Revised optical air mass tables and approximation formula

Fritz Kasten and Andrew T. Young

Fritz Kasten is with Deutscher Wetterdienst, Meteorologisches Observatorium Hamburg, D-2000 Hamburg, Federal Republic of Germany; A. T. Young is with San Diego State University, Department of Astronomy, San Diego, California 92182-0334.
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We correct an error in a widely used air mass table by recalculating the values on the basis of the ISO Standard Atmosphere (1972) and revise its approximation formula.

In 1965, Kasten[1] published new tables of the air mass function and a simple approximation formula that have been widely used, particularly in meteorology and solar energy. The relative optical air mass $m(\gamma)$ at solar elevation γ is defined as

$$m(\gamma) = m_{\mathrm{abs}}(\gamma)/m_{\mathrm{abs}}(90°), \tag{1}$$

where the absolute optical air mass $m_{\mathrm{abs}}(\gamma)$ is

$$m_{\mathrm{abs}}(\gamma) = \rho_0 \int_0^\infty (\rho/\rho_0)\{1 - [1 + 2\delta_0(1 - \rho/\rho_0)] \times [\cos\gamma/(1 + h/R)]^2\}^{-1/2} dh, \tag{2}$$

and h = height above mean sea level,
$\rho = \rho(h)$ = air density at height h,
ρ_0 = air density at $h = 0$,
$\delta_0 = n_0 - 1$,
n_0 = refractive index for air at 0.7-μm wavelength at $h = 0$,
R = mean earth radius.

As $m_{\mathrm{abs}}(\gamma)$ contains the optical parameter δ_0, it is not a purely mechanical quantity. Therefore, we call $m_{\mathrm{abs}}(\gamma)$ the absolute optical air mass and $m(\gamma)$ the relative optical air mass.

The relative optical air mass is used by many different disciplines that deal with atmospheric transmission, including astronomy, atmospheric optics, geophysics, meteorology, and solar energy. Unfortunately, these different fields employ conflicting terminology and notation. The vertical distance above sea level (or, more strictly, above the geoid) is called altitude by meteorologists but height by astronomers, who use altitude to denote angular distance above the horizon. To avoid ambiguity, we, therefore, avoid altitude in favor of height, even though this is not the standard term in the meteorological community. This quantity is often denoted by z in geophysics and meteorology; but unfortunately astronomers use this symbol for zenith distance, the complement of astronomical altitude (which we call elevation throughout this paper to avoid ambiguity). As preferred astronomical usage is to give air mass as a function of zenith distance z, we must avoid z to avoid confusing our astronomical readers. As height is *hauteur* in French and *Höhe* in German, the symbol h seems natural, although h is used by astronomers for the quantity called hour angle, which is used to compute zenith distance. Furthermore, *Höhe* in German also means astronomical altitude or elevation angle. Evidently, there is no common notation or terminology that will make everyone happy. We tried to choose terms and symbols that will least confuse a potential reader, although they are not standard in some fields. But we see no really satisfactory solution to this problem.

The $m(\gamma)$ table in Ref. 1 has been computed from the air density profile of the ARDC Model Atmosphere, 1959,[2] with $\rho_0 = 1.22500$ kg/m^3, $\delta_0 = 2.76 \times 10^{-4}$, and $R = 6.371229 \times 10^6$ m, and a function $f(\gamma)$ for approximating $m(\gamma)$ was found:

$$f(\gamma) = [\sin\gamma + a \cdot (\gamma + b)^{-c}]^{-1}, \tag{3}$$

for γ in degrees. The constants a, b, and c were determined by the method of least squares of the relative errors, using a standard differential correction method for these nonlinear parameters; the values found were $a = 0.1500$, $b = 3.885°$, and $c = 1.253$.

The integrand of Eq. (2) becomes indefinite at $\gamma = 0$ as h approaches 0. In this case, the integration was performed by a special procedure, described in Appendix 1 of Kasten.[1] Unfortunately, an error seems to have crept into the computations. The value at the horizon, 36.2648, seems to be ~5% too small.

