

# Towards an ideal Particle Swarm Optimizer for multidimensional functions

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## Abstract

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## 1 Introduction

The global optimization problem is usually defined as:

$$x^* = \arg \min_{x \in S} f(x) \quad (1)$$

with  $S$ :

$$S = [a_1, b_1] \otimes [a_2, b_2] \otimes \dots [a_n, b_n]$$

The function  $f$  is considered continuous and differentiable function formulated as  $f : S \rightarrow R, S \subset R^n$ . This problem finds application in a variety of objective problems from the real world such as problems of physics [1, 2, 3], chemistry [4, 5, 6], economics [7, 8], medicine [9, 10] etc. Global optimization methods are grouped into two broad categories: deterministic and stochastic methods. The most common methods of the first category are the so called Interval methods [11, 12], where the set  $S$  is divided iteratively in subregions and some subregions that not contain the global solution are discarded using some pre defined criteria. In stochastic methods, the finding of the global minimum is guided by randomness. In these methods there is no guarantee to find the global minimum but they constitute the vast majority of global optimization methods, mainly due to the simplicity in their implementation. There have been proposed many stochastic methods by various researchers such as Controlled Random Search methods [13, 14, 15], Simulated Annealing methods [16, 17, 18], Differential Evolution methods [19, 20], Particle Swarm Optimization methods [21, 22, 23], Ant Colony Optimization [24, 25], Genetic algorithms [26, 27, 28] etc. Also,

many works have been apperead utilizing the modern GPU processing units [29, 30, 31].

The method of Particle Swarm Optimization is method based on a population of candidate solutions that called also particles. These particles have two basic characteristics: their position at any given time  $\vec{x}$  and the speed  $\vec{u}$  at which they move. The purpose of the method is to move the particles repetitively and their next position is calculated as a function not only of their position but also of the best position they had in the past as well as the best position of the population. The method was sucessfully used in a variety of scientific and practical problems from areas such as physics [32, 33], chemistry [34, 35], medicine [36, 37], economics [38] etc. Due to its high popularity, the method has received a number of interventions in recent years, such as combination with the mutation mechanism [39, 40, 41], improved initialization of the velocity vector [42], hybrid techniques [43, 44], parallel techniques [46, 47, 48], methods aim to improve the velocity calculation [49, 50, 51] etc. This text introduces three distinct modifications to the original method, which drastically improve the time required to find the total minimum by reducing the required number of function evaluations. These modifications cover a large part of the method: the speed calculation, a new method of avoiding running local search methods, and a new adaptive termination rule. The proposed modifications were applied to a number of problems from the relevant literature and compared with similar techniques and the results are presented below.

The rest of this article is didived as follows: in section 2 the initial method and the proposed modifications are discussed, in section 3 the experiments are listed and finally in section 4 some conclusions and guidelines for future improvements are presented.

## 2 Method description

The base algorithm of PSO and the proposed modifications are outlinen in detail in the following subsections. The discussion starts with a new mechanism that calculates the velocity of the populations, continoues with a discarding procedure used minimize the number of local searches performed and ends with a discussion about the new stopping rule proposed here.

### 2.1 The base algorithm

The base algorithm is listed below with the peridical application of local search method in order to enhance the estimation of the global minimum, ie. at every iteration a decision with probability  $p_l$  is made in order to apply a local search procedure to some of the particles. Usually, this probability is small for example 0.05 (5%).

#### 1. Initialization.

- (a) **Set** iter = 0 (iteration counter).

