

Peskin Solutions: Chapter 4

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1 Problem 4.1

(a)

$$M = {}_{in} \langle 0 | 0 \rangle_{out} = \lim_{T \rightarrow \infty(1-i\epsilon)} \langle 0 | e^{-iH(2T)} | 0 \rangle$$

$$\lim_{T \rightarrow \infty(1-i\epsilon)} e^{-iH(2T)} | 0 \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \sum_n e^{-iE_n(2T)} | n \rangle \langle n | 0 \rangle \approx \lim_{T \rightarrow \infty(1-i\epsilon)} e^{-iE_0(2T)} | \Omega \rangle \langle \Omega | 0 \rangle$$

$$M = \lim_{T \rightarrow \infty(1-i\epsilon)} e^{-iE_0(2T)} | \langle \Omega | 0 \rangle |^2$$

From P.87, we have

$$1 = \langle \Omega | \Omega \rangle = \left(| \langle 0 | \Omega \rangle |^2 e^{-iE_0(2T)} \right)^{-1} \langle 0 | U(T, t_0) U(t_0, -T) | 0 \rangle$$

So,

$$M = \lim_{T \rightarrow \infty(1-i\epsilon)} \langle 0 | U(T, t_0) U(t_0, -T) | 0 \rangle$$

$$P(0) = |M|^2 = \lim_{T \rightarrow \infty(1-i\epsilon)} \left| \left\langle 0 \left| T \exp \left\{ -i \int d^4x \mathcal{H}_{int} \right\} \right| 0 \right\rangle \right|^2$$

(b)

In the expansion of the exponential, those terms propotional to $j, j^3 \dots$ will vanish because they cannot contract completely, so the expansion is

$$1 - \frac{1}{2} \int d^4x j(x) \phi(x) \int d^4y j(y) \phi(y) + O(j^4)$$

$$M = 1 - \frac{1}{2} \int d^4x \int d^4y j(x) j(y) \langle 0 | T \phi(x) \phi(y) | 0 \rangle + O(j^4)$$

Assume $x^0 > y^0$,

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

$$M = 1 - \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \tilde{j}(p) \tilde{j}(-p) + O(j^4)$$

$$\text{If } \tilde{j}(p) \tilde{j}(-p) = |\tilde{j}(p)|^2$$

$$P(0) = |M|^2 = 1 - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2 + O(j^4)$$

So, $\lambda = \langle N \rangle$.

(c)

Feynman diagrams are some line segments.

(d)

$$P = |\text{out} \langle \vec{k} | 0 \rangle_{\text{in}}|^2$$

$$M = 1 + i \int d^4x j(x) \langle \vec{k} | \phi(x) | 0 \rangle = i \int d^4x j(x) e^{ip \cdot x} = i \tilde{j}(p)$$

So, for one particle, the first term is

$$P = |M|^2 = |\tilde{j}(p)|^2$$

The n-th term is

$$\frac{(-1)^n i}{(2n+1)!} \int d^4x_1 \dots d^4x_{2n+1} j(x_1) \dots j(x_{2n+1}) \langle \vec{k} | T \phi(x_1) \phi(x_2) \dots \phi(x_{2n+1}) | 0 \rangle$$

$$= \frac{(-1)^n i}{(2n+1)!} (2n+1)(2n-1) \dots 1 \int d^4x_1 e^{ik \cdot x_1} j(x_1) \left(\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2 \right)^n$$

$$= \frac{(-1)^n i}{2^n n!} \tilde{j}(k) \lambda^n$$

$$P = \left| \sum_{n=0}^{\infty} \frac{(-\lambda/2)^n}{n!} i \tilde{j}(k) \right|^2 = |\tilde{j}(k)|^2 e^{-\lambda}$$

(e)

In the final state, different momentum distribution should be summed over the probabilities.

$$P = \frac{1}{n!} \int \frac{d^3 k_1 \cdots d^3 k_n}{(2\pi)^{3n} 2^n E_{\mathbf{k}_1} \cdots E_{\mathbf{k}_n}} \left| \left\langle \mathbf{k}_1 \cdots \mathbf{k}_n \left| T \exp \left\{ i \int d^4 x j(x) \phi_I(x) \right\} \right| 0 \right\rangle \right|^2$$

the $\frac{1}{n!}$ represents the symmetry of exchanging \vec{k}_i and \vec{k}_j .

