

PDFs and Lattice calculation

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1 Basic logic

We calculate 3pt correlation on lattice, then extract PDFs from 3pt correlation.

2 Deduction

2.1 What is PDFs

sth like

$$\langle H(P_z) | \bar{\psi}(z) \gamma^t W(z, 0) \psi(0) | H(P_z) \rangle$$

*[check out Peskin 18.5.]

2.2 Calculate PDFs through 3pt correlation

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \hat{O}_H(\vec{x}, t_{sep}) \hat{O}(\vec{y}, t; z) \hat{O}_H^\dagger(0, 0) \right| \Omega \right\rangle$$

in which \hat{O}_H is projection operator, and

$$\hat{O}(\vec{y}, t; z) = \bar{\psi}(z + \vec{y}, t) \gamma^t W(z + \vec{y}, t; \vec{y}, t) \psi(\vec{y}, t)$$

therefore, **ignore t variables first for convenience,**

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \hat{O}_H(\vec{x}) \sum_H \int \frac{d^3\vec{p}'}{(2\pi)^3} |H_{\vec{p}'}\rangle \langle H_{\vec{p}'}| \hat{O}(\vec{y}; z) \right. \right. \\ \left. \left. \sum_{H'} \int \frac{d^3\vec{p}''}{(2\pi)^3} |H'_{\vec{p}''}\rangle \langle H'_{\vec{p}''}| \hat{O}_H^\dagger(0) \right| \Omega \right\rangle$$

with spatial translation operator:

$$\hat{O}_H(\vec{x}) = e^{-i\hat{\vec{p}}\cdot\vec{x}} \hat{O}_H e^{i\hat{\vec{p}}\cdot\vec{x}}$$

so,

$$\begin{aligned}
3\text{pt} &= \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \hat{O}_H \cdot e^{i\hat{p}\cdot\vec{x}} \sum_H \int \frac{d^3\vec{p}'}{(2\pi)^3} |H_{\vec{p}'}\rangle \langle H_{\vec{p}'}| e^{-i\hat{p}\cdot\vec{y}} \cdot \hat{O}(0; z) \cdot e^{i\hat{p}\cdot\vec{y}} \right. \right. \\
&\quad \left. \left. \sum_{H'} \int \frac{d^3\vec{p}''}{(2\pi)^3} |H'_{\vec{p}''}\rangle \langle H'_{\vec{p}''}| \hat{O}_H^\dagger(0) \right| \Omega \right\rangle \\
&= \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \hat{O}_H \cdot \sum_H \int \frac{d^3\vec{p}'}{(2\pi)^3} e^{i\vec{p}'\cdot\vec{x}} |H_{\vec{p}'}\rangle \langle H_{\vec{p}'}| e^{-i\vec{p}'\cdot\vec{y}} \cdot \hat{O}(0; z) \right. \right. \\
&\quad \left. \left. \sum_{H'} \int \frac{d^3\vec{p}''}{(2\pi)^3} e^{i\vec{p}''\cdot\vec{y}} |H'_{\vec{p}''}\rangle \langle H'_{\vec{p}''}| \hat{O}_H^\dagger(0) \right| \Omega \right\rangle
\end{aligned}$$

do the integral of x and y , then we get,

$$\begin{aligned}
3\text{pt} &= \left\langle \Omega \left| \hat{O}_H \cdot \sum_H \int d^3\vec{p}' \delta(\vec{p}' - \vec{p}) |H_{\vec{p}'}\rangle \langle H_{\vec{p}'}| \hat{O}(0; z) \sum_{H'} \int d^3\vec{p}'' \delta(\vec{p}' - \vec{p}'') |H'_{\vec{p}''}\rangle \langle H'_{\vec{p}''}| \hat{O}_H^\dagger(0) \right| \Omega \right\rangle \\
&= \sum_{H'} \sum_H \langle \Omega | \hat{O}_H | H_{\vec{p}} \rangle \langle H_{\vec{p}} | \hat{O}(0; z) | H'_{\vec{p}'} \rangle \langle H'_{\vec{p}'} | \hat{O}_H^\dagger(0) | \Omega \rangle
\end{aligned}$$

then projection operator \hat{O}_H will select the hadron with specific quantum numbers, like for π^+ , $\hat{O}_{\pi^+} = \bar{d}\gamma^5 u$.