- (b) **Set** the number of particles  $m$ .
  - (c) **Set** the maximum number of iterations allowed  $\text{iter}_{\max}$
  - (d) **Set** the local search rate  $p_l \in [0, 1]$ .
  - (e) **Initialize** randomly the positions of the  $m$  particles  $x_1, x_2, \dots, x_m$ , with  $x_i \in S \subset R^n$
  - (f) **Initialize** randomly the velocities of the  $m$  particles  $u_1, u_2, \dots, u_m$ , with  $u_i \in S \subset R^n$
  - (g) **For**  $i = 1..m$  do  $p_i = x_i$ . The  $p_i$  vector are the best located values for every particle  $i$ .
  - (h) **Set**  $p_{\text{best}} = \arg \min_{i \in 1..m} f(x_i)$
2. **Termination Check.** Check for termination. If termination criteria are met then stop.
  3. **For**  $i = 1..m$  **Do**
    - (a) **Update** the velocity  $u_i$  as a function of  $u_i$ ,  $p_i$  and  $p_{\text{best}}$
    - (b) **Update** the position  $x_i = x_i + u_i$
    - (c) **Set**  $r \in [0, 1]$  a random number. If  $r \leq p_m$  then  $x_i = \text{LS}(x_i)$ , where  $\text{LS}(x)$  is a local search procedure.
    - (d) **Evaluate** the fitness of the particle  $i$ ,  $f(x_i)$
    - (e) **If**  $f(x_i) \leq f(p_i)$  then  $p_i = x_i$
  4. **End For**
  5. **Set**  $p_{\text{best}} = \arg \min_{i \in 1..m} f(x_i)$
  6. **Set**  $\text{iter} = \text{iter} + 1$ .
  7. **Goto** Step 2

## 2.2 Velocity calculation

The algorithm of subsection 2.1 calculates at every iteration the new position  $x_i$  is calculated using the old position  $x_i$  and the associated velocity  $u_i$  according to the scheme:

$$x_i = x_i + u_i \quad (2)$$

Typically the velocity is calculated as a combination of the old velocity and the best located values  $p_i$  and  $p_{\text{best}}$  and may be defined as:

$$u_i = \omega u_i + r_1 c_1 (p_i - x_i) + r_2 c_2 (p_{\text{best}} - x_i) \quad (3)$$

where

1. The parameters  $r_1$ ,  $r_2$  are random numbers with  $r_1 \in [0, 1]$  and  $r_2 \in [0, 1]$ .

2. The constant number  $c_1, c_2$  are in the range  $[1, 2]$ .
3. The variable  $\omega$  is called inertia, with  $\omega \in [0, 1]$ .

The inertia was proposed by Shi and Eberhart [21]. They argued that high values of the inertia coefficient cause better exploration of the search area while small values of this variable are used when we want to achieve better local research around promising areas for the global minimum. The value of the inertia factor generally starts with large values and decreases with the repetition. This article proposes a new adaptive technique for the inertia parameter and it this mechanism is compared against three others from the relevant literature.

### 2.2.1 Random inertia

The inertia calculation is proposed in [52] and it is defined as

$$\omega_{\text{iter}} = 0.5 + \frac{r}{2} \quad (4)$$

where  $r$  a random number with  $r \in [0, 1]$ . This inertia calculation will be called I1 in the tables with the experimental results.

### 2.2.2 Linear time varying inertia ( min version)

This inertia schema has been proposed in used in various studies [52, 53, 54] and it is defined as:

$$\omega_{\text{iter}} = \frac{\text{iter}_{\text{max}} - \text{iter}}{\text{iter}_{\text{max}}} (\omega_{\text{max}} - \omega_{\text{min}}) + \omega_{\text{min}} \quad (5)$$

where  $\omega_{\text{min}}$  is the minimum value of inertia and  $\omega_{\text{max}}$  the maximum value for inertia. This inertia calculation will be called I2 in the tables with the experimental results.

### 2.2.3 Linear time varying inertia ( max version )

This method is proposed in [55, 56] and it is defined as:

$$\omega_{\text{iter}} = \frac{\text{iter}_{\text{max}} - \text{iter}}{\text{iter}_{\text{max}}} (\omega_{\text{min}} - \omega_{\text{max}}) + \omega_{\text{max}} \quad (6)$$

This inertia calculation will be called I3 in the tables with the experimental results.

