Linear Algebra for SVD

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Learning Plan

- Matrix Basics
  - Symmetric Matrices
  - Vectors
  - Matrix Transformations
- What are eigenvectors and eigenvalues?

Matrix Basics

The matrix is a 1999 science fiction action film....

Just kidding. A matrix looks like this:

$$\begin{bmatrix} 1 & 3 \\ -2 & 1.5 \end{bmatrix}$$

Or it can look something like this:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

You may have had some exposure to

them. You can add them, subtract them, and multiply them. But try not to worry about these specifics too much.

$$\begin{bmatrix} 1 & 3 \\ -2 & 1.5 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 5.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -3 & -1 \\ 1 & 3 & 5 \end{bmatrix}$$

Scalars

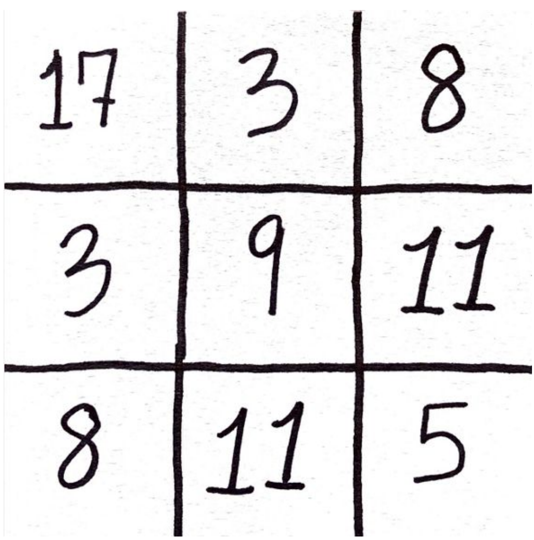
Matrices--collections of values--exist in contrast to *scalars*, which are the numbers we’re used to: a single number on its own. You can also multiply a matrix by a scalar, which is the same as multiplying each entry of the matrix by the scalar.

Symmetric Matrices

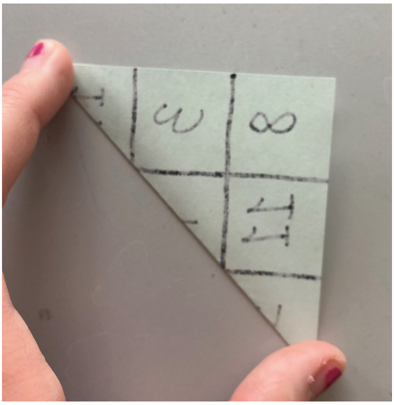
Square matrices (also called *n* x *n* matrices) have specific properties that help us in linear algebra. Even more specific than that, *symmetric matrices* are a special type of square matrix that we can use to help us understand the SVD.

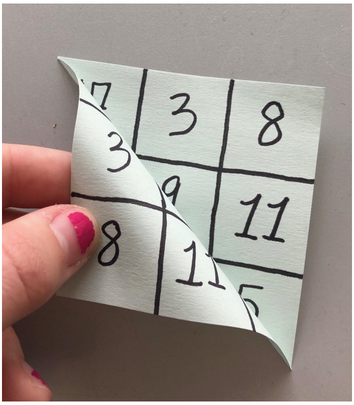
Symmetric matrices are symmetric along their diagonal entries.

Here's an example: I drew this matrix on a sticky note.



When I fold it along the diagonal, the symmetrical entries touch each other:





In other words, if I were to take the matrix's columns and make them rows, it would still be the same matrix.

The process of swapping columns with rows is called taking the *transpose*. The transpose of A is written as A^T.

So, for a symmetric matrix, $\mathbf{A=A^T}$.

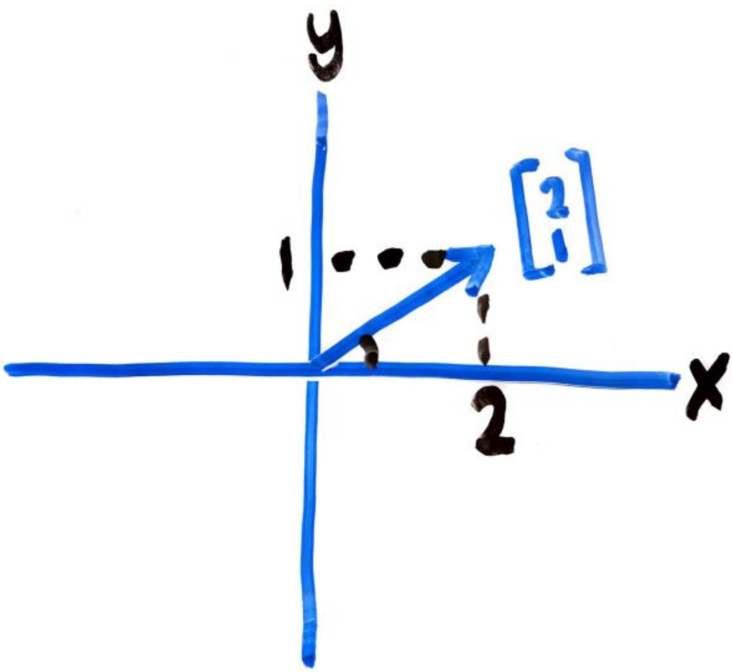
Before we go any further, let's answer the question:

