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A New Table and Approximation Formula for the Relative Optical Air Mass

By

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Summary. The mostly used table of the relative optical air mass m as function of solar altitude γ (or zenith angle $\theta = 90° - \gamma$) was computed by BEMPORAD [*1, 2, 3*] using values of the vertical air density profile up to 10 km height as they were known about the turn of the century, and taking the refractive index of the air at ground level $n_0 = 1.000293$; this index refers to light of wavelength 0.54 μ (peak of the visible spectrum) and to air of temperature 0 °C and pressure 1013.25 mb.

A new table is presented which is computed from the air density profile of the ARDC Model Atmosphere, 1959 [*4*], up to 84 km height; $n_0 = 1.000276$ is taken corresponding to air of 15 °C and 1013.25 mb (ground level values of the ARDC Model Atmosphere, 1959) and to the wavelength 0.7 μ. This wavelength is more representative for the whole solar spectrum than 0.54 μ because it divides the solar spectrum into two parts of equal energy.

A new approximation formula for the relative optical air mass m as function of solar altitude γ [deg] is presented also. The equation is $m(\gamma) = 1/[\sin \gamma + a(\gamma + b)^{-c}]$ in which $a = 0.1500$, $b = 3.885$, $c = 1.253$ are empirical constants. These constants were calculated from the new tabulated values of $m(\gamma)$ by successive approximation applying the method of least squares of the relative errors to obtain each approximation.

The values of the relative optical air mass calculated from the approximation formula are in very good agreement with the tabulated values. The deviation is less than 0.1% for $\gamma > 4$ deg. The highest deviation, 1.25%, occurs at $\gamma = 0.5$ deg.

The approximation formula can be applied as well to the old BEMPORAD table and to the table of the relative optical water vapor mass computed by SCHNAIDT [*5*]; then the constants have the values $a = 0.6556$, $b = 6.379$, $c = 1.757$, and $a = 0.0548$, $c = 2.650$, $c = 1.452$, respectively.

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Zusammenfassung. Die am meisten gebräuchliche Tabelle der relativen optischen Luftmasse m als Funktion der Sonnenhöhe γ (oder Zenitdistanz $\theta = 90° - \gamma$) wurde von BEMPORAD [1, 2, 3] berechnet, und zwar unter Benutzung von Werten des vertikalen Luftdichteprofils bis zu 10 km Höhe, wie sie um die Jahrhundertwende bekannt waren, und mit einem Brechungsindex der Luft im Bodenniveau $n_0 = 1.000293$; dieser Index bezieht sich auf Licht der Wellenlänge 0.54 μ (Maximum des sichtbaren Spektrums) und auf Luft der Temperatur 0° C und des Drucks 1013.25 mb.

Im nachfolgenden wird eine neue Tabelle vorgelegt, die aus dem Luftdichteprofil der ARDC Modell-Atmosphäre 1959 [4] bis 84 km Höhe berechnet wurde; es wurde $n_0 = 1.000276$ genommen entsprechend Luft von 15° C und 1013.25 mb (Bodenniveau-Werte der ARDC Modell-Atmosphäre 1959) und einer Wellenlänge von 0.7 μ. Diese Wellenlänge ist repräsentativer für das ganze Sonnenspektrum als 0.54 μ, weil sie das Sonnenspektrum in zwei Teile gleicher Energie teilt.

Eine neue Näherungsformel für die relative optische Luftmasse m als Funktion der Sonnenhöhe γ [grad] wird ebenfalls vorgelegt. Die Gleichung lautet $m(\gamma) = 1/[\sin \gamma + a (\gamma + b)^{-c}]$, wobei $a = 0.1500$, $b = 3.885$, $c = 1.253$ empirische Konstanten sind. Diese Konstanten wurden aus den neu tabellierten Werten von $m(\gamma)$ durch sukzessive Approximation berechnet, indem die Methode der kleinsten Quadrate der relativen Fehler angewandt wurde, um jede Näherung zu erhalten.

Die mit der Näherungsformel berechneten Werte der relativen optischen Luftmasse sind in sehr guter Übereinstimmung mit den tabellierten Werten. Die Abweichung ist kleiner als 0.1% für $\gamma > 4$ grad. Die höchste Abweichung, 1.25%, tritt bei $\gamma = 0.5$ Grad auf.

Die Näherungsformel kann ebensogut auf die alte Bemporad-Tabelle und auf die Tabelle der relativen optischen Wasserdampfmasse, die von SCHNAIDT [5] berechnet wurde, angewandt werden; die Konstanten haben dann die Werte $a = 0.6556$, $b = 6.379$, $c = 1.757$, bzw. $a = 0.0548$, $b = 2.650$, $c = 1.452$.

Résumé. La table calculée par BEMPORAD [1, 2, 3] est la plus utilisée pour établir la masse optique relative de l'air m en fonction de la hauteur du soleil γ (ou angle au zénith $\theta = 90° - \gamma$). Cette table est basée sur des profils verticaux de la densité de l'air jusqu'à 10 km d'altitude tels qu'ils étaient connus au début du siècle. L'indice de réfraction au sol utilisé est $n_0 = 1{,}000293$. Cet indice se rapporte à une longueur d'ondes de la lumière de 0,54 μ (maximum du spectre visible), à une température de 0 °C et à une pression de 1013,25 mb.

On présente ici une nouvelle table calculée, elle, selon le profil de densité de l'air résultant du modèle atmosphérique ARDC 1959 [4] et atteignant 84 km d'altitude. Dans ce cas, $n_0 = 1{,}000276$ correspondant à de l'air ayant au sol 15,0 °C de température et 1013,25 mb de pression (valeurs au sol du modèle atmosphérique ARDC 1959) et à une longueur d'ondes de 0,7 μ. Cette longueur d'ondes est plus représentative que 0,54 μ pour le spectre solaire entier, car elle divise le dit spectre solaire en deux parties d'égales énergies. On présente également une formule permettant de calculer approximativement la masse optique relative de l'air m en fonction de la hauteur du soleil γ (en degrés). Cette formule a la forme:

$$m(\gamma) = 1/[\sin \gamma + a (\gamma + b)^{-c}]$$

où $a = 0{,}1500$, $b = 3{,}885$ et $c = 1{,}253$, a, b et c étant des constantes empiriques. Ces dernières furent calculées par approximations successives en partant

des valeurs des nouvelles tables donnant $m(\gamma)$. Pour ce faire, on a utilisé la méthode du plus petit carré de l'erreur relative afin d'obtenir chaque approximation. Les valeurs de la masse optique relative de l'air, calculées par la dite formule d'approximation, correspondent parfaitement à celles de la table. La différence en est inférieure à 0,1% pour $\gamma > 4$ degrés. La plus grande différence est de 1,25% et se rencontre pour $\gamma = 0{,}5°$.