### 2.2.4 Proposed technique

The proposed mechanism is a stochastic one as I1 and it is defined as:

$$\omega_{\text{iter}} = \frac{1}{4 + \frac{r}{2}} \quad (7)$$

where  $r$  being a random number in  $[0, 1]$ . This mechanism will be called IP in the experiemental results.

### 2.3 The discarding procedure

The method in each iteration performs under conditional a series of local searches. However, these searches will often either lead to local minima already found or locate values far below the global minimum. This means that much of the computing time will be wasted on actions that could have been avoided. In order to be able to group points that would lead by local search to the same local minimum, we introduce the concept of cluster, which refers to a set of points that are believed, under some asymptotic considerations, to belong to the same region of attraction of the function. The region of attraction for a local minimum  $x^*$  is defined as:

$$A(x^*) = \{x : x \in S \subset R^n, \text{LS}(x) = x^*\} \quad (8)$$

where  $\text{LS}(x)$  is a local search procedure that starts from a given point  $x$  and terminates when a local minimum is discovered. The discarding procedure suggested here prevents the method from starting a local search from a point  $x$  if that point belongs to the same region of attraction with other points. This procedure is composed by two two major parts:

- The first is the typical distance that is calculated after every local search and it is given by

$$r_C = \frac{1}{M} \sum_{i=1}^M \|x_i - x_{iL}\| \quad (9)$$

where the local search procedure  $\text{LS}(x)$  initiates from  $x_i$  and  $x_{iL}$  is the outcome of  $\text{LS}(x_i)$ . If a point  $x$  is close enough to an already discovered local minima then it is highly possible that the point belongs to the so called region of attraction of the minima. .... **edo na poume pos akoma kai an den kanei local search to kanoume update se periptosi pou exei kopsei polla elaxista idi**

- The second element is a gradient check performed between a candidate starting point and an already discovered local minimum. The value of the objective function  $f(x)$  near to an already discovered local minimum can be calculated using:

$$f(x) \simeq f(z) + \frac{1}{2} (x - z)^T B (x - z) \quad (10)$$

where  $B$  is the Hessian matrix at the minimum  $z$ . By taking gradients in both sides of Equation 10 we obtain:

$$\nabla f(x) \simeq B (x - z) \quad (11)$$

Of course equation 11 holds for any other point  $y$  near to  $z$

$$\nabla f(y) \simeq B (y - z) \quad (12)$$

By subtracting the equation 12 from 11 and by multiplying with  $(x - y)^T$  we have the following equation:

$$(x - y)^T (\nabla f(x) - \nabla f(y)) \simeq (x - y)^T B (x - y)^T > 0 \quad (13)$$

The distance  $r_C$  is used with conjunction with equation 13 to reject a point. The method rejects points from being start points if they are close enough to some other located local minima and in the same time equation 13 holds. A measure to identify if a point is close enough to some other located local minima is the distance  $r_C$  calculated after every local minimization.

## 2.4 Stopping rule

### 2.4.1 Doublebox method

### 2.4.2 Proposed technique

## 3 Experiments

In order to determine the effectiveness of the proposed modifications, a series of experiments were performed on known functions from the relevant literature [57, 58]. The choice of these functions was made as they are widely used in the literature by many researchers [59, 60, 61, 62], they have quite a complex structure and in many cases have a large number of dimensions that make them ideal for study and testing.

The experiments were divided into two major categories. In the first category all the schemes for the Differential Weight were tested using the termination rule of Equation ?? and in the second category the same schemes were tested using the proposed termination criterion. Also, after every successful termination the local optimization method BFGS [63] was applied in order to get even closer to the global minimum.

### 3.1 Test functions

The description of the test functions used in the experiments has as follows:

- **Bf1** (Bohachevsky 1) function defined as:

$$f(x) = x_1^2 + 2x_2^2 - \frac{3}{10} \cos(3\pi x_1) - \frac{4}{10} \cos(4\pi x_2) + \frac{7}{10}$$

with  $x \in [-100, 100]^2$ . The value of global minimum is 0.0.

