NOTES ON BARNSLEY FERN

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1. Affine transformations

An *affine transformation* on the plane is a mapping T that preserves collinearity and ratios of distances: given two points A and B, if C is the middle of the segment $[A, B]$ then $T(C)$ is the middle of the segment $[T(A), T(B)]$. Translations, contractions, expansions, reflections, rotations, are all particular cases of affine transformations. In general, an affine transformation T maps a point (x, y) to a point (x', y') such that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

for some reals a, b, c, d, e and f, that is:

$$\begin{aligned} x' &= ax + by + e \\ y' &= cx + dy + f \end{aligned}$$

In other words, T is the composition of a *linear transformation* with coefficients a, b, c and d, and a translation with coefficients e and f.

T is totally determined by the image of three noncollinear points P_1, P_2 and P_3. Indeed, set $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$, $T(P_1) = (x'_1, y'_1)$, $T(P_2) = (x'_2, y'_2)$ and $T(P_3) = (x'_3, y'_3)$. Then a, b and e are the solutions of the system of equations

$$\begin{aligned} x_1 a + y_1 b + e &= x'_1 \\ x_2 a + y_2 b + e &= x'_2 \\ x_3 a + y_3 b + e &= x'_3 \end{aligned}$$

and c, d and f are the solutions of the system of equations

$$\begin{aligned} x_1 c + y_1 d + f &= y'_1 \\ x_2 c + y_2 d + f &= y'_2 \\ x_3 c + y_3 d + f &= y'_3 \end{aligned}$$

Both systems of equation have a unique solution iff

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \neq 0$$

which is equivalent, since a determinant does not change if one line is subtracted from another line, to

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} \neq 0$$

which is equivalent to $(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1) \neq 0$, which is equivalent to $\frac{y_2 - y_1}{x_2 - x_1} \neq \frac{y_3 - y_1}{x_3 - x_1}$, that is, P_1, P_2 and P_3 are noncollinear.

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2. Barnsley fern

Barnsley fern is a *fractal*, a *scale invariant* part of the plane, resembling the real fern `Black Spleenwort`, defined from four affine contractions, that is, affine transformations T such that for all points A and B of the plane, $d(T(A),T(B)) < d(A,B)$. These transformations are:

- T_1, which maps the fern to its yellow part, by projecting all points of the fern on the y-axis before applying a contraction of factor 0.16, sending the point (x,y) to the point (x',y') such that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0.16 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- T_2, which maps the fern to its green part, by mapping the tip of the fern to itself and the tips of the red and blue leaflets to the tips of the largest left and right green leaflets, respectively, sending the point (x,y) to the point (x',y') such that

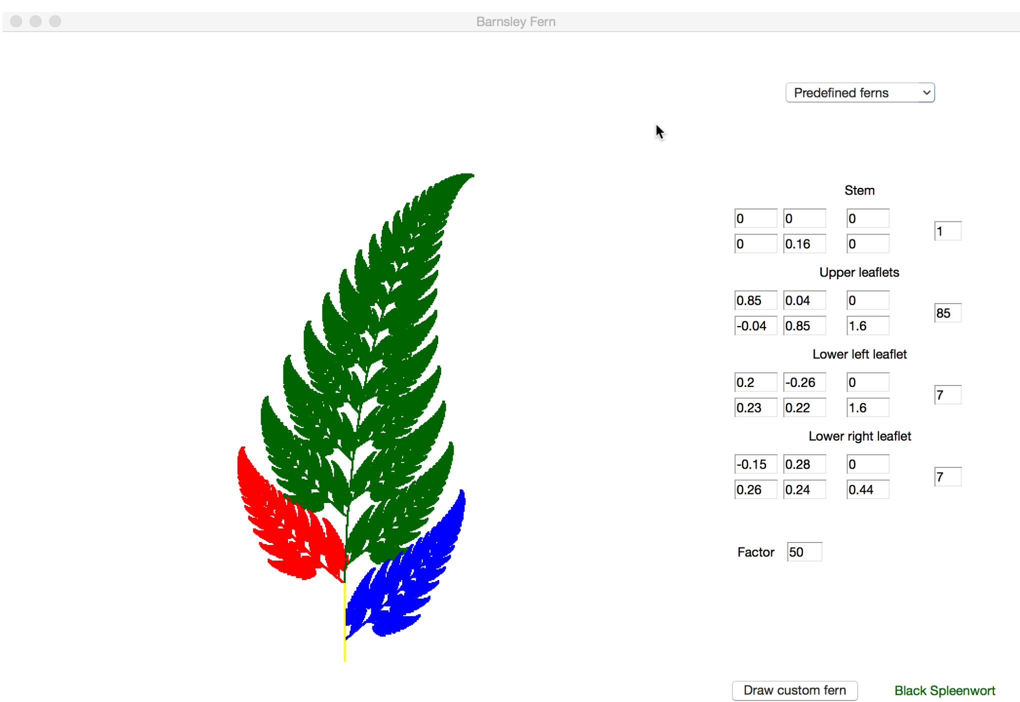
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1.6 \end{pmatrix}$$

