Fit Functions for Correlators on Lattice

Basics

Renormalization

We take the relativistic normalization of state $\left|H_{\vec{p}}\right\rangle$

$$\langle H_{\vec{p}'}|H_{\vec{p}}\rangle = (2E_{\vec{p}})(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')$$

where the Dirac delta function satisfies

$$\int d^4x\, e^{ik\cdot x} = (2\pi)^4\delta^{(4)}(k)$$

Identity

Therefore, the identity operator is defined as

$$I = \sum_{H'}\int \frac{d^3\vec{p}'}{2E_{\vec{p}'}(2\pi)^3}|H'_{\vec{p}'}\rangle\langle H'_{\vec{p}'}|$$

which satisfies

$$\langle H_{\vec{p}}|I|H_{\vec{p}}\rangle = \int \frac{d^3\vec{p}'}{2E_{\vec{p}'}(2\pi)^3}\langle H_{\vec{p}}|H_{\vec{p}'}\rangle\langle H_{\vec{p}'}|H_{\vec{p}}\rangle = \int \frac{d^3\vec{p}'}{2E_{\vec{p}'}(2\pi)^3}(2E_{\vec{p}})^2(2\pi)^6\delta^{(3)}(\vec{p}-\vec{p}')\delta^{(3)}(\vec{p}-\vec{p}')$$
$$= (2E_{\vec{p}})(2\pi)^3\delta^{(3)}(\vec{0})$$

Time and spatial translation

The time and spatial translation are defined as

$$\hat{O}_H(\vec{x},t_{\text{sep}}) = e^{-i\vec{p}\cdot\vec{x}}\hat{O}_H(\vec{0},t_{\text{sep}})e^{i\vec{p}\cdot\vec{x}}$$
$$\hat{O}_H(\vec{0},t_{\text{sep}}) = e^{\hat{H}t}\hat{O}_H(\vec{0},0)e^{-\hat{H}t}$$

note we did Wick rotation $it_{\text{M}} = t_{\text{E}}$ on lattice, so all the time below are in the Euclidean space.

Periodic boundary condition

If we take the periodic boundary condition into the consideration, we can just replace the exponential decay factor as

$$e^{-E_n t_{\text{sep}}} \rightarrow e^{-E_n t_{\text{sep}}} + e^{-E_n(L_t - t_{\text{sep}})}$$

with L_t is the lattice size in the time direction.

Two-point correlator

Correlator on Lattice

$$C_{\text{2pt}} = \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}\langle\Omega|\hat{O}_H(\vec{x},t_{\text{sep}})\hat{O}_H^{\dagger}(\vec{0},0)|\Omega\rangle$$

After inserting the identity operator and doing the spatial translation, we have

$$C_{\text{2pt}} = \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}} \sum_{H'} \int \frac{d^3\vec{p}'}{2E_{\vec{p}'}(2\pi)^3} \langle\Omega|\hat{O}_H(\vec{0}, t_{\text{sep}})|H'_{\vec{p}'}\rangle e^{i\vec{p}'\cdot\vec{x}} \langle H'_{\vec{p}'}|\hat{O}^\dagger_H(\vec{0}, 0)|\Omega\rangle$$
$$= \sum_{H'} \int \frac{d^3\vec{p}'}{2E_{\vec{p}'}} \langle\Omega|\hat{O}_H(\vec{0}, t_{\text{sep}})|H'_{\vec{p}'}\rangle \delta^{(3)}(\vec{p}-\vec{p}') \langle H'_{\vec{p}'}|\hat{O}^\dagger_H(\vec{0}, 0)|\Omega\rangle$$

then do the time translation and integrate the momentum to get

$$C_{\text{2pt}} = \sum_{H'} \frac{1}{2E_{\vec{p}}} \langle\Omega|\hat{O}_H(\vec{0}, 0)|H'_{\vec{p}}\rangle e^{-E_{\vec{p}}t_{\text{sep}}} \langle H'_{\vec{p}}|\hat{O}^\dagger_H(\vec{0}, 0)|\Omega\rangle$$

Within all hadron states H', only those who has the same quantum numbers as the projection operator $\hat{O}_H$ survive, so

$$C_{\text{2pt}} = \sum_{E_n} \frac{1}{2E_n} e^{-E_n t_{\text{sep}}} \langle\Omega|\hat{O}_H(\vec{0}, 0)|E_n\rangle \langle E_n|\hat{O}^\dagger_H(\vec{0}, 0)|\Omega\rangle$$

where all $\left|E_n\right\rangle$ states have 3-momentum $\vec{p}$.

