

Peskin Solutions: Chapter 3

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1 Problem 3.1

(a)

$$[L^i, L^j] = \frac{1}{4} \epsilon^{ilm} \epsilon^{jst} [J^{lm}, J^{st}] = \frac{i}{4} \epsilon^{ilm} \epsilon^{jst} (g^{ms} J^{lt} - g^{ls} J^{mt} - g^{mt} J^{ls} + g^{lt} J^{ms})$$

The four terms in the bracket are equal after switching the indexes, and $g^{ms} = -1$ when $m = s \in \{1, 2, 3\}$ so we got

$$[L^i, L^j] = i \epsilon^{ilm} \epsilon^{jst} g^{ms} J^{lt} = -i \epsilon^{mil} \epsilon^{mtj} J^{lt} = -i (\delta_i^t \delta_l^j - \delta_i^j \delta_l^t) J^{lt} = -i J^{ji}$$

$$i \epsilon^{ijk} L^k = \frac{i}{2} \epsilon^{ijk} \epsilon^{klm} J^{lm} = \frac{i}{2} (J^{ij} - J^{ji}) = -i J^{ji}$$

$$[L^i, L^j] = i \epsilon^{ijk} L^k$$

$$[K^i, K^j] = [J^{0i}, J^{0j}] = i (g^{i0} J^{0j} - g^{00} J^{ij} - g^{ij} J^{00} + g^{0j} J^{i0}) = -i J^{ij} = -i \epsilon^{ijk} L^k$$

$$[L^i, K^j] = \frac{1}{2} \epsilon_{imn} [J^{mn}, J^{0j}] = \frac{1}{2} \epsilon_{imn} (g^{nj} K^m - g^{mj} K^n) = i \epsilon_{ijk} K^k$$

$$[J_+^i, J_-^j] = \frac{1}{4} ([L^i, L^j] - i[L^i, K^j] + i[K^i, L^j] + [K^i, K^j]) = 0$$

$$[J_+^i, J_+^j] = \frac{1}{4} ([L^i, L^j] + i[L^i, K^j] + i[K^i, L^j] - [K^i, K^j]) = \frac{1}{2} (i \epsilon^{ijk} L^k - \epsilon^{ijk} K^k) = i \epsilon^{ijk} J_+^k$$

$$[J_+^i, J_+^j] = i \epsilon^{ijk} J_-^k$$

(b)

Once we get the expression of $\hat{\vec{L}}$ and $\hat{\vec{K}}$, we get a set of generators $J^{\mu\nu}$ of Lorentz group, also we get \hat{J}_+ and \hat{J}_- , each of them is a set of generators of rotation group.

when $(j_+, j_-) = (\frac{1}{2}, 0)$, $\hat{J}_+^i = \frac{\sigma^i}{2}$ and $\hat{J}_-^i = 0$

$$L^i = (J_+^i + J_-^i) = \frac{1}{2}\sigma^i$$

$$K^i = -i(J_+^i - J_-^i) = -\frac{i}{2}\sigma^i$$

$$\phi \rightarrow (1 - i\theta^i \frac{\sigma^i}{2} - \beta^i \frac{\sigma^i}{2})$$

This is the transformation of ψ_L , eq.(3.37).

(c)

Need more thinking ...

2 Problem 3.2

We know that $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, so,

$$i\sigma^{\mu\nu}q_\nu = (g^{\mu\nu} - \gamma^\mu\gamma^\nu)(p' - p)_\nu = (p' - p)^\mu - (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)p'_\nu + \gamma^\mu\gamma^\nu p_\nu = -(p' + p)^\mu + \not{p}'\gamma^\mu + \gamma^\mu\not{p}$$

According to the Dirac equation,

$$\bar{u}(p')[\not{p}'\gamma^\mu + \gamma^\mu\not{p}]u(p) = \bar{u}(p')[2m\gamma^\mu]u(p)$$

3 Problem 3.3

(a)

$$\not{k}_0 u_{R0} = \not{k}_0 \not{k}_1 u_{L0} = \gamma^\mu k_{0\mu} \gamma^\nu k_{1\nu} u_{L0} = \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} k_{0\mu} k_{1\nu} u_{L0} = g^{\mu\nu} k_{0\mu} k_{1\nu} u_{L0} = 0$$

$$\not{p} u_L(p) = \frac{1}{\sqrt{2p \cdot k_0}} \not{p} \not{p} u_{R0} = \frac{1}{\sqrt{2p \cdot k_0}} p^2 u_{R0} = 0$$

for the same reason,

$$\not{p} u_R(p) = 0$$

(b)

We know that u_{L0} is the left-handed spinor for a fermion with momentum k_0 , so $m = 0$ and $\not{k}_0 u_{L0} = 0$.