What is a vector?

A vector a way to express a magnitude and a direction, such as velocity. We express this in a very similar way to a matrix, but it only has one column:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We can visualize this vector on a graph:



Back to Matrices

Cool. Now that we talked about vectors, we can talk about some of the other cool things that matrices can do.

Matrices may look intimidating, but they are just a way to express information. They are key tools for us to use in linear algebra. For example:

$$x - 3y = -8$$
$$3x + 2y = 9$$

$$\downarrow$$

Variable Vector

Coefficient Matrix

$$\begin{bmatrix} 1 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 9 \end{bmatrix}$$

Solution Vector

$$\downarrow$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} x + \begin{bmatrix} -3 \\ 2 \end{bmatrix} y = \begin{bmatrix} -8 \\ 9 \end{bmatrix}$$

$$x - 3y = -8$$
$$3x + 2y = 9$$

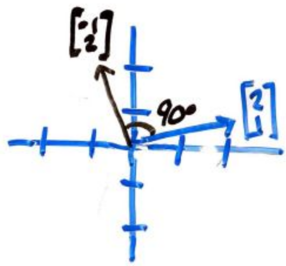
We can use a matrix to express transformations:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This matrix, when multiplied by a vector, rotates that vector θ (theta) degrees. We can express this:

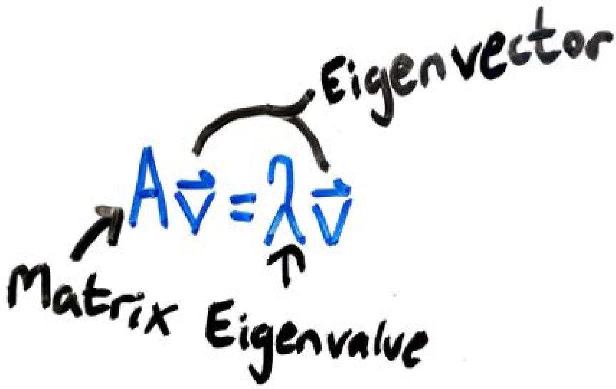
$$R_{90^\circ} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$R_{90^\circ}\vec{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



Eigenvalues & Eigenvectors

Simply put, eigenvalues and eigenvectors answer the following equation:



In what case can multiplying a vector v by the matrix A be the same as multiplying v by a scalar λ (lambda)?

Do not get scared away by new terminology and the notation.

An eigenvalue is a scalar that represents what a matrix does as it acts on a specific vector - called an eigenvector. Here is an example:

$$\begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix} \qquad \begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\lambda\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix} \quad \lambda=3 \qquad \lambda\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix} \quad \lambda=-1$$

The eigenvectors of $\begin{bmatrix}1&2\\2&1\end{bmatrix}$ are $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ with corresponding eigenvalues 3 and -1.

And the same thing written out in the Av = λ v format:

