PDFs and Lattice calculation

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I. BASIC LOGIC

We calculate 3pt correlation on lattice, then extract PDFs from 3pt correlation.

II. DEDUCTION

A. What is PDFs

sth like

$$\left\langle H\left(P_z\right)\left|\bar{\psi}(z)\gamma^t W(z,0)\psi(0)\right|H\left(P_z\right)\right\rangle$$

*[check out Peskin 18.5.]

B. Calculate PDFs through 3pt correlation

$$3\text{pt}=\int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}}\int d^3\vec{y}\left\langle\Omega\left|\hat{O}_H\left(\vec{x},t_{sep}\right)\hat{O}(\vec{y},t;z)\hat{O}_H^{\dagger}(0,0)\right|\Omega\right\rangle$$

in which $\hat{O}_H$ is projection operator, and

$$\hat{O}(\vec{y},t;z)=\bar{\psi}(z+\vec{y},t)\gamma^t W(z+\vec{y},t;\vec{y},t)\psi(\vec{y},t)$$

therefore, ignore t variables first for convenience,

$$3\text{pt}=\int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}}\int d^3\vec{y}\left\langle\Omega\left|\hat{O}_H\left(\vec{x}\right)\sum_H\int\frac{d^3\vec{p'}}{(2\pi)^3}|H_{\vec{p'}}><H_{\vec{p'}}|\hat{O}(\vec{y};z)\right.\right.$$

$$\left.\left.\sum_{H'}\int\frac{d^3\vec{p''}}{(2\pi)^3}|H'_{\vec{p''}}><H'_{\vec{p''}}|\hat{O}_H^{\dagger}(0)\right|\Omega\right\rangle$$

with spatial translation operator:

$$\hat{O}_H(\vec{x})=e^{-i\hat{\vec{p}}\cdot\vec{x}}\hat{O}_H e^{i\hat{\vec{p}}\cdot\vec{x}}$$

so,

$$3\text{pt}=\int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}}\int d^3\vec{y}\left\langle\Omega\left|\hat{O}_H\cdot e^{i\hat{\vec{p}}\cdot\vec{x}}\sum_H\int\frac{d^3\vec{p'}}{(2\pi)^3}|H_{\vec{p'}}><H_{\vec{p'}}|e^{-i\hat{\vec{p}}\cdot\vec{y}}\cdot\hat{O}(0;z)\cdot e^{i\hat{\vec{p}}\cdot\vec{y}}\right.\right.$$

$$\sum_{H'}\int\frac{d^3\vec{p''}}{(2\pi)^3}|H'_{\vec{p''}}><H'_{\vec{p''}}|\hat{O}^\dagger_H(0)\bigg|\Omega\bigg\rangle$$

$$=\int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}}\int d^3\vec{y}\bigg\langle\Omega\bigg|\hat{O}_H\cdot\sum_H\int\frac{d^3\vec{p'}}{(2\pi)^3}e^{i\vec{p'}\cdot\vec{x}}|H_{\vec{p'}}><H_{\vec{p'}}|e^{-i\vec{p'}\cdot\vec{y}}\cdot\hat{O}(0;z)$$

$$\sum_{H'}\int\frac{d^3\vec{p''}}{(2\pi)^3}e^{i\vec{p''}\cdot\vec{y}}|H'_{\vec{p''}}><H'_{\vec{p''}}|\hat{O}^\dagger_H(0)\bigg|\Omega\bigg\rangle$$

do the integral of x and y, then we get,

$$3\text{pt}=\bigg\langle\Omega\bigg|\hat{O}_H\cdot\sum_H\int d^3\vec{p'}\delta(\vec{p'}-\vec{p})|H_{\vec{p'}}><H_{\vec{p'}}|\hat{O}(0;z)\sum_{H'}\int d^3\vec{p''}\delta(\vec{p'}-\vec{p''})|H'_{\vec{p''}}><H'_{\vec{p''}}|\hat{O}^\dagger_H(0)\bigg|\Omega\bigg\rangle$$

$$=\sum_{H'}\sum_H<\Omega|\hat{O}_H|H_{\vec{p}}><H_{\vec{p}}|\hat{O}(0;z)|H'_{\vec{p}}><H'_{\vec{p}}|\hat{O}^\dagger_H(0)|\Omega>$$

then projection operator $\hat{O}_H$ will select the hadron with specific quantum numbers, like for π^+, $\hat{O}_{\pi^+}=\bar{d}\gamma^5 u$.

