Peskin Solutions: Chapter 9

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I. HOW TO USE THE FUNCTIONAL METHOD TO GET PROPOGATOR.

According to (9.34), generating functional $Z[J] = \int \mathcal{D}\phi[\exp(i\int d^4x\mathcal{L}) \cdot \exp(i\int d^4xJ(x)\phi(x))]$. Then change the variable to get $Z[J] = \int \mathcal{D}\phi'[\exp(i\int d^4x\mathcal{L}') \cdot \exp(-\frac{i}{2}\int d^4xJ\hat{O}^{-1}J)]$. Here the current term is irrelavant to ϕ', so $Z[J] = Z_0 \cdot \exp(-\frac{i}{2}\int d^4xJ\hat{O}^{-1}J)$, and the functional derivatives will be applied on the current term.

II. PROBLEM 9.1

(a)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + (\partial_\mu\phi^* - ieA_\mu\phi^*)(\partial^\mu\phi + ieA^\mu\phi) - m^2\phi^*\phi = \mathcal{L}_A + \mathcal{L}_\phi + \mathcal{L}_I$$

The $\mathcal{L}_A$ is just free E-M field, so the propogator is the propogator of photon.

The $\mathcal{L}_\phi = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi = \partial_\mu(\phi^*\partial^\mu\phi) - \phi^*\partial_\mu(\partial^\mu\phi) - m^2\phi^*\phi$, because the differential term in the Lagrangian density makes no difference, we got $\mathcal{L}_\phi = -\phi^*\partial_\mu(\partial^\mu\phi) - m^2\phi^*\phi = \phi^*(-\partial^2 - m^2)\phi = \phi^*\hat{T}\phi$.

With generating functional method, we have $\mathcal{L}_\phi + \eta^*\phi + \phi^*\eta$ in the $Z[J]$, then do a shift $\phi \to \phi' = \phi + \hat{T}^{-1}\eta$, we got $\mathcal{L}_\phi + \eta^*\phi + \phi^*\eta = \mathcal{L}_{\phi'} - \eta^*\hat{T}^{-1}\eta$. If G is the Green function of $\hat{T}$, then $\mathcal{L}_{\phi'} - \eta^*\hat{T}^{-1}\eta = \mathcal{L}_{\phi'} - \eta^*(iG * \eta)$,

$$Z[\eta,\eta^*] = Z_0 \cdot \exp[-i\int d^4xd^4y\ \eta^*(x)iG(x-y)\eta(y)]$$

$$\text{prop} = -\frac{\delta}{\delta\eta^*}\frac{\delta}{\delta\eta}\exp[-i\int d^4xd^4y\ \eta^*(x)iG(x-y)\eta(y)] = -G$$

After two functional derivatives, we will find the propogator is exactly the $-G$.

So the propogator of ϕ and ϕ^* is $\frac{i}{p^2-m^2+i\epsilon}$. (How to calculate the Green function of $\hat{T}$ - Check Eq.(2.57) in Peskin)

$$\hat{T}^{-1}\eta(x) = i\int d^4y\ G(x-y)\eta(y)$$

$$\hat{T}G(x-y) = (-\partial^2 - m^2)G(x-y) = -i\delta(x-y)$$

FT to get,

$$(p^2 - m^2)\tilde{G}(p) = -i$$

$$-\tilde{G}(p) = \frac{i}{p^2 - m^2}$$

Then comes to vertices, $\mathcal{H}_I = -\mathcal{L}_I$ (P. 289 in Peskin), theoretically we should check Eq.(4.31) and do the contraction to get Feynman rules, but here we can just look at $\exp[i\int \mathcal{L}_I]$, here $\mathcal{L}_I = ieg^{\mu\nu}(\partial_\mu\phi^* A_\nu\phi - A_\mu\phi^*\partial_\nu\phi) + e^2 g^{\mu\nu} A_\mu\phi^* A_\nu\phi$, then $i\mathcal{L}_I = -ieg^{\mu\nu}(-i\partial_\mu\phi^* A_\nu\phi + A_\mu\phi^* i\partial_\nu\phi) + ie^2 g^{\mu\nu} A_\mu\phi^* A_\nu\phi$.

There are three terms, let's throw those fields away and turn $i\partial\phi$ to $p_\phi\phi$, $-i\partial\phi^*$ to $p_{\phi^*}\phi^*$, here p's are along particle/anti-particle lines, besides, the third term has two A fields, which are commutative, so there should be a factor 2 for the $AA\phi^*\phi$ vertex.