Cette formule d'approximation peut aussi très bien être utilisée pour calculer des valeurs se rapprochant soit de l'ancienne table de BEMPORAD, soit de la table établie par SCHNAIDT [5] pour la masse optique relative de la vapeur d'eau. Les constantes empiriques sont alors: $a = 0{,}6556$, $b = 6{,}379$, $c = 1{,}757$ pour la table de BEMPORAD, respectivement $a = 0{,}0548$, $b = 2{,}650$ et $c = 1{,}452$ pour celle de SCHNAIDT.

1. Introduction

Extraterrestrial monochromatic solar radiation $I_{0\lambda}$ is attenuated along its path through the atmosphere according to

$$I_\lambda = I_{0\lambda} \exp\left[-a_\lambda\, m(\gamma)\right] \tag{1}$$

where I_λ = monochromatic direct solar radiation received at the earth surface at solar altitude γ, a_λ = monochromatic extinction coefficient of the whole atmosphere at vertical incidence ($\gamma = 90$ deg), and $m(\gamma)$ = relative optical air mass at solar altitude γ, which is a measure of the extension of the optical path with respect to the path at $\gamma = 90$ deg.

The *absolute optical air mass* m_{abs} is defined by

$$m_{\text{abs}} = \int_0^\infty \rho \,\mathrm{d}s \tag{2}$$

where $\mathrm{d}s$ = geometrical path element of the light ray from the sun, and ρ = air density at $\mathrm{d}s$. When the sun is in the zenith, the light path goes straight downward and $\mathrm{d}s$ equals the height element $\mathrm{d}h$. Thus

$$m_{\text{abs, v}} = \int_0^\infty \rho \,\mathrm{d}h\,. \tag{3}$$

This is the mass of a vertical air column of unit cross section, or its weight divided by the acceleration of gravity. Since weight per unit area = pressure, we have

$$\int_0^\infty \rho \,\mathrm{d}h = p_0/g_0 \tag{4}$$

where p_0 and g_0 = air pressure and acceleration of gravity, respectively, at ground level. Eq. (4) is often written

$$\int_0^\infty \rho \,\mathrm{d}h = \rho_0\, H \tag{5}$$

where ρ_0 = air density at ground level and $H = p_0/(g_0\, \rho_0)$ = scale height (height of a homogeneous atmosphere of density ρ_0).

The *relative optical air mass m* is defined as

$$m = \frac{m_{\mathrm{abs}}}{m_{\mathrm{abs,\,v}}} = \frac{\int_0^\infty \rho \, \mathrm{d}s}{\int_0^\infty \rho \, \mathrm{d}h} = \frac{\int_0^\infty \rho \, \mathrm{d}s}{\rho_0 H} . \tag{6}$$

Absolute and relative air mass mainly depend on the solar altitude γ. Neglecting the curvature of the atmosphere and atmospheric refraction, $\mathrm{d}s = \mathrm{d}h/\sin\gamma$ and, therefore, from eq. (6):

$$m = \frac{1}{\sin\gamma} . \tag{7}$$

This approximations holds for $\gamma > 30$ deg within 0.25%. Considering the curvature of the atmosphere, the expression for $\mathrm{d}s$ is more complicated and leads to

$$m = \frac{1}{\rho_0 H} \int_0^\infty \frac{\rho \, \mathrm{d}h}{\sqrt{1 - \left(\frac{R}{R+h}\right)^2 \cos^2\gamma}} \tag{8}$$

where R = mean earth radius and h = height above the earth surface. Taking into account atmospheric refraction, eq. (8) finally becomes

$$m = \frac{1}{\rho_0 H} \int_0^\infty \frac{\rho \, \mathrm{d}h}{\sqrt{1 - \left(\frac{R}{R+h}\right)^2 \left(\frac{n_0}{n}\right)^2 \cos^2\gamma}} \tag{9}$$

where n = refractive index at height h, n_0 = refractive index at ground level ($h = 0$), and now γ = apparent (observed) solar altitude (see e. g. Linke [*6*]). Bemporad [*1, 2, 3*] used this equation and the then known values of the air density distribution with height to calculate the relative optical air mass m as a function of γ. These tables, sometimes called Bemporad function, are still generally used, cf. the handbooks [*7, 8, 9*].

Hulburt [*10*] computed relative optical air masses assuming the air density to decrease exponentially with height and neglecting the ray curvature. Link and Sekera [*11*] used vertical air density profiles as known about 1940 from balloon soundings, sound propagation, and twilight measurements to compute tables of m separately for mean summer and winter conditions. Rossow [*12*] gave analytical formulas to compute m under the assumptions of a constant temperature gradient in the troposphere, an isothermal stratosphere, and the ray curvature being neglected.

Recently, new values of the air density up to great heights have become available through radiosonde, rocket, and satellite measurements, and are compiled in the so-called Standard Atmospheres. From the ARDC Model Atmosphere, 1959, ELY [13] calculated absolute air masses m_{abs} using the integral in eq. (8), i. e. neglecting atmospheric refraction. We have now computed new values of the optical air mass, also using the ARDC Model Atmospere, 1959 but taking atmospheric refraction into account.

