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Problem 1. *Compute the following greatest common divisor:*
(a) gcd(12,8)
(b) gcd(36,84)
(c) gcd(120,98)

Solution.
(a) gcd(12,8)=4
(b) gcd(36,84)=12
(c) gcd(120,98)=2
□

Problem 2. *Using the Euclidean Algorithm to find the greatest common divisor for the numbers in Question1.*

Solution.
(a) gcd(12,8)=gcd(8,4)=gcd(4,0)=4
(b) gcd(36,84)=gcd(36,12)=gcd(12,0)=12
(c) gcd(120,98)=gcd(98,22)=gcd(22,10)=gcd(10,2)=gcd(2,0)=2
□

Problem 3. *Determine the following statements are true or false:*
(a)If gcd(a,b)=1 and gcd(a,c)=1, then gcd(b,c)=1
(b)If gcd(a,b)=1 and gcd(a,c)=1, then gcd(a, bc)=1

Solution.
(a) false. Counterexample: a=5,b=4,c=2, gcd(a,b)=gcd(a,c)=1 but gcd(b,c)=2
(b) true. Proof
by Bezout's identity, there exist s,t,u,v such that

$$\begin{cases} sa + tb = 1 \\ ua + vc = 1 \end{cases}$$

So $(sa + tb)(ua + vc) = 1$. It then gives $(sau + svc + tbu)a + (tv)bc = 1$, which implies that spc(a,bc)=1.
Hence gcd(a,bc)=1
□

Problem 4. *Prove that if gcd(x,y)=1, then gcd(x+y,x-y)=1 or 2*

Solution.
Proof by cases.
$\because gcd(x, y) = 1$
$\therefore spc(x, y) = 1, \exists t, u \in \mathbb{Z}$ *such that* $tx + uy = 1$
it then gives $\frac{t+u}{2}(x + y) + \frac{t-u}{2}(x - y) = 1$

(i) If ($t \equiv 1$ *mod* 2 and $u \equiv 1$ *mod* 2) or ($t \equiv 0$ *mod* 2 and $u \equiv 0$ *mod* 2) = true, then $\frac{t+u}{2}$ and $\frac{t-u}{2}$ are integers. Then $spc(x+y, x-y) = 1$

(ii) If ($t \equiv 1$ *mod* 2 and $u \equiv 1$ *mod* 2) or ($t \equiv 0$ *mod* 2 and $u \equiv 0$ *mod* 2) = false, then $\frac{t+u+1}{2}$ and $\frac{t-u+1}{2}$ are integers.

Suppose $\exists n, m \in \mathbb{Z}$ such that $n(x+y) + m(x-y) = 1$. Then $(n+m)x + (n-m)y = 1$. Substracting $(n+m)x+(n-m)y = 1$ and $tx+uy = 1$, we have $(n+m-t)x+(n-m-u)y = 0$.

$\because$ gcd(x,y)=1, then $x \not| y$ and $y \not| x$

$\therefore rx + qy = 0, (r, q \in \mathbb{Z}) \implies r = q = 0$

$\therefore \begin{cases} n+m=t \\ n-m=u \end{cases}$, it gives $t+u = 2n$, $t-u = 2m$, contradicting our assumption that $t+u$ and $t-u$ are not both even numbers.

Hence $\nexists n, m \in \mathbb{Z}$. i.e. spc(x+y,x-y)=2

□

Problem 5. *Use the Euclidean Algorithm to compute gcd(120,84), and then find the integer a and b such that gcd(120,84)=120a+84b*

Solution.

gcd(120,84)=gcd(84,36)=gcd(36,12)=gcd(12,0)=12

In the Euclidean Algorithm, we have $120 = 1 \times 84 + 36$ and $84 = 2 \times 36 + 12$. By these two equalities we have

$$\begin{aligned} 12 &= 84 - 2 \times 36 \\ &= 84 - 2 \times (120 - 84) \\ &= 3 \times 84 - 2 \times 120 \end{aligned}$$

So a=-2,b=3

□

Problem 6. *Prove that for any $n \in \mathbb{Z}$, gcd(n,n+1)=1. Conclude that if a prime p divides n, then p cannot divide n+1*

Solution.

$gcd(n, n+1) = spc(n, n+1) = (n+1) - n = 1$

If $p|n$ then $\exists m$ such that $n = pm$.

Then $gcd(n+1, p) = spc(pm+1, p) = pm + 1 - pm = 1$. i.e. p cannot divide n+1

□

Problem 7. *If the equation gcd(n,m)=gcd(n+m,n-m) is true? Prove it if it is true, otherwise, give a counter example.*

Solution. Counterexample: gcd(5,3)=1, but gcd(5+3,5-3)=2 □

Problem 8. *Suppose n is even and gcd(n,m)=5, show that m is odd*

Solution. Proof by contradiction

Assume that m is even. Then $2|n$ and $2|m$, $5|n$ and $5|m$. Therefore, $10|n$ and $10|m$. $10 > 5$ which contradicts the assumption that gcd(n,m)=5

Hence m is odd □