$$3\text{pt} = \langle \Omega | \hat{O}_H(0, t_{\text{sep}}) | H_{\vec{p}} \rangle \langle H_{\vec{p}} | \hat{O}(0, t; z) | H_{\vec{p}} \rangle \langle H_{\vec{p}} | \hat{O}_H^\dagger(0) | \Omega \rangle$$

$$= \langle \Omega | \hat{O}_H(0, t_{\text{sep}}) | H_{\vec{p}} \rangle \cdot \text{PDFs} \cdot \langle H_{\vec{p}} | \hat{O}_H^\dagger(0) | \Omega \rangle$$

In order to get PDFs, we also need 2pt correlation function:

$$2\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \langle \Omega | \hat{O}_H(\vec{x}, t_{\text{sep}}) \hat{O}_H^\dagger(0, 0) | \Omega \rangle$$

$$= \langle \Omega | \hat{O}_H(0, t_{\text{sep}}) | H_{\vec{p}} \rangle \langle H_{\vec{p}} | \hat{O}_H^\dagger(0, 0) | \Omega \rangle$$

Pay attention here $|H_{\vec{p}}\rangle$ is a superposition of Hamiltonian operator instead of eigenstate. Therefore, we have the expression below (we did Wick rotation on the lattice, so $it_M = t_E$)

$$3\text{pt} = \sum_{m,n} \langle \Omega | \hat{O}_H(0, 0) e^{-\hat{H}t_{\text{sep}}} | E_n \rangle \langle E_n | e^{\hat{H}t} \hat{O}(0, 0; z) e^{-\hat{H}t} | E_m \rangle \langle E_m | \hat{O}_H^\dagger(0) | \Omega \rangle$$

$$= \sum_{m,n} e^{-E_n t_{\text{sep}}} e^{E_n t} e^{-E_m t} \langle \Omega | \hat{O}_H(0,0) | E_n \rangle \langle E_n | \hat{O}(0,0;z) | E_m \rangle \langle E_m | \hat{O}_H^\dagger(0) | \Omega \rangle$$

same for 2pt,

$$2\text{pt} = \sum_n e^{-E_n t_{\text{sep}}} \langle \Omega | \hat{O}_H(0,0) | E_n \rangle \langle E_n | \hat{O}_H^\dagger(0,0) | \Omega \rangle$$

so,

$$\text{PDFs} = \frac{3\text{pt}}{2\text{pt}} = \langle H_{\vec{p}} | \hat{O}(0,0;z) | H_{\vec{p}} \rangle \left(1 + e^{-\Delta E t} + e^{-\Delta E(t_{\text{sep}}-t)} + e^{-\Delta E t_{\text{sep}}} \right)$$

2.3 Calculate 3pt on lattice

For example,

$$\text{Projection operator } \pi^+ : \hat{O}_{\pi^+}(\vec{x}, t_{\text{sep}}) = \bar{d}(\vec{x}, t_{\text{sep}}) \gamma^5 u(\vec{x}, t_{\text{sep}})$$

$$\hat{O}_{\pi^+}^\dagger(\vec{x}, t_{\text{sep}}) = -\bar{u}(\vec{x}, t_{\text{sep}}) \gamma^5 d(\vec{x}, t_{\text{sep}})$$

$$\text{Quasi-PDF operator } u : \hat{O}(\vec{y}, t; z) = \bar{u}(z + \vec{y}, t) \gamma^t W(z + \vec{y}, t; \vec{y}, t) u(\vec{y}, t)$$

$$\begin{aligned} 3\text{pt} &= \int d^3 \vec{x} e^{-i\vec{p} \cdot \vec{x}} \int d^3 \vec{y} \langle \Omega | \hat{O}_H(\vec{x}, t_{\text{sep}}) \hat{O}(\vec{y}, t; z) \hat{O}_H^\dagger(0,0) | \Omega \rangle \\ &= \int d^3 \vec{x} e^{-i\vec{p} \cdot \vec{x}} \int d^3 \vec{y} \langle \Omega | \bar{d}(\vec{x}, t_{\text{sep}}) \gamma^5 u(\vec{x}, t_{\text{sep}}) \bar{u}(z + \vec{y}, t) \gamma^t W(z + \vec{y}, t; \vec{y}, t) u(\vec{y}, t) \\ &\quad \cdot (-\bar{u}(0,0) \gamma^5 d(0,0)) | \Omega \rangle \end{aligned}$$