$$3\text{pt}=<\Omega|\hat{O}_H(0,t_{sep})|H_{\vec{p}}><H_{\vec{p}}|\hat{O}(0,t;z)|H_{\vec{p}}><H_{\vec{p}}|\hat{O}^\dagger_H(0)|\Omega>$$

$$=<\Omega|\hat{O}_H(0,t_{\text{sep}})|H_{\vec{p}}>\cdot\text{PDFs}\cdot<H_{\vec{p}}|\hat{O}^\dagger_H(0)|\Omega>$$

In order to get PDFs, we also need 2pt correlation function:

$$2\text{pt}=\int d^3\vec{x}e^{-i\vec{p}\cdot\vec{x}}<\Omega|\hat{O}_H\left(\vec{x},t_{sep}\right)\hat{O}^\dagger_H(0,0)|\Omega>$$

$$=<\Omega|\hat{O}_H(0,t_{\text{sep}})|H_{\vec{p}}><H_{\vec{p}}|\hat{O}^\dagger_H(0,0)|\Omega>$$

Pay attention here $|H_{\vec{p}}>$ is a superposition of Hamiltonian operator instead of eigenstate. Therefore, we have the expression below (we did Wick rotation on the lattice, so $it_M=t_E$)

$$3\text{pt}=\sum_{m,n}<\Omega|\hat{O}_H(0,0)e^{-\hat{H}t_{\text{sep}}}|E_n><E_n|e^{\hat{H}t}\hat{O}(0,0;z)e^{-\hat{H}t}|E_m><E_m|\hat{O}^\dagger_H(0)|\Omega>$$

$$=\sum_{m,n}e^{-E_nt_{\text{sep}}}e^{E_nt}e^{-E_mt}<\Omega|\hat{O}_H(0,0)|E_n><E_n|\hat{O}(0,0;z)|E_m><E_m|\hat{O}^\dagger_H(0)|\Omega>$$

same for 2pt,

$$2\text{pt} = \sum_n e^{-E_n t_{\text{sep}}} < \Omega|\hat{O}_H(0,0)|E_n >< E_n|\hat{O}_H^\dagger(0,0)|\Omega >$$

For convenience, we define,

$$z_n^\dagger =< E_n|\hat{O}_H^\dagger(0,0)|\Omega >$$

$$z_n =< \Omega|\hat{O}_H(0,0)|E_n >$$

$$O_{nm} =< E_n|\hat{O}(0,0;z)|E_m >$$

preserve only lowest two energy states, we got

$$3\text{pt} \approx z_0^2 O_{00} e^{-E_0 t_{\text{sep}}} + z_0^\dagger z_1 O_{01} e^{-E_0 t_{\text{sep}}} e^{-\Delta E t} + z_1^\dagger z_0 O_{10} e^{-E_1 t_{\text{sep}}} e^{\Delta E t} + z_1^2 O_{11} e^{-E_1 t_{\text{sep}}}$$

$$2\text{pt} \approx z_0^2 e^{-E_0 t_{\text{sep}}} + z_1^2 e^{-E_1 t_{\text{sep}}} = z_0^2 e^{-E_0 t_{\text{sep}}}(1 + c_1 e^{-\Delta E t_{\text{sep}}})$$

so, assume $O_{01} = O_{10}$, we have fit function

$$\frac{3\text{pt}}{2\text{pt}} = \frac{1}{1 + c_1 e^{-\Delta E t_{\text{sep}}}} \cdot [O_{00} + \frac{z_0^\dagger z_1}{z_0^2} O_{01} e^{-\Delta E t} + \frac{z_1^\dagger z_0}{z_0^2} O_{10} e^{-\Delta E t_{\text{sep}}} e^{\Delta E t} + \frac{z_1^2}{z_0^2} O_{11} e^{-\Delta E t_{\text{sep}}}]$$

