INTRODUCTION TO

FUNCTIONAL PROGRAMMING

OUTLINE

- Fixed Point
- β-Normal Form
- Church-Rosser theorem
- Reduction Strategies

TERM EQUATIONS

- β-equivalence allows us to formulate term equations
  - $(\lambda x . M) N =_\beta [x \mapsto N] M$
- What to do if the equation is recursive?
  - $F M =_\beta M N$

Find F that $\forall M, N, L$: $\lambda \vdash F M N L =_\beta M L (N L)$

$$F M N L =_\beta M L (N L)$$
$$F M N L =_\beta (\lambda l . M l (N l)) L$$
$$F M N =_\beta \lambda l . M l (N l)$$
$$F M =_\beta \lambda n . \lambda l . M l (n l)$$
$$F =_\beta \lambda m . \lambda n . \lambda l . m l (n l)$$

FIXED POINT THEOREM

Theorem

$\forall \lambda -$ term F there exists a fixed point:

$\forall F \in \Lambda \,.\, \exists X \in \Lambda \,.\, \lambda \vdash F\,X =_\beta X$

Proof

Consider $W = \lambda x \,.\, F(x\,x)$ and $X = W\,W$.

Then $X = W\,W = (\lambda x \,.\, F(x\,x))\,W =_\beta F\,(W\,W) = F\,X$

FIXED POINT COMBINATOR THEOREM

Theorem

$\forall \lambda -$ term F, there exists a fixed point combinator $\boldsymbol{Y}$

such that $\forall F \in \Lambda \,.\, \lambda \vdash F\,(\boldsymbol{Y}\,F) =_\beta \boldsymbol{Y}\,F$

Proof

Consider $\boldsymbol{Y} = \lambda f\,.\,(\lambda x\,.\,f(x\,x))(\lambda x\,.\,f(x\,x))$

Then $\boldsymbol{Y}\,F =_\beta (\lambda x\,.\,F(x\,x))(\lambda x\,.\,F(x\,x)) =_\beta F\,((\lambda x\,.\,F(x\,x))(\lambda x\,.\,F(x\,x))) =_\beta F\,(\boldsymbol{Y}\,F)$

Y-COMBINATOR INTRODUCES RECURSION IN λ-CALCULUS

Factorial as an equation

$$fac = \lambda n \,.\, if\,(iszro\; n)\; 1\; (mult\; n\; (fac\; (pred\; n)))$$

Rewrite this as an application:

$$fac = (\lambda f\, n \,.\, if\,(iszro\; n)\; 1\; (mult\; n\; (f\; (pred\; n))))\; fac$$

Now we can see that fac is a fixed point of fac'

$$fac' = (\lambda f\, n \,.\, if\,(iszro\; n)\; 1\; (mult\; n\; (f\; (pred\; n))))$$

Y-COMBINATOR INTRODUCES RECURSION IN λ-CALCULUS

$$
\begin{aligned}
fac\,3 &= (\mathbf{Y}\,fac')\,3 \\
&= fac'\,(\mathbf{Y}\,fac')3 \\
&= if\,(iszro\,3)\,1\,(mult\,3\,((\mathbf{Y}\,fac')\,(pred\,3))) \\
&= mult\,3\,((\mathbf{Y}\,fac')\,2) \\
&= mult\,3\,(fac'\,(\mathbf{Y}\,fac')\,2) \\
&= mult\,3\,(mult\,2\,((\mathbf{Y}\,fac')\,1)) \\
&= mult\,3\,(mult\,2\,(mult\,1\,(\mathbf{Y}\,fac')\,0))) \\
&= mult\,3\,(mult\,2\,(mult\,1\,1)) \\
&= 6
\end{aligned}
$$

EXERCISE: Y-COMBINATOR

- $Y = \lambda f \,.\, (\lambda x \,.\, f\,(x\,x))\,(\lambda x \,.\, f\,(x\,x))$
- $Y\,F = F\,(Y\,F)$
- But not $Y\,F \twoheadrightarrow_\beta F\,(Y\,F)$ or $F\,(Y\,F) \twoheadrightarrow_\beta Y\,F$
  - Prove this
- Prove that this holds for $\Theta = A\,A$, where $A = \lambda x\,y \,.\, y\,(x\,x\,y)$

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β-REDUCTION RELATION

$$M \twoheadrightarrow_\beta M \qquad \text{(refl)}$$

$$M \rightarrow_\beta N \Rightarrow M \twoheadrightarrow_\beta N \qquad \text{(ini)}$$

$$M \twoheadrightarrow_\beta N, N \twoheadrightarrow_\beta L \Rightarrow M \twoheadrightarrow_\beta L \qquad \text{(trans)}$$

$$(\lambda x\,.\,x)\,((\lambda yz\,.\,y)\,(\lambda p\,.\,p)) \twoheadrightarrow_\beta (\lambda x\,.\,x)\,((\lambda yz\,.\,y)\,(\lambda p\,.\,p))$$

$$(\lambda x\,.\,x)\,((\lambda yz\,.\,y)\,(\lambda p\,.\,p)) \twoheadrightarrow_\beta ((\lambda yz\,.\,y)\,(\lambda p\,.\,p))$$

$$(\lambda x\,.\,x)\,((\lambda yz\,.\,y)\,(\lambda p\,.\,p)) \twoheadrightarrow_\beta (\lambda z\,p\,.\,p)$$

β-EQUIVALENCE RELATION

- Intuition: two terms are β-equivalent if there is a sequence of $\rightarrow_\beta$ arrows linking them together
- Prove that $\boldsymbol{K}\,\boldsymbol{I} =_\beta \boldsymbol{I}\,\boldsymbol{I}\,\boldsymbol{K}_*$
  - $\boldsymbol{K} =_\beta \lambda xy\,.\,x$
  - $\boldsymbol{K}_* =_\beta \lambda xy\,.\,y$
  - $\boldsymbol{I} =_\beta \lambda x\,.\,x$

$$M \twoheadrightarrow_\beta N \Rightarrow M =_\beta N \qquad \text{(ini)