For example, the tables of Link and Neuzil[3] give 38.16 for the horizontal air mass in the 1962 U.S. Standard Atmosphere, which is very similar to the 1959 ARDC model used by Kasten. Snider and Goldman[4] give 38.10 for the 1962 model, in good agreement with Link and Neuzil. Treve,[5] using the 1959 ARDC model, got 38.11 at 0.55 μm and 38.08 at 0.70 μm for the horizontal relative air mass.

Internal evidence for the error at the horizon in Ref. 1 is given by the first differences of the old table, which increase smoothly toward the horizon but suddenly decrease in the last interval. This is a well known method for detecting errors in tables. One might have expected the error to show up as a large residual in the least-squares fitting of Eq. (3); but the point at the horizon has high leverage, so the largest residual is actually at the adjacent point in the table. Because of this, and the obscurity of most other modern air mass calculations, the error escaped notice until recently.

These discrepancies led us to recalculate the whole optical air mass table. We have taken the opportunity to replace the ARDC Model Atmosphere 1959 with the ISO Standard Atmosphere,[6] which is based on the International Civil Aviation Organization (ICAO) 1964 and U.S. 1962 Standard Atmospheres. For heights up to 32 km, the ISO Standard Atmosphere is identical to the present Standard Atmospheres of ICAO and the World Meteorological Organization. The older ARDC Model Atmosphere 1959 had generally higher temperatures and consequently lower densities than the newer standards in the stratosphere above 20 km.[7]

Moreover, the ISO Standard Atmosphere has two numerical advantages over the ARDC Model Atmosphere 1959: the height resolution is twice as fine, and the air densities are given to one more decimal place, both of which increase the accuracy of the numerical integration of Eq. (2). The constants ρ_0 and δ_0 remain unchanged, while the nominal earth radius $R = 6.356766 \times 10^6$ m is taken from Ref. 6.

In evaluating Eq. (2), finer steps $\Delta\gamma$ of solar elevation γ were chosen than in Ref. 1 (see Table 1). Several numerical integration methods were used, giving results identical with-

Table I. Step Widths $\Delta\gamma$ of Elevation γ Used in Computing the Optical Air Masses

Range of γ (deg)	$\Delta\gamma$ (deg)
0–20	0.1
20–30	0.2
30–55	0.5
55–90	1.0

Table II. Absolute Optical Air Mass $m_{abs}(\gamma)$ in kg/m^2, Relative Optical Air Mass $m(\gamma) = m_{abs}(\gamma)/m_{abs}(90°)$, and Relative Error $r(\gamma)$, Eq. (4), as Functions of Elevation Angle γ in Degrees Computed from the ISO Standard Atmosphere

