$\boxed{1}$ $S = \lambda f g x . f x (g x)$

$S (\lambda x. M)(\lambda x. N) \stackrel{?}{=} \lambda x. M N$

$S(\lambda x. M)(\lambda x. N) = \lambda x. \underbrace{(\lambda x. M) x}_{M} \underbrace{((\lambda x. N) x)}_{N} =$

$= \lambda x. M N$

substitutitions are correct as $(\lambda x. M)$ and $(\lambda x. N)$ surely dont contain free ~~...~~ x variables (they're abstracted by x) so all closed occurences of x in N and M remain closed by the same abstraction and all free are replaced with x itself

$\boxed{2}$

$xor = \lambda a.\ a\ (not)\ (id)$

When a is `true`, this is `not`.
When a is `false`, this is `id`.
Without other terms:

$xor = \lambda axyz.\ a(x\ z\ y)\ (x\ y\ z)$

xor false false = xor false false

and
xor true true = xor true true

and
$xor\ true\ false = (\lambda axyz.\ a(xzy)(xyz))$
$(\lambda xy.\ x)\ (\lambda xy.\ y) =$
$= (\lambda xyz.\ (\lambda xy.\ x)\ (x\ z\ y)\ (x\ y\ z))(\lambda xy.\ y) =$
$= (\lambda xyz.\ x\ z\ y)\ (\lambda xy.\ y) =$
$= \lambda yz.\ (\lambda xy.\ y)\ z\ y = \lambda yz.\ y$
$xor\ false\ true = (\lambda xyz.\ (\lambda xy.y)(xzy)(xyz))(\lambda xy.x) =$
$= (\lambda xyz.\ x\ y\ z)(\lambda xy.\ x) = \lambda yz.y$

$\boxed{3}$ $\text{isZero} = \lambda n.\, n\,(\text{and false})(\text{true}) =$

$\text{succ} = \lambda n f x.\, f\,(n\, f\, x)$

$\text{pair} = \lambda x y p.\; p\, x\, y$

$\text{fst} = \lambda p.\; p\ \text{true}$

$\text{snd} = \lambda p.\; p\ \text{false}$

$\text{fst}\,(\text{pair}\ a\, b) = (\lambda p.\, p\ \text{true})(\lambda p.\, p\, a\, b) =$

$= (\lambda p.\, p\, a\, b)\ \text{true} = \text{true}\ a\, b = a$

step = λp. if (fst p) (pair false zero)
(pair false (succ $ snd p)))

$\boxed{\text{pred} = \lambda n.\ \text{snd}\ (n\ \text{step}\ \$\ \text{pair true zero})}$

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$$\text{mult} = \lambda m n f.\ m(n\ f) =$$

$$\underset{\eta}{=} \lambda m n f x.\ m(n\ f)\ x$$

~~i.e. we created a~~

I.E. if m generates a function repeated iteravlly m times from other function then to repeat a function mn times we need to repeat m times a function repeated n times.

$$(\lambda m n f.\ m(n f))(\lambda x.x) =$$

$$= \lambda n f.\ (\lambda x.x)(n f) = \lambda n f.\ n f$$

$$\lambda m. (\lambda m n f.\ m(n f))\ m\ (\lambda x.x) =$$

$$= \lambda m f.\ m((\lambda x.x) f) = \lambda m f.\ m f$$

~~(pair false zero)(pair false))~~

~~step (pair true zero) false~~

~~false =~~

$\boxed{4}$

step = λp. pair (succ (fst p))

(mult (succ (fst p)) (snd p))

$\boxed{\text{fact} = \lambda n.\ \text{snd}\ (n\ \text{step}\ (\text{pair zero one}))}$

Step generates from pair (n, n!) next pair (n+1, (n+1)!).

Applied n times from (0, 1) it generates (n, n!).

fact three = snd (three step zero one) =

= snd (step (step (step (pair zero one)))) =

= snd (step (step (pair one one))) =

= snd (step (pair two two)) = six