- **Bf2** (Bohachevsky 2) function defined as:

$$f(x) = x_1^2 + 2x_2^2 - \frac{3}{10} \cos(3\pi x_1) \cos(4\pi x_2) + \frac{3}{10}$$

with  $x \in [-50, 50]^2$ . The value of the global minimum is 0.0.

- **Branin** function. The function is defined by  $f(x) = (x_2 - \frac{5.1}{4\pi^2}x_1^2 + \frac{5}{\pi}x_1 - 6)^2 + 10(1 - \frac{1}{8\pi})\cos(x_1) + 10$  with  $-5 \leq x_1 \leq 10$ ,  $0 \leq x_2 \leq 15$ . The value of global minimum is 0.397887. with  $x \in [-10, 10]^2$ . The value of global minimum is -0.352386.

- **CM** function. The Cosine Mixture function is given by the equation

$$f(x) = \sum_{i=1}^n x_i^2 - \frac{1}{10} \sum_{i=1}^n \cos(5\pi x_i)$$

where  $x \in [-1, 1]^n$ . For our experiments we used  $n = 4$ .

- **Camel** function. The function is given by

$$f(x) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4, \quad x \in [-5, 5]^2$$

- **Easom** function. The function is given by the equation

$$f(x) = -\cos(x_1) \cos(x_2) \exp\left((x_2 - \pi)^2 - (x_1 - \pi)^2\right)$$

with  $x \in [-100, 100]^2$ .

- **Exponential** function, defined as:

$$f(x) = -\exp\left(-0.5 \sum_{i=1}^n x_i^2\right), \quad -1 \leq x_i \leq 1$$

The global minimum is located at  $x^* = (0, 0, \dots, 0)$  with value  $-1$ . In our experiments we used this function with  $n = 2, 4, 8, 16, 32$ .

- **Goldstein and Price function**

The function is given by the equation

$$\begin{aligned} f(x) = & \left[1 + (x_1 + x_2 + 1)^2\right. \\ & (19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)] \times \\ & [30 + (2x_1 - 3x_2)^2 \\ & (18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)] \end{aligned}$$

With  $x \in [-2, 2]^2$ . The global minimum is located at  $x^* = (0, -1)$  with value 3.0

- **Griewank2** function. The function is given by

$$f(x) = 1 + \frac{1}{200} \sum_{i=1}^2 x_i^2 - \prod_{i=1}^2 \frac{\cos(x_i)}{\sqrt{(i)}}, \quad x \in [-100, 100]^2$$

The global minimum is located at the  $x^* = (0, 0, \dots, 0)$  with value 0.

- **Gkls** function.  $f(x) = \text{Gkls}(x, n, w)$ , is a function with  $w$  local minima, described in [64] with  $x \in [-1, 1]^n$  and  $n$  a positive integer between 2 and 100. The value of the global minimum is -1 and in our experiments we have used  $n = 2, 3$  and  $w = 50, 100$ .

- **Hansen** function.  $f(x) = \sum_{i=1}^5 i \cos[(i-1)x_1 + i] \sum_{j=1}^5 j \cos[(j+1)x_2 + j]$ ,  $x \in [-10, 10]^2$ .

