Peskin Solutions: Chapter 2

Jinchen

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1 Problem 2.1

(a)

We know the Euler-Lagrange eq. as below,

$$\mathcal{L} = -\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}$$

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right)$$

And obviously,

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = 0$$

$$\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} = \delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho$$

So,

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\frac{1}{4}\partial_\mu(2F^{\mu\nu} - 2F^{\nu\mu}) = -\partial_\mu F^{\mu\nu} = 0$$

The above eq. are Maxwell's equations, when $\mu = 0$, we got $\nabla\cdot\vec{E} = 0$, when $\mu = i$, we got $\partial_t\vec{E} = \nabla\times\vec{B}$.

(b)

We know the energy-momentum tensor can be calculated as,

$$T^\mu{}_\nu \equiv \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\nu\phi - \mathcal{L}\delta^\mu{}_\nu$$

Here we use A_λ as ϕ and from (a) we know $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = -F^{\mu\lambda}$.

So,

$$T^{\mu\nu} = \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} - F^{\mu\lambda} \partial^\nu A_\lambda$$

This expression is not symmetric under the exchange of μ and ν, so we add another term.

$$\hat{T}^{\mu\nu} = \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} - F^{\mu\lambda} \partial^\nu A_\lambda + \partial_\lambda (F^{\mu\lambda} A^\nu)$$

From (a) we know $\partial_\lambda F^{\mu\lambda} = 0$, so we got,

$$\hat{T}^{\mu\nu} = \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} + F^{\mu\lambda} F_\lambda{}^\nu$$

Now it is symmetric under the exchange of μ and ν.

$$\hat{T}^{00} = \left(-\frac{1}{2} F^{0i} F^{0i} + \frac{1}{4} F^{ij} F^{ij} \right) + F^{0i} F^{0i}$$

$$F_{\rho\sigma} F^{\rho\sigma} = 2(\vec{B}^2 - \vec{E}^2)$$

$$\hat{T}^{00} = \frac{1}{2} (\vec{B}^2 + \vec{E}^2)$$

And,

$$\hat{T}^{0i} = F^{0\lambda} F_\lambda{}^i = -F^{0j} F^{ji}$$

2 Problem 2.2

(a)

From the expression of action, we know that $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$.
So,

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi^*$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi^*)} = \partial_t \phi$$

And the canonical commutation relations are as below,

$$[\phi(\vec{x}), \pi(\vec{y})] = [\phi(t, \vec{x}), \pi(t, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$$

Heisenberg equation of motion is,

$$i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$$

So,

$$i\frac{\partial}{\partial t}\phi(x) = [\phi(t,\vec{x}), H(t,\vec{x}')] = \int d^3x'[\phi(t,\vec{x}), \pi^*(t,\vec{x}')\pi(t,\vec{x}')] = i\pi^*(x)$$

$$i\frac{\partial}{\partial t}\pi^*(x) = \int d^3x'([\pi^*, \nabla'\phi^* \cdot \nabla'\phi] + m^2[\pi^*, \phi^*\phi])$$

And we noticed that $[\pi^*, \nabla'\phi^* \cdot \nabla'\phi] = [\pi^*, \nabla'\phi^*] \cdot \nabla'\phi = \nabla'[\pi^*, \phi^*] \cdot \nabla'\phi$,

$$i\frac{\partial}{\partial t}\pi^*(x) = (-i)\int d^3x'\{\nabla'\delta^{(3)}(\vec{x}-\vec{x}') \cdot \nabla'\phi(t,\vec{x}')\} - im^2\phi(x)$$

$$\nabla'\delta^{(3)}(\vec{x}-\vec{x}') \cdot \nabla'\phi(t,\vec{x}') = \nabla'\{\delta^{(3)}(\vec{x}-\vec{x}') \cdot \nabla'\phi(t,\vec{x}')\} - \delta^{(3)}(\vec{x}-\vec{x}')\nabla'^2\phi(t,\vec{x}')$$

Because $\delta^{(3)}(\vec{x}-\vec{x}') = 0$ when $\vec{x}'$ goes to the boundary at infinity, after integral the first term was cancelled, then we got,

$$i\frac{\partial}{\partial t}\pi^*(x) = i(\nabla^2 - m^2)\phi(x)$$

So we got,

$$\frac{\partial^2}{\partial^2 t}\phi(x) = (\nabla^2 - m^2)\phi(x)$$

$$(\partial^2 + m^2)\phi(x) = 0$$

This is the K-G equation.