2. Refractive Index of the Air for Solar Radiation

A gas of density ρ has a refractive index n for monochromatic light which is given by

$$\frac{1}{\rho}\,\frac{n^2-1}{n^2+2} = \text{const.} \tag{10}$$

Since n is little greater than 1, we can set up

$$n = 1 + \delta \tag{11}$$

with $0 < \delta \ll 1$. Substitution of eq. (11) into eq. (10), neglecting quadratic δ-terms, yields

$$\frac{\delta}{\rho} \equiv \frac{n-1}{\rho} = \text{const}. \tag{12}$$

If the refractive index n_0 for a certain density ρ_0, at ground level for instance, is known, the constant in eq. (12) is determined:

$$\frac{n-1}{\rho} = \frac{n_0-1}{\rho_0} \tag{13}$$

or

$$n = 1 + \delta_0\,\frac{\rho}{\rho_0} \tag{14}$$

where according to eq. (11), δ_0 is defined by $n_0 = 1 + \delta_0$. In the air mass formula eq. (9) we have to know the quantity $(n_0/n)^2$. Using eq. (14) and again neglecting quadratic terms of δ and δ_0, one obtains:

$$\left(\frac{n_0}{n}\right)^2 = 1 + 2\,\delta_0\left(1 - \frac{\rho}{\rho_0}\right). \tag{15}$$

Now the dependence of n on the wavelength λ of the incident light must be investigated. BEMPORAD computed his air mass table mainly for astronomical purposes; the air mass was used as a measure of the decrease of the visual brightness of stars with increasing zenith distance. Therefore the refractive index for light of wavelength $\lambda = 0.54\ \mu$, the peak of the visible spectrum, was selected which amounts to $n_0 = 1.000293$ for air of temperature 0 °C and pressure 760 mm Hg (1023.25 mb).

Considering the air mass with regard to the total solar energy transmitted through the atmosphere, the mean refractive index for visible light only is not appropriate but has to be replaced by a value representing the whole solar spectrum. A representative wavelength for the total solar spectrum is that wavelength λ_s which divides the spectrum into two parts of equal energy. For any black-body radiator the relation

$$T \lambda_s = 4.10 \times 10^3 \tag{16}$$

holds (HOFMANN [*14*]) where T is the temperature in °K and λ_s in μ. The sun can be conceived as a black body of temperature 5793 °K (GEIGER [*15*]) which, according to eq. (16), yields a representative wavelength λ_s of the total solar radiation of approximately 0.7 μ.

CAMPEN, CUNNINGHAM, and PLANK [*16*] tabulated the quantity

$$N = 10^6 (n - 1) \equiv 10^6 \delta \tag{17}$$

as a function of wavelength of light and temperature of the transmitting air for ground pressure $p_0 = 1013.25$ mb. The ARDC Model Atmosphere, 1959 assumes a temperature at ground level of 15 °C and also a ground level pressure of 1013.25 mb. For this temperature and $\lambda_s = 0.7\ \mu$, one finds from the table of CAMPEN et al. $N_0 = 276$. To give more decimal places would be meaningless because of the approximated value of the wavelength λ_s; however, the rounded value $N_0 = 276$ covers the refraction in the wavelength range from 0.64 through 0.73 μ anyway. Thus we take $\delta_0 = 2.76 \times 10^{-4}$ or

$$n_0 = 1.000276 \tag{18}$$

as the refractive index of the atmosphere at ground level, which is representative of the total energy spectrum of the direct solar radiation.

3. Integration of the Air Mass Formula

The air mass formula eq. (9) is rewritten by replacing $(n_0/n)^2$ from eq. (15) and $\rho_0 H$ from eq. (5):

$$m(\gamma) = \left(\frac{1}{\int_0^\infty \rho \, \mathrm{d}h}\right) \int_0^\infty \frac{\rho \, \mathrm{d}h}{\sqrt{1 - \left\{1 + 2\delta_0\left[1 - \left(\frac{\rho}{\rho_0}\right)\right]\right\}\left\{\left(\frac{R}{R+h}\right)\cos\gamma\right\}^2}} \tag{19}$$

where ρ_0 = air density at ground level of the ARDC Model Atmosphere, 1959, $= 1.2250$ kg m^{-3} and R = mean earth radius = 6371.229 km (HEISKANEN and PEOPLES [*17*]). The integration was performed with solar altitude γ as parameter, using the air densities ρ tabulated in the ARDC Model Atmosphere, 1959. The integration was done on the basis of Simpson's rule by means of an electronic computer. The programming system used allows for an accuracy of 12 significant digits. As the upper limit of the integral, $h = 84$ km was chosen; at this height, $\rho \approx 10^{-5}$ kg m^{-3}, which is one power of ten lower than the last decimal of $\rho_0 = 1.2250$ kg m^{-3} given by the ARDC atmosphere. The step width in height

h was chosen as $\Delta h = 0.1$ km for $0 \leqslant h \leqslant 19.6$ km, $\Delta h = 0.2$ km for $19.6 \leqslant h \leqslant 50$ km, and $\Delta h = 0.5$ km for $50 \leqslant h \leqslant 84$ km.

The absolute vertical air mass $m_{\mathrm{abs,v}}$ was also computed by numerical integration on the basis of this model and amounts to

$$m_{\mathrm{abs,\,v}} = \int_0^{84\mathrm{km}} \rho \,\mathrm{d}h = 10330.7 \;\mathrm{kg}\; m^{-2}. \tag{20}$$

This value is slightly lower than the exact mass of the atmosphere per unit area which is

$$\frac{p_0}{g_0} = 10332.3 \;\mathrm{kg}\; m^{-2} \tag{21}$$

where $p_0 = 1013.25$ mb = ground level pressure at 15 °C and $g_0 = 980.665$ cm sec^{-2} = acceleration of gravity at ground level and 45 deg geographical latitude. Nevertheless, the value given in eq. (20) was used for $\int_0^\infty \rho \,\mathrm{d}h$ in eq. (19) because it is consistent with the computation of the other integral; in particular, m (90 deg) = 1 is guaranteed by using this value.

Table 1. *Step Widths $\Delta\gamma$ of Solar Altitude γ Used in Computation of the Relative Optical Air Mass $m(\gamma)$*

γ-range [deg]	$\Delta\gamma$ [deg]
0—3	0.5
3—20	0.1
20—30	0.2
30—50	0.5
50—75	1
75—90	5

To facilitate comparison of the results with BEMPORAD's table of air masses as published by MÖLLER [8], the solar altitude γ, which is the parameter in the second integral in eq. (19), was varied by the same step widths as used in that table, see Table 1.

