PDFs and Lattice calculation

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1 Basic logic

We calculate 3pt correlation on lattice, then extract PDFs from 3pt correlation.

2 Deduction

2.1 What is PDFs

sth like

$$\left\langle H\left(P_z\right)\left|\bar{\psi}(z)\gamma^t W(z,0)\psi(0)\right| H\left(P_z\right)\right\rangle$$

*[check out Peskin 18.5.]

2.2 Calculate PDFs through 3pt correlation

$$3\mathrm{pt}=\int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}}\int d^3\vec{y}\left\langle\Omega\left|\hat{O}_H\left(\vec{x},t_{sep}\right)\hat{O}(\vec{y},t;z)\hat{O}_H^{\dagger}(0,0)\right|\Omega\right\rangle$$

in which $\hat{O}_H$ is projection operator, and

$$\hat{O}(\vec{y},t;z)=\bar{\psi}(z+\vec{y},t)\gamma^t W(z+\vec{y},t;\vec{y},t)\psi(\vec{y},t)$$

therefore, ignore t variables first for convenience,

$$3\mathrm{pt}=\int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}}\int d^3\vec{y}\left\langle\Omega\left|\hat{O}_H\left(\vec{x}\right)\sum_H\int\frac{d^3\vec{p'}}{(2\pi)^3}|H_{\vec{p'}}><H_{\vec{p'}}|\hat{O}(\vec{y};z)\right.\right.$$
$$\left.\left.\sum_{H'}\int\frac{d^3\vec{p''}}{(2\pi)^3}|H'_{\vec{p''}}><H'_{\vec{p''}}|\hat{O}_H^{\dagger}(0)\right|\Omega\right\rangle$$

with spatial translation operator:

$$\hat{O}_H(\vec{x})=e^{-i\hat{\vec{p}}\cdot\vec{x}}\hat{O}_H e^{i\hat{\vec{p}}\cdot\vec{x}}$$

so,

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \middle| \hat{O}_H \cdot e^{i\hat{\vec{p}}\cdot\vec{x}} \sum_H \int \frac{d^3\vec{p'}}{(2\pi)^3} |H_{\vec{p'}} >< H_{\vec{p'}}| e^{-i\hat{\vec{p}}\cdot\vec{y}} \cdot \hat{O}(0;z) \cdot e^{i\hat{\vec{p}}\cdot\vec{y}} \right.$$

$$\left. \sum_{H'} \int \frac{d^3\vec{p''}}{(2\pi)^3} |H'_{\vec{p''}} >< H'_{\vec{p''}}| \hat{O}_H^\dagger(0) \middle| \Omega \right\rangle$$

$$= \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \middle| \hat{O}_H \cdot \sum_H \int \frac{d^3\vec{p'}}{(2\pi)^3} e^{i\vec{p'}\cdot\vec{x}} |H_{\vec{p'}} >< H_{\vec{p'}}| e^{-i\vec{p'}\cdot\vec{y}} \cdot \hat{O}(0;z) \right.$$

$$\left. \sum_{H'} \int \frac{d^3\vec{p''}}{(2\pi)^3} e^{i\vec{p''}\cdot\vec{y}} |H'_{\vec{p''}} >< H'_{\vec{p''}}| \hat{O}_H^\dagger(0) \middle| \Omega \right\rangle$$

do the integral of x and y, then we get,

$$3\text{pt} = \left\langle \Omega \middle| \hat{O}_H \cdot \sum_H \int d^3\vec{p'} \delta(\vec{p'} - \vec{p}) |H_{\vec{p'}} >< H_{\vec{p'}}| \hat{O}(0;z) \sum_{H'} \int d^3\vec{p''} \delta(\vec{p'} - \vec{p''}) |H'_{\vec{p''}} >< H'_{\vec{p''}}| \hat{O}_H^\dagger(0) \middle| \Omega \right\rangle$$

$$= \sum_{H'} \sum_H < \Omega | \hat{O}_H | H_{\vec{p}} >< H_{\vec{p}} | \hat{O}(0;z) | H'_{\vec{p}} >< H'_{\vec{p}} | \hat{O}_H^\dagger(0) | \Omega >$$

then projection operator $\hat{O}_H$ will select the hadron with specific quantum numbers, like for π^+, $\hat{O}_{\pi^+} = \bar{d}\gamma^5 u$.

$$3\text{pt} =< \Omega | \hat{O}_H(0, t_{sep}) | H_{\vec{p}} >< H_{\vec{p}} | \hat{O}(0, t; z) | H_{\vec{p}} >< H_{\vec{p}} | \hat{O}_H^\dagger(0) | \Omega >$$

$$=< \Omega | \hat{O}_H(0, t_{\text{sep}}) | H_{\vec{p}} > \cdot \text{PDFs} \cdot < H_{\vec{p}} | \hat{O}_H^\dagger(0) | \Omega >$$

In order to get PDFs, we also need 2pt correlation function:

$$2\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} < \Omega | \hat{O}_H(\vec{x}, t_{sep}) \hat{O}_H^\dagger(0, 0) | \Omega >$$

$$=< \Omega | \hat{O}_H(0, t_{\text{sep}}) | H_{\vec{p}} >< H_{\vec{p}} | \hat{O}_H^\dagger(0, 0) | \Omega >$$