Correlator in software

To construct two-point correlators in softwares, like Chroma, QLUA and PyQUDA, we take the π^+ as an example.

The projection operator is

$$\hat{O}_{\pi^+}(\vec{x}, t) = \bar{d}(\vec{x}, t)\gamma^5 u(\vec{x}, t)$$
$$\hat{O}^\dagger_{\pi^+}(\vec{x}, t) = -\bar{u}(\vec{x}, t)\gamma^5 d(\vec{x}, t)$$

so the two-point correlator in the coordinate space is

$$\langle\Omega|\hat{O}_{\pi^+}(\vec{x}, t_{\text{sep}})\hat{O}^\dagger_{\pi^+}(\vec{0}, 0)|\Omega\rangle = -\langle\Omega|\bar{d}(\vec{x}, t_{\text{sep}})\gamma^5 u(\vec{x}, t_{\text{sep}})\bar{u}(\vec{0}, 0)\gamma^5 d(\vec{0}, 0)|\Omega\rangle$$
$$= \text{tr}\left[d(\vec{0}, 0)\bar{d}(\vec{x}, t_{\text{sep}})\gamma^5 u(\vec{x}, t_{\text{sep}})\bar{u}(\vec{0}, 0)\gamma^5\right] = \text{tr}\left[S_d(\vec{0}, 0; \vec{x}, t_{\text{sep}})\gamma^5 S_u(\vec{x}, t_{\text{sep}}; \vec{0}, 0)\gamma^5\right]$$

note the exchange of quark fields gives an extra minus sign. Then use the hermiticity relation [see Gatteringer P136 Eq.(6.31)]

$$S_d(\vec{0}, 0; \vec{x}, t_{\text{sep}}) = \gamma^5 S^\dagger_d(\vec{x}, t_{\text{sep}}; \vec{0}, 0)\gamma^5$$

Fit function

If we define the overlap factors as

$$z_n = \langle\Omega|\hat{O}_H(\vec{0}, 0)|E_n\rangle$$
$$z^\dagger_n = \langle E_n|\hat{O}^\dagger_H(\vec{0}, 0)|\Omega\rangle$$

then we got the fit function for two-point correlator as

$$C_{\text{2pt}} = \sum_n \frac{z_n \cdot z_n^\dagger}{2E_n} e^{-E_n t_{\text{sep}}}$$

Distribution Amplitude

Correlator on Lattice

$$C_{\text{DA}} = \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}} \langle \Omega | j_z(\vec{x}, t_{\text{sep}}) \hat{O}_H^\dagger(\vec{0}, 0) | \Omega \rangle$$

where j_z is a non-local current with separation z.
Similar to the local two-point correlator above, after inserting the identity operator and doing the spatial-time translation, we have

$$C_{\text{DA}} = \sum_{E_n} \frac{1}{2E_n} e^{-E_n t_{\text{sep}}} \langle \Omega | j_z(\vec{0}, 0) | E_n \rangle \langle E_n | \hat{O}_H^\dagger(\vec{0}, 0) | \Omega \rangle$$

Fit function

Take the same definition of overlap factors, we got

$$C_{\text{DA}} = \sum_n \frac{z_n^\dagger}{2E_n} \langle \Omega | j_z(\vec{0}, 0) | E_n \rangle\, e^{-E_n t_{\text{sep}}}$$

