TOPOLOGY, SUMMER 2025

ANON

Contents

1. Introduction

Problems primarily from [?lee2011manifolds].

2. Topological Spaces

2.1. Topologies.

Problem 1. *Suppose M is a set and d, d' are two different metrics on M. Prove that d, d' generate the same topology on M iff $\forall x \in M, \forall r > 0, \exists r_1, r_2 > 0 : B_{r_1}^{d'}(x) \subseteq B_r^d(x)$ and $B_{r_2}^d(x) \subseteq B_r^{d'}(x)$*

Proof. $\implies$ d, d' generate the same topology so, $B_r^d(x)$ is open in d' topology.
Thus by definition of openness, $\forall x \in B_r^d(x), \exists r_1 > 0 : B_{r_1}^{d'}(x) \subseteq B_r^d(x)$
Similar reasoning yields, $B_{r_2}^d(x) \subseteq B_r^{d'}(x)$

$\impliedby$

Strategy: Show open set in d topology is open in d' topology and vice versa to prove that they generate the same topology.Please feel free to add new insights :)

Let U be an open set in d topology.So, $B_r^d(x) \subseteq U$ and we also have $\forall r > 0, \exists r_1 > 0 : B_{r_1}^{d'}(x) \subseteq B_r^d(x) \implies B_{r_1}^{d'}(x) \subseteq U$ .Thus, U is open set in d' topology. Similar reasoning yields V is open set in d topology given that V is open set in d' topology. $\square$

Problem 2. *Let (M, d) be a metric space, let c be a positive real number, and define a new metric d' on M by $d'(x, y) = c \cdot d(x, y)$. Prove that d and d' generate the same topology on M.*

Proof. Let U be an open subset in d topology.
Now, $B_r^d(x) \subseteq U, B_r^d(x) = \{y \in M | d(x, y) < r\}$
Define $r_1 = r \cdot c, c > 0$

Strategy: Same as above. We leverage the relation between the metrics.

$B_{r_1}^{d'}(x) = \{y \in M | d'(x, y) < r_1\} = \{y \in M | c \cdot d(x, y) < r_1\} = \{y \in M | d(x, y) < \dfrac{r_1}{c} = r\} = B_r^d(x)$

Thus, $B_{r_1}^{d'}(x) \subseteq U$, so U is open in d' topology.

With similar reasoning, for an open subset V in d' topology, V is open in d topology.Therefore, d, d' generate the same topology on M. $\square$

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Problem 3. *Define a metric d' on $\mathbb{R}^n$ by $d'(x,y) = \max\{|x_1 - y_1, \ldots, |x_n - y_n|\}$. Show that the Euclidean metric and d' generate the same topology on $\mathbb{R}^n$.*

2.2. Convergence and Continuity.

2.3. Hausdorff Spaces.

2.4. Bases and Countability.

2.5. Manifolds.

2.6. Problems.

3. New Spaces From Old

3.1. Subspaces.

3.2. Product Spaces.

3.3. Disjoint Union Spaces.

3.4. Quotient Spaces.

3.5. Adjunction Spaces.

3.6. Topological Groups and Group Actions.

3.7. Problems.

References

[1] John M. Lee, *Introduction to Topological Manifolds*, 2nd ed., Graduate Texts in Mathematics, Springer, New York, NY, 2011. Originally published: 28 December 2010.