Peskin Solutions: Chapter 3

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1 Problem 3.1

(a)

$$[L^i, L^j] = \frac{1}{4}\epsilon^{ilm}\epsilon^{jst}[J^{lm}, J^{st}] = \frac{i}{4}\epsilon^{ilm}\epsilon^{jst}(g^{ms}J^{lt} - g^{ls}J^{mt} - g^{mt}J^{ls} + g^{lt}J^{ms})$$

The four terms in the braket are equal after switching the indexes, and $g^{ms} = -1$ when $m = s \in \{1, 2, 3\}$ so we got

$$[L^i, L^j] = i\epsilon^{ilm}\epsilon^{jst}g^{ms}J^{lt} = -i\epsilon^{mil}\epsilon^{mtj}J^{lt} = -i(\delta_i^t\delta_l^j - \delta_i^j\delta_l^t)J^{lt} = -iJ^{ji}$$

$$i\epsilon^{ijk}L^k = \frac{i}{2}\epsilon^{ijk}\epsilon^{klm}J^{lm} = \frac{i}{2}(J^{ij} - J^{ji}) = -iJ^{ji}$$

$$[L^i, L^j] = i\epsilon^{ijk}L^k$$

$$[K^i, K^j] = [J^{0i}, J^{0,j}] = i(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}) = -iJ^{ij} = -i\epsilon^{ijk}L^k$$

$$[L^i, K^j] = \frac{1}{2}\epsilon_{imn}[J^{mn}, J^{0j}] = \frac{1}{2}\epsilon_{imn}(g^{nj}K^m - g^{mj}K^n) = i\epsilon_{ijk}K^k$$

$$[J_+^i, J_-^j] = \frac{1}{4}([L^i, L^j] - i[L^i, K^j] + i[K^i, L^j] + [K^i, K^j]) = 0$$

$$[J_+^i, J_+^j] = \frac{1}{4}([L^i, L^j] + i[L^i, K^j] + i[K^i, L^j] - [K^i, K^j]) = \frac{1}{2}(i\epsilon^{ijk}L^k - \epsilon^{ijk}K^k) = i\epsilon^{ijk}J_+^k$$

$$[J_+^i, J_+^j] = i\epsilon^{ijk}J_-^k$$

(b)

Once we get the expression of $\hat{\vec{L}}$ and $\hat{\vec{K}}$, we get a set of generators $J^{\mu\nu}$ of Lorentz group, also we get $\hat{\vec{J}}_+$ and $\hat{\vec{J}}_-$, each of them is a set of generators of rotation group.

when $(j_+, j_-) = (\frac{1}{2}, 0)$, $\hat{J}^i_+ = \frac{\sigma^i}{2}$ and $\hat{J}^i_- = 0$

$$L^i = \left(J^i_+ + J^i_-\right) = \frac{1}{2}\sigma^i$$

$$K^i = -\mathrm{i}\left(J^i_+ - J^i_-\right) = -\frac{\mathrm{i}}{2}\sigma^i$$

$$\phi \to (1 - i\theta^i \frac{\sigma^i}{2} - \beta^i \frac{\sigma^i}{2})$$

This is the transformation of ψ_L, eq.(3.37).

(c)

Need more thinking ...

2 Problem 3.2

We know that $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, so,

$$i\sigma^{\mu\nu}q_\nu = (g^{\mu\nu} - \gamma^\mu\gamma^\nu)(p' - p)_\nu = (p' - p)^\mu - (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)p'_\nu + \gamma^\mu\gamma^\nu p_\nu = -(p' + p)^\mu + \not{p}'\gamma^\mu + \gamma^\mu\not{p}$$

According to the Dirac equation,

$$\bar{u}(p')[\not{p}'\gamma^\mu + \gamma^\mu\not{p}]u(p) = \bar{u}(p')[2m\gamma^\mu]u(p)$$

3 Problem 3.3

(a)

$$\not{k}_0 u_{R0} = \not{k}_0\not{k}_1 u_{L0} = \gamma^\mu k_{0\mu}\gamma^\nu k_{1\nu} u_{L0} = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}k_{0\mu}k_{1\nu}u_{L0} = g^{\mu\nu}k_{0\mu}k_{1\nu}u_{L0} = 0$$

$$\not{p}u_L(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}\not{p}u_{R0} = \frac{1}{\sqrt{2p\cdot k_0}}p^2 u_{R0} = 0$$

for the same reason,

$$\not{p}u_R(p) = 0$$

(b)

We know that u_{L0} is the left-handed spinor for a fermion with momentum k_0, so $m = 0$ and $\not{k}_0 u_{L0} = 0$.