So,

$$For\ \phi^* A\phi : -ie(p+p')^\mu$$

$$For\ AA\phi^*\phi : 2ie^2 g^{\mu\nu}$$

Theoretically,

$$<\phi\phi^*|S|\gamma> = <\phi\phi^*|T\int d^4x i\mathcal{L}_I|\gamma> = <\phi\phi^*|T\int d^4x(-ie)g^{\mu\nu}(-i\partial_\mu\phi^* A_\nu\phi + A_\mu\phi^* i\partial_\nu\phi)|\gamma>$$

and

$$<\phi\phi^*|S|\gamma\gamma> = <\phi\phi^*|T\int d^4x i\mathcal{L}_I|\gamma\gamma> = <\phi\phi^*|T\int d^4x(ie^2)g^{\mu\nu} A_\mu\phi^* A_\nu\phi|\gamma\gamma>$$

give the Feynman rules of two kinds of vertex with contractions.

(b)

With Eq.(4.84), m_e is ignored, then,

$$(\frac{d\sigma}{d\Omega})_{c.m.} = \frac{|\vec{p}_\phi|}{32(2\pi)^2 E_e^2 \cdot 2E_e}\frac{1}{4}\Sigma|\mathcal{M}(ee\to\phi^*\phi)|^2$$

The outlines of ϕ and ϕ^* are 1, the Feynman diagram looks similar to the diagram in P.131.

$$i\mathcal{M} = (-ie)^2\bar{v}(k_2)\gamma^\mu u(k_1)\frac{-ig_{\mu\nu}}{s+i\epsilon}(p_1-p_2)^\nu = ie^2\bar{v}(k_2)(\not{p_1}-\not{p_2})u(k_1)\frac{1}{s+i\epsilon}$$

$$\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}|^2 = \sum_{\text{spin}}\frac{e^4}{4s^2}\bar{v}(k_2)(\not{p_1}-\not{p_2})u(k_1)\bar{u}(k_1)(\not{p_1}-\not{p_2})v(k_2)$$

$$= \sum_{\text{spin}}\frac{e^4}{4s^2}\text{tr}(v(k_2)\bar{v}(k_2)(\not{p_1}-\not{p_2})u(k_1)\bar{u}(k_1)(\not{p_1}-\not{p_2}))$$

$$= \frac{e^4}{4s^2}\text{tr}(\not{k_2}(\not{p_1}-\not{p_2})\not{k_1}(\not{p_1}-\not{p_2}))$$

$$= \frac{e^4}{4s^2}\left[8(k_1\cdot p_1 - k_1\cdot p_2)(k_2\cdot p_1 - k_2\cdot p_2) - 4(k_1\cdot k_2)(p_1-p_2)^2\right]$$

Choose a specific frame,

$$k_1 = (E, 0, 0, E), \quad p_1 = (E, p\sin\theta, 0, p\cos\theta)$$

$$k_2 = (E, 0, 0, -E), \quad p_2 = (E, -p\sin\theta, 0, -p\cos\theta)$$

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 = \frac{e^4 p^2}{2E^2}\sin^2\theta$$

$ee \to \mu\mu$ is Eq.(5.11)
So,

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{c.m.}} = \frac{1}{2(2E)^2}\frac{p}{8(2\pi)^2 E}\left(\frac{1}{4}\sum |\mathcal{M}|^2\right) = \frac{\alpha^2}{8s}\left(1 - \frac{m^2}{E^2}\right)^{3/2}\sin^2\theta$$

(c)

Two diagrams because there are two kinds of vertex which are listed in (a). Because the minus signs in the $\mathcal{L}_I$ are all absorbed in vertex, and there is no Fermion field, the sign between two vertices is $+$. That's why the two diagrams should be added.