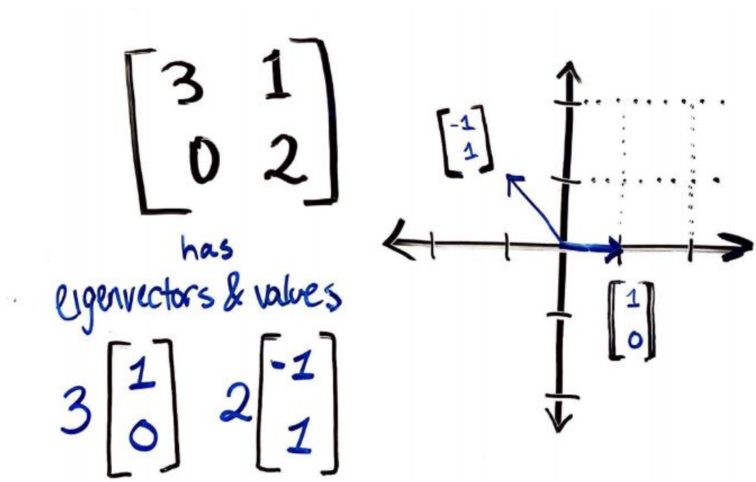
$$\overset{\text{matrix A}}{\begin{bmatrix}1&2\\2&1\end{bmatrix}}\overset{\text{Eigenvector } v_1}{\begin{bmatrix}1\\1\end{bmatrix}}=\underset{\text{Eigenvalue } \lambda_1}{3}\begin{bmatrix}1\\1\end{bmatrix} \qquad \overset{\text{matrix A}}{\begin{bmatrix}1&2\\2&1\end{bmatrix}}\overset{\text{Eigenvector } v_2}{\begin{bmatrix}1\\-1\end{bmatrix}}=\underset{\text{Eigenvalue } \lambda_2}{-1}\begin{bmatrix}1\\-1\end{bmatrix}$$

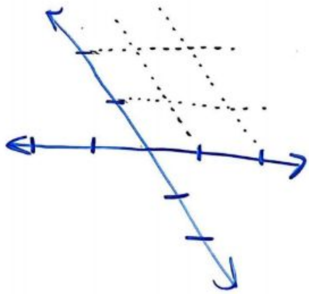
Multiplying matrix **A** times the eigenvector $\mathbf{v_1}$ is the same as multiplying **3**$\mathbf{v_1}$. Multiplying matrix **A** times the eigenvector $\mathbf{v_2}$ is the same as multiplying **-1**$\mathbf{v_2}$.

WHY DO THEY MATTER?

Eigenvalues and eigenvectors are important because they allow us to turn the complicated ordeal of matrix multiplication into something simple--scalar multiplication.

We can rewrite our matrices, traditionally in reference to the *x* and *y* coordinate axes, in terms of any set of axes by multiplying them.





$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

The coordinate system is also called the *basis*, so this is known as a *change of basis*.

Using eigenvectors as the basis makes our matrix into something wonderful--a *diagonal matrix*. Diagonal matrices are easier to work with, especially when it comes to multiplication. It may be weird to look at, but don't shy away from using an *eigenbasis* to make math easier!

Quick Recap

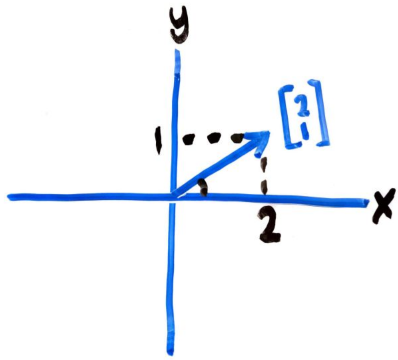
So we have gone over a ton of information in a short amount of time. Let's recap!

First of all, matrices are a way to express information in an organized manner, and can act as transformations:

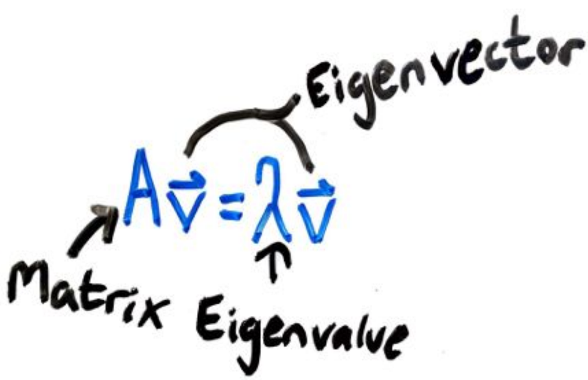
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Vectors are something with a magnitude and a direction, and are written a lot like a matrix:



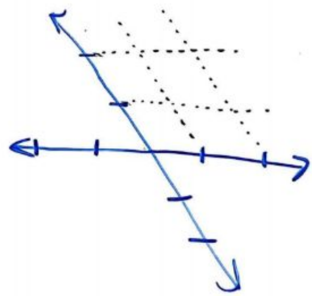
We can how matrices “act” on a vector as an eigenvalue:



Eigenvalues and eigenvectors make math much easier for us, letting us rewrite matrices and perform operations faster.

Change of Basis

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$



Got all that? From here, we recommend looking at our SVD Zine, which applies all the linear algebra learned here in a really interesting application.

For further resources and questions, make sure to check out our website:

www.griffithstites.com/Linearity-Zine/

Thank you so much for reading!