The computation of m (0 deg) requires special attention because for the lower limit $h = 0$ of the second integral in eq. (19),the integrand becomes infinite in the case of $\gamma = 0$, so that numerical integration is not possible. This difficulty is overcome by the method outlined in Appendix 1.

4. Results of the Numerical Integration

The results of the new computations are listed in Table 2. They are rounded off to the fourth decimal place. The fourth decimal is uncertain and should be considered as a rounding digit only. For the air densities of the ARDC Model Atmosphere are given with four decimals, and the air mass formula eq. (19) contains practically a ratio of two densities which doubles the relative error. — The absolute optical air masses can be obtained from Table 2 by multiplying the values by the factor $m_{\mathrm{abs,\,v}} = 10330.7$ kg m^{-2}.

For comparison, Table 3 shows the relative optical air masses for a few selected solar altitudes from the present computations, after BEMPORAD, and according to ELY. ELY calculated *absolute* air masses and presents them graphically. Therefore, the obtainable values are rather inaccurate;

Table 2. *Relative Optical Air Mass m (γ) as a Function of Solar Altitude* γ, *Computed on the Basis of the ARDC Model Atmosphere, 1959.*
Underline indicates figure was smaller than 5 beforing rounding

γ [deg]	m (γ)	γ [deg]	m (γ)	γ [deg]	m (γ)
0	36.2648	7	7.7334	11.6	4.8602
0.5	31.3898	7.1	7.6364	11.7	4.8210
1	26.3150	7.2	7.5417	11.8	4.7825
1.5	22.4570	7.3	7.4492	11.9	4.7446
2	19.4601	7.4	7.3589	12	4.7073
2.5	17.0884	7.5	7.2707	12.1	4.6706
3	15.1796	7.6	7.1845	12.2	4.6345
3.1	14.8427	7.7	7.1003	12.3	4.5989
3.2	14.5189	7.8	7.0179	12.4	4.5639
3.3	14.2076	7.9	6.937$\underline{5}$	12.5	4.5294
3.4	13.9080	8	6.8587	12.6	4.4954
3.5	13.6196	8.1	6.7817	12.7	4.4619
3.6	13.3419	8.2	6.7064	12.8	4.4290
3.7	13.0743	8.3	6.6327	12.9	4.396$\underline{5}$
3.8	12.8162	8.4	6.5605	13	4.364$\underline{5}$
3.9	12.5673	8.5	6.4899	13.1	4.3329
4	12.3271	8.6	6.4207	13.2	4.3019
4.1	12.0951	8.7	6.3530	13.3	4.2712
4.2	11.8710	8.8	6.2866	13.4	4.2410
4.3	11.654$\underline{5}$	8.9	6.2216	13.5	4.2113
4.4	11.4451	9	6.1579	13.6	4.1819
4.5	11.2426	9.1	6.095$\underline{5}$	13.7	4.1530
4.6	11.0466	9.2	6.0343	13.8	4.124$\underline{5}$
4.7	10.8568	9.3	5.9743	13.9	4.0963
4.8	10.6731	9.4	5.9154	14	4.0686
4.9	10.4950	9.5	5.8577	14.1	4.0412
5	10.3224	9.6	5.8011	14.2	4.0142
5.1	10.1551	9.7	5.7456	14.3	3.9876
5.2	9.9927	9.8	5.6911	14.4	3.9613
5.3	9.8352	9.9	5.6376	14.5	3.9354
5.4	9.6822	10	5.5851	14.6	3.9098
5.5	9.5337	10.1	5.5336	14.7	3.8846
5.6	9.3894	10.2	5.4829	14.8	3.8597
5.7	9.2492	10.3	5.4332	14.9	3.8351
5.8	9.1129	10.4	5.3844	15	3.8108
5.9	8.9804	10.5	5.336$\underline{5}$	15.1	3.7868
6	8.8514	10.6	5.2894	15.2	3.7632
6.1	8.7260	10.7	5.2431	15.3	3.7398
6.2	8.6039	10.8	5.1976	15.4	3.7168
6.3	8.4850	10.9	5.1529	15.5	3.6940
6.4	8.3692	11	5.1089	15.6	3.6715
6.5	8.2564	11.1	5.0657	15.7	3.6493
6.6	8.1464	11.2	5.0232	15.8	3.6274
6.7	8.0392	11.3	4.9814	15.9	3.6058
6.8	7.9347	11.4	4.9403	16	3.5844
6.9	7.8328	11.5	4.8999	16.1	3.5632

Table 2 (continued)

