1 Question 1

Proof.

Prove this by contradiction.

Suppose $\{T\mathbf{v}_1, \ldots, T\mathbf{v}_k\}$ is linearly dependent in W. That is, $\exists \alpha_1, \ldots, \alpha_k$ not all zero such that:

$$\sum_{i=1}^{k} \alpha_i T\mathbf{v}_i = \mathbf{0}_W$$

Then

$$T\left(\sum_{i=1}^{k} \alpha_i \mathbf{v}_i\right) = \mathbf{0}_W$$

So $\sum_{i=1}^{k} \alpha_i \mathbf{v}_i \in \text{Ker}(T)$

$\because \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are linearly independent

$\therefore$

$$\sum_{i=1}^{k} \alpha_i \mathbf{v}_i \neq \mathbf{0}_V$$

However, $\because T$ is injective. That means, $\text{Ker}(T) = \{\mathbf{0}_V\}$.

$\therefore$Contradiction. $\alpha_1, \ldots, \alpha_k$ do not exist.

Hence, $\{T\mathbf{v}_1, \ldots, T\mathbf{v}_k\}$ is linearly independent in W.

$\blacksquare$

2 Question 2

Proof.

$\forall \mathbf{w} \in W$
$\because T$ is surjective.
$\therefore \exists \mathbf{v} \in V$ such that

$$\mathbf{w} = T(\mathbf{v}) \tag{1}$$

$\because \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ spans V.
$\therefore \exists \alpha_1, \dots, \alpha_k$ such that

$$\mathbf{v} = \sum_{i=1}^{k} \alpha_i \mathbf{v}_i \tag{2}$$

By (1) and (2),

$$\begin{aligned} \mathbf{w} &= T(\mathbf{v}) \\ &= T(\sum_{i=1}^{k} \alpha_i \mathbf{v}_i) \\ &= \sum_{i=1}^{k} \alpha_i T(\mathbf{v}_i) \end{aligned} \tag{3}$$

Hence, $\{T\mathbf{v}_1, \dots, T\mathbf{v}_k\}$ spans W.

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3 Question 3

Proof.

$\because T : V \to W$ is an isomorphism.
$\therefore T$ is bijective. Then the following three conditions are satisfied.

$$\forall \mathbf{v}_1 \in V, \mathbf{v}_2 \in V, \mathbf{v}_1 \neq \mathbf{v}_2 \Longrightarrow T(\mathbf{v}_1) \neq T(\mathbf{v}_2) \tag{1}$$

$$\forall \mathbf{w} \in W, \exists \mathbf{v} \in V, \text{such that } \mathbf{w} = T(\mathbf{v}) \tag{2}$$

$$\mathrm{Ker}(T) = \{\mathbf{0}_V\} \tag{3}$$

$\because Span(\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}) = V$
$\therefore$ By (2), $\forall \mathbf{w} \in W, \exists \alpha_1, \ldots, \alpha_n, \mathbf{w} = T(\sum_{i=1}^n \alpha_i \mathbf{v}_i) = \sum_{i=1}^n \alpha_i T(\mathbf{v}_i)$. So

$$Span(\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}) = W \tag{4}$$

Suppose $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is linearly dependent.
That is, $\exists \alpha_1, \ldots, \alpha_n$ such that $\sum_{i=1}^n \alpha_i T(\mathbf{v}_i) = \mathbf{0}_W$.

$$\begin{gathered} \sum_{i=1}^n \alpha_i T(\mathbf{v}_i) = \mathbf{0}_W \\ T\left(\sum_{i=1}^n \alpha_i \mathbf{v}_i\right) = \mathbf{0}_W \end{gathered} \tag{5}$$

$\sum_{i=1}^n \alpha_i \mathbf{v}_i \neq \mathbf{0}_V$. But $\mathrm{Ker}(T) = \{\mathbf{0}_V\}$. Contradiction!
$\therefore$

$$\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\} \text{ is linearly independent} \tag{6}$$

Hence, $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a basis of W.

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4 Question 4

Proof.

Let $dim(V) = dim(W) = n$.