$$i\Pi_1^{\mu\nu} = e^2 \int \frac{d^4k}{(2\pi)^4}(2k+q)^\mu \frac{1}{k^2 - m^2 + i\epsilon}(2k+q)^\nu \frac{1}{(k+q)^2 - m^2 + i\epsilon}$$

$$i\Pi_2^{\mu\nu} = -2e^2 g^{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k+q)^2 - m^2 + i\epsilon}$$

add togeter, get

$$i\Pi^{\mu\nu} = -e^2 \int \frac{d^4k}{(2\pi)^4}\frac{2g^{\mu\nu}(k^2 - m^2) - (2k+q)^\mu(2k+q)^\nu}{(k^2 - m^2)((k+q)^2 - m^2)}$$

$$\frac{1}{(k^2 - m^2)((k+q)^2 - m^2)} = \int_0^1 dx \frac{1}{[(k + (1-x)q)^2 + xq^2 - x^2q^2 - m^2]^2}$$

change the variable, $l = k + (1-x)q$, with Eq.(6.45),

$$\text{numerator} = g^{\mu\nu} l^2 + 2g^{\mu\nu}(1-x)^2 q^2 - 2g^{\mu\nu} m^2 - (2x-1)^2 q^\mu q^\nu$$

do the Wick rotation, $l^0 = il_E^0$ and $l^i = l_E^i$, so we have $d^4l = id^4l_E$ and $l^2 = -l_E^2$,

$$i\Pi^{\mu\nu} = -ie^2 \int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4}\frac{-g^{\mu\nu}l_E^2 + 2g^{\mu\nu}(1-x)^2q^2 - 2g^{\mu\nu}m^2 - (2x-1)^2q^\mu q^\nu}{[l_E^2 + m^2 + x^2q^2 - xq^2]^2}$$

$$= -ie^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} [\frac{-g^{\mu\nu} l_E^2}{(l_E^2+\Delta)^2} + \frac{2g^{\mu\nu}(1-x)^2 q^2 - 2g^{\mu\nu} m^2 - (2x-1)^2 q^\mu q^\nu}{(l_E^2+\Delta)^2}]$$

use dimensional regularization, with Eq.(7.85) and Eq.(7.86),

$$i\Pi^{\mu\nu} = -ie^2 \int_0^1 dx[(2g^{\mu\nu}(1-x)^2 q^2 - 2g^{\mu\nu} m^2 - (2x-1)^2 q^\mu q^\nu) \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

$$-g^{\mu\nu} \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma\left(2-\frac{d}{2}-1\right)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}-1}]$$

$$= -ie^2 \int_0^1 dx \frac{1}{(4\pi)^{d/2}} (\frac{1}{\Delta})^{2-d/2} \Gamma(2-d/2)[(2g^{\mu\nu}(1-x)^2 q^2 - (2x-1)^2 q^\mu q^\nu) - g^{\mu\nu} \frac{d}{2-d}(x^2 q^2 - x q^2)]$$

set $d = 4 - \epsilon$ with $\epsilon \to 0$,

$$i\Pi^{\mu\nu} = \frac{-ie^2}{(4\pi)^2} \int_0^1 dx (\frac{\epsilon}{2} - \log\Delta - \gamma + \log(4\pi))[(g^{\mu\nu}(2x-2)(2x-1)q^2 - (2x-1)^2 q^\mu q^\nu)]$$

Because $\int_0^1 dx(\frac{2}{\epsilon} - \log\Delta - \gamma + \log(4\pi))(2x-1) = \int_0^1 dx \frac{2}{\epsilon}(2x-1) = 0$, we have

$$i\Pi^{\mu\nu} = \frac{-ie^2}{(4\pi)^2} \int_0^1 dx (\frac{\epsilon}{2} - \log\Delta - \gamma + \log(4\pi))(2x-1)^2 [(g^{\mu\nu} q^2 - q^\mu q^\nu)]$$

with MS-bar scheme,

$$\Pi(q^2) = \frac{-\alpha}{4\pi} \int_0^1 dx(-\log\Delta)(2x-1)^2$$

If we adopt $-q^2 >> m^2$,

$$\Pi(q^2) = \frac{-\alpha}{4\pi} \int_0^1 dx(-\log(x-x^2) - \log(-q^2))(2x-1)^2 \to \frac{-\alpha}{12\pi} \log(-q^2)$$

while looking at Eq.(7.90), $\int_0^1 dx x(1-x) = \frac{1}{6}$, we know for $e+e-$ pair,

$$\Pi(q^2) \to \frac{-\alpha}{3\pi} \log(-q^2)$$

which is four times as our results.

III. PROBLEM 9.2

(a)

In Ch.9.1, because of superposition principle, the we have

$$U\left(x_a, x_b; T\right)=\sum_{\text{all paths}} e^{i \cdot(\text { phase })}=\int \mathcal{D} x(t) e^{i \cdot(\text { phase })}$$

$$\left\langle x_b\left|e^{-i H T / \hbar}\right| x_a\right\rangle=U\left(x_a, x_b; T\right)=\int \mathcal{D} x(t) e^{i S[x(t)] / \hbar}$$

Here if we treat H as a matrix in the linear space constructed by eigenstates $|q\rangle$, the trace can be rewritten as $Z=\int d^d q\,\langle q|e^{-\beta H}|q\rangle$. However, we do not know how to write it as a functional integral yet because we do not know what is the "action" here.