- T_3, which maps the fern to its red part, by mapping the bottom of the stem of the fern to the bottom of the stem of the red leaflet and the tips of the red and blue leaflets to the tips of the largest left and right leaflets of the red leaflet, respectively, sending the point (x,y) to the point (x',y') such that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1.6 \end{pmatrix}$$

- T_4, which maps the fern to its blue part, by mapping the bottom of the stem of the fern to the bottom of the stem of the blue leaflet and the tips of the red and blue leaflets to the tips of the largest left and right leaflets of the blue leaflet, respectively, sending the point (x,y) to the point (x',y') such that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0.44 \end{pmatrix}$$



3. Banach's fixed point theorem

Denote by d the distance between two points of the plane, $\mathbb{R}^2$. An *iterated function system* (IFS) is a finite sequence of contracting affine transformations.

If F is the set of points that make up Barnsley fern, then $T_1\langle F\rangle \cup T_2\langle F\rangle \cup T_3\langle F\rangle \cup T_4\langle F\rangle = F$, capturing the fact that Barnsley fern is the *attractor* of the IFS (T_1, T_2, T_3, T_4) defined in the previous section: it is the unique *fixed point* of the operator T that maps a compact subset X of $\mathbb{R}^2$, that is, a bounded closed subset X of $\mathbb{R}^2$, to $T_1\langle X\rangle \cup T_2\langle X\rangle \cup T_3\langle X\rangle \cup T_4\langle X\rangle$. The existence and unicity of F follows from Banach's fixed point theorem:

> Let K be a complete metric space and let $T : K \to K$ be a contraction. Then T has a unique fixed point. Moreover, for all points X in K, $T^n(X)$ converges to this fixed point when n tends towards infinity.

To apply the theorem, we take for K the set of compact subsets of $\mathbb{R}^2$ and for the distance between two members of K, the *Hausdorff distance* d_H: given two compact subsets X_1 and X_2 of $\mathbb{R}^2$, the Hausdorff distance between X_1 and X_2 is the least real number δ such that every point in X_1 is at a distance of at most δ of some point in X_2, and every point in X_2 is at a distance of at most δ of some point in X_1:

$$d_H(X_1, X_2) = \max\bigl(\max_{x_1\in X_1} \min_{x_2\in X_2} d(x_1, x_2), \max_{x_2\in X_2} \min_{x_1\in X_1} d(x_1, x_2)\bigr)$$

For the theorem to apply, we need to verify that if $(T_1, \dots, T_n)$ is an IFS, then the function T that maps a compact subset X of $\mathbb{R}^2$ to $T_1\langle X\rangle \cup \cdots \cup T_n\langle X\rangle$ is a contraction. It suffices to show the following: if for all $i \in \{1, \dots, n\}$, T_i has a contraction factor of $r_i \in [0, 1)$, then T has a contraction factor equal to $\max(r_1, \dots, r_n)$, that is: if for all $i \in \{1, \dots, n\}$ and members x_1 and x_2 of $\mathbb{R}$, $d(T_i(x_1), T_i(x_2))$ is at most equal to $r_i d(x_1, x_2)$ then for all compact subsets X_1 and X_2 of $\mathbb{R}^2$, $d_H(T(X_1), T(X_2))$ is at most equal to $\max(r_1, \dots, r_n)d_H(X_1, X_2)$. This follows from the two lemmas that follow.

Lemma 1. *Let T be an affine contraction and $r \in [0, 1)$ be a contraction factor for T. Then the mapping, still denoted T, that maps a compact subset X of $\mathbb{R}^2$ to $\{T(x) \mid x \in X\}$ is a contraction of contraction factor r.*

Proof. Let X_1 and X_2 be two compact subsets of $\mathbb{R}^2$. It suffices to show that $d_H(T(X_1), T(X_2)) \leq r d_H(X_1, X_2)$, that is: for all $u \in X_1$, $\min_{x_2\in X_2} d(T(u), T(x_2)) \leq r d_H(X_1, X_2)$. Let $u \in X_1$ be given. Since for all $v \in X_2$, $d(T(u), T(v)) \leq r d(u, v)$, it follows that for all $v \in X_2$, $\min_{x_2\in X_2} d(T(u), T(x_2)) \leq r d(u, v)$, hence

$$\min_{x_2\in X_2} d(T(u), T(x_2)) \leq r \min_{x_2\in X_2} d(u, x_2)) \leq r \max_{x_1\in X_1} \min_{x_2\in X_2} d(x_1, x_2) \leq r d_H(X_1, X_2),$$

and we are done. □

Lemma 2. *Let A, B, C and D be compact subsets of $\mathbb{R}^2$. Then $d_H(A \cup B, C \cup D) \leq \max(d_H(A, C), d_H(B, D))$.*