* compared with 1, drop the $e^{-\Delta E t_{\text{sep}}}$ term in the denominator

$$\approx O_{00} + \frac{z_0^\dagger z_1}{z_0^2} O_{01} e^{-\Delta E t} + \frac{z_1^\dagger z_0}{z_0^2} O_{10} e^{-\Delta E t_{\text{sep}}} e^{\Delta E t} + \frac{z_1^2}{z_0^2} O_{11} e^{-\Delta E t_{\text{sep}}}$$

$$= O_{00} \cdot [1 + a_1(e^{-\Delta E(t_{\text{sep}}-t)} + e^{-\Delta E t}) + a_2 e^{-\Delta E t_{\text{sep}}}]$$

also, the FH fit function is

$$\Sigma(t_{\text{sep}}) = \sum_{t=n}^{t_{\text{sep}}-n} \frac{3\text{pt}}{2\text{pt}} = (O_{00} + O_{00} a_2 e^{-\Delta E t_{\text{sep}}}) \cdot (t_{\text{sep}} + 1 - 2n) + O_{00} a_1 e^{-\Delta E t_{\text{sep}}} \sum_t e^{\Delta E t} + O_{00} a_1 \sum_t e^{-\Delta E t}$$

$$\sum_{t=n}^{t_{\text{sep}}-n} e^{\Delta E t} = e^{\Delta E n} \frac{1 - e^{\Delta E(t_{\text{sep}}+1-2n)}}{1 - e^{\Delta E}} \approx \frac{e^{\Delta E n}}{1 - e^{\Delta E}}$$

$$\Sigma(t_{\text{sep}}) = (O_{00} + O_{00} a_2 e^{-\Delta E t_{\text{sep}}}) \cdot (t_{\text{sep}} + 1 - 2n) + \frac{O_{00} a_1 e^{-\Delta E(t_{\text{sep}}-n)}}{1 - e^{\Delta E}} + \frac{O_{00} a_1 e^{-\Delta E n}}{1 - e^{-\Delta E}}$$

$$= (O_{00} + O_{00} a_2 e^{-\Delta E t_{\text{sep}}}) \cdot (t_{\text{sep}} + 1 - 2n) + \frac{O_{00} a_1 e^{-\Delta E(t_{\text{sep}}-n+1)}}{e^{-\Delta E} - 1} + \frac{O_{00} a_1 e^{-\Delta E n}}{1 - e^{-\Delta E}}$$

$$\text{FH} = \Sigma(t_{\text{sep}} + 1) - \Sigma(t_{\text{sep}}) = O_{00} + O_{00} a_2 e^{-\Delta E t_{\text{sep}}}[(t_{\text{sep}} + 2 - 2n)e^{-\Delta E} - (t_{\text{sep}} + 1 - 2n)] + O_{00} a_1 e^{-\Delta E(t_{\text{sep}}-n+1)}$$

$$\text{FH} = O_{00} \cdot [1 + a_1' e^{-\Delta E t_{\text{sep}}} + a_2' \cdot t_{\text{sep}} \cdot e^{-\Delta E t_{\text{sep}}}]$$

the O_{00} is PDFs.

