Peskin Solutions: Chapter 4

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1 Problem 4.1

(a)

$$M ={}_{in}< 0|0 >_{out}= \lim_{T\to\infty(1-i\epsilon)} < 0|e^{-iH(2T)}|0 >$$

$$\lim_{T\to\infty(1-i\epsilon)} e^{-iH(2T)}|0 >= \lim_{T\to\infty(1-i\epsilon)} \Sigma_n e^{-iE_n(2T)}|n >< n|0 >\approx \lim_{T\to\infty(1-i\epsilon)} e^{-iE_0(2T)}|\Omega >< \Omega|0 >$$

$$M = \lim_{T\to\infty(1-i\epsilon)} e^{-iE_0(2T)}| < \Omega|0 > |^2$$

From P.87, we have

$$1 = \langle\Omega \mid \Omega\rangle = \left(|\langle 0 \mid \Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} \langle 0 \left|U\left(T, t_0\right) U\left(t_0, -T\right)\right| 0\rangle$$

So,

$$M = \lim_{T\to\infty(1-i\epsilon)} \langle 0 \left|U\left(T, t_0\right) U\left(t_0, -T\right)\right| 0\rangle$$

$$P(0) = |M|^2 = \lim_{T\to\infty(1-i\epsilon)} \left|\left\langle 0 \left|T \exp\left\{-i\int \mathrm{d}^4 x\mathcal{H}_{\text{int}}\right\}\right| 0\right\rangle\right|^2$$

(b)

In the expansion of the exponential, those terms propotional to j, j^3 ... will vanish because they cannot contract completely, so the expansion is

$$1 - \frac{1}{2}\int d^4xj(x)\phi(x)\int d^4yj(y)\phi(y) + O(j^4)$$

$$M = 1 - \frac{1}{2}\int d^4x\int d^4yj(x)j(y) < 0|T\phi(x)\phi(y)|0 > +O(j^4)$$

Assume $x^0 > y^0$,

$$< 0|T\phi(x)\phi(y)|0 >= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)}$$

$$M = 1 - \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\tilde{j}(p)\tilde{j}(-p) + O(j^4)$$

If $\tilde{j}(p)\tilde{j}(-p) = |\tilde{j}(p)|^2$

$$P(0) = |M|^2 = 1 - \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\tilde{j}(p)|^2 + O(j^4)$$

So, $\lambda =< N >$.

(c)

Feynman diagrams are some line segments.

(d)

$$P = |_{out} < \vec{k}|0 >_{in}|^2$$

$$M = 1 + i\int d^4x j(x) < \vec{k}|\phi(x)|0 >= i\int d^4x j(x)e^{ip\cdot x} = i\tilde{j}(p)$$

So, for one particle, the first term is

$$P = |M|^2 = |\tilde{j}(p)|^2$$

The n-th term is

$$\frac{(-1)^n i}{(2n+1)!}\int d^4x_1...d^4x_{2n+1}j(x_1)...j(x_{2n+1}) < \vec{k}|T\phi(x_1)\phi(x_2)...\phi(x_{2n+1})|0 >$$

$$= \frac{(-1)^n i}{(2n+1)!}(2n+1)(2n-1)...1\int d^4x_1 e^{ik\cdot x_1}j(x_1)(\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}|\tilde{j}(p)|^2)^n$$

$$= \frac{(-1)^n i}{2^n n!}\tilde{j}(k)\lambda^n$$

$$P = \left|\sum_{n=0}^{\infty}\frac{(-\lambda/2)^n}{n!}i\tilde{j}(k)\right|^2 = |\tilde{j}(k)|^2 e^{-\lambda}$$

(e)

In the final state, different momentum distribution should be summed over the probabilities.