Proof. It suffices to show that $d_H(A \cup B, C \cup D) \leq d_H(A, C)$ or $d_H(A \cup B, C \cup D) \leq d_H(B, D)$, that is: for all $u \in A \cup B$, $\min_{y\in C\cup D} d(u, y) \leq d_H(A, C)$ or $\min_{y\in C\cup D} d(u, y) \leq d_H(B, D)$. Let $u \in A \cup B$ be given. Since $C \cup D$ is compact, let $v \in C \cup D$ be such that $d(u, v) = \min_{y\in C\cup D} d(u, y)$. So $d(u, v) \leq \min_{y\in C} d(u, y)$ and $d(u, v) \leq \min_{y\in D} d(u, y)$. Moreover, either $u \in A$, in which case $\min_{y\in C} d(u, y) \leq \max_{x\in A} \min_{y\in C} d(x, y) = d_H(A, C)$, or $u \in B$, in which case $\min_{y\in D} d(u, y) \leq \max_{x\in B} \min_{y\in D} d(x, y) = d_H(B, D)$, and we are done. □

Let us now prove Banach's fixed point theorem, denoting by d the distance on the complete metric space K, and by r the contraction factor of T. For unicity, assume for a contradiction that X_1 and X_2 are two distinct fixed points of T. Then $T(X_1) - T(X_2) = X_2 - X_1$, so $d(T(X_1), T(X_2)) = d(X_1, X_2) \neq 0$. Moreover, since T is a contraction, $d(T(X_1), T(X_2)) \leq r d(X_1, X_2)$ for some $r \in [0, 1)$; contradiction indeed. Let X_0 be a member of K and for all $n \in \mathbb{N}$, set $X_{n+1} = T(X_n)$. To complete the proof, it suffices to show that $(X_n)_{n\in\mathbb{N}}$ converges to a member X of K, and that X is a fixed point of T. If $X_1 = X_0$ then we are done, so suppose otherwise. As K is complete, in order to show that $(X_n)_{n\in\mathbb{N}}$ converges, it suffices to verify that $(X_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, that is: for all $\epsilon > 0$, there exists $p \in \mathbb{N}$ such that for $m > p$ and $n > m$, $d(X_n, X_m) < \epsilon$. First note that for all $n \geq 1$, $d(X_{n+1}, X_n) = d(T(X_n), T(X_{n-1})) \leq r d(X_n, X_{n-1})$, so for all $n \in \mathbb{N}$, $d(X_{n+1}, X_n) \leq r^n d(X_1, X_0)$. Then note that for all $m \in \mathbb{N}$ and $n > m$,

$$\begin{aligned}
d(X_n, X_m) &= d((X_n, X_{n-1}) + (X_{n-1}, X_{n-2}) + \cdots + (X_{m+1}, X_m)) \\
&\leq d(X_n, X_{n-1}) + d(X_{n-1}, X_{n-2}) + \cdots + d(X_{m+1}, X_m) \\
&\leq (r^{n-1} + r^{n-2} + \cdots + r^m) d(X_1, X_0) \\
&\leq \frac{r^m}{1 - r} d(X_1, X_0)
\end{aligned}$$

Since $r < 1$, this implies that $(X_n)_{n\in\mathbb{N}}$ is indeed a Cauchy sequence. Now observe that T is continuous, and actually uniformly continuous: for all $\epsilon > 0$ and $X, X' \in K$, if $d(X, X') < \epsilon$ then $d(T(X), T(X')) \leq rd(X, X') < r\epsilon < \epsilon$. Since T is continuous and $(X_n)_{n\in\mathbb{N}}$ converges to some member of K, say X, $(T(X_n))_{n\in\mathbb{N}}$ converges to $T(X)$. Then $T(X) = \lim_{n\to\infty} T(X_n) = \lim_{n\to\infty} X_{n+1} = X$, completing the proof of the theorem.

4. Practical application

The fixed point of the contraction $T : K \to K$ of Banach's fixed point theorem can be obtained by successive applications of T starting from an arbitrary point of K.