γ (degrees)	$m_{abs}(\gamma)$	$m(\gamma)$	$r(\gamma)$ (%)
0.0	394428	38.0868	0.432
0.1	378596	36.5581	0.243
0.2	363727	35.1223	0.099
0.3	349750	33.7726	-0.009
0.4	336598	32.5026	-0.087
0.5	324210	31.3064	-0.141
0.6	312531	30.1786	-0.177
0.7	301509	29.1144	-0.197
0.8	291099	28.1091	-0.206
0.9	281257	27.1588	-0.205
1.0	271944	26.2595	-0.198
1.1	263125	25.4079	-0.186
1.2	254765	24.6006	-0.171
1.3	246834	23.8349	-0.152
1.4	239305	23.1078	-0.133
1.5	232151	22.4170	-0.113
1.6	225348	21.7601	-0.092
1.7	218873	21.1349	-0.072
1.8	212708	20.5395	-0.052
1.9	206831	19.9720	-0.033
2.0	201226	19.4308	-0.015
2.1	195877	18.9143	0.002
2.2	190767	18.4209	0.018
2.3	185884	17.9493	0.032
2.4	181213	17.4983	0.045
2.5	176743	17.0667	0.057
2.6	172463	16.6534	0.069
2.7	168361	16.2573	0.078
2.8	164428	15.8775	0.087
2.9	160654	15.5131	0.094
3.0	157031	15.1633	0.101
3.1	153551	14.8273	0.107
3.2	150207	14.5043	0.111
3.3	146990	14.1937	0.115
3.4	143896	13.8949	0.118
3.5	140916	13.6072	0.120
3.6	138047	13.3301	0.122
3.7	135281	13.0630	0.123
3.8	132615	12.8056	0.124
3.9	130042	12.5572	0.124
4.0	127560	12.3174	0.123
4.1	125162	12.0859	0.122
4.2	122846	11.8623	0.121
4.3	120608	11.6462	0.120
4.4	118444	11.4372	0.118
4.5	116350	11.2350	0.116
4.6	114324	11.0394	0.114
4.7	112362	10.8500	0.111
4.8	110463	10.6665	0.108
4.9	108622	10.4887	0.105
5.0	106837	10.3164	0.102
5.1	105107	10.1493	0.099
5.2	103428	9.9872	0.095
5.3	101799	9.8299	0.092
5.4	100217	9.6772	0.089
5.5	98681	9.5289	0.086
5.6	97189	9.3848	0.082
5.7	95739	9.2448	0.078
5.8	94329	9.1087	0.075
5.9	92959	8.9763	0.071
6.0	91625	8.8475	0.067
6.1	90327	8.7222	0.064
6.2	89064	8.6002	0.060
6.3	87834	8.4815	0.057
6.4	86637	8.3658	0.053
6.5	85470	8.2531	0.049
6.6	84332	8.1433	0.046
6.7	83224	8.0363	0.044
6.8	82143	7.9319	0.040
6.9	81088	7.8300	0.036
7.0	80060	7.7307	0.033
7.1	79056	7.6338	0.030
7.2	78076	7.5392	0.027
7.3	77119	7.4468	0.024
7.4	76185	7.3566	0.021
7.5	75272	7.2684	0.017
7.6	74380	7.1823	0.014
7.7	73509	7.0982	0.012
7.8	72657	7.0159	0.009
7.9	71824	6.9355	0.007
8.0	71010	6.8568	0.003
8.1	70213	6.7799	0.001
8.2	69433	6.7046	-0.001
8.3	68670	6.6310	-0.003
8.4	67924	6.5589	-0.005
8.5	67193	6.4883	-0.008
8.6	66477	6.4192	-0.010
8.7	65776	6.3515	-0.012
8.8	65089	6.2852	-0.014
8.9	64416	6.2202	-0.017
9.0	63757	6.1565	-0.019
9.1	63111	6.0942	-0.020
9.2	62478	6.0330	-0.022
9.3	61857	5.9730	-0.025
9.4	61248	5.9142	-0.027
9.5	60650	5.8565	-0.029
9.6	60065	5.8000	-0.029
9.7	59490	5.7445	-0.030
9.8	58926	5.6900	-0.033
9.9	58372	5.6365	-0.035
10.0	57829	5.5841	-0.035
10.1	57295	5.5325	-0.038
10.2	56771	5.4820	-0.038
10.3	56257	5.4323	-0.040
10.4	55752	5.3835	-0.041
10.5	55255	5.3356	-0.042
10.6	54768	5.2885	-0.043
10.7	54289	5.2422	-0.045
10.8	53818	5.1968	-0.045
10.9	53355	5.1521	-0.046
11.0	52900	5.1081	-0.048
11.1	52453	5.0649	-0.049
11.2	52013	5.0225	-0.049
11.3	51580	4.9807	-0.050
11.4	51155	4.9396	-0.052
11.5	50736	4.8992	-0.052
11.6	50325	4.8595	-0.052
11.7	49920	4.8204	-0.053
11.8	49521	4.7819	-0.054
11.9	49129	4.7440	-0.055
12.0	48743	4.7067	-0.056
12.1	48363	4.6700	-0.057
12.2	47989	4.6339	-0.057
12.3	47621	4.5984	-0.056
12.4	47258	4.5633	-0.059
12.5	46901	4.5288	-0.059
12.6	46549	4.4949	-0.058
12.7	46203	4.4614	-0.060
12.8	45861	4.4285	-0.059
12.9	45525	4.3960	-0.060
13.0	45194	4.3640	-0.060
13.1	44867	4.3325	-0.060
13.2	44545	4.3014	-0.061
13.3	44228	4.2708	-0.061
13.4	43916	4.2406	-0.062
13.5	43608	4.2108	-0.063
13.6	43304	4.1815	-0.062
13.7	43004	4.1526	-0.062
13.8	42709	4.1241	-0.062
13.9	42418	4.0959	-0.064
14.0	42130	4.0682	-0.063
14.1	41847	4.0408	-0.064
14.2	41567	4.0138	-0.064
14.3	41292	3.9872	-0.064
14.4	41020	3.9610	-0.062
14.5	40751	3.9350	-0.065