Three-point correlator

Correlator on Lattice

$$C_{\text{3pt}} = \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y}\, \langle \Omega | \hat{O}_H(\vec{x}, t_{\text{sep}})\, j(\vec{y}, \tau)\, \hat{O}_H^\dagger(\vec{0}, 0) | \Omega \rangle$$

in which $j(\vec{y}, \tau)$ is the insertion current.
After inserting the identity operator and doing the spatial translation, we have

$$\begin{aligned} C_{\text{3pt}} &= \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \sum_{H'} \int \frac{d^3\vec{p}'}{2E_{\vec{p}'}(2\pi)^3} \sum_{H''} \int \frac{d^3\vec{p}''}{2E_{\vec{p}''}(2\pi)^3} \\ &\times \langle \Omega | \hat{O}_H(\vec{0}, t_{\text{sep}}) | H'_{\vec{p}'} \rangle e^{i\vec{p}'\cdot\vec{x}} e^{-i\vec{p}'\cdot\vec{y}} \langle H'_{\vec{p}'} |\, j(\vec{0}, \tau) | H''_{\vec{p}''} \rangle e^{i\vec{p}''\cdot\vec{y}} \langle H''_{\vec{p}''} | \hat{O}_H^\dagger(\vec{0}, 0) | \Omega \rangle \\ &= \sum_{H'} \int \frac{d^3\vec{p}'}{2E_{\vec{p}'}} \sum_{H''} \int \frac{d^3\vec{p}''}{2E_{\vec{p}''}} \\ &\times \langle \Omega | \hat{O}_H(\vec{0}, t_{\text{sep}}) | H'_{\vec{p}'} \rangle \delta^{(3)}(\vec{p} - \vec{p}') \langle H'_{\vec{p}'} |\, j(\vec{0}, \tau) | H''_{\vec{p}''} \rangle \delta^{(3)}(\vec{p}' - \vec{p}'') \langle H''_{\vec{p}''} | \hat{O}_H^\dagger(\vec{0}, 0) | \Omega \rangle \end{aligned}$$

$$= \sum_{H',H''} \frac{1}{2E_{\vec{p}}} \frac{1}{2E_{\vec{p}}} \langle \Omega | \hat{O}_H(\vec{0}, t_{\text{sep}}) | H'_{\vec{p}} \rangle \langle H'_{\vec{p}} | \, j(\vec{0}, \tau) | H''_{\vec{p}} \rangle \langle H''_{\vec{p}} | \hat{O}^\dagger_H(\vec{0}, 0) | \Omega \rangle$$

$$= \sum_{H',H''} \frac{1}{2E_{\vec{p}}} \frac{1}{2E_{\vec{p}}} \langle \Omega | \hat{O}_H(\vec{0}, 0) | H'_{\vec{p}} \rangle e^{-H' t_{\text{sep}}} e^{H'\tau} \langle H'_{\vec{p}} | \, j(\vec{0}, \tau) | H''_{\vec{p}} \rangle e^{-H''\tau} \langle H''_{\vec{p}} | \hat{O}^\dagger_H(\vec{0}, 0) | \Omega \rangle$$

Note that the factor $\frac{1}{2E_{\vec{p}}} \frac{1}{2E_{\vec{p}}}$ comes from H' and H'', so the two energy denominators can be different.

Within all hadron states H' and H'', only those who has the same quantum numbers as the projection operator $\hat{O}_H$ survive, so

$$C_{\text{3pt}} = \sum_{n,m} \frac{1}{4E_n E_m} \langle \Omega | \hat{O}_H(\vec{0}, 0) | E_n \rangle e^{-E_n(t_{\text{sep}} - \tau)} \langle E_n | \, j(\vec{0}, 0) | E_m \rangle e^{-E_m \tau} \langle E_m | \hat{O}^\dagger_H(\vec{0}, 0) | \Omega \rangle$$

Correlator in software (quasi-PDF as an example)

