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CSE: 122 Homework 4B

1 Homework 4B

1.1 Problem 1

Equation:

$$b_k = \frac{b_{k-1}}{1 + b_{k-1}}$$

Work for Explicit Formula

$$b_{k-1} = \frac{b_{k-2}}{1 + b_{k-2}}$$

$$b_k = \frac{\frac{b_{k-2}}{1+b_{k-2}}}{1 + \frac{b_{k-2}}{1+b_{k-2}}} = \frac{\frac{b_{k-2}}{1+b_{k-2}}}{\frac{1+2b_{k-2}}{1+b_{k-2}}}$$

$$b_k = \frac{b_{k-2}}{1 + 2b_{k-1}}$$

$$b_{k-2} = \frac{b_{k-3}}{1 + b_{k-3}}$$

$$b_k = \frac{\frac{b_{k-3}}{1+b_{k-3}}}{1 + \frac{2b_{k-3}}{1+b_{k-3}}} = \frac{\frac{b_{k-3}}{1+b_{k-3}}}{\frac{1+3b_{k-3}}{1+b_{k-3}}}$$

$$b_k = \frac{b_{k-3}}{1 + 3b_{k-3}}$$

$$b_k = \frac{b_{k-n}}{1 + nb_{k-n}}$$

$$b_{k-n} = b_0 \Rightarrow k - n = 0 \Rightarrow k = n$$

$$b_k = \frac{b_{k-k}}{1 + kb_{k-k}} = \frac{b_0}{1 + k(b_0)}$$

$$\mathbf{b_k = \frac{1}{1 + k}}$$

Proof by Induction
Equation

$$b_n = \frac{b_{n-1}}{1 + b_{n-1}} = \frac{1}{1 + n}, \text{where } b_0 = 1$$

Base Case: If n = 0

$$b_0 = \frac{1}{1+0} = 1$$

Assumtpion: If n = k, then

$$b_k = \frac{b_{k-1}}{1+b_{k-1}} = \frac{1}{1+k}$$

Prove: If n = k + 1, then

$$b_{k+1} = \frac{b_k}{1+b_k} = \frac{1}{2+k}$$

Proof:

$$b_{k+1} = \frac{b_k}{1+b_k} = \frac{\frac{1}{1+k}}{1+\frac{1}{1+k}} = \frac{b_k}{1+b_k} = \frac{\frac{1}{1+k}}{\frac{2+k}{1+k}} = \frac{1}{2+k}$$

1.2 Problem 2

Equation:

$$T(n) = T(\frac{n}{2}) + b, T(1) = a$$

Work for Explicit Formula:

$$T(\frac{n}{2}) = T(\frac{n}{4}) + b$$
$$T(n) = T(\frac{n}{4}) + 2b$$
$$T(\frac{n}{4}) = T(\frac{n}{8}) + b$$
$$T(n) = T(\frac{n}{8}) + 3b$$
$$\mathbf{T(n) = T(\frac{n}{2^k}) + kb}$$
$$T(\frac{n}{2^k}) = T(1) \Rightarrow \frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow log_2(n) = k$$
$$T(n) = T(\frac{2^k}{2^k}) + log_2(n)b$$

Explicit Equation: $T(n) = a + log_2(n)b$

Big-O: $O(log(n))$

1.3 Problem 3

Equation:

$$T(n) = 2T(\frac{n}{2}) + b, T(1) = a$$

Work for Explicit Formula:

$$T(\frac{n}{2}) = 2T(\frac{n}{4}) + b$$
$$T(n) = 2 * (T(\frac{n}{4}) + b) + b = 2T(\frac{n}{4}) + 2b + b$$
$$T(\frac{n}{4}) = 2T(\frac{n}{8}) + b$$
$$T(n) = 8T(\frac{n}{8}) + 4b + 2b + b$$
$$\mathbf{T(n) = 2^k T(\frac{n}{2^k}) + 2^{k-1} b}$$
$$T(\frac{n}{2^k}) = T(1) \Rightarrow \frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow log_2(n) = k$$
$$T(n) = 2^{log_2(n)} T(\frac{2^k}{2^k}) + 2^{log_2(n)-1} b$$
$$\textbf{Explicit Equation: } T(n) = na + \frac{n}{2} b$$
$$\textbf{Big-O: } O(n)$$

1.4 Problem 4

Equation:

$$T(n) = T(n-1) + n^k, T(1) = a$$

Work for Explicit Formula:

$$T(n-1) = T((n-1)-1) + (n-1)^k$$
$$T(n) = T(n-2) + (n-1)^k + n^k$$
$$T(n-2) = T(n-3) + (n-2)^k$$
$$T(n) = T(n-3) + (n-2)^k + (n-1)^k + n^k$$
$$T(n-3) = T(n-4) + (n-3)^k$$
$$T(n) = T(n-4) + (n-3)^k + (n-2)^k + (n-1)^k + n^k$$
$$\mathbf{T(n) = T(n-j) + \sum_{i=0}^{j-1}(n-i)^k}$$
$$T(n-j) = T(1) \Rightarrow n-j = 1 \Rightarrow j = n-1$$
$$T(n) = T(n-n+1) + \sum_{i=0}^{(n-1)-1}(n-i)^k$$
$$\textbf{Explicit Equation: } T(n) = a + \sum_{i=0}^{n-2}(n-i)^k$$

Prove $\mathbf{T(n) = a + \sum_{i=0}^{n-2}(n-i)^k}$ is $\mathbf{O(n^{k+1})}$

$$a + \sum_{i=0}^{n-2}(n-i)^k = a + n^k + (n-1)^k + ... + 2^k$$
$$< an^k + n^k + n^k + ... + n^k \text{ } (n-1 \text{ times}) = an^k + n * n^k - n^k$$
$$= (a-1)n^k + n^{k+1} < a * n^{k+1} + n^{k+1} = a * n^{k+1}$$
$$\mathbf{a + \sum_{i=0}^{n-2}(n-i)^k \leq a * n^{k+1}}$$
$$n_0 = 1$$
$$c = a$$
$$\textbf{Big-O: } O(n^{k+1})$$