Pay attention here $|H_{\vec{p}} >$ is a superposition of Hamiltonian operator instead of eigenstate. Therefore, we have the expression below (we did Wick rotation on the lattice, so $it_M = t_E$)

$$3\text{pt} = \sum_{m,n} < \Omega | \hat{O}_H(0,0) e^{-\hat{H}t_{\text{sep}}} | E_n >< E_n | e^{\hat{H}t} \hat{O}(0, 0; z) e^{-\hat{H}t} | E_m >< E_m | \hat{O}_H^\dagger(0) | \Omega >$$

$$= \sum_{m,n} e^{-E_n t_{\text{sep}}} e^{E_n t} e^{-E_m t} < \Omega|\hat{O}_H(0,0)|E_n >< E_n|\hat{O}(0,0;z)|E_m >< E_m|\hat{O}_H^\dagger(0)|\Omega >$$

same for 2pt,

$$2\text{pt} = \sum_n e^{-E_n t_{\text{sep}}} < \Omega|\hat{O}_H(0,0)|E_n >< E_n|\hat{O}_H^\dagger(0,0)|\Omega >$$

so,

$$\text{PDFs} = \frac{3\text{pt}}{2\text{pt}} = \left\langle H_{\vec{p}}|\hat{O}(0,0;z)|H_{\vec{p}}\right\rangle \left(1 + e^{-\Delta E t} + e^{-\Delta E(t_{\text{sep}}-t)} + e^{-\Delta E t_{sep}}\right)$$

2.3 Calculate 3pt on lattice

For example,

$$\text{Projection operator } \pi^+ : \hat{O}_{\pi^+}(\vec{x}, t_{\text{sep}}) = \bar{d}(\vec{x}, t_{\text{sep}})\gamma^5 u(\vec{x}, t_{\text{sep}})$$

$$\hat{O}_{\pi^+}^\dagger(\vec{x}, t_{\text{sep}}) = -\bar{u}(\vec{x}, t_{\text{sep}})\gamma^5 d(\vec{x}, t_{\text{sep}})$$

$$\text{Quasi-PDF operator } u : \hat{O}(\vec{y}, t; z) = \bar{u}(z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t)u(\vec{y}, t)$$

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \hat{O}_H(\vec{x}, t_{sep}) \hat{O}(\vec{y}, t; z) \hat{O}_H^\dagger(0,0) \right| \Omega \right\rangle$$

$$= \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \bar{d}(\vec{x}, t_{\text{sep}})\gamma^5 u(\vec{x}, t_{\text{sep}}) \bar{u}(z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t)u(\vec{y}, t) \right.\right.$$

$$\left.\left. \cdot(-\bar{u}(0,0)\gamma^5 d(0,0)) \right| \Omega \right\rangle$$

Add trace, then move the d at the end to the beginning. Notice here in the spinor space, this moving just move the column vector forward with trace, so no minus sign, while the d is a dirac field, containing generation annihilation operator (or say it is Grassmann number), so this moving will contribute a minus sign.

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \text{tr}[d(0,0)\bar{d}(\vec{x}, t_{\text{sep}})\gamma^5 u(\vec{x}, t_{\text{sep}}) \right.\right.$$

$$\left.\left. \bar{u}(z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t)u(\vec{y}, t)\bar{u}(0,0)\gamma^5] \right| \Omega \right\rangle$$

*[Wick theorem Gattringer P109 (5.36), 2 d fields contract, 4 u have 2 kinds of contraction]

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \{$$

$$< \Omega|\text{tr}[S_d(0,0;\vec{x},t_{\text{sep}})\gamma^5 S_u(\vec{x},t_{\text{sep}};z+\vec{y},t)\gamma^t W(z+\vec{y},t;\vec{y},t)S_u(\vec{y},t;0,0)\gamma^5]|\Omega >$$

$$- < \Omega|\text{tr}[S_d(0,0;\vec{x},t_{\text{sep}})\gamma^5 S_u(\vec{x},t_{\text{sep}};0,0)\gamma^5] \cdot \text{tr}[S_u(\vec{y},t;z+\vec{y},t)\gamma^t W(z+\vec{y},t;\vec{y},t)]|\Omega >\}$$

two terms in the integral represent two diagrams, take the first one as an example.

$$\int d^3\vec{y} < \Omega|\text{tr}[\underline{\int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}}\gamma^5 S_d(0,0;\vec{x},t_{\text{sep}})\gamma^5 S_u(\vec{x},t_{\text{sep}};z+\vec{y},t)}\gamma^t W(z+\vec{y},t;\vec{y},t)S_u(\vec{y},t;0,0)]|\Omega >$$

in which, red part is sequential source, and the underlined part is sequential propagator.

*[we need to avoid calculating all to all propagator, like $S_u(\vec{x},t_{\text{sep}};z+\vec{y},t)$ (x and y are both integrated)]

$$\int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}}\gamma^5 S_d\left(0,0;\vec{x},t_{\text{sep}}\right)\gamma^5 S_u\left(\vec{x},t_{\text{sep}};z+\vec{y},t\right)$$

$$= \gamma^5[\int d^3\vec{x} S_u\left(z+\vec{y},t;\vec{x},t_{\text{sep}}\right)e^{i\vec{p}\cdot\vec{x}}\gamma^5 S_d\left(\vec{x},t_{\text{sep}};0,0\right)\gamma^5]^\dagger\gamma^5$$

here we used

$$\gamma^5 S^\dagger(x;y)\gamma^5 = S(y;x)$$

*[Gattringer P136 (6.31)]

the $\dagger$ here acts on spinor and color indices.

So, sequential propagator:

$$\int d^3\vec{x}\cdot S_u\left(z+\vec{y},t;\vec{x},t_{\text{sep}}\right)e^{i\vec{p}\cdot\vec{x}}\gamma^5 S_d\left(\vec{x},t_{\text{sep}};0,0\right)\gamma^5$$