We would like to construct three-point function operator in software, take the π^+ as an example

$$\hat{O}_{\pi^+}(\vec{x}, t) = \bar{d}(\vec{x}, t) \gamma^5 u(\vec{x}, t)$$
$$\hat{O}^\dagger_{\pi^+}(\vec{x}, t) = -\bar{u}(\vec{x}, t) \gamma^5 d(\vec{x}, t)$$

the non-local quasi-PDF operator of u quark is

$$j(\vec{y}, \tau; z) = \bar{u}(z + \vec{y}, \tau) \gamma^t W(z + \vec{y}, \tau; \vec{y}, \tau) u(\vec{y}, \tau)$$

so the three-point function is

$$C_{\text{3pt}} = \int d^3\vec{x} \, e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \, \langle \Omega | \hat{O}_{\pi^+}(\vec{x}, t_{\text{sep}}) \, j(\vec{y}, \tau; z) \, \hat{O}^\dagger_{\pi^+}(\vec{0}, 0) | \Omega \rangle$$

$$= \int d^3\vec{x} \, e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y} \, \langle \Omega | \left[\bar{d}(\vec{x}, t_{\text{sep}}) \gamma^5 u(\vec{x}, t_{\text{sep}}) \right] j(\vec{y}, \tau; z) \left[-\bar{u}(\vec{0}, 0) \gamma^5 d(\vec{0}, 0) \right] | \Omega \rangle$$

Take trace and use Wick theorem [Gatteringer P109 Eq.(5.36)], note that there are 2 contractions of 4 u fields. So, we have

$$C_{\text{3pt}} = \int d^3\vec{x} \, e^{-i\vec{p}\cdot\vec{x}} \int d^3\vec{y}$$
$$\times \Big\{ \langle \Omega | \text{tr} \left[S_d(0; \vec{x}, t_{\text{sep}}) \gamma^5 S_u(\vec{x}, t_{\text{sep}}; z + \vec{y}, \tau) \gamma^t W(z + \vec{y}, \tau; \vec{y}, \tau) S_u(\vec{y}, \tau; 0) \gamma^5 \right] | \Omega \rangle$$
$$- \langle \Omega | \text{tr} \left[S_d(0; \vec{x}, t_{\text{sep}}) \gamma^5 S_u(\vec{x}, t_{\text{sep}}; 0) \gamma^5 \right] \cdot \text{tr} \left[S_u(\vec{y}, \tau; z + \vec{y}, \tau) \gamma^t W(z + \vec{y}, \tau; \vec{y}, \tau) \right] | \Omega \rangle \Big\}$$

two terms in the integral represent two diagrams, take the first one as an example. We have

$$\int d^3\vec{y} \, \langle \Omega | \text{tr} \left[\underline{\int d^3\vec{x} \, e^{-i\vec{p}\cdot\vec{x}} \gamma^5 S_d(0; \vec{x}, t_{\text{sep}}) \gamma^5 S_u(\vec{x}, t_{\text{sep}}; z + \vec{y}, \tau)} \, \gamma^t W(z + \vec{y}, \tau; \vec{y}, \tau) S_u(\vec{y}, \tau; 0) \right] | \Omega \rangle$$

in which the red part is sequential source, and the underlined part is the sequential propagator.

* We need to avoid calculating all to all propagator, like $S_u(\vec{x}, t_{\text{sep}}; z + \vec{y}, \tau)$, where x and y are both integrated, that's why we define the sequential source and sequential propagator.

Use the hermiticity relation, we have

$$\int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}} \left[\gamma^5 S_d(0;\vec{x},t_{\rm sep})\gamma^5\right] S_u(\vec{x},t_{\rm sep};z+\vec{y},\tau)$$
$$= \int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}} S_d^\dagger(\vec{x},t_{\rm sep};0)\left[\gamma^5 S_u^\dagger(z+\vec{y},\tau;\vec{x},t_{\rm sep})\gamma^5\right]$$

where $\dagger$ acts on both the spinor and color indices.