continued

in a precision of 10^{-5} except for very low solar elevations ($\gamma <$ 0.4°). Therefore, we turned to stepwise analytical integration, which is possible when the air density ρ is expressible as a linear function of height h. Thanks to the fine height resolution of the ISO Standard Atmosphere, linear interpolation between any two adjacent tabulated values $\rho(h)$ has a relative error of $<10^{-4}$.

This stepwise analytical integration, which is described in detail in Appendix 1 of Ref. 1 for the special case of $\gamma = 0$, was performed for each γ. Table II gives the results; for $\gamma \geq 0.4$, the values are identical to those obtained by numerical integration. For completeness, this table gives not only the relative optical air masses $m(\gamma)$ but also the absolute optical air masses $m_{abs}(\gamma)$ from which the relative values were derived, according to Eq. (1).

From the values in Table II, the constants in the approximating function, Eq. (3), were recalculated, using the nonlinear least-squares program GAUSSFIT, written by Jefferys *et al.*[8] By setting the estimated variances of the data proportional to the squares of the tabular values, the sum of squares of the relative errors is minimized. This is equivalent to the procedure described in Appendix 2 of Ref. 1. The resulting

Table II, continued

γ (degrees)	$m_{abs}(\gamma)$	$m(\gamma)$	$r(\gamma)$ (%)
14.6	40486	3.9095	-0.063
14.7	40225	3.8842	-0.065
14.8	39967	3.8593	-0.065
14.9	39713	3.8347	-0.065
15.0	39462	3.8105	-0.063
15.1	39213	3.7865	-0.065
15.2	38969	3.7629	-0.064
15.3	38727	3.7395	-0.065
15.4	38488	3.7165	-0.064
15.5	38252	3.6937	-0.065
15.6	38020	3.6713	-0.063
15.7	37790	3.6491	-0.063
15.8	37563	3.6271	-0.065
15.9	37339	3.6055	-0.063
16.0	37117	3.5841	-0.064
16.1	36898	3.5630	-0.063
16.2	36682	3.5421	-0.063
16.3	36469	3.5215	-0.063
16.4	36258	3.5011	-0.063
16.5	36049	3.4810	-0.062
16.6	35843	3.4611	-0.062
16.7	35639	3.4414	-0.063
16.8	35438	3.4220	-0.062
16.9	35239	3.4028	-0.061
17.0	35043	3.3838	-0.062
17.1	34848	3.3650	-0.062
17.2	34656	3.3465	-0.061
17.3	34466	3.3281	-0.062
17.4	34278	3.3100	-0.061
17.5	34093	3.2921	-0.060
17.6	33909	3.2743	-0.062
17.7	33728	3.2568	-0.061
17.8	33548	3.2395	-0.059
17.9	33371	3.2223	-0.061
18.0	33195	3.2054	-0.059
18.1	33021	3.1886	-0.060
18.2	32850	3.1720	-0.060
18.3	32680	3.1556	-0.059
18.4	32511	3.1394	-0.058
18.5	32345	3.1233	-0.059
18.6	32181	3.1074	-0.059
18.7	32018	3.0917	-0.058
18.8	31857	3.0762	-0.056
18.9	31697	3.0608	-0.056
19.0	31540	3.0455	-0.058
19.1	31384	3.0305	-0.056
19.2	31229	3.0156	-0.055
19.3	31076	3.0008	-0.056
19.4	30925	2.9862	-0.055
19.5	30775	2.9717	-0.056