(b)

From (a) we know that $\phi(x)$ is a solution of K-G equation, and noticed that,

$$\partial^2 e^{\pm ip\cdot x} = (\partial_t^2 - \nabla^2)e^{\pm(iEt - i\vec{p}\cdot\vec{x})} = (-E^2 + |\vec{p}|^2)e^{\pm ip\cdot x}$$

So $\phi(x)$ is the linear combination of $e^{\pm ip\cdot x}$.

On the other hand,

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}})$$

$$e^{iHt}a_{\vec{p}}e^{-iHt} = a_{\vec{p}}e^{-iE_{\vec{p}}t}$$

$$e^{iHt}a^\dagger_{\vec{p}}e^{-iHt} = a^\dagger_{\vec{p}}e^{-iE_{\vec{p}}t}$$

Here the operators for positive and negative frequence are no need to be conjugate with each other, so we have,

$$\phi(\vec{x},t) = e^{iHt}\phi(\vec{x})e^{-iHt} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{-ip\cdot x} + b^\dagger_{\vec{p}}e^{ip\cdot x}\right)\bigg|_{p^0=E_{\vec{p}}}$$

$$\pi^*(x) = \frac{\partial}{\partial t}\phi(x) = i\int \frac{d^3p}{(2\pi)^3}\frac{\sqrt{E_{\vec{p}}}}{\sqrt{2}}\left(-a_{\vec{p}}e^{-ip\cdot x} + b^\dagger_{\vec{p}}e^{ip\cdot x}\right)\bigg|_{p^0=E_{\vec{p}}}$$

$$\phi^* = \phi^\dagger$$

So, we can use $a_{\vec{p}}$ and $b_{\vec{p}}$ to express H.

$$H = \int \frac{d^3p}{(2\pi)^3}\frac{E_{\vec{p}}}{2}(a_{\vec{p}}a^\dagger_{\vec{p}} + b^\dagger_{\vec{p}}b_{\vec{p}} + a^\dagger_{\vec{p}}a_{\vec{p}} + b_{\vec{p}}b^\dagger_{\vec{p}}) = \int \frac{d^3p}{(2\pi)^3}E_{\vec{p}}(b^\dagger_{\vec{p}}b_{\vec{p}} + a^\dagger_{\vec{p}}a_{\vec{p}}) + \int d^3pE_{\vec{p}}\delta^{(3)}(0)$$

(c)

$$Q = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}(a^\dagger_{\vec{p}}a_{\vec{p}} - b_{\vec{p}}b^\dagger_{\vec{p}}) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}(a^\dagger_{\vec{p}}a_{\vec{p}} - b^\dagger_{\vec{p}}b_{\vec{p}}) - \frac{1}{2}\int d^3p\delta^{(3)}(0)$$

(d)

Waiting for more thinking...

3 Problem 2.3

We know that the form of $D(x-y)$ is invarient under the Lorentz transformations, so we can assume $(x-y)^\mu = (0,0,0,r)$.

$$D(x-y) = \int \frac{p^2 Sin(\theta)dpd\theta d\phi}{(2\pi)^3}\frac{1}{2\sqrt{m^2+p^2}}e^{iprCos(\theta)} = \frac{1}{8\pi^2}\int_0^\infty dp\frac{p^2}{\sqrt{m^2+p^2}}\int_0^\pi d\theta\ Sin(\theta)\ e^{iprCos(\theta)}$$

$$D(x-y) = \frac{1}{4\pi^2 r}\int_0^\infty dp\frac{p}{\sqrt{m^2+p^2}}\ Sin(pr)$$