γ [deg]	m (γ)	γ [deg]	m (γ)	γ [deg]	m (γ)
16.2	3.5424	22.2	2.6306	35	1.7398
16.3	3.5217	22.4	2.6087	35.5	1.7186
16.4	3.5014	22.6	2.5871	36	1.6980
16.5	3.4812	22.8	2.5659	36.5	1.6780
16.6	3.4613	23	2.5450	37	1.6587
16.7	3.4416	23.2	2.5245	37.5	1.6398
16.8	3.4222	23.4	2.5044	38	1.6216
16.9	3.4030	23.6	2.4846	38.5	1.6038
17	3.3840	23.8	2.4652	39	1.5865
17.1	3.3652	24	2.4461	39.5	1.5698
17.2	3.3467	24.2	2.4273	40	1.5535
17.3	3.3283	24.4	2.4088	40.5	1.5376
17.4	3.3102	24.6	2.3907	41	1.5222
17.5	3.2923	24.8	2.3728	41.5	1.5072
17.6	3.2745	25	2.3552	42	1.4926
17.7	3.2570	25.2	2.3379	42.5	1.4784
17.8	3.2397	25.4	2.3209	43	1.4646
17.9	3.2225	25.6	2.3042	43.5	1.4511
18	3.2056	25.8	2.2877	44	1.4380
18.1	3.1888	26	2.2715	44.5	1.4252
18.2	3.1722	26.2	2.2555	45	1.4128
18.3	3.1558	26.4	2.2398	45.5	1.4007
18.4	3.1396	26.6	2.2243	46	1.3888
18.5	3.1235	26.8	2.2091	46.5	1.3773
18.6	3.1076	27	2.1941	47	1.3661
18.7	3.0919	27.2	2.1793	47.5	1.3552
18.8	3.0763	27.4	2.1648	48	1.3445
18.9	3.0609	27.6	2.1505	48.5	1.3341
19	3.0457	27.8	2.1363	49	1.3240
19.1	3.0306	28	2.1224	49.5	1.3141
19.2	3.0157	28.2	2.1087	50	1.3045
19.3	3.0009	28.4	2.0952	51	1.2859
19.4	2.9863	28.6	2.0819	52	1.2682
19.5	2.9719	28.8	2.0688	53	1.2514
19.6	2.9576	29	2.0559	54	1.2354
19.7	2.9434	29.2	2.0431	55	1.2202
19.8	2.9294	29.4	2.0306	56	1.2057
19.9	2.9155	29.6	2.0182	57	1.1918
20	2.9017	29.8	2.0060	58	1.1787
20.2	2.8746	30	1.9939	59	1.1662
20.4	2.8481	30.5	1.9645	60	1.1543
20.6	2.8220	31	1.9361	61	1.1430
20.8	2.7965	31.5	1.9087	62	1.1322
21	2.7714	32	1.8822	63	1.1220
21.2	2.7469	32.5	1.8565	64	1.1123
21.4	2.7227	33	1.8317	65	1.1031
21.6	2.6991	33.5	1.8076	66	1.0944
21.8	2.6758	34	1.7843	67	1.0862
22	2.6530	34.5	1.7617	68	1.0784

Table 2 (continued)

γ [deg]	m (γ)	γ [deg]	m (γ)	γ [deg]	m (γ)
69	1.0710	73	1.0456	85	1.0038
70	1.0640	74	1.0402	90	1.0000
71	1.0575	75	1.0352		
72	1.0513	80	1.0154		

they were converted into relative air masses by dividing them by $p_0/g_0 = 10332.2$ kg m^{-2}. The function $1/\sin\gamma$ which is the relative optical air mass of a plane-parallel non-refracting atmosphere is also listed.

As expected, the differences are greatest at low solar altitudes anp diminish for increasing γ; from $\gamma = 40$ deg on, the values are practically identical. The air masses from all three computations[1] are always smaller

Table 3. *Relative Optical Air Mass $m(\gamma)$ for Selected Solar Altitudes γ after* KASTEN, BEMPORAD, *and* ELY, *and for a Plane-Parallel Non-Refracting Atmosphere ($1/\sin\gamma$). Last Decimal Places of* ELY'*s Values Are Uncertain*

		m (γ) after		
γ [deg]	KASTEN	BEMPORAD	ELY	1/sin γ
0	36.265	39.7	35.8	∞
5	10.322	10.40	10.54	11.474
10	5.585	5.600	5.66	5.764
20	2.902	2.904	2.95	2.924
30	1.994	1.995	2.01	2.000
40	1.553	1.553	1.55	1.556
50	1.304	1.304	1.30	1.305
60	1.154	1.154	1.15	1.155

than $1/\sin\gamma$ because the geometrical path of the sun rays is shorter in a curved than in a plane-parallel atmosphere of the same thickness, although a certain extension of the rays due to refraction is taken into account in KASTEN's and BEMPORAD's values.

Table 3 shows, furthermore, that up to a solar altitude of about 30 deg, the air masses from the present computations are always smaller, whereas ELY's air masses are always greater than BEMPORAD's, with the exception of $\gamma = 0$ deg where the air mass after ELY is even smaller than the air mass after KASTEN.

One should expect our air masses to be always greater than ELY's because both the air masses are based on the same model atmosphere, but in the case of our air masses the light path is extended due to refraction. This expectation is verified only for $\gamma = 0$ deg and for low solar altitudes up to about 4 deg (not listed in Table 3). Indeed, at low solar altitudes refraction is most significant and influential on the air mass. But ELY's values for higher solar altitudes appear too large. Possibly the step widths

[1] Keeping in mind that the third decimal places of ELY's values are rather inaccurate.

in height chosen by ELY for the numerical integration are slightly too large. It can be visualized that too large step widths in the SIMPSON method

Table 4. *Contribution of the Single Layers of the Atmosphere to the Total Absolute Air Mass, in %, for Different Solar Altitudes γ. $h = 0 - 9.6$ km: "Troposphere", $h = 9.6 - 34$ km: "Stratosphere," $h = 34 - 84$ km: "Mesosphere"*

	Height of air layers [km]					
γ [deg]	0 — 9.6	9.6 — 12.4	12.4 — 22	22 — 34	34 — 50	50 — 84
0	88.3	4.9	5.7	1.0	0.1	0.0
0.5	86.7	5.6	6.4	1.2	0.1	0.0
1	84.6	6.4	7.5	1.4	0.1	0.0
1.5	82.7	7.1	8.4	1.6	0.2	0.0
2	81.1	7.6	9.3	1.8	0.2	0.0
2.5	79.8	8.1	9.9	2.0	0.2	0.0
3	78.7	8.4	10.5	2.2	0.2	0.0
3.5	77.8	8.7	11.0	2.3	0.2	0.0
4	77.1	8.9	11.4	2.4	0.2	0.0
4.5	76.5	9.0	11.7	2.5	0.2	0.1
5	76.0	9.1	12.0	2.6	0.2	0.1
5.5	75.6	9.2	12.2	2.7	0.2	0.1
6	75.2	9.3	12.4	2.7	0.3	0.1
6.5	75.0	9.3	12.5	2.8	0.3	0.1
7	74.7	9.4	12.6	2.9	0.3	0.1
7.5	74.5	9.4	12.8	2.9	0.3	0.1
8	74.3	9.5	12.9	2.9	0.3	0.1
8.5	74.2	9.5	12.9	3.0	0.3	0.1
9	74.1	9.5	13.0	3.0	0.3	0.1
9.5	74.0	9.5	13.1	3.0	0.3	0.1
10	73.9	9.5	13.1	3.1	0.3	0.1
11	73.7	9.6	13.2	3.1	0.3	0.1
12	73.6	9.6	13.3	3.1	0.3	0.1
13	73.5	9.6	13.3	3.2	0.3	0.1
14	73.4	9.6	13.4	3.2	0.3	0.1
15	73.4	9.6	13.4	3.2	0.3	0.1
16	73.3	9.7	13.4	3.2	0.3	0.1
18	73.2	9.7	13.5	3.2	0.3	0.1
20	73.1	9.7	13.5	3.2	0.4	0.1
25	73.0	9.7	13.5	3.3	0.4	0.1
30	73.0	9.7	13.5	3.3	0.4	0.1
35	72.9	9.7	13.6	3.3	0.4	0.1
40	72.9	9.7	13.6	3.3	0.4	0.1
50	72.9	9.7	13.6	3.3	0.4	0.1
60	72.9	9.7	13.6	3.3	0.4	0.1
90	72.9	9.7	13.6	3.3	0.4	0.1