$$P = \frac{1}{n!}\int \frac{\mathrm{d}^3k_1\cdots\mathrm{d}^3k_n}{(2\pi)^{3n}2^nE_{\mathbf{k}_1}\cdots E_{\mathbf{k}_n}}|\left\langle \mathbf{k}_1\cdots\mathbf{k}_n \middle| T\exp\left\{i\int \mathrm{d}^4xj(x)\phi_I(x)\right\}\middle| 0\right\rangle|^2$$

the $\frac{1}{n!}$ represents the symmetry of exchanging $\vec{k_i}$ and $\vec{k_j}$.
The first term of M is

$$\frac{i^n}{n!}\int d^4x_1...d^4x_nj(x_1)...j(x_n) < \vec{k_1}...\vec{k_n}|T\phi(x_1)...\phi(x_n)|0 >= \frac{i^n}{n!}\tilde{j}(k_1)...\tilde{j}(k_n)$$

the (m+1)-th term of M is

$$\frac{i^{n+2m}}{(n+2m)!}\frac{(n+2m)!}{2^mm!}\tilde{j}(k_1)...\tilde{j}(k_n)\int\frac{d^3p_1...d^3p_m}{(2\pi)^{3m}2^mE_{p_1}...E_{p_m}}|\tilde{j}(p_1)|^2...|\tilde{j}(p_m)|^2$$

$$= i^n\tilde{j}(k_1)...\tilde{j}(k_n)(\frac{-\lambda}{2})^m\frac{1}{m!}$$

$$P = \frac{1}{n!}\int \frac{\mathrm{d}^3k_1\cdots\mathrm{d}^3k_n}{(2\pi)^{3n}2^nE_{\mathbf{k}_1}\cdots E_{\mathbf{k}_n}}|i^n\tilde{j}(k_1)...\tilde{j}(k_n)e^{-\frac{\lambda}{2}}|^2 = \frac{\lambda^n}{n!}e^{-\lambda}$$

(e)

$$\Sigma_{n=0}^{\infty}P(n) = \Sigma_{n=0}^{\infty}\frac{\lambda^n}{n!}\cdot\exp(-\lambda) = 1$$

$$\Sigma_{n=0}^{\infty}nP(n) = \Sigma_{n=1}^{\infty}nP(n) = \lambda\exp(-\lambda)\Sigma_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!} = \lambda$$

from the above equation,

$$\Sigma_{n=1}^{\infty}\frac{n\lambda^n}{n!} = \lambda\cdot e^{\lambda}$$

apply $\lambda\frac{d}{d\lambda}$ to both sides, then we get

$$\Sigma_{n=1}^{\infty}\frac{n^2\lambda^n}{n!} = (\lambda^2+\lambda)\cdot e^{\lambda}$$

$$< (N- < N >)^2 >= (\lambda^2+\lambda) - \lambda^2 = \lambda$$

2 Problem 4.2

The decay process is $\Phi \to \phi + \phi$, lifetime of Φ is $\tau = \frac{1}{\Gamma}$, $\Gamma = \int d\Gamma$.
From (4.86), we know the decay rate formula,

$$\int \mathrm{d}\Gamma = \frac{1}{2M} \int \frac{\mathrm{d}^3 p_1 \, \mathrm{d}^3 p_2}{(2\pi)^6} \frac{1}{4E_{\mathbf{p}_1} E_{\mathbf{p}_2}} \left| \mathcal{M} \left(\Phi(0) \to \phi\left(p_1\right) \phi\left(p_2\right) \right) \right|^2 (2\pi)^4 \delta^{(4)} \left(p_\Phi - p_1 - p_2 \right)$$

$$\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | S | \mathbf{k}_\mathcal{A} \mathbf{k}_\mathcal{B} \rangle = \lim_{T \to \infty} \left\langle \mathbf{p}_1 \mathbf{p}_2 \cdots \left| e^{-iH(2T)} \right| \mathbf{k}_\mathcal{A} \mathbf{k}_\mathcal{B} \right\rangle$$

$$\lim_{T \to \infty(1-i\epsilon)} \left(\mathbf{p}_1 \cdots \mathbf{p}_n \left| e^{-iH(2T)} \right| \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \right\rangle_0$$

$$\propto \lim_{T \to \infty(1-i\epsilon)} {}_0\left\langle \mathbf{p}_1 \cdots \mathbf{p}_n \left| T\left(\exp\left[-i \int_{-T}^{T} dt H_I(t) \right] \right) \right| \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \right\rangle_0$$

