PDFs and Lattice calculation

Yushan and Jinchen

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1 Basic logic

We calculate 3pt correlation on lattice, then extract PDFs from 3pt correlation.

2 Deduction

2.1 What is PDFs

sth like

$$\left\langle H\left(P_z\right)\left|\bar{\psi}(z)\gamma^t W(z,0)\psi(0)\right| H\left(P_z\right)\right\rangle$$

*[check out Peskin 18.5.]

2.2 Calculate PDFs through 3pt correlation

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \hat{O}_H\left(\vec{x}, t_{sep}\right) \hat{O}(\vec{y}, t; z) \hat{O}_H^\dagger(0,0) \right| \Omega \right\rangle$$

in which $\hat{O}_H$ is projection operator, and

$$\hat{O}(\vec{y}, t; z) = \bar{\psi}(z+\vec{y}, t)\gamma^t W(z+\vec{y}, t; \vec{y}, t)\psi(\vec{y}, t)$$

therefore, ignore t variables for convenient,

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \hat{O}_H\left(\vec{x}\right) \sum_H \int \frac{d^3\vec{p'}}{(2\pi)^3} |H_{\vec{p'}}><H_{\vec{p'}}| \hat{O}(\vec{y}; z) \right.\right.$$
$$\left.\left. \sum_{H'} \int \frac{d^3\vec{p''}}{(2\pi)^3} |H'_{\vec{p''}}><H'_{\vec{p''}}| \hat{O}_H^\dagger(0) \right| \Omega \right\rangle$$

with spatial translation operator:

$$\hat{O}_H(\vec{x}) = e^{-i\hat{\vec{p}}\cdot\vec{x}} \hat{O}_H e^{i\hat{\vec{p}}\cdot\vec{x}}$$

so,

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \hat{O}_H \cdot e^{i\hat{\vec{p}}\cdot\vec{x}} \sum_H \int \frac{d^3\vec{p'}}{(2\pi)^3} |H_{\vec{p'}} >< H_{\vec{p'}}| e^{-i\hat{\vec{p}}\cdot\vec{y}} \cdot \hat{O}(0;z) \cdot e^{i\hat{\vec{p}}\cdot\vec{y}} \right.\right.$$

$$\left.\left. \sum_{H'} \int \frac{d^3\vec{p''}}{(2\pi)^3} |H'_{\vec{p''}} >< H'_{\vec{p''}}| \hat{O}^\dagger_H(0) \right| \Omega \right\rangle$$

$$= \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \hat{O}_H \cdot \sum_H \int \frac{d^3\vec{p'}}{(2\pi)^3} e^{i\vec{p'}\cdot\vec{x}} |H_{\vec{p'}} >< H_{\vec{p'}}| e^{-i\vec{p'}\cdot\vec{y}} \cdot \hat{O}(0;z) \right.\right.$$

$$\left.\left. \sum_{H'} \int \frac{d^3\vec{p''}}{(2\pi)^3} e^{i\vec{p''}\cdot\vec{y}} |H'_{\vec{p''}} >< H'_{\vec{p''}}| \hat{O}^\dagger_H(0) \right| \Omega \right\rangle$$

do the integral of x and y, then we get,

$$3\text{pt} = \left\langle \Omega \left| \hat{O}_H \cdot \sum_H \int d^3\vec{p'} \delta(\vec{p'} - \vec{p}) |H_{\vec{p'}} >< H_{\vec{p'}}| \hat{O}(0;z) \sum_{H'} \int d^3\vec{p''} \delta(\vec{p'} - \vec{p''}) |H'_{\vec{p''}} >< H'_{\vec{p''}}| \hat{O}^\dagger_H(0) \right| \Omega \right\rangle$$

$$= \sum_{H'} \sum_H < \Omega | \hat{O}_H | H_{\vec{p}} >< H_{\vec{p}} | \hat{O}(0;z) | H'_{\vec{p}} >< H'_{\vec{p}} | \hat{O}^\dagger_H(0) | \Omega >$$

then projection operator $\hat{O}_H$ will select the hadron with specific quantum numbers, like for π^+, $\hat{O}_{\pi^+} = \bar{d}\gamma^5 u$.

$$3\text{pt} =< \Omega | \hat{O}_H(0, t_{sep}) | H_{\vec{p}} >< H_{\vec{p}} | \hat{O}(0, t; z) | H_{\vec{p}} >< H_{\vec{p}} | \hat{O}^\dagger_H(0) | \Omega >$$

$$=< \Omega | \hat{O}_H(0, t_{\text{sep}}) | H_{\vec{p}} > \cdot \text{PDFs} \cdot < H_{\vec{p}} | \hat{O}^\dagger_H(0) | \Omega >$$