Pick one basis respectively for V and W:

$$\mathcal{B}_V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\},\ \mathcal{B}_W = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$$

Define a linear transformation $T : V \to W$,

First define $T : \mathcal{B}_V \to \mathcal{B}_W$:

$$\forall i = 1, \dots, n,\ \mathbf{v}_i = \mathbf{w}_i \tag{1}$$

Since

$$\forall \mathbf{v} \in V, \exists \alpha_1, \dots, \alpha_n, \text{ such that } \mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

So define

$$T(\mathbf{v}) := \sum_{i=1}^{n} \alpha_i T(\mathbf{v}_i) = \sum_{i=1}^{n} \alpha_i \mathbf{w}_i$$

$\because$

$$\forall \mathbf{w} \in W, \exists \beta_1, \dots, \beta_n, \text{ such that } \mathbf{w} = \sum_{i=1}^{n} \beta_i \mathbf{w}_i$$

$\therefore$

$$\forall \mathbf{w} \in W, \mathbf{w} = \sum_{i=1}^{n} \beta_i T(\mathbf{v}_i) = T(\sum_{i=1}^{n} \beta_i \mathbf{v}_i)$$

that is, T is surjective.

Suppose T is not injective. That is

$$\exists \mathbf{v} \in V, \mathbf{v} \neq \mathbf{0}_V, T(\mathbf{v}) = \mathbf{0}_W$$

$\because \exists \alpha_1, \dots, \alpha_n$ not all zero, such that $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$

$\therefore$

$$T(\mathbf{v}) = \sum_{i=1}^{n} \alpha_i \mathbf{w}_i = \mathbf{0}_W$$

that is, $\mathcal{B}_W$ is linearly dependent. Contradiction!

$\therefore T$ must be injective.

Hence, T is bijective, which is an isomorphism.

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5 Question 5

$\because (1,3,0,0,0)$ and $(0,0,1,1,1)$ are independent
and $Span(\{(1,3,0,0,0),(0,0,1,1,1)\}) = \text{Nul}(T)$
$\therefore dim(\text{Nul}(T)) = 2$
So
$dim(\text{Nul}(T)) + dim(\text{Image}(T)) = 2 + dim(\text{Image}(T)) \leq 2 + dim(\mathbf{F}^2) = 4$
However, $dim(\text{Nul}(T)) + dim(\text{Image}(T)) = dim(\mathbb{F}^5) = 5$. Contradiction!
Hence, there is no such a linear transformation.

6 Question 6

Proof.

(a)

$\because \forall x \in \mathbb{F}, y \in \mathbb{F}, \mathbf{v}_1 \in V, \mathbf{v}_2 \in V,$

$$\begin{aligned}\alpha T(x\mathbf{v}_1 + y\mathbf{v}_2) &= \alpha(T(x\mathbf{v}_1 + y\mathbf{v}_2)) \\ &= \alpha(xT(\mathbf{v}_1) + yT(\mathbf{v}_2)) \\ &= x(\alpha T)(\mathbf{v}_1) + y(\alpha T)(\mathbf{v}_2) \in W\end{aligned}$$

$\therefore \alpha T \in Hom_{\mathbb{F}}(V, W)$.

$\because \forall x \in \mathbb{F}, y \in \mathbb{F}, \mathbf{v}_1 \in V, \mathbf{v}_2 \in V,$

$$\begin{aligned}(T + S)(x\mathbf{v}_1 + y\mathbf{v}_2) &= T(x\mathbf{v}_1 + y\mathbf{v}_2) + S(x\mathbf{v}_1 + y\mathbf{v}_2) \\ &= xT(\mathbf{v}_1) + yT(\mathbf{v}_2) + xS(\mathbf{v}_1) + yS(\mathbf{v}_2) \\ &= x(T + S)(\mathbf{v}_1) + y(T + S)(\mathbf{v}_2) \in W\end{aligned}$$

$\therefore (T + S) \in Hom_{\mathbb{F}}(V, W)$.

(b)

Define $T_0(\mathbf{v}) = \mathbf{0}_W, \forall \mathbf{v} \in V$.

$\because \forall T \in Hom_{\mathbb{F}}(V, W), \forall \mathbf{v} \in V, (T + T_0)(\mathbf{v}) = T(\mathbf{v})$

that is $(T + T_0) = T$

$\therefore$the additive identity in $Hom_{\mathbb{F}}(V, W)$ is T_0.

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