We know $S = i + iT$, and

$$\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | iT | \mathbf{k}_\mathcal{A} \mathbf{k}_\mathcal{B} \rangle = (2\pi)^4 \delta^{(4)} \left(k_\mathcal{A} + k_\mathcal{B} - \sum p_f \right) \cdot i\mathcal{M} \left(k_\mathcal{A}, k_\mathcal{B} \to p_f \right)$$

Φ and ϕ are real scalar fields, so they satisfy K-G eq., with Feynman rules in P.115, we can calculate $\mathcal{M}$ by

$$i\mathcal{M} = ({}_0 < \phi\phi | T\exp(-i \int d^4 x \mu \Phi\phi\phi) | \Phi >_0)_{connected,amputated}$$

$\mathcal{H}_I = \mu\Phi\phi\phi$, the lowest order in μ is

$$i\mathcal{M} = -i\mu({}_0 < \phi\phi | \int d^4 x (T\Phi\phi\phi) | \Phi >_0)_{connected,amputated}$$

After contraction,

$$i\mathcal{M} = -i\mu * 2\delta(p_\Phi - p_1 - p_2)$$

the factor 2 is because ϕs have two ways of contraction, also we can calculate with Feynman rules in P.115, the diagram is one vertex with three external solid lines, here $\int d^4 x$ will also be included in the Feynman rules.

the vertex is $-i\mu$, the external solid line is 1, and because two ways of contraction refer to same diagram, there will be an extra factor 2.

With the expression of $\mathcal{M}$, we get

$$\Gamma = \frac{1}{2} \cdot \frac{2\mu^2}{M} \int \frac{\mathrm{d}^3 p_1 \, \mathrm{d}^3 p_2}{(2\pi)^6} \frac{1}{4E_{\mathbf{p}_1} E_{\mathbf{p}_2}} (2\pi)^4 \delta \left(M - E_{\mathbf{p}_1} - E_{\mathbf{p}_2} \right) \delta^{(3)} \left(\mathbf{p}_1 + \mathbf{p}_2 \right)$$

the factor $\frac{1}{2}$ is accounted for the exchange of two ϕ in the final state. Notice that when calculating $\mathcal{M}$/Feynman diagrams, we treat each ϕ operator differently.

$$\Gamma = \frac{\mu^2}{M} \int \frac{\mathrm{d}^3 p_1}{(2\pi)^2} \frac{1}{4E_{\mathbf{p}_1}^2} \delta \left(M - 2E_{\mathbf{p}_1} \right) = \frac{\mu^2}{8\pi M} \left(1 - \frac{4m^2}{M^2} \right)^{1/2}$$

Notice here $\delta(M - 2E_{\mathbf{p}_1}) = \frac{1}{2}\delta(E_{\mathbf{p}_1} - \frac{M}{2})$.

3 Problem 4.3

(a) Firstly, here the propagator is the contraction of two fields in the interaction picture, and when $\lambda = 0$, $H = \Sigma H_i$, so each field Φ^i satisfies K-G equation separately, the contraction is the standard K-G propagator.

We have $\mathcal{H}_I = \frac{\lambda}{4}(\Sigma_i(\Phi^i)^2)^2 = \frac{\lambda}{4}(\Sigma_i(\Phi^i)^4 + 2\Sigma_{i>j}(\Phi^i)^2(\Phi^j)^2)$, if the vertex has four same fields, then one diagram represents 4! contraction terms, if the vertex has two kinds of fields, then one diagram represents $2*2$ different contractions, adding the extra factor 2 in $\mathcal{H}_I$, totally 2^3 terms.

Therefore, vertex of 4 Φ^i has value $-i\frac{\lambda}{4}*4! = -6i\lambda$, and vertex of 2 kinds Φ^i and Φ^j has value $-i\frac{\lambda}{4}*2*2*2 = -2i\lambda$.