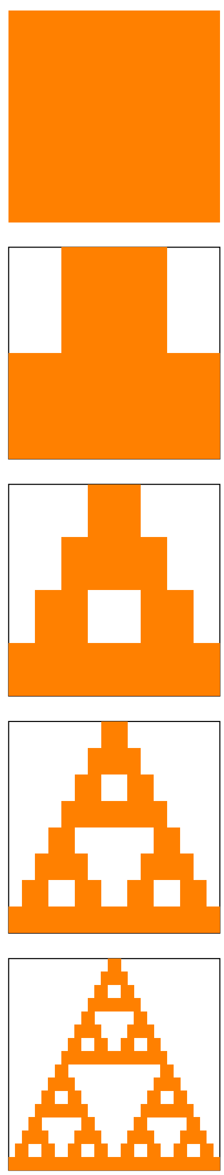
Let us see this in action with Sierpinsky triangle, which is obtained from an iterated function system consisting of three affine contractions, which are the composition of a contraction of factor 0.5, hence a linear transformation defined by the matrix

$$\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

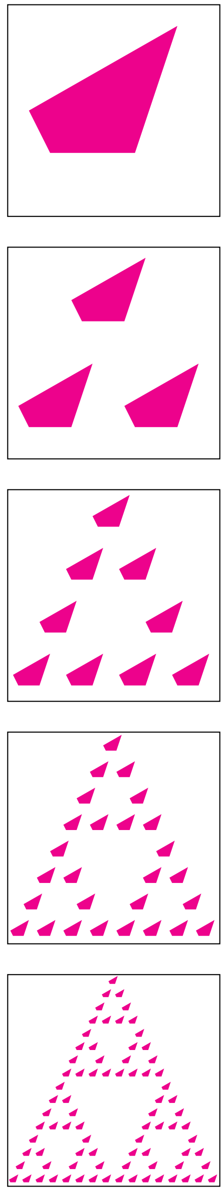
and one of the following three translations (taking the lower left corner of the triangle as origin):

- identity (for the lower left subtriangle),
- a translation by $(0.5, 0)$ (for the lower right subtriangle)
- a translation by $(0.25, 0.5)$ (for the top subtriangle)

Here is what is obtained in 4 iterations, starting with the unit square with the origin as lower left corner:



And here is what is obtained in 4 iterations, starting with some particular part of the unit square:



Any compact subset of $\mathbb{R}^2$ can be used as a starting point. It does not have to be included in the unit square; it can simply be an arbitrary point p of $\mathbb{R}^2$—indeed, $\{p\}$ is compact. Let us consider again Barnsley fern F. If we chose as starting point an arbitrary point p of $\mathbb{R}^2$ that belongs to F, then F is the topological closure of the union of

- $\{p\}$,
- $T_1\langle\{p\}\rangle$, $T_2\langle\{p\}\rangle$, $T_3\langle\{p\}\rangle$, $T_4\langle\{p\}\rangle$,
- $T_1\langle T_1\langle\{p\}\rangle\rangle$, $T_1\langle T_2\langle\{p\}\rangle\rangle$, $T_1\langle T_3\langle\{p\}\rangle\rangle$, $T_1\langle T_4\langle\{p\}\rangle\rangle$,
  $T_2\langle T_1\langle\{p\}\rangle\rangle$, $T_2\langle T_2\langle\{p\}\rangle\rangle$, $T_2\langle T_3\langle\{p\}\rangle\rangle$, $T_2\langle T_4\langle\{p\}\rangle\rangle$,
  $T_3\langle T_1\langle\{p\}\rangle\rangle$, $T_3\langle T_2\langle\{p\}\rangle\rangle$, $T_3\langle T_3\langle\{p\}\rangle\rangle$, $T_3\langle T_4\langle\{p\}\rangle\rangle$,
  $T_4\langle T_1\langle\{p\}\rangle\rangle$, $T_4\langle T_2\langle\{p\}\rangle\rangle$, $T_4\langle T_3\langle\{p\}\rangle\rangle$, $T_4\langle T_4\langle\{p\}\rangle\rangle$,
- $\ldots$

In practice, we start from $p_0 = (0,0)$, which is the point at the bottom of the stem of the fern. For all $n > 0$, we generate at stage n a point p_n, in such a way that for all $n > 0$,:

- p_{n+1} is $T_1(p_n)$ with probability 0.01,
- p_{n+1} is $T_2(p_n)$ with probability 0.85,
- p_{n+1} is $T_3(p_n)$ with probability 0.07,
- p_{n+1} is $T_4(p_n)$ with probability 0.07.

so that the number of points drawn in the yellow, green, red and blue parts of the fern are in proportion of those probabilities, respectively.

As for the factor, it is used to appropriately scale the image of the fern.

COMP9021 Principles of Programming