1 Burgers' Example

A basic example of an IVP for the inviscid Burgers' equation is presented to illustrate the utility of weak solutions. The inviscid Burgers' equation is given by:

$$u_t + uu_x = 0 \tag{1}$$

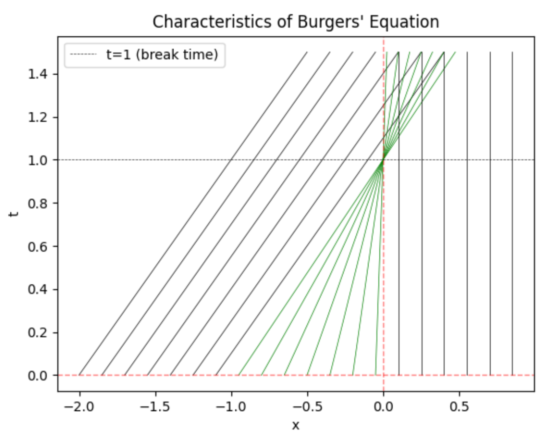
and in "conservation form" is written as:

$$u_t + (f(u))_x = 0 \tag{2}$$

With the initial values given by $u(x,0) = g(x)$:

$$g(x) = \begin{cases} 1, & x < -1, \\ -x, & -1 \leq x \leq 0, \\ 0, & x > 0 \end{cases} \tag{3}$$

A strong solution can be found for $t < 1$ by method of characteristics.



$$u(x,t) = \begin{cases} 1, & x < t-1, \\ \frac{-x}{1-t}, & t-1 < x \leq 0, \\ 0, & x > 0 \end{cases} \qquad t < 1$$

At $t \geq 1$, the solution becomes multi-valued. A weak solution is required for $t \geq 1$ to allow for solutions that may not be differentiable at all points (x,t).

Definition 1 (Weak Solution)**.**

$$u \in L^\infty(\mathbb{R} \times [0, \infty))$$

u is bounded and measurable.

u is a weak solution to (1) - (3) if

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t + f(u)\phi_x \, dx \, dt = -\int_{-\infty}^\infty u(x,0)\phi(x,0) \, dx \tag{4}$$

$\forall \phi \in C_c^1(\mathbb{R} \times [0, \infty))$.

ϕ is a test function that is continuous in space and time and has **compact support**.

Theorem 1 (Rankine-Hugoniot Condition)**.** *If a solution u is discontinuous at (x_0, t_0), then the Rankine-Hugoniot condition must be satisfied:*

$$f(u_L) - f(u_R) = S(u_L - u_R) \tag{5}$$

For the example given by (1) - (3): $u_L = 1$, $u_R = 0$, $f(u) = \frac{1}{2}u^2$.

$$f(u_L) - f(u_R) = \frac{1}{2} - 0 = \frac{1}{2} = S(u_L - u_R) = S$$

$$\frac{x - x_0}{t - t_0} = S \implies \frac{x}{t-1} = \frac{1}{2} \implies t = 2x + 1$$

Thus the full weak solution is given by:

$$u(x,t) = \begin{cases} 1, & x < t - 1 \\ \frac{-x}{1-t}, & t - 1 < x < 0 \\ 0, & x > 0 \end{cases} \qquad t < 1 \tag{6}$$

$$u(x,t) = \begin{cases} 1, & 2x + 1 < t \\ 0, & 2x + 1 \geq t \end{cases} \qquad t \geq 1 \tag{7}$$