$$k_0 u_{L0} = 0 \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2E \\ 2E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} u_{L0} = 0$$

$$u_{L0} = (0, \sqrt{2E}, 0, 0)^T$$

$$u_{R0} = (0, 0, -\sqrt{2E}, 0)^T$$

We have $u_L(p) = \frac{1}{\sqrt{2pcdotk_0}} \not{p} u_{R0}$ and $u_R(p) = \frac{1}{\sqrt{2pcdotk_0}} \not{p} u_{L0}$

$$u_L(p) = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} -(p_0 + p_3) \\ -(p_1 + ip_2) \\ 0 \\ 0 \end{pmatrix}$$

$$u_R(p) = \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} 0 \\ 0 \\ -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix}$$

(c)

$$s(p, q) = \bar{u}_R(p) u_L(q) = \frac{(p_1 + ip_2)(q_0 + q_3) - (q_1 + iq_2)(p_0 + p_3)}{\sqrt{(p_0 + p_3)(q_0 + q_3)}}$$

$$t(p, q) = \bar{u}_L(p) u_R(q) = \frac{(q_1 - iq_2)(p_0 + p_3) - (p_1 - ip_2)(q_0 + q_3)}{\sqrt{(p_0 + p_3)(q_0 + q_3)}}$$

So $s(p, q) = -s(q, p)$ and $t(p, q) = (s(q, p))^*$

4 Problem 3.4

(a)

5 Problem 3.5

(a)

$$\delta\left(\partial_\mu \phi^* \partial^\mu \phi\right) = -\mathrm{i}\left(\partial_\mu \chi^* \sigma^2 \epsilon^\dagger\right)\partial^\mu \phi + \left(\partial_\mu \phi^*\right)\left(-\mathrm{i}\epsilon^T \sigma^2 \partial^\mu \chi\right)$$

$$\delta\left(F^* F\right) = \mathrm{i}\left(\partial_\mu \chi^\dagger\right)\bar{\sigma}^\mu \epsilon F - \mathrm{i}F^* \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi$$

(b)

$$\delta(\Delta\mathcal{L}) = -\mathrm{i}m\epsilon^T \sigma^2 \chi F - \mathrm{i}m\phi\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi + \frac{1}{2}\mathrm{i}m\left[\epsilon^T F + \epsilon^\dagger \left(\sigma^2\right)^T \left(\sigma^\mu\right)^T \partial_\mu \phi\right]\sigma^2 \chi$$
$$+\frac{1}{2}\mathrm{i}m\chi^T \sigma^2 \left[\epsilon F + \sigma^\mu \left(\partial_\mu \phi\right)\sigma^2 \epsilon^*\right] + \mathrm{c.c}$$

6 Problem 3.6

(a) We need to find the normalization coefficients of all 16 elements.

$$tr[\gamma^0\gamma^0] = 4$$

$$tr[\gamma^i\gamma^i] = -4$$

So, there are γ^0 and $i\gamma^i$ in the Γ^A

(b)

Multiply equation at left by $(\bar{u}_2\Gamma^F u_3)(\bar{u}_4\Gamma^E u_1)$.

Also, notice that $\bar{u}_i\Gamma u_j$ is a 1×1 number, so the order can be changed as you want; and $(\bar{u}_i\Gamma u_i) = tr(\bar{u}_i\Gamma u_i) = tr(\Gamma)$. With these equations, we can derive the equation we need.

(c)

Use the results of (b), we can get it easily.

7 Problem 3.7

(a)

$$P\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})P = \frac{\mathrm{i}}{2}\bar{\psi}(t,-\mathbf{x})\gamma^0\left[\gamma^\mu,\gamma^\nu\right]\gamma^0\psi(t,-\mathbf{x})$$

$$\gamma^0[\gamma^0,\gamma^i]\gamma^0 = -[\gamma^0,\gamma^i]$$

$$\gamma^0[\gamma^i,\gamma^j]\gamma^0 = [\gamma^i,\gamma^j]$$

Notice that $\hat{T}\gamma^\mu\hat{T} = (\gamma^\mu)^*$.

(b)

$$\phi(\vec{x},t) = e^{iHt}\phi(\vec{x})e^{-iHt} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{-ip\cdot x} + b^\dagger_{\vec{p}}e^{ip\cdot x}\right)\Big|_{p^0=E_{\vec{p}}}$$

$$Pa_{\vec{p}}P = a_{-\vec{p}}$$

So,

$$P\phi(\vec{x},t)P = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{-\vec{p}}e^{-ip\cdot x} + b^\dagger_{-\vec{p}}e^{ip\cdot x}\right)|_{p^0=E_{\vec{p}}}$$

Replace the variable $-\vec{p}$ with $\vec{p}$,

$$P\phi(\vec{x},t)P = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{-i(p_0t+\vec{p}\cdot\vec{x})} + b^\dagger_{\vec{p}}e^{i(p_0t+\vec{p}\cdot\vec{x})}\right)|_{p^0=E_{\vec{p}}} = \phi(-\vec{x},t)$$

$$C\phi(x)C = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(a^\dagger_{\vec{p}}e^{ip\cdot x} + b_{\vec{p}}e^{-ip\cdot x}) = \phi^*(x)$$