It is true that you can use eigenstates $|p\rangle$ to write $Z=\int \frac{d^d p}{(2\pi)^d}\langle p|e^{-\beta H}|p\rangle$, but you will meet $\langle p_2|\hat{x}|p_1\rangle = \int d^d q\; e^{-i(p_2-p_1)q}\cdot q$

Recall the equations about location and momentum eigenstates,

$$1=\int \mathrm{d}^d q|q\rangle\langle q|=\int \frac{\mathrm{d}^d p}{(2 \pi)^d}|p\rangle\langle p|$$

$$|q\rangle=\int \frac{\mathrm{d}^d p}{(2 \pi)^d} e^{-i p \cdot q}|p\rangle$$

$$\langle p | q\rangle=e^{-i p \cdot q}$$

.

Then we suppose that $H=\frac{p^2}{2m}+V(q)$, and separate Z as

$$e^{-\beta H}=e^{-\epsilon H} \cdot e^{-\epsilon H} \ldots$$

$$Z=\int d^d q(\prod_{j=1}^{N-1} \int d^d q_j)\langle q|e^{-\epsilon H}|q_{N-1}\rangle \ldots\langle q_{i+1}|e^{-\epsilon H}|q_i\rangle \ldots\langle q_1|e^{-\epsilon H}|q\rangle$$

$$\langle q_{i+1}|e^{-\epsilon H}|q_i\rangle=e^{-\epsilon V(q_i)}\langle q_{i+1}|e^{-\epsilon \frac{p^2}{2 m}}|q_i\rangle=e^{-\epsilon V(q_i)} \int \frac{\mathrm{d}^d p_i}{(2 \pi)^d} \frac{\mathrm{d}^d p_{i+1}}{(2 \pi)^d}\langle q_{i+1}|p_{i+1}\rangle\langle p_{i+1}|e^{-\epsilon \frac{p^2}{2 m}}|p_i\rangle\langle p_i|q_i\rangle$$

$$=e^{-\epsilon V(q_i)} \int \frac{\mathrm{d}^d p_i}{(2 \pi)^d}\langle q_{i+1}|p_i\rangle e^{-\epsilon \frac{p_i^2}{2 m}}\langle p_i|q_i\rangle=e^{-\epsilon V(q_i)} \int \frac{\mathrm{d}^d p_i}{(2 \pi)^d} e^{i p_i \cdot(q_{i+1}-q_i)} \cdot e^{-\epsilon \frac{p_i^2}{2 m}}$$

$$=\left[\frac{m}{2 \pi \epsilon}\right]^{d / 2} e^{-m(q_{i+1}-q_i)^2 / 2 \epsilon} \cdot e^{-\epsilon V(q_i)}$$

here we can find that $q \rightarrow q_1 \rightarrow \cdots \rightarrow q_{N-1} \rightarrow q$ is a loop route, so we set $q_0=q_N=q$ and we multiply all terms together to get

$$Z=\left[\frac{m}{2 \pi \epsilon}\right]^{N d / 2} \prod_{i=0}^{N-1} \int \mathrm{d}^d q_i \exp \left[-\frac{m\left(q_{i+1}-q_i\right)^2}{2 \epsilon}-\epsilon V\left(q_i\right)\right]$$

$$=\left[\frac{m}{2 \pi \epsilon}\right]^{N d / 2}(\prod_{i=0}^{N-1} \int \mathrm{d}^d q_i) \exp \left[-\beta \sum_{i=0}^{N-1} \frac{1}{N}(\frac{m\left(q_{i+1}-q_i\right)^2}{2 \epsilon^2}+V\left(q_i\right))\right]$$

When $N \to \infty$, define $\int \mathcal{D}q = \lim_{N\to\infty}(\left[\frac{m}{2\pi\epsilon}\right]^{Nd/2}(\prod_{i=0}^{N-1}\int \mathrm{d}^d q_i))$, so we get

$$Z = \int \mathcal{D}q \exp[-\beta \oint dt L_E[q(t)]]$$

$$L_E[q(t)] = \frac{m}{2}(\frac{dq}{dt})^2 + V(q(t))$$