Fit function

If we define the overlap factors as

$$z_n = \langle\Omega|\hat{O}_H(\vec{0},0)|E_n\rangle$$
$$z_n^\dagger = \langle E_n|\hat{O}_H^\dagger(\vec{0},0)|\Omega\rangle$$

define matrix elements as

$$O_{nm} = \langle E_n|\, j(\vec{0},0)|E_m\rangle$$

then we got the fit function for three-point correlator as

$$C_{\rm 3pt} = \sum_{n,m} \frac{z_n O_{nm} z_m^\dagger}{4E_n E_m} \cdot e^{-E_n(t_{\rm sep}-\tau)} e^{-E_m\tau}$$

However, since the current that we use has γ^t structure, the ground state matrix element is

$$O_{00} = 2E_0 \cdot h^0(z)$$

where $h^0(z)$ is the bare matrix element of quasi-PDF.

Ratio

Definition

$$R(t_{\rm sep},\tau) = \frac{C_{\rm 3pt}(t_{\rm sep},\tau)}{C_{\rm 2pt}(t_{\rm sep})}$$

Fit function

Using the fit functions of two-point and three-point functions above, we have

$$R(t_{\rm sep},\tau) = \frac{\sum_{n,m} \frac{z_n O_{nm} z_m^\dagger}{4E_n E_m} \cdot e^{-E_n(t_{\rm sep}-\tau)} e^{-E_m\tau}}{\sum_n \frac{z_n\cdot z_n^\dagger}{2E_n} e^{-E_n t_{\rm sep}}}$$

If we just keep 2 states, then we have approximation

$$\sum_n \frac{z_n\cdot z_n^\dagger}{2E_n} e^{-E_n t_{\rm sep}} \approx \frac{z_0^2}{2E_0} e^{-E_0 t_{\rm sep}}\left(1 + c_1 e^{-\Delta E t_{\rm sep}}\right)$$

where $c_1 = \frac{E_0}{E_1}\frac{z_1^2}{z_0^2}$ and energy gap $\Delta E = E_1 - E_0$, and the ratio is approximated as

$$R(t_{\text{sep}}, \tau) \approx \frac{1}{2E_0}\frac{O_{00} + a_1\left(e^{-\Delta E\tau} + e^{-\Delta E(t_{\text{sep}}-\tau)}\right) + a_2 e^{-\Delta E_{t_{\text{sep}}}}}{1 + c_1 e^{-\Delta E t_{\text{sep}}}}$$

where $a_1 = \frac{E_0}{E_1}\frac{z_0 O_{01} z_1^\dagger}{z_0^2} = \frac{E_0}{E_1}\frac{z_1 O_{10} z_0^\dagger}{z_0^2}$ and $a_2 = \frac{E_0^2}{E_1^2}\frac{z_1^2 O_{11}}{z_0^2}$.

Feynman-Hellmann correlator

Definition

$$FH(t_{\text{sep}}, \tau_{\text{cut}}, d\tau) = \left(\sum_{\tau=\tau_{\text{cut}}}^{\tau=t_{\text{sep}}+d\tau-\tau_{\text{cut}}} R(t_{\text{sep}}+d\tau, \tau) - \sum_{\tau=\tau_{\text{cut}}}^{\tau=t_{\text{sep}}-\tau_{\text{cut}}} R(t_{\text{sep}}, \tau)\right) / d\tau$$

Fit function

Firstly, let's derive the fit function of the summation as

$$S(t_{\text{sep}}, \tau_{\text{cut}}) = \sum_{\tau=\tau_{\text{cut}}}^{\tau=t_{\text{sep}}-\tau_{\text{cut}}} C_{3\text{pt}}(t_{\text{sep}}, \tau) = \sum_{\tau=\tau_{\text{cut}}}^{\tau=t_{\text{sep}}-\tau_{\text{cut}}} \sum_{n,m} \frac{z_n O_{nm} z_m^\dagger}{4E_n E_m} \cdot e^{-E_n(t_{\text{sep}}-\tau)} e^{-E_m\tau}$$