- **Hartman 3** function. The function is given by

$$f(x) = - \sum_{i=1}^4 c_i \exp \left( - \sum_{j=1}^3 a_{ij} (x_j - p_{ij})^2 \right)$$

with  $x \in [0, 1]^3$  and  $a = \begin{pmatrix} 3 & 10 & 30 \\ 0.1 & 10 & 35 \\ 3 & 10 & 30 \\ 0.1 & 10 & 35 \end{pmatrix}$ ,  $c = \begin{pmatrix} 1 \\ 1.2 \\ 3 \\ 3.2 \end{pmatrix}$  and

$$p = \begin{pmatrix} 0.3689 & 0.117 & 0.2673 \\ 0.4699 & 0.4387 & 0.747 \\ 0.1091 & 0.8732 & 0.5547 \\ 0.03815 & 0.5743 & 0.8828 \end{pmatrix}$$

- **Hartman 6** function.

$$f(x) = - \sum_{i=1}^4 c_i \exp \left( - \sum_{j=1}^6 a_{ij} (x_j - p_{ij})^2 \right)$$

with  $x \in [0, 1]^6$  and  $a = \begin{pmatrix} 10 & 3 & 17 & 3.5 & 1.7 & 8 \\ 0.05 & 10 & 17 & 0.1 & 8 & 14 \\ 3 & 3.5 & 1.7 & 10 & 17 & 8 \\ 17 & 8 & 0.05 & 10 & 0.1 & 14 \end{pmatrix}$ ,  $c = \begin{pmatrix} 1 \\ 1.2 \\ 3 \\ 3.2 \end{pmatrix}$

and

$$p = \begin{pmatrix} 0.1312 & 0.1696 & 0.5569 & 0.0124 & 0.8283 & 0.5886 \\ 0.2329 & 0.4135 & 0.8307 & 0.3736 & 0.1004 & 0.9991 \\ 0.2348 & 0.1451 & 0.3522 & 0.2883 & 0.3047 & 0.6650 \\ 0.4047 & 0.8828 & 0.8732 & 0.5743 & 0.1091 & 0.0381 \end{pmatrix}$$

- **Potential** function. The molecular conformation corresponding to the global minimum of the energy of N atoms interacting via the Lennard-Jones potential[65] is used as a test case here. The function to be minimized is given by:

$$V_{LJ}(r) = 4\epsilon \left[ \left( \frac{\sigma}{r} \right)^{12} - \left( \frac{\sigma}{r} \right)^6 \right] \quad (14)$$

In the current experiments three different cases were studied:  $N = 3, 4, 5$

- **Rastrigin** function. The function is given by

$$f(x) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2), \quad x \in [-1, 1]^2$$

The global minimum is located at  $x^* = (0, 0)$  with value -2.0.



- **Rosenbrock function.**

This function is given by

$$f(x) = \sum_{i=1}^{n-1} \left( 100 (x_{i+1} - x_i^2)^2 + (x_i - 1)^2 \right), \quad -30 \leq x_i \leq 30.$$

The global minimum is located at the  $x^* = (0, 0, \dots, 0)$  with  $f(x^*) = 0$ . In our experiments we used this function with  $n = 4, 8, 16$ .

- **Shekel 7 function.**

$$f(x) = - \sum_{i=1}^7 \frac{1}{(x - a_i)(x - a_i)^T + c_i}$$

with  $x \in [0, 10]^4$  and  $a = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 \\ 3 & 7 & 3 & 7 \\ 2 & 9 & 2 & 9 \\ 5 & 3 & 5 & 3 \end{pmatrix}$ ,  $c = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \\ 0.6 \\ 0.3 \end{pmatrix}$ . The value of

global minimum is -10.342378.

- **Shekel 5 function.**

$$f(x) = - \sum_{i=1}^5 \frac{1}{(x - a_i)(x - a_i)^T + c_i}$$

with  $x \in [0, 10]^4$  and  $a = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 \\ 3 & 7 & 3 & 7 \end{pmatrix}$ ,  $c = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \end{pmatrix}$ . The value of

global minimum is -10.107749.

