ali.

FUNCTION	**STATIC**	**ALI**	**PROPOSED**
BF1	1142	1431	847
BF2	1164	1379	896
BRANIN	984	816	707
CM4	3590	7572	2079
CAMEL	1094	18849	685
EASOM	1707	2014	1327
EXP2	532	323	449
EXP4	2421	1019	1494
EXP8	15750	3670	5632
EXP16	160031	15150	21416
EXP32	320039	152548	77936
GKLS250	784	944	614
GKLS2100	772	1531	599(0.97)
GKLS350	1906(0.93)	3263	1275(0.93)
GKLS3100	1883	3539	1373
GOLDSTEIN	988	818	769
GRIEWANK2	1299(0.97)	1403	883(0.93)
HANSEN	2398	2968	1400
HARTMAN3	1448	836	1050
HARTMAN6	9489(0.97)	4015(0.97)	4667(0.80)
POTENTIAL3	90027	89776	21824
POTENTIAL4	120387(0.97)	120405(0.33)	45705(0.97)
POTENTIAL5	150073	150104	83342
RASTRIGIN	1246	1098(0.93)	871
ROSENBROCK4	6564	9695	4499
ROSENBROCK8	44240	72228	13959
ROSENBCROK16	160349(0.90)	160538(0.60)	53594
SHEKEL5	5524	3810	3057(0.83)
SHEKEL7	5266	3558	2992(0.87)
SHEKEL10	5319	3379	3076
TEST2N4	4200	1980	2592
TEST2N5	7357	2957	4055
TEST2N6	12074	4159	5836
TEST2N7	18872	5490	7904
SINU4	3270	1855	2216
SINU8	23108	6995	8135
SINU16	160092	36044	30943
SINU32	213757(0.70)	160536(0.53)	83369(0.80)
TEST30N3	1452	1732	959
TEST30N4	1917	2287	1378
TOTAL	1564515(0.97)	1062714(0.96)	506404(0.98)

Table 2: Experiments with the proposed termination rule.

FUNCTION	**STATIC**	**ALI**	**PROPOSED**
BF1	996	1124	889
BF2	926	1026	816
BRANIN	878	900	730
CM4	1148(0.70)	1991	1103
CAMEL	1049	904(0.93)	846
EASOM	447	448	446
EXP2	470	461	467
EXP4	915	903	892
EXP8	1797	3558	1796
EXP16	3578	7082	3521
EXP32	7082	14125	7022
GKLS250	498	576	493
GKLS2100	533	884(0.97)	515
GKLS350	823	1130(0.93)	814(0.97)
GKLS3100	858	1495(0.97)	829(0.93)
GOLDSTEIN	945	993	915
GRIEWANK2	947	921	826
HANSEN	2104	1949	1479
HARTMAN3	1017	1005	952
HARTMAN6	4679(0.90)	3744(0.97)	3128(0.87)
POTENTIAL3	21473	2284	8197
POTENTIAL4	44191(0.43)	3098(0.33)	24659(0.97)
POTENTIAL5	75910	3443	52664
RASTRIGIN	841	994	777
ROSENBROCK4	4934	7192	3300
ROSENBROCK8	29583	49696	10907
ROSENBCROK16	160349	160538(0.60)	38315
SHEKEL5	4389(0.97)	4266	2839(0.83)
SHEKEL7	3905	3685	2668
SHEKEL10	4049	3548	2629
TEST2N4	2785	2275	2221
TEST2N5	4481	3170	3122
TEST2N6	6852	4286	4296
TEST2N7	11971	5701	6267
SINU4	2322	1987	1755
SINU8	9990	6156	5113
SINU16	6892	3628(0.97)	16905
SINU32	7235(0.80)	7438(0.83)	7218
TEST30N3	1033	1098	951
TEST30N4	1355	1444	1285
TOTAL	432610(0.98)	321166(0.96)	224567(0.99)

3.2 Experimental results

4 Conclusions

Compliance with Ethical Standards

All authors declare that they have no has no conict of interest.

Acknowledgments

The experiments of this research work was performed at the high performance computing system established at Knowledge and Intelligent Computing Laboratory, Dept of Informatics and Telecommunications, University of Ioannina, acquired with the project "Educational Laboratory equipment of TEI of Epirus" with MIS 5007094 funded by the Operational Programme "Epirus" 2014-2020, by ERDF and national finds.

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