

1 Homework 4B

1.1 Problem 1

Equation:

$$b_k = \frac{b_{k-1}}{1 + b_{k-1}}$$

Work for Explicit Formula

$$\begin{aligned} b_{k-1} &= \frac{b_{k-2}}{1 + b_{k-2}} \\ b_k &= \frac{\frac{b_{k-2}}{1+b_{k-2}}}{1 + \frac{b_{k-2}}{1+b_{k-2}}} = \frac{\frac{b_{k-2}}{1+b_{k-2}}}{\frac{1+2b_{k-2}}{1+b_{k-2}}} \\ b_k &= \frac{b_{k-2}}{1 + 2b_{k-1}} \\ b_{k-2} &= \frac{b_{k-3}}{1 + b_{k-3}} \\ b_k &= \frac{\frac{b_{k-3}}{1+b_{k-3}}}{1 + \frac{2b_{k-3}}{1+b_{k-3}}} = \frac{\frac{b_{k-3}}{1+b_{k-3}}}{\frac{1+3b_{k-3}}{1+b_{k-3}}} \\ b_k &= \frac{b_{k-3}}{1 + 3b_{k-3}} \\ b_k &= \frac{b_{k-n}}{1 + nb_{k-n}} \\ b_{k-n} = b_0 &\Rightarrow k - n = 0 \Rightarrow k = n \\ b_k &= \frac{b_{k-k}}{1 + kb_{k-k}} = \frac{b_0}{1 + k(b_0)} \\ \mathbf{b_k} &= \frac{1}{1 + \mathbf{k}} \end{aligned}$$

Proof by Induction

Equation

$$b_n = \frac{b_{n-1}}{1 + b_{n-1}} = \frac{1}{1 + n}, \text{ where } b_0 = 1$$

Base Case: If $n = 0$

$$b_0 = \frac{1}{1+0} = 1$$

Assumption: If $n = k$, then

$$b_k = \frac{b_{k-1}}{1+b_{k-1}} = \frac{1}{1+k}$$

Prove: If $n = k + 1$, then

$$b_{k+1} = \frac{b_k}{1+b_k} = \frac{1}{2+k}$$

Proof:

$$b_{k+1} = \frac{b_k}{1+b_k} = \frac{\frac{1}{1+k}}{1+\frac{1}{1+k}} = \frac{b_k}{1+b_k} = \frac{\frac{1}{1+k}}{\frac{2+k}{1+k}} = \frac{1}{2+k}$$

1.2 Problem 2

Equation:

$$T(n) = T\left(\frac{n}{2}\right) + b, T(1) = a$$

Work for Explicit Formula:

$$T\left(\frac{n}{2}\right) = T\left(\frac{n}{4}\right) + b$$

$$T(n) = T\left(\frac{n}{4}\right) + 2b$$

$$T\left(\frac{n}{4}\right) = T\left(\frac{n}{8}\right) + b$$

$$T(n) = T\left(\frac{n}{8}\right) + 3b$$

$$T(n) = T\left(\frac{n}{2^k}\right) + kb$$

$$T\left(\frac{n}{2^k}\right) = T(1) \Rightarrow \frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow \log_2(n) = k$$

$$T(n) = T\left(\frac{2^k}{2^k}\right) + \log_2(n)b$$

Explicit Equation: $T(n) = a + \log_2(n)b$

Big-O: $O(\log(n))$

1.3 Problem 3

Equation:

$$T(n) = 2T\left(\frac{n}{2}\right) + b, T(1) = a$$

Work for Explicit Formula:

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + b$$

$$T(n) = 2 * \left(T\left(\frac{n}{4}\right) + b\right) + b = 2T\left(\frac{n}{4}\right) + 2b + b$$

$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + b$$

$$T(n) = 8T\left(\frac{n}{8}\right) + 4b + 2b + b$$

$$\mathbf{T(n) = 2^k T\left(\frac{n}{2^k}\right) + 2^{k-1}b}$$

$$T\left(\frac{n}{2^k}\right) = T(1) \Rightarrow \frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow \log_2(n) = k$$

$$T(n) = 2^{\log_2(n)} T\left(\frac{2^k}{2^k}\right) + 2^{\log_2(n)-1}b$$

$$\mathbf{Explicit\ Equation: } T(n) = na + \frac{n}{2}b$$

$$\mathbf{Big-O: } O(n)$$

1.4 Problem 4

Equation:

$$T(n) = T(n-1) + n^k, T(1) = a$$

Work for Explicit Formula:

$$\begin{aligned}
T(n-1) &= T((n-1)-1) + (n-1)^k \\
T(n) &= T(n-2) + (n-1)^k + n^k \\
T(n-2) &= T(n-3) + (n-2)^k \\
T(n) &= T(n-3) + (n-2)^k + (n-1)^k + n^k \\
T(n-3) &= T(n-4) + (n-3)^k \\
T(n) &= T(n-4) + (n-3)^k + (n-2)^k + (n-1)^k + n^k
\end{aligned}$$

$$\mathbf{T}(\mathbf{n}) = \mathbf{T}(\mathbf{n} - \mathbf{j}) + \sum_{\mathbf{i}=0}^{\mathbf{j}-1} (\mathbf{n} - \mathbf{i})^{\mathbf{k}}$$

$$T(n-j) = T(1) \Rightarrow n-j = 1 \Rightarrow j = n-1$$

$$T(n) = T(n-n+1) + \sum_{i=0}^{(n-1)-1} (n-i)^k$$

$$\textbf{Explicit Equation: } T(n) = a + \sum_{i=0}^{n-2} (n-i)^k$$

Prove $\mathbf{T}(\mathbf{n}) = \mathbf{a} + \sum_{\mathbf{i}=0}^{\mathbf{n}-2} (\mathbf{n} - \mathbf{i})^{\mathbf{k}}$ **is** $\mathbf{O}(\mathbf{n}^{\mathbf{k}+1})$

$$\begin{aligned}
a + \sum_{i=0}^{n-2} (n-i)^k &= a + n^k + (n-1)^k + \dots + 2^k \\
&< an^k + n^k + n^k + \dots + n^k \text{ (n-1 times)} = an^k + n * n^k - n^k \\
&= (a-1)n^k + n^{k+1} < a * n^{k+1} + n^{k+1} = a * n^{k+1}
\end{aligned}$$

$$\mathbf{a} + \sum_{\mathbf{i}=0}^{\mathbf{n}-2} (\mathbf{n} - \mathbf{i})^{\mathbf{k}} \leq \mathbf{a} * \mathbf{n}^{\mathbf{k}+1}$$

$$n_0 = 1$$

$$c = a$$

$$\textbf{Big-O: } O(n^{k+1})$$