- **Shekel 10 function.**

$$f(x) = - \sum_{i=1}^{10} \frac{1}{(x - a_i)(x - a_i)^T + c_i}$$

with  $x \in [0, 10]^4$  and  $a = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 \\ 3 & 7 & 3 & 7 \\ 2 & 9 & 2 & 9 \\ 5 & 5 & 3 & 3 \\ 8 & 1 & 8 & 1 \\ 6 & 2 & 6 & 2 \\ 7 & 3.6 & 7 & 3.6 \end{pmatrix}$ ,  $c = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.4 \\ 0.4 \\ 0.6 \\ 0.3 \\ 0.7 \\ 0.5 \\ 0.6 \end{pmatrix}$ . The value

of global minimum is -10.536410.

- **Sinusoidal** function. The function is given by

$$f(x) = - \left( 2.5 \prod_{i=1}^n \sin(x_i - z) + \prod_{i=1}^n \sin(5(x_i - z)) \right), \quad 0 \leq x_i \leq \pi.$$

The global minimum is located at  $x^* = (2.09435, 2.09435, \dots, 2.09435)$  with  $f(x^*) = -3.5$ . In our experiments we used  $n = 4, 8, 16, 32$  and  $z = \frac{\pi}{6}$  and the corresponding functions are denoted by the labels SINU4, SINU8, SINU16 and SINU32 respectively.

- **Test2N** function. This function is given by the equation

$$f(x) = \frac{1}{2} \sum_{i=1}^n x_i^4 - 16x_i^2 + 5x_i, \quad x_i \in [-5, 5].$$

The function has  $2^n$  in the specified range and in our experiments we used  $n = 4, 5, 6, 7$ . The corresponding values of global minimum is -156.664663 for  $n = 4$ , -195.830829 for  $n = 5$ , -234.996994 for  $n = 6$  and -274.163160 for  $n = 7$ .

- **Test30N** function. This function is given by

$$f(x) = \frac{1}{10} \sin^2(3\pi x_1) \sum_{i=2}^{n-1} \left( (x_i - 1)^2 (1 + \sin^2(3\pi x_{i+1})) \right) + (x_n - 1)^2 (1 + \sin^2(2\pi x_n))$$

with  $x \in [-10, 10]$ , with  $30^n$  local minima in the searc space. For our experiments we used  $n = 3, 4$ .

### 3.2 Experimental setup

particles=100, generations=100, local search rate=5%

Table 1: Experiments with the double box stopping rule, without gradient check.

FUNCTION	I1	I2	I3	IP
BF1	6807	6866	6712	6757
BF2	6102	6150	6057	6207
BRANIN	4551	4596	4470	4435
CM4	9814	10101	9580	9342
CAMEL	5055	5202	4897	5004
EASOM	2975	2788	3014	3000
EXP2	4436	4541	4377	4543
EXP4	5443	5562	5331	5290
EXP8	5682	5754	5614	5504
EXP16	5707	5799	5638	5526
EXP32	5871	5797	5769	5659
GKLS250	3973	3906	3971	3921
GKLS2100	4009	3862	4073	3958
GKLS350	4558	3965	4525(0.97)	4266
GKLS3100	4701(0.87)	4266	4361(0.90)	4465
GOLDSTEIN	10259	9145	7945	7625
GRIEWANK2	5932	6194	5700	5915
HANSEN	6386	6260	5688(0.97)	5874
HARTMAN3	4681	4694	4625	4675
HARTMAN6	14245	14091	13793	13825
POTENTIAL3	7219	7206	7532	7234
POTENTIAL4	38053	37924	38421	38897
POTENTIAL5	15196	14459	15708	15358
RASTRIGIN	5915	5797	5944(0.83)	5844
ROSENBROCK4	91574	101485	117512	76367
ROSENBROCK8	66648	61974	58831	41591
ROSENBCROK16	62029	54550	63406	55800
SHEKEL5	9119	10271(0.97)	8975	8538
SHEKEL7	9197	9831	9638	8732
SHEKEL10	10417	10449	9373	9721(0.90)
TEST2N4	8512	8272	8884	7992
TEST2N5	5793	5704	5511(0.90)	5515
TEST2N6	9797(0.93)	9731	9657(0.83)	9666(0.97)
TEST2N7	6435(0.80)	6659	6713(0.77)	5990(0.87)
SINU4	7567	7774	7334	7063
SINU8	9882	10083	9643	9331
SINU16	12750	12947	12569	12207
SINU32	20164	21112	19684(0.90)	19239
TEST30N3	6388	7942	5934	5855
TEST30N4	7611	9251	6385	8284
<b>TOTAL</b>	<b>531453(0.99)</b>	<b>532690(0.99)</b>	<b>543794(0.98)</b>	<b>475015(0.99)</b>