$$k_0 u_{L0} = 0 \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2E \\ 2E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} u_{L0} = 0$$

$$u_{L0} = (0, \sqrt{2E}, 0, 0)^T$$

$$u_{R0} = (0, 0, -\sqrt{2E}, 0)^T$$

We have $u_L(p) = \frac{1}{\sqrt{2p_0 + p_3}} \not{p} u_{R0}$ and $u_R(p) = \frac{1}{\sqrt{2p_0 + p_3}} \not{p} u_{L0}$

$$u_L(p) = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} -(p_0 + p_3) \\ -(p_1 + ip_2) \\ 0 \\ 0 \end{pmatrix}$$

$$u_R(p) = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} 0 \\ 0 \\ -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix}$$

(c)

$$s(p, q) = \bar{u}_R(p) u_L(q) = \frac{(p_1 + ip_2)(q_0 + q_3) - (q_1 + iq_2)(p_0 + p_3)}{\sqrt{(p_0 + p_3)(q_0 + q_3)}}$$

$$t(p, q) = \bar{u}_L(p) u_R(q) = \frac{(q_1 - iq_2)(p_0 + p_3) - (p_1 - ip_2)(q_0 + q_3)}{\sqrt{(p_0 + p_3)(q_0 + q_3)}}$$

So $s(p, q) = -s(q, p)$ and $t(p, q) = (s(q, p))^*$

4 Problem 3.4

(a)

5 Problem 3.5

(a)

$$\delta(\partial_\mu \phi^* \partial^\mu \phi) = -i \left(\partial_\mu \chi^* \sigma^2 \epsilon^\dagger \right) \partial^\mu \phi + (\partial_\mu \phi^*) \left(-i \epsilon^T \sigma^2 \partial^\mu \chi \right)$$

$$\delta(F^* F) = i \left(\partial_\mu \chi^\dagger \right) \bar{\sigma}^\mu \epsilon F - i F^* \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi$$

(b)

$$\begin{aligned} \delta(\Delta \mathcal{L}) &= -im \epsilon^T \sigma^2 \chi F - im \phi \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi + \frac{1}{2} im \left[\epsilon^T F + \epsilon^\dagger (\sigma^2)^T (\sigma^\mu)^T \partial_\mu \phi \right] \sigma^2 \chi \\ &\quad + \frac{1}{2} im \chi^T \sigma^2 \left[\epsilon F + \sigma^\mu (\partial_\mu \phi) \sigma^2 \epsilon^* \right] + \text{c.c} \end{aligned}$$

6 Problem 3.6

(a) We need to find the normalization coefficients of all 16 elements.

$$\text{tr}[\gamma^0 \gamma^0] = 4$$

$$\text{tr}[\gamma^i \gamma^i] = -4$$

So, there are γ^0 and $i\gamma^i$ in the Γ^A

(b)

Multiply equation at left by $(\bar{u}_2 \Gamma^F u_3)(\bar{u}_4 \Gamma^E u_1)$.

Also, notice that $\bar{u}_i \Gamma u_j$ is a 1×1 number, so the order can be changed as you want; and $(\bar{u}_i \Gamma u_i) = \text{tr}(\bar{u}_i \Gamma u_i) = \text{tr}(\Gamma)$. With these equations, we can derive the equation we need.

(c)

Use the results of (b), we can get it easily.

7 Problem 3.7

(a)

$$P \bar{\psi}(t, \mathbf{x}) \sigma^{\mu\nu} \psi(t, \mathbf{x}) P = \frac{i}{2} \bar{\psi}(t, -\mathbf{x}) \gamma^0 [\gamma^\mu, \gamma^\nu] \gamma^0 \psi(t, -\mathbf{x})$$

$$\gamma^0 [\gamma^0, \gamma^i] \gamma^0 = -[\gamma^0, \gamma^i]$$

$$\gamma^0[\gamma^i, \gamma^j]\gamma^0 = [\gamma^i, \gamma^j]$$

Notice that $\hat{T}\gamma^\mu\hat{T} = (\gamma^\mu)^*$.

(b)

$$\phi(\vec{x}, t) = e^{iHt}\phi(\vec{x})e^{-iHt} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}e^{-ip \cdot x} + b_{\vec{p}}^\dagger e^{ip \cdot x} \right) \Big|_{p^0=E_{\vec{p}}}$$

$$Pa_{\vec{p}}P = a_{-\vec{p}}$$

So,

$$P\phi(\vec{x}, t)P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{-\vec{p}}e^{-ip \cdot x} + b_{-\vec{p}}^\dagger e^{ip \cdot x} \right) \Big|_{p^0=E_{\vec{p}}}$$

Replace the variable $-\vec{p}$ with \vec{p} ,

$$P\phi(\vec{x}, t)P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}e^{-i(p_0t+\vec{p} \cdot \vec{x})} + b_{\vec{p}}^\dagger e^{i(p_0t+\vec{p} \cdot \vec{x})} \right) \Big|_{p^0=E_{\vec{p}}} = \phi(-\vec{x}, t)$$

$$C\phi(x)C = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} (a_{\vec{p}}^\dagger e^{ip \cdot x} + b_{\vec{p}} e^{-ip \cdot x}) = \phi^*(x)$$