The first term of M is

$$\frac{i^n}{n!} \int d^4 x_1 \cdots d^4 x_n j(x_1) \cdots j(x_n) \langle \vec{k}_1 \cdots \vec{k}_n | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle = \frac{i^n}{n!} \tilde{j}(k_1) \cdots \tilde{j}(k_n)$$

the (m+1)-th term of M is

$$\begin{aligned} & \frac{i^{n+2m}}{(n+2m)!} \frac{(n+2m)!}{2^m m!} \tilde{j}(k_1) \cdots \tilde{j}(k_n) \int \frac{d^3 p_1 \cdots d^3 p_m}{(2\pi)^{3m} 2^m E_{p_1} \cdots E_{p_m}} |\tilde{j}(p_1)|^2 \cdots |\tilde{j}(p_m)|^2 \\ & = i^n \tilde{j}(k_1) \cdots \tilde{j}(k_n) \left(\frac{-\lambda}{2} \right)^m \frac{1}{m!} \end{aligned}$$

$$P = \frac{1}{n!} \int \frac{d^3 k_1 \cdots d^3 k_n}{(2\pi)^{3n} 2^n E_{\mathbf{k}_1} \cdots E_{\mathbf{k}_n}} |i^n \tilde{j}(k_1) \cdots \tilde{j}(k_n) e^{-\frac{\lambda}{2}}|^2 = \frac{\lambda^n}{n!} e^{-\lambda}$$

(e)

$$\Sigma_{n=0}^{\infty} P(n) = \Sigma_{n=0}^{\infty} \frac{\lambda^n}{n!} \cdot \exp(-\lambda) = 1$$

$$\Sigma_{n=0}^{\infty} n P(n) = \Sigma_{n=1}^{\infty} n P(n) = \lambda \exp(-\lambda) \Sigma_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} = \lambda$$

from the above equation,

$$\Sigma_{n=1}^{\infty} \frac{n \lambda^n}{n!} = \lambda \cdot e^{\lambda}$$

apply $\lambda \frac{d}{d\lambda}$ to both sides, then we get

$$\Sigma_{n=1}^{\infty} \frac{n^2 \lambda^n}{n!} = (\lambda^2 + \lambda) \cdot e^{\lambda}$$

$$\langle (N - \langle N \rangle)^2 \rangle = (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

2 Problem 4.2

The decay process is $\Phi \rightarrow \phi + \phi$, lifetime of Φ is $\tau = \frac{1}{\Gamma}$, $\Gamma = \int d\Gamma$.

From (4.86), we know the decay rate formula,

$$\int d\Gamma = \frac{1}{2M} \int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \frac{1}{4E_{\mathbf{p}_1} E_{\mathbf{p}_2}} |\mathcal{M}(\Phi(0) \rightarrow \phi(p_1) \phi(p_2))|^2 (2\pi)^4 \delta^{(4)}(p_\Phi - p_1 - p_2)$$

$$\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | S | \mathbf{k}_A \mathbf{k}_B \rangle = \lim_{T \rightarrow \infty} \langle \mathbf{p}_1 \mathbf{p}_2 \cdots | e^{-iH(2T)} | \mathbf{k}_A \mathbf{k}_B \rangle$$

$$\lim_{T \rightarrow \infty(1-i\epsilon)} \left(\mathbf{p}_1 \cdots \mathbf{p}_n | e^{-iH(2T)} | \mathbf{p}_A \mathbf{p}_B \right)_0$$

$$\propto \lim_{T \rightarrow \infty(1-i\epsilon)} {}_0 \left\langle \mathbf{p}_1 \cdots \mathbf{p}_n \left| T \left(\exp \left[-i \int_{-T}^T dt H_I(t) \right] \right) \right| \mathbf{p}_A \mathbf{p}_B \right\rangle_0$$

We know $S = i + iT$, and

$$\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | iT | \mathbf{k}_A \mathbf{k}_B \rangle = (2\pi)^4 \delta^{(4)} \left(k_A + k_B - \sum p_f \right) \cdot i \mathcal{M}(k_A, k_B \rightarrow p_f)$$

Φ and ϕ are real scalar fields, so they satisfy K-G eq., with Feynman rules in P.115, we can calculate \mathcal{M} by

$$i\mathcal{M} = ({}_0 \langle \phi \phi | T \exp(-i \int d^4x \mu \Phi \phi \phi) | \Phi >_0 \rangle_{connected, amputated})$$

$\mathcal{H}_I = \mu \Phi \phi \phi$, the lowest order in μ is

$$i\mathcal{M} = -i\mu ({}_0 \langle \phi \phi | \int d^4x (T \Phi \phi \phi) | \Phi >_0 \rangle_{connected, amputated})$$

After contraction,

$$i\mathcal{M} = -i\mu * 2\delta(p_\Phi - p_1 - p_2)$$

the factor 2 is because ϕ s have two ways of contraction, also we can calculate with Feynman rules in P.115, the diagram is one vertex with three external solid lines, here $\int d^4x$ will also be included in the Feynman rules.

the vertex is $-i\mu$, the external solid line is 1, and because two ways of contraction refer to same diagram, there will be an extra factor 2.