Table 2: Experiments with the double box stopping rule, with the gradient check enabled.

FUNCTION	I1	I2	I3	IP
BF1	3296	3038	3063	3003
BF2	2922	2762	2863	2845
BRANIN	2562	2641	2538	2564
CM4	3569	4277	2944	3230
CAMEL	2646	2854	2467	2577
EASOM	2490	2390	2479	2464
EXP2	2377	2489	2261	2315
EXP4	2456	2669	2282	2389
EXP8	2429	2671	2268	2385
EXP16	2358	2569	2227	2326
EXP32	2337	2533	2248	2312
GKLS250	2394	2535	2274	2321
GKLS2100	2384	2511	2267	2333
GKLS350	2492	2410	2212	2339
GKLS3100	2800(0.90)	2708	2648(0.83)	2571
GOLDSTEIN	3161	3701	3166	2799
GRIEWANK2	3910	4520	3543	3641
HANSEN	4409	4268	3755	4325
HARTMAN3	2423	2518	2374	2425
HARTMAN6	3913	4390	4199(0.93)	3700
POTENTIAL3	3951	4093	4482	4021
POTENTIAL4	18555	19559	19506	18691
POTENTIAL5	8771	8397	9677	9154
RASTRIGIN	3111	3244(0.97)	3031	3146
ROSENBROCK4	9729	12980	11453	8587
ROSENBROCK8	4987	6738	4688	5512
ROSENBCROK16	4410	5939	4553	4002
SHEKEL5	3906	4095	3203	3495
SHEKEL7	3119	3965	2950	3528
SHEKEL10	3497(0.97)	4464	3142(0.97)	3353
TEST2N4	3468(0.97)	4059	4167	3881(0.93)
TEST2N5	3318(0.97)	3786	2926(0.90)	3157(0.97)
TEST2N6	4523(0.93)	5046(0.93)	5537(0.83)	4066(0.97)
TEST2N7	3364(0.80)	4191(0.90)	4183(0.80)	3315(0.87)
SINU4	3173	3807	2610	3004
SINU8	3055	3742	2592	2857
SINU16	3160	3746	3854	3290
SINU32	6613	7377	6327	6450
TEST30N3	5129	6367	5605	4451
TEST30N4	5649	6441	6074	5543
<b>TOTAL</b>	<b>162816(0.99)</b>	<b>182490(0.99)</b>	<b>164638(0.98)</b>	<b>158367(0.99)</b>

Table 3: Experiments with the proposed box stopping rule, without the gradient check.