C. Calculate 3pt on lattice

For example,

$$\text{Projection operator } \pi^+ : \hat{O}_{\pi^+}(\vec{x}, t_{\text{sep}}) = \bar{d}(\vec{x}, t_{\text{sep}})\gamma^5 u(\vec{x}, t_{\text{sep}})$$

$$\hat{O}^\dagger_{\pi^+}(\vec{x}, t_{\text{sep}}) = -\bar{u}(\vec{x}, t_{\text{sep}})\gamma^5 d(\vec{x}, t_{\text{sep}})$$

$$\text{Quasi-PDF operator } u : \hat{O}(\vec{y}, t; z) = \bar{u}(z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t) u(\vec{y}, t)$$

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \left\langle \Omega \left| \hat{O}_H(\vec{x}, t_{sep}) \hat{O}(\vec{y}, t; z) \hat{O}^\dagger_H(0, 0) \right| \Omega \right\rangle$$

$$= \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \langle \Omega \left| \bar{d}(\vec{x}, t_{\text{sep}})\gamma^5 u(\vec{x}, t_{\text{sep}}) \bar{u}(z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t) u(\vec{y}, t) \right.$$

$$\cdot(-\bar{u}(0, 0)\gamma^5 d(0, 0)) \big| \Omega \big\rangle$$

Add trace, then move the d at the end to the beginning. Notice here in the spinor space, this moving just move the column vector forward with trace, so no minus sign, while the d is a dirac field, containing generation annihilation operator (or say it is Grassmann number), so this moving will contribute a minus sign.

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \langle \Omega \left| \text{tr}[d(0, 0)\bar{d}(\vec{x}, t_{\text{sep}})\gamma^5 u(\vec{x}, t_{\text{sep}}) \right.$$

$$\bar{u}(z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t) u(\vec{y}, t)\bar{u}(0, 0)\gamma^5] \big| \Omega \big\rangle$$

*[Wick theorem Gattringer P109 (5.36), 2 d fields contract, 4 u have 2 kinds of contraction]

$$3\text{pt} = \int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \{$$

$$< \Omega | \text{tr}[S_d(0, 0; \vec{x}, t_{\text{sep}})\gamma^5 S_u(\vec{x}, t_{\text{sep}}; z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t) S_u(\vec{y}, t; 0, 0)\gamma^5] | \Omega >$$

$$- < \Omega | \text{tr}[S_d(0, 0; \vec{x}, t_{\text{sep}})\gamma^5 S_u(\vec{x}, t_{\text{sep}}; 0, 0)\gamma^5] \cdot \text{tr}[S_u(\vec{y}, t; z + \vec{y}, t)\gamma^t W(z + \vec{y}, t; \vec{y}, t)] | \Omega > \}$$

two terms in the integral represent two diagrams, take the first one as an example.

$$\int d^3\vec{y} < \Omega|\mathrm{tr}[\underline{\int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \gamma^5 S_d(0,0;\vec{x},t_{\text{sep}})\gamma^5 S_u(\vec{x},t_{\text{sep}};z+\vec{y},t)}\gamma^t W(z+\vec{y},t;\vec{y},t)S_u(\vec{y},t;0,0)]|\Omega >$$

in which, red part is sequential source, and the underlined part is sequential propagator.
*[we need to avoid calculating all to all propagator, like $S_u(\vec{x},t_{\text{sep}};z+\vec{y},t)$ (x and y are both integrated)]

$$\int d^3\vec{x} e^{-i\vec{p}\cdot\vec{x}} \gamma^5 S_d\left(0,0;\vec{x},t_{\text{sep}}\right)\gamma^5 S_u\left(\vec{x},t_{\text{sep}};z+\vec{y},t\right)$$

$$= \gamma^5[\int d^3\vec{x} S_u\left(z+\vec{y},t;\vec{x},t_{\text{sep}}\right) e^{i\vec{p}\cdot\vec{x}} \gamma^5 S_d\left(\vec{x},t_{\text{sep}};0,0\right)\gamma^5]^\dagger \gamma^5$$

here we used

$$\gamma^5 S^\dagger(x;y)\gamma^5 = S(y;x)$$

*[Gattringer P136 (6.31)]
the † here acts on spinor and color indices.
So, sequential propagator:

$$\int d^3\vec{x} \cdot S_u\left(z+\vec{y},t;\vec{x},t_{\text{sep}}\right) e^{i\vec{p}\cdot\vec{x}} \gamma^5 S_d\left(\vec{x},t_{\text{sep}};0,0\right)\gamma^5$$