would yield too large absolute air masses, and the error would increase as the solar altitude decreases so that the relative air masses, $m = \frac{m_{\text{abs}}}{m_{\text{abs, v}}}$, also come out too large. For very low solar altitudes, $\gamma < 4$ deg, this effect is hidden, because here the refraction makes the air masses after KASTEN greater than the air masses after ELY.

The fact that our air masses are always smaller than the BEMPORAD values can be explained, apart from differences between the air densities used, by the lower refractive index at ground level used in our computations.

For numerical integration of eq. (19) by the electronic computer, the atmosphere had to be divided into six layers with the boundaries of 0, 9.6, 12.4, 22, 34, 50, and 84 km height. This procedure permitted calculation of the percentage of the total absolute air mass which is contributed by each of the single layers (Table 4). For each solar altitude γ, the lowest layer, which roughly represents the troposphere, yields more than 70%, and the atmosphere between 34 and 84 km (mesosphere) contributes 0.5% or less to the total absolute air mass. The layers from 9.6 through 34 km represent approximately the stratosphere. With increasing solar altitude, the percentage contributed by the troposphere decreases, while that of all the other layers increases but, above $\gamma = 30$ deg, all percentages remain constant.

5. Approximation Formula for the Relative Optical Air Mass

The analytical form of the relative optical air mass is given by the integral in eq. (19), which cannot be expressed by elementary functions because the air density $\rho\,(h)$ is not an analytical function but is given by a table. Therefore it is desirable to express eq. (19) by an empirical approximation formula containing elementary function only, so that the air mass can easily be computed numerically.

Table 5. *Numerical Values of the Constants a, b, and c in the Approximation Formula* $f\,(\gamma) = 1/[\sin\gamma + a\,(\gamma + b)^{-c}]$ *for the Relative Optical Air Mass after* KASTEN *and after* BEMPORAD, *and for the Relative Optical Water Vapor Mass after* SCHNAIDT

	Air mass after KASTEN	Air mass after BEMPORAD	Water vapor mass after SCHNAIDT
$a =$	0.1500	0.6556	0.05480
$b =$	3.885	6.379	2.650
$c =$	1.253	1.757	1.452

The approximation formula $f\,(\gamma)$ must meet the following requirements: 1. $f\,(\gamma)$ approaches $1/\sin\gamma$ for increasing solar altitude γ; 2. $f\,(0)$ has a finite value. The first postulate suggests a function which contains $1/\sin\gamma$ and some kind of correction term. The second postulate requires this correction term to compensate for $1/\sin\gamma$ tending toward infinity for $\gamma \to 0$.

Several trial calculations showed that it is impossible to approximate $m\,(\gamma)$ by a function with only two constants if reasonable accuracy is wanted. Finally, the function

$$f(\gamma) = \frac{1}{\sin\gamma + a(\gamma + b)^{-c}} \tag{22}$$

was selected. The constants a, b, and c have to be determined empirically from the tabulated $(m; \gamma)$ pairs. The conventional Gauss method of least squares is applicable only to equations which are linear in the coefficients to be determined, but a non-linear method can be used if approximative

Table 6. *Relative Deviations* $r_i = (f_i - m_i)/m_i$ [%] *of the Values* f_i *Approximated by Eq. (22) with the Corresponding Constants from Table 5, with Respect to the Tabulated Values* m_i *of the Air Mass after* KASTEN (K) *and after* BEMPORAD (B), *and of the Water Vapor Mass after* SCHNAIDT (S)

γ [deg]	(K)	(B)	(S)	γ [deg]	(K)	(B)	(S)
0.0	0.68	—0.21	0.03	25	—0.06	0.03	—0.01
0.5	—1.25	0.22	—2.96	26	—0.06	—0.02	—0.00
1.0	—0.02	0.19	0.10	27	—0.06	0.05	—0.01
1.5	0.36	0.07	—0.01	28	—0.06	0.05	0.02
2.0	0.41	—0.04	0.08	29	—0.06	0.07	0.01
2.5	0.35	—0.09	—0.11	30	—0.06	0.01	0.03
3.0	0.26	—0.14	—0.09	32	—0.06	0.07	—
3.5	0.17	—0.13	—0.05	34	—0.06	0.06	—
4.0	0.10	—0.15	—0.07	36	—0.06	0.04	—
4.5	0.05	—0.10	0.03	38	—0.06	0.07	—
5.0	0.01	—0.10	—0.00	40	—0.06	0.05	—
5.5	—0.02	—0.05	—0.03	42	—0.06	0.06	—
6.0	—0.05	—0.05	0.01	44	—0.06	0.01	—
6.5	—0.06	—0.06	0.02	46	—0.06	—0.00	—
7.0	—0.07	—0.04	0.04	48	—0.06	0.04	—
7.5	—0.08	0.01	0.05	50	—0.06	0.04	—
8.0	—0.08	0.02	0.05	52	—0.06	0.09	—
8.5	—0.09	0.06	0.04	54	—0.06	0.03	—
9.0	—0.09	—0.00	0.04	56	—0.06	0.05	—
9.5	—0.09	0.13	0.05	58	—0.06	0.05	—
10	—0.09	0.06	0.04	60	—0.06	0.01	—
11	—0.08	0.08	0.06	62	—0.06	0.01	—
12	—0.08	0.09	0.05	64	—0.06	0.01	—
13	—0.08	0.08	0.03	66	—0.06	0.02	—
14	—0.07	0.08	0.02	68	—0.06	0.01	—
15	—0.07	0.08	—0.02	70	—0.06	—0.02	—
16	—0.07	0.10	—0.01	72	—0.06	—0.08	—
17	—0.07	0.07	—0.00	74	—0.06	—0.00	—0.01
18	—0.06	0.07	0.03	76	—0.05	—0.07	—0.01
19	—0.06	0.05	0.05	78	—0.06	0.01	—0.01
20	—0.06	0.07	0.06	80	—0.06	0.02	—0.01
21	—0.06	0.08	0.04	82	—0.05	—0.04	—0.01
22	—0.06	0.09	0.03	84	—0.05	—0.07	—0.01
23	—0.06	0.08	0.00	86	—0.05	—0.02	—0.01
24	—0.06	0.07	—0.01	88	—0.05	—0.02	—0.01
				90	—0.05	—0.02	—0.01