For $n = m$ part, we got

$$\sum_{\tau=\tau_{\text{cut}}}^{\tau=t_{\text{sep}}-\tau_{\text{cut}}} \frac{z_n O_{nm} z_m^\dagger}{4E_n E_m} \cdot e^{-E_n(t_{\text{sep}}-\tau)} e^{-E_m\tau} = (t_{\text{sep}} - 2\tau_{\text{cut}} + 1) \cdot \frac{z_n O_{nn} z_n^\dagger}{4E_n^2} \cdot e^{-E_n t_{\text{sep}}}$$

For $n \neq m$ part, we got

$$\sum_{\tau=\tau_{\text{cut}}}^{\tau=t_{\text{sep}}-\tau_{\text{cut}}} \frac{z_n O_{nm} z_m^\dagger}{4E_n E_m} \cdot e^{-E_n(t_{\text{sep}}-\tau)} e^{-E_m\tau} = \sum_{\tau=\tau_{\text{cut}}}^{\tau=t_{\text{sep}}-\tau_{\text{cut}}} \frac{z_n O_{nm} z_m^\dagger}{4E_n E_m} \cdot e^{-E_n t_{\text{sep}}} e^{\Delta_{nm}\tau}$$

$$= \frac{z_n O_{nm} z_m^\dagger}{4E_n E_m} \cdot e^{-E_n t_{\text{sep}}} e^{\Delta_{nm}\tau_{\text{cut}}} \frac{1 - e^{\Delta_{nm}(t_{\text{sep}}-2\tau_{\text{cut}}+1)}}{1 - e^{\Delta_{nm}}}$$

where $\Delta_{nm} = E_n - E_m$.

If we preserve the first two energy states,

$$\frac{S(t_{\text{sep}}, \tau_{\text{cut}})}{C_{2\text{pt}}(t_{\text{sep}})} = \frac{1}{2E_0}\frac{(t_{\text{sep}} - 2\tau_{\text{cut}} + 1)O_{00}\left(1 + b_1 e^{-\Delta E t_{\text{sep}}}\right) + b_2 + b_3 e^{-\Delta E t_{\text{sep}}}}{1 + c_1 e^{-\Delta E t_{\text{sep}}}}$$

Thus, the one-state fit function is

$$\frac{S(t_{\mathrm{sep}},\tau_{\mathrm{cut}})}{C_{\mathrm{2pt}}(t_{\mathrm{sep}})}=\frac{1}{2E_0}(t_{\mathrm{sep}}-2\tau_{\mathrm{cut}}+1)O_{00}+b_2$$

the two-state fit function is

$$\frac{S(t_{\mathrm{sep}},\tau_{\mathrm{cut}})}{C_{\mathrm{2pt}}(t_{\mathrm{sep}})}=\frac{1}{2E_0}\frac{(t_{\mathrm{sep}}-2\tau_{\mathrm{cut}}+1)O_{00}\left(1+b_1e^{-\Delta E t_{\mathrm{sep}}}\right)+b_2+b_3e^{-\Delta E t_{\mathrm{sep}}}}{1+c_1e^{-\Delta E t_{\mathrm{sep}}}}$$

Accordingly, the one-state fit function for FH correlator is

$$FH(t_{\mathrm{sep}},\tau_{\mathrm{cut}},d\tau)=\left[\frac{S(t_{\mathrm{sep}}+d\tau,\tau_{\mathrm{cut}})}{C_{\mathrm{2pt}}(t_{\mathrm{sep}}+d\tau)}-\frac{S(t_{\mathrm{sep}},\tau_{\mathrm{cut}})}{C_{\mathrm{2pt}}(t_{\mathrm{sep}})}\right]/d\tau=\frac{O_{00}}{2E_0}$$

and the two-state fit function for FH correlator is

$$FH(t_{\mathrm{sep}},\tau_{\mathrm{cut}},d\tau)=\left[\frac{S(t_{\mathrm{sep}}+d\tau,\tau_{\mathrm{cut}})}{C_{\mathrm{2pt}}(t_{\mathrm{sep}}+d\tau)}-\frac{S(t_{\mathrm{sep}},\tau_{\mathrm{cut}})}{C_{\mathrm{2pt}}(t_{\mathrm{sep}})}\right]/d\tau$$