Add trace, then move the d at the end to the beginning. Notice here in the spinor space, this moving just move the column vector forward with trace, so no minus sign, while the d is a dirac field, containing generation annihilation operator (or say it is Grassmann number), so this moving will contribute a minus sign.

$$\begin{aligned} 3\text{pt} &= \int d^3 \vec{x} e^{-i\vec{p} \cdot \vec{x}} \int d^3 \vec{y} \langle \Omega | \text{tr}[d(0,0) \bar{d}(\vec{x}, t_{\text{sep}}) \gamma^5 u(\vec{x}, t_{\text{sep}}) \\ &\quad \bar{u}(z + \vec{y}, t) \gamma^t W(z + \vec{y}, t; \vec{y}, t) u(\vec{y}, t) \bar{u}(0,0) \gamma^5] | \Omega \rangle \end{aligned}$$

*[Wick theorem Gattringer P109 (5.36), 2 d fields contract, 4 u have 2 kinds of contraction]

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \{$$

$$< \Omega | \text{tr}[S_d(0, 0; \vec{x}, t_{\text{sep}}) \gamma^5 S_u(\vec{x}, t_{\text{sep}}; z + \vec{y}, t) \gamma^t W(z + \vec{y}, t; \vec{y}, t) S_u(\vec{y}, t; 0, 0) \gamma^5] | \Omega >$$

$$- < \Omega | \text{tr}[S_d(0, 0; \vec{x}, t_{\text{sep}}) \gamma^5 S_u(\vec{x}, t_{\text{sep}}; 0, 0) \gamma^5] \cdot \text{tr}[S_u(\vec{y}, t; z + \vec{y}, t) \gamma^t W(z + \vec{y}, t; \vec{y}, t)] | \Omega > \}$$

two terms in the integral represent two diagrams, take the first one as an example.

$$\int d^3\vec{y} < \Omega | \text{tr}[\underbrace{\int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \gamma^5 S_d(0, 0; \vec{x}, t_{\text{sep}}) \gamma^5 S_u(\vec{x}, t_{\text{sep}}; z + \vec{y}, t) \gamma^t W(z + \vec{y}, t; \vec{y}, t) S_u(\vec{y}, t; 0, 0)}_{\text{sequential propagator}}] | \Omega >$$

in which, red part is sequential source, and the underlined part is sequential propagator.

*[we need to avoid calculating all to all propagator, like $S_u(\vec{x}, t_{\text{sep}}; z + \vec{y}, t)$ (x and y are both integrated)]

$$\begin{aligned} & \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \gamma^5 S_d(0, 0; \vec{x}, t_{\text{sep}}) \gamma^5 S_u(\vec{x}, t_{\text{sep}}; z + \vec{y}, t) \\ &= \gamma^5 \left[\int d^3\vec{x} S_u(z + \vec{y}, t; \vec{x}, t_{\text{sep}}) e^{i\vec{p}\cdot\vec{x}} \gamma^5 S_d(\vec{x}, t_{\text{sep}}; 0, 0) \gamma^5 \right]^\dagger \gamma^5 \end{aligned}$$

here we used

$$\gamma^5 S^\dagger(x; y) \gamma^5 = S(y; x)$$

*[Gattringer P136 (6.31)]

the \dagger here acts on spinor and color indices.

So, sequential propagator:

$$\int d^3\vec{x} \cdot S_u(z + \vec{y}, t; \vec{x}, t_{\text{sep}}) e^{i\vec{p}\cdot\vec{x}} \gamma^5 S_d(\vec{x}, t_{\text{sep}}; 0, 0) \gamma^5$$