In order to get PDFs, we also need 2pt correlation function:

$$2\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} < \Omega | \hat{O}_H(\vec{x}, t_{sep}) \, \hat{O}^\dagger_H(0,0) | \Omega >$$

$$=< \Omega | \hat{O}_H(0, t_{\text{sep}}) | H_{\vec{p}} >< H_{\vec{p}} | \hat{O}^\dagger_H(0,0) | \Omega >$$

so,

$$\frac{3\text{pt}}{2\text{pt}} = \left\langle H_{\vec{p}} | \hat{O}(0,0;z) | H_{\vec{p}} \right\rangle \left(1 + e^{-\Delta E t} + e^{-\Delta E (t_{\text{sep}} - t)} + e^{-\Delta E t_{sep}} \right)$$

2.3 Calculate 3pt on lattice

For example,

$$\text{Projection operator } \pi^+ : \hat{O}_{\pi^+}(\vec{x}, t_{\text{sep}}) = \bar{d}(\vec{x}, t_{\text{sep}})\gamma^5 u(\vec{x}, t_{\text{sep}})$$

$$\hat{O}^\dagger_{\pi^+}(\vec{x}, t_{\text{sep}}) = -\bar{u}(\vec{x}, t_{\text{sep}})\gamma^5 d(\vec{x}, t_{\text{sep}})$$

$$\text{Quasi-PDF operator } u : \hat{O}(\vec{y}, t; z) = \bar{u}(z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t)u(\vec{y}, t)$$

$$\text{3pt} = \int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \hat{O}_H(\vec{x}, t_{sep}) \hat{O}(\vec{y}, t; z) \hat{O}^\dagger_H(0, 0) \right| \Omega \right\rangle$$

$$= \int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \langle \Omega | \bar{d}(\vec{x}, t_{\text{sep}})\gamma^5 u(\vec{x}, t_{\text{sep}}) \bar{u}(z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t)u(\vec{y}, t)$$

$$\cdot(-\bar{u}(0, 0)\gamma^5 d(0, 0)) | \Omega \rangle$$

Add trace, then move the d at the end to the beginning. Notice here in the spinor space, this moving just move the column vector forward with trace, so no minus sign, while the d is a dirac field, containing generation annihilation operator (or say it is Grassmann number), so this moving will contribute a minus sign.

$$\text{3pt} = \int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \langle \Omega | \text{tr}[d(0, 0)\bar{d}(\vec{x}, t_{\text{sep}})\gamma^5 u(\vec{x}, t_{\text{sep}})$$

$$\bar{u}(z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t)u(\vec{y}, t)\bar{u}(0, 0)\gamma^5] | \Omega \rangle$$

*[Wick theorem Gattringer P109 (5.36), 2 d fields contract, 4 u have 2 kinds of contraction]

$$\text{3pt} = \int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y}\{$$

$$\langle \Omega | \text{tr}[S_d(0, 0; \vec{x}, t_{\text{sep}})\gamma^5 S_u(\vec{x}, t_{\text{sep}}; z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t)S_u(\vec{y}, t; 0, 0)\gamma^5] | \Omega \rangle$$

$$- \langle \Omega | \text{tr}[S_d(0, 0; \vec{x}, t_{\text{sep}})\gamma^5 S_u(\vec{x}, t_{\text{sep}}; 0, 0)\gamma^5] \cdot \text{tr}[S_u(\vec{y}, t; z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t)] | \Omega \rangle\}$$

two terms in the integral represent two diagrams, take the first one as an example.

$$\int d^3\vec{y} \left\langle \Omega \left| \text{tr}[\underline{\int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}}\gamma^5 S_d(0, 0; \vec{x}, t_{\text{sep}})\gamma^5} S_u(\vec{x}, t_{\text{sep}}; z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t)S_u(\vec{y}, t; 0, 0)] \right| \Omega \right\rangle$$

in which, red part is sequential source, and the underlined part is sequential propagator.

*[we need to avoid calculating all to all propagator, like $S_u(\vec{x}, t_{\text{sep}}; z+\vec{y}, t)$ (x and y are both integrated)]

$$\int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}}\gamma^5 S_d\left(0,0;\vec{x},t_{\text{sep}}\right)\gamma^5 S_u\left(\vec{x},t_{\text{sep}};z+\vec{y},t\right)$$

$$=\gamma^5[\int d^3\vec{x}S_u\left(z+\vec{y},t;\vec{x},t_{\text{sep}}\right)e^{i\vec{p}\cdot\vec{x}}\gamma^5 S_d\left(\vec{x},t_{\text{sep}};0,0\right)\gamma^5]^\dagger\gamma^5$$

here we used

$$\gamma^5 S^\dagger(x;y)\gamma^5 = S(y;x)$$

the † here acts on spinor and color indices.

So, sequential propagator:

$$\int d^3\vec{x}\cdot S_u\left(z+\vec{y},t;\vec{x},t_{\text{sep}}\right)e^{i\vec{p}\cdot\vec{x}}\gamma^5 S_d\left(\vec{x},t_{\text{sep}};0,0\right)\gamma^5$$