values a_0, b_0, and c_0, the so-called zero approximations, are known (BAUR [*18*]), see Appendix 2.

By means of this method, the relative optical air mass computed on the basis of the air densities of the ARDC Model Atmosphere, 1959 was approximated by formula (22). The same formula was used to approximate the relative optical air mass according to BEMPORAD's table and the relative optical water vapor mass computed by SCHNAIDT [*5*] by an equation analogous to eq. (19). SCHNAIDT did not compute water vapor masses for solar altitudes $\gamma > 30$ deg, but assumed these values equal to $1/\sin\gamma$. Consequently, the water vapor masses were approximated up to $\gamma = 30$ deg only, but values of $1/\sin\gamma$ in the range 74 deg $\leqslant \gamma \leqslant$ 90 deg were included to make sure that the approximation function approaches $1/\sin\gamma$ for high solar altitudes γ. The values of the constants a, b, and c for the three approximations are listed in Table 5.

Table 6 contains the relative deviations $r_i = (f_i - m_i)/m_i$ [%] of the approximated values f_i with respect to the tabulated air masses or water vapor masses m_i. Generally, the relative deviations are of the order of 0.1% or less. Only for $\gamma \leqslant 4$ deg do errors up to 1% occur, with the exception of $\gamma = 0.5$ deg where the error reaches 1.25% for the air mass and 3% for the water vapor mass. The course of r_i with γ is very smooth for the air masses computed in this paper, whereas the r_i referring to the BEMPORAD and SCHNAIDT values scatter. This is considered to be a consequence of the lower numerical accuracy of the last decimal places in the tables of BEMPORAD and SCHNAIDT who did not have an electronic computer at their disposal.

6. Conclusions

Table 2 is proposed as new standard table of the relative optical air mass $m(\gamma)$ for terrestrial solar radiation. Eq. (22) with $a = 0.1500$, $b = 3.885$, and $c = 1.253$ is a convenient approximation formula of high accuracy to calculate $m(\gamma)$ for any solar altitude γ [deg]. The same formula may be used to approximate the old BEMPORAD function and the relative optical water vapor mass after SCHNAIDT by substituting the corresponding constants a, b, and c from Table 5.

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Appendix 1

Computation of the Air Mass for Solar Altitude 0

The second integral in eq. (19) is split into two parts, from 0 to 0.4 km and from 0.4 to 84 km. The second part is integrated numerically. Considering the first part, inspection of the ARDC Model Atmosphere, 1959 shows that the air density ρ can be interpolated linearly within a step width of $\Delta h = 0.1$ km for the range $0 \leq h \leq 0.4$ km. Denoting the lower limit of the interval Δh by

h_L, the upper limitby h_U, and the corresponding air densities by ρ_L, ρ_U, we have

$$\frac{\rho - \rho_L}{h - h_L} = \frac{\rho_U - \rho_L}{h_U - h_L} \tag{23}$$

With the definition

$$\frac{\rho_U - \rho_L}{h_U - h_L} = \rho' \tag{24}$$

eq. (23) becomes

$$\rho = (\rho_L - \rho' h_L) + \rho' h. \tag{25}$$

The right integral in the air mass formula eq. (19), taken over the interval $h_L \ldots h_U$ and specialized for $\gamma = 0$, is now called J and can be rewritten as

$$J = \int_{h_L}^{h_U} \frac{\rho\,(R + h)\,\mathrm{d}h}{\sqrt{(R + h)^2 - \left[1 + 2\,\delta_0 \left(1 - \frac{\rho}{\rho_0}\right)\right] R^2}}\,. \tag{26}$$

Substituting from eq. (25) for ρ gives, after some algebraic transformations,

$$J = \int_{h_L}^{h_U} \frac{(A\,h^2 + B\,h + C)\,\mathrm{d}h}{\sqrt{a\,h^2 + b\,h + c}} \tag{27}$$

where the new symbols stand for the following constants:

$$\begin{aligned}
A &= \rho',\\
B &= (\rho_L - \rho' h_L) + \rho' R,\\
C &= (\rho_L - \rho\ h_L)\,R,\\
a &= 1,\\
b &= 2\,R\,[1 + \delta_0\,(\rho'/\rho_0)\,R],\\
c &= R^2 - \{1 + 2\,\delta_0\,[1 - (\rho_L - \rho' h_L)/\rho_0]\}\,R^2.
\end{aligned}$$

The integral eq. (27) can be solved exactly; making use of $a = 1$,

$$\begin{aligned}
J = {} & \left(\frac{A}{2}h + \frac{1}{4}(4\,B - 3\,A\,b)\right)\sqrt{h^2 + b\,h + c}\\
& - \left(\frac{A c}{2} + \frac{b}{8}(4\,B - 3\,A\,b) - C\right)\cdot ln\left|\,h + \frac{b}{2} + \sqrt{h^2 + b\,h + c}\,\right|
\end{aligned} \tag{28}$$

which has to be taken between the limits h_L and h_U.