19.6	30627	2.9574	-0.055
19.7	30480	2.9432	-0.056
19.8	30335	2.9292	-0.055
19.9	30191	2.9153	-0.055
20.0	30049	2.9016	-0.053
20.2	29768	2.8745	-0.053
20.4	29493	2.8479	-0.054
20.6	29224	2.8219	-0.052
20.8	28959	2.7964	-0.050
21.0	28700	2.7713	-0.051
21.2	28445	2.7467	-0.052
21.4	28196	2.7226	-0.050
21.6	27951	2.6990	-0.047
21.8	27710	2.6757	-0.049
22.0	27474	2.6529	-0.049
22.2	27242	2.6305	-0.048
22.4	27014	2.6086	-0.044
22.6	26791	2.5870	-0.044
22.8	26571	2.5658	-0.043
23.0	26355	2.5449	-0.045
23.2	26143	2.5245	-0.041
23.4	25935	2.5043	-0.044
23.6	25730	2.4846	-0.040
23.8	25529	2.4651	-0.042
24.0	25331	2.4460	-0.041
24.2	25136	2.4272	-0.041
24.4	24945	2.4088	-0.037
24.6	24757	2.3906	-0.038
24.8	24572	2.3727	-0.039
25.0	24390	2.3552	-0.035
25.2	24211	2.3379	-0.035
25.4	24035	2.3209	-0.033
25.6	23861	2.3041	-0.035
25.8	23691	2.2876	-0.036
26.0	23523	2.2714	-0.035
26.2	23358	2.2555	-0.031
26.4	23195	2.2398	-0.030
26.6	23035	2.2243	-0.031
26.8	22877	2.2091	-0.029
27.0	22722	2.1941	-0.028
27.2	22569	2.1793	-0.029
27.4	22418	2.1647	-0.031
27.6	22270	2.1504	-0.028
27.8	22124	2.1363	-0.027
28.0	21979	2.1224	-0.026
28.2	21838	2.1087	-0.025
28.4	21698	2.0952	-0.024
28.6	21560	2.0819	-0.023
28.8	21424	2.0688	-0.022
29.0	21290	2.0558	-0.025
29.2	21158	2.0431	-0.022
29.4	21028	2.0305	-0.023
29.6	20900	2.0181	-0.024
29.8	20773	2.0059	-0.022
30.0	20649	1.9939	-0.020
30.5	20344	1.9645	-0.019
31.0	20050	1.9361	-0.019
31.5	19766	1.9087	-0.015
32.0	19492	1.8821	-0.017
32.5	19226	1.8565	-0.012
33.0	18968	1.8316	-0.014
33.5	18719	1.8076	-0.010
34.0	18478	1.7843	-0.008
34.5	18244	1.7617	-0.008
35.0	18018	1.7398	-0.008
35.5	17798	1.7186	-0.005
36.0	17585	1.6980	-0.005
36.5	17378	1.6780	-0.006
37.0	17177	1.6586	-0.005
37.5	16982	1.6398	-0.003
38.0	16793	1.6215	-0.003
38.5	16609	1.6038	0.001
39.0	16430	1.5865	-0.001
39.5	16256	1.5698	0.005
40.0	16088	1.5535	0.006
40.5	15924	1.5376	0.004
41.0	15764	1.5222	0.006
41.5	15608	1.5072	0.007
42.0	15457	1.4926	0.008
42.5	15310	1.4784	0.009
43.0	15167	1.4645	0.005
43.5	15028	1.4511	0.011
44.0	14892	1.4380	0.012
44.5	14759	1.4252	0.011
45.0	14631	1.4128	0.014
45.5	14505	1.4006	0.010
46.0	14383	1.3888	0.011
46.5	14264	1.3773	0.012
47.0	14147	1.3661	0.014
47.5	14034	1.3552	0.017
48.0	13924	1.3445	0.015
48.5	13816	1.3341	0.015
49.0	13711	1.3240	0.018
49.5	13609	1.3141	0.018
50.0	13509	1.3045	0.021
50.5	13412	1.2951	0.022
51.0	13317	1.2859	0.020
51.5	13224	1.2769	0.016
52.0	13134	1.2682	0.019
52.5	13046	1.2597	0.020
53.0	12960	1.2514	0.021
53.5	12876	1.2433	0.022
54.0	12794	1.2354	0.023
54.5	12714	1.2277	0.024
55.0	12636	1.2202	0.027
56.0	12486	1.2057	0.028
57.0	12343	1.1918	0.021
58.0	12207	1.1787	0.025
59.0	12077	1.1662	0.026