With the expression of \mathcal{M} , we get

$$\Gamma = \frac{1}{2} \cdot \frac{2\mu^2}{M} \int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \frac{1}{4E_{\mathbf{p}_1} E_{\mathbf{p}_2}} (2\pi)^4 \delta(M - E_{\mathbf{p}_1} - E_{\mathbf{p}_2}) \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2)$$

the factor $\frac{1}{2}$ is accounted for the exchange of two ϕ in the final state. **Notice that when calculating \mathcal{M} /Feynman diagrams, we treat each ϕ operator differently.**

$$\Gamma = \frac{\mu^2}{M} \int \frac{d^3p_1}{(2\pi)^2} \frac{1}{4E_{\mathbf{p}_1}^2} \delta(M - 2E_{\mathbf{p}_1}) = \frac{\mu^2}{8\pi M} \left(1 - \frac{4m^2}{M^2} \right)^{1/2}$$

Notice here $\delta(M - 2E_{\mathbf{p}_1}) = \frac{1}{2}\delta(E_{\mathbf{p}_1} - \frac{M}{2})$.

3 Problem 4.3

(a) Firstly, here the propagator is the contraction of two fields in the interaction picture, and when $\lambda = 0$, $H = \Sigma H_i$, so each field Φ^i satisfies K-G equation separately, the contraction is the standard K-G propagator.

We have $\mathcal{H}_I = \frac{\lambda}{4}(\Sigma_i(\Phi^i)^2)^2 = \frac{\lambda}{4}(\Sigma_i(\Phi^i)^4 + 2\Sigma_{i>j}(\Phi^i)^2(\Phi^j)^2)$, if the vertex has four same fields, then one diagram represents 4! contraction terms, if the vertex has two kinds of fields, then one diagram represents 2*2 different contractions, adding the extra factor 2 in \mathcal{H}_I , totally 2^3 terms.

Therefore, vertex of 4 Φ^i has value $-i\frac{\lambda}{4} * 4! = -6i\lambda$, and vertex of 2 kinds Φ^i and Φ^j has value $-i\frac{\lambda}{4} * 2 * 2 * 2 = -2i\lambda$.

For $\Phi^1\Phi^2 \rightarrow \Phi^1\Phi^2$, to the leading order of λ ,

$$i\mathcal{M} = \frac{-i\lambda}{4} \langle 0 | \Phi^1\Phi^2 | \int d^4x ((\Phi^1)^2 + (\Phi^2)^2)^2 | \Phi^1\Phi^2 \rangle_{connected, amputated}$$

$$((\Phi^1)^2 + (\Phi^2)^2)^2 = \Phi^1\Phi^1\Phi^1\Phi^1 + 2 * \Phi^1\Phi^1\Phi^2\Phi^2 + \Phi^2\Phi^2\Phi^2\Phi^2$$

here only the mixed term survived, so $\mathcal{M} = -6i\lambda$, the diagram is a vertex with 4 external solid lines. With Eq.(4.84)

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{cm}} |\mathcal{M}(p_A, p_B \rightarrow p_1, p_2)|^2$$

we know that Φ^1 and Φ^2 have same mass, if the mass is ignorable compared to $E_{c.m.}$, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{c.m.}} = \frac{9\lambda^2}{16\pi^2 E_{c.m.}}$$

Same thing for other two decay channels.

(b)

Because of the rotation symmetry of $\vec{\Phi}$, we can assume when $\vec{\Phi} = (\Phi^i = 0, \Phi^N = v)$, the potential energy $V = V_{min} = -\frac{1}{2}\mu^2\nu^2 + \frac{\lambda}{4}\nu^4$, the derivative $\frac{\partial V}{\partial \nu} = \nu(\lambda\nu^2 - \mu^2) = 0$, so we get $\nu = \frac{\mu}{\sqrt{\lambda}}$.

Apply the new coordinates $\Phi^i = \pi^i$ and $\Phi^N = \nu + \sigma$, plus $\Pi^i = \dot{\Phi}^i$, we can get the Lagrangian density,

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \pi^k)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 - \frac{1}{2} (2\mu^2) \sigma^2 - \sqrt{\lambda} \mu \sigma^3 - \sqrt{\lambda} \mu \sigma \pi^k \pi^k - \frac{\lambda}{4} \sigma^4 - \frac{\lambda}{2} \sigma^2 (\pi^k \pi^k) - \frac{\lambda}{4} (\pi^k \pi^k)^2$$

from the above equation, we can find that π^k are $N - 1$ massless K-G fields, σ is a K-G field with mass $\sqrt{2}\nu$, their propagators have the same form as the K-G propagator.

σ field propagator:

$$\frac{i}{k^2 - 2\nu^2 + i\epsilon}$$

π^k field propagator:

$$\frac{i\delta_{ij}}{k^2 + i\epsilon}$$

vertex of 3 σ fields:

$$-6i\sqrt{\lambda}\mu = -6i\lambda\nu$$

factor 6 is because the exchange of 3 σ .

vertex of σ , π^i and π^j :

$$-2i\sqrt{\lambda}\mu\delta_{ij} = -2i\lambda\nu\delta_{ij}$$

factor 2 is accounted for the exchange of two π , and δ_{ij} is accounted for $\pi^k\pi^k = \sum_{i=1}^{N-1}(\pi^i)^2$.

Other Feynman rules are same things.