FUNCTION	I1	I2	I3	IP
BF1	5305	5326	5240	5209
BF2	4760	4841	4750	4856
BRANIN	3599	3703	3520	3443
CM4	7674	7835	7430	7057
CAMEL	3996	4131	3864	3825
EASOM	2370	2292	2425	2478
EXP2	3528	3613	3455	3675
EXP4	4292	4350	4178	4020
EXP8	4579	4632	4515	4278
EXP16	4576	4637	4505	4236
EXP32	4692	4771	4588	4296
GKLS250	3105	3065	3115	3024
GKLS2100	3193	3049	3193	3099
GKLS350	3308	3000	3560	3401
GKLS3100	2935(0.97)	2777	3158(0.83)	3088
GOLDSTEIN	5534	5595	5332	5265
GRIEWANK2	4225	4332	4413	4489
HANSEN	3865	3762	3824	3769
HARTMAN3	3724	3770	3714	3705
HARTMAN6	11901(0.97)	11829(0.97)	11386	10573
POTENTIAL3	5910	5850	6134	6501
POTENTIAL4	30880	30570	31180	30682
POTENTIAL5	12021	11643	12521	13475
RASTRIGIN	4583	4595	4625	4360
ROSENBROCK4	58299	61266	55759	35517
ROSENBROCK8	31778	30888	30989	22055
ROSENBROCK16	32719	30503	30957	24478
SHEKEL5	6806	7047(0.97)	6636	6233
SHEKEL7	6807	7001	6626	6270
SHEKEL10	6774	6987	6583	6534
TEST2N4	6111	6127	5909	5893
TEST2N5	4455(0.97)	4558	4372(0.97)	4271(0.93)
TEST2N6	7446(0.97)	7419	7218(0.87)	7122(0.93)
TEST2N7	4992(0.90)	5057	4888(0.83)	4680(0.90)
SINU4	5948	6043	5750	5229
SINU8	7965	8095	7778	6963
SINU16	10121	10252	9968	9219
SINU32	16093	16509	15663	14478
TEST30N3	4331	4953	4230	3957
TEST30N4	6290	6341	4288	4717
<b>TOTAL</b>	<b>361490(0.99)</b>	<b>363013(0.99)</b>	<b>352239(0.99)</b>	<b>310420(0.99)</b>

Table 4: Experiments with the proposed box stopping rule, with the gradient check enabled.

FUNCTION	I1	I2	I3	IP
BF1	2276	2379	2266	2250
BF2	2157	2274	2098	2191
BRANIN	2132	2178	2051	2170
CM4	3098	3717	2538	2791
CAMEL	2198	2335	1974	2058
EASOM	2007	2011	2031	2084
EXP2	1952	2030	1842	1861
EXP4	2046	2266	1877	1909
EXP8	1990	2240	1849	1879
EXP16	1944	2110	1828	1838
EXP32	1953	2126	1859	1867
GKLS250	1982	2079	1850	1900
GKLS2100	1983	2064	1859	1891
GKLS350	1882	1944	1793	1831
GKLS3100	1898	1909(0.97)	1850(0.83)	1833(0.83)
GOLDSTEIN	2523	2670	2110	2164
GRIEWANK2	2893	2885	2791	2681
HANSEN	2766	2879	2731	2804
HARTMAN3	1988	2093	1949	2015
HARTMAN6	3366	3871(0.97)	2767	3133
POTENTIAL3	3312	3487	3613	3892
POTENTIAL4	15392	16390	16223	17497
POTENTIAL5	7109	7104	7732	8477
RASTRIGIN	2591	2648	2474	2732
ROSENBROCK4	8023	12179	4433	6025
ROSENBROCK8	4376	6081	2721	3314
ROSENBCROK16	3643	4954	2746	2485
SHEKEL5	2849	3296	2274	2390
SHEKEL7	2696	3294	2262	2283
SHEKEL10	2624	3251	2338(0.93)	2359
TEST2N4	2536	2637	2427	2782
TEST2N5	2266(0.97)	2336(0.97)	2163(0.90)	2342(0.90)
TEST2N6	2724(0.93)	2832	2694(0.80)	3133(0.90)
TEST2N7	2283(0.80)	2370	2279(0.80)	2585(0.90)
SINU4	2789	3245	2228	2436
SINU8	2601	3151	2233	2348
SINU16	2721	3086	2443	2624
SINU32	4652	5135	4086	4089
TEST30N3	3031	3349	3007	2562
TEST30N4	3747	3797	3250	3237
<b>TOTAL</b>	<b>126999(0.99)</b>	<b>142682(0.99)</b>	<b>115539(0.98)</b>	<b>122742(0.99)</b>

### 3.3 Experimental results

## 4 Conclusions

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