Eq. (28) was evaluated by the electronic computer for each pair h_L, h_U of the four h-intervals 0 . . . 0.1, 0.1 . . . 0.2, 0.2 . . . 0.3, and 0.3 . . . 0.4 km; the sum of these four values constitutes the integral from 0 to 0.4 km. This is added to the integral from 0.4 to 84 km integrated numerically. Dividing by $m_{\mathrm{abs,v}}$ gives the relative optical air mass m (0 deg) $= 36.2648$.

The exact integration is possible only in the lower heights of the atmosphere because linear interpolation of the air density is not permissible at higher levels; a trial calculation for $\gamma = 90$ deg gives an absolute air mass of 10356.3 kg m^{-2} instead of 10330.7 kg m^{-2} as computed by the SIMPSON method [eq. (20)].

Appendix 2

Determination of the Constants in the Approximation Formula by a Modified Least Squares Method

Let a_0, b_0, and c_0 be the so-called zero approximations, and $a_1 = a_0 + \alpha$, $b_1 = b_0 + \beta$, and $c_1 = c_0 + \zeta$ be the first approximations of the constants, where α, β, and ζ are the "corrections," which have to be small. Then the function $f^{(1)}(\gamma)$ given by eq. (22), but with the constants of the first approximation, can be extended into a TAYLOR series:

$$f^{(1)}(\gamma; a_0 + \alpha, b_0 + \beta, c_0 + \zeta) = f^{(0)}(\gamma; a_0, b_0, c_0) + \frac{\partial f^{(0)}}{\partial a}\alpha + \frac{\partial f^{(0)}}{\partial b}\beta + \frac{\partial f^{(0)}}{\partial c}\zeta + \dots \quad (29)$$

Now we set up the "error equations"

$$f_i^{(1)} - m_i = v_i \qquad (i = 1, \dots, n) \quad (30)$$

where m_i = true value of the air mass for γ_i, v_i = "error," and n = number of (m_i, γ_i) pairs used in the approximation. Substituting eq. (29) into eqs. (30) we obtain

$$\frac{\partial f_i^{(0)}}{\partial a}\alpha + \frac{\partial f_i^{(0)}}{\partial b}\beta + \frac{\partial f_i^{(0)}}{\partial c}\zeta + (f_i^{(0)} - m_i) = v_i \,. \quad (31)$$

With the abbreviations

$$\frac{\partial f_i^{(0)}}{\partial a} = A_i\,; \quad \frac{\partial f_i^{(0)}}{\partial b} = B_i\,; \quad \frac{\partial f_i^{(0)}}{\partial c} = C_i\,; \quad f_i^{(0)} - m_i = L_i \quad (32)$$

eqs. (31) become

$$A_i\,\alpha + B_i\,\beta + C_i\,\zeta + L_i = v_i. \quad (33)$$

These equations are linear in α, β, ζ so that the wellknown Gauss method can now be applied: The eqs. (33) are first squared and then summed; partial differentiation with respect to α, β, ζ and equating to zero makes the sum of the squares of the errors, $\Sigma_i\, v_i^2$, a minimum. The three following so-called "normal equations" result:

$$\begin{aligned} [AA]\,\alpha + [AB]\,\beta + [AC]\,\zeta + [AL] &= 0 \\ [AB]\,\alpha + [BB]\,\beta + [BC]\,\zeta + [BL] &= 0 \\ [AC]\,\alpha + [BC]\,\beta + [CC]\,\zeta + [CL] &= 0 \end{aligned} \quad (34)$$

where the symbol $[AA]$ stands for $\Sigma_i\, A_i\, A_i$, $[AB]$ stands for $\Sigma_i\, A_i\, B_i$, etc. From this system of three equations, the "corrections" α, β, ζ can be determined so that the first approximations a_1, b_1, c_1 are known. Now the whole process is iterated with a_1, b_1, c_1 taking the place of a_0, b_0, c_0.

In both the conventional and the "nonlinear" least squares methods, the absolute errors v_i are more or less evenly distributed over the whole range of the function to be approximated. This is disadvantageous for functions as the air mass $m(\gamma)$ where the absolute accuracy of the given values is not constant over the whole range of solar altitude γ. Therefore, we modify the method outlined above by introducing the *relative* errors

$$r_i = \frac{v_i}{m_i} \qquad (i = 1, \ldots, n) \tag{35}$$

into eqs. (31):

$$\frac{1}{m_i}\frac{\partial f_i^{(0)}}{\partial a}\alpha + \frac{1}{m_i}\frac{\partial f_i^{(0)}}{\partial b}\beta + \frac{1}{m_i}\frac{\partial f_i^{(0)}}{\partial c}\zeta + \frac{f_i^{(0)} - m_i}{m_i} = r_i\,. \tag{36}$$

Using the abbreviations

$$\frac{1}{m_i}\frac{\partial f_i^{(0)}}{\partial a} = A_i^*,\ \frac{1}{m_i}\frac{\partial f_i^{(0)}}{\partial b} = B_i^*,\ \frac{1}{m_i}\frac{\partial f_i^{(0)}}{\partial c} = C_i^*,\ \frac{f_i^{(0)} - m_i}{m_i} = L_i^* \tag{37}$$

eqs. (36) take the form

$$A_i^*\,\alpha + B_i^*\,\beta + C_i^*\,\zeta + L_i^* = r_i \tag{38}$$

which are analogous to eqs. (33). They lead to a system of normal equations analogous to the system eqs. (34) but with the asteric attached to the respective quantities.

This method, which may be called "method of least squares of the *relative* errors applied to equations being non-linear in the coefficients," was programmed for computation on the electronic computer. The "corrections" α, β, ζ obtained after one approximation cycle are added to the coefficients a, b, c resulting from the cycle before, and these new a, b, c are used to begin the next iteration. The calculations are iterated until the next cycle does not improve the coefficients obtained in the previous cycle. The resulting "last" approximations of a, b, c are rounded to four significant digits and given in Table 5.

With these final constants, the function $f(\gamma)$ is computed once more according to eq. (22) for all γ being used in the foregoing least squares calculations, and the relative "errors" $r_i = (f_i - m_i)/m_i$ determined. They are the relative deviations of the air masses approximated by eq. (22) from the tabulated air masses and are listed in Table 6.

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