continued

values are: $a = 0.50572$, $b = 6.07995°$, $c = 1.6364$. These values are much closer than the former ones to the constants determined from Bemporad's classical air mass tables, namely, $a = 0.6556$, $b = 6.379°$, and $c = 1.757$ (see Ref. 1).

The new constants give a greatly improved fit to the tabular values over the whole range of the table, because the one bad value at the horizon had corrupted the whole least-squares fit. For example, even near the zenith the error of the new fit is less than half of that of the old one. Table II gives the relative errors

$$r(\gamma) = [f(\gamma) - m(\gamma)]/m(\gamma) \tag{4}$$

of this new fit, in percent. The maximum relative error of the approximating formula is now <0.5%, several times smaller than in Ref. 1.

The tables and approximating function given here use the apparent elevation rather than that calculated from positions and times without refraction corrections, as argument, γ. As emphasized by Young,[9] substantial errors will be incurred if unrefracted elevations are used by mistake. The distinction between true and apparent elevation must be respected carefully if the accuracy of Table II and Eq. (3) is to be utilized.

Table II, continued

γ (degrees)	$m_{abs}(\gamma)$	$m(\gamma)$	$r(\gamma)$ (%)
60.0	11954	1.1543	0.027
61.0	11837	1.1430	0.028
62.0	11726	1.1322	0.025
63.0	11620	1.1220	0.026
64.0	11519	1.1123	0.027
65.0	11424	1.1031	0.027
66.0	11334	1.0944	0.029
67.0	11248	1.0862	0.034
68.0	11168	1.0784	0.035
69.0	11091	1.0710	0.033
70.0	11019	1.0640	0.028
71.0	10951	1.0575	0.032
72.0	10888	1.0513	0.027
73.0	10828	1.0456	0.033
74.0	10772	1.0402	0.031
75.0	10721	1.0352	0.032
76.0	10672	1.0305	0.027
77.0	10628	1.0262	0.027
78.0	10587	1.0223	0.033
79.0	10549	1.0187	0.034
80.0	10515	1.0154	0.032
81.0	10485	1.0124	0.028
82.0	10458	1.0098	0.031
83.0	10434	1.0075	0.032
84.0	10413	1.0055	0.031
85.0	10396	1.0038	0.030
86.0	10381	1.0024	0.027
87.0	10370	1.0014	0.033
88.0	10362	1.0006	0.029
89.0	10358	1.0002	0.034
90.0	10356	1.0000	0.029

Also, these calculations refer to the well mixed components of the molecular atmosphere and are not applicable to components (like ozone and aerosols) that are not uniformly mixed.

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