For $\Phi^1\Phi^2 \to \Phi^1\Phi^2$, to the leading order of λ,

$$i\mathcal{M} = \frac{-i\lambda}{4}({}_0\!< \Phi^1\Phi^2|\int d^4x((\Phi^1)^2 + (\Phi^2)^2)^2|\Phi^1\Phi^2 >_0)_{connected,amputated}$$

$$((\Phi^1)^2 + (\Phi^2)^2)^2 = \Phi^1\Phi^1\Phi^1\Phi^1 + 2*\Phi^1\Phi^1\Phi^2\Phi^2 + \Phi^2\Phi^2\Phi^2\Phi^2$$

here only the mixed term survived, so $\mathcal{M} = -6i\lambda$, the diagram is a vertex with 4 external solid lines. With Eq.(4.84)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} = \frac{1}{2E_{\mathcal{A}}2E_{\mathcal{B}}\left|v_{\mathcal{A}} - v_{\mathcal{B}}\right|}\frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{\mathrm{cm}}}\left|\mathcal{M}\left(p_{\mathcal{A}}, p_{\mathcal{B}} \to p_1, p_2\right)\right|^2$$

we know that Φ^1 and Φ^2 have same mass, if the mass is ignorable compared to $E_{c.m.}$, we have

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{c.m.}} = \frac{9\lambda^2}{16\pi^2 E_{c.m.}}$$

Same thing for other two decay channels.

(b)

Because of the rotation symmetry of $\vec{\Phi}$, we can assume when $\vec{\Phi} = (\Phi^i = 0, \Phi^N = v)$, the potential energy $V = V_{min} = -\frac{1}{2}\mu^2\nu^2 + \frac{\lambda}{4}\nu^4$, the derivative $\frac{\partial V}{\partial \nu} = \nu(\lambda\nu^2 - \mu^2) = 0$, so we get $\nu = \frac{\mu}{\sqrt{\lambda}}$.

Apply the new coordinates $\Phi^i = \pi^i$ and $\Phi^N = \nu + \sigma$, plus $\Pi^i = \dot{\Phi}^i$, we can get the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\pi^k\right)^2 + \frac{1}{2}\left(\partial_\mu\sigma\right)^2 - \frac{1}{2}\left(2\mu^2\right)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu\sigma\pi^k\pi^k - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\sigma^2\left(\pi^k\pi^k\right) - \frac{\lambda}{4}\left(\pi^k\pi^k\right)^2$$

from the above equation, we can find that π^k are $N-1$ massless K-G fields, σ is a K-G field with mass $\sqrt{2}\nu$, their propagators have the same form as the K-G propagator.

σ field propagator:

$$\frac{i}{k^2 - 2\mu^2 + i\epsilon}$$

π^k field propagator:

$$\frac{i\delta_{ij}}{k^2 + i\epsilon}$$

vertex of 3 σ fields:

$$-6i\sqrt{\lambda}\mu = -6i\lambda\nu$$

factor 6 is because the exchange of 3 σ.
vertex of σ, π^i and π^j:

$$-2i\sqrt{\lambda}\mu\delta_{ij} = -2i\lambda\nu\delta_{ij}$$

factor 2 is accounted for the exchange of two π, and δ_{ij} is accounted for $\pi^k\pi^k = \Sigma_{i=1}^{N-1}(\pi^i)^2$.

Other Feynman rules are same things.

(c)

Because the vertex of σ, π^i and π^j has δ_{ij}, so for $\pi^i\pi^1 \to \pi^2\pi^2$, only the first and the fourth diagram are not vanished.

All fields here are K-G fields, so the first diagram is:

$$(-2i\lambda\nu\delta_{ij}) \cdot \frac{i}{(p_1 + p_2)^2 - 2\mu^2 + i\epsilon} \cdot (-2i\lambda\nu\delta_{kl})$$

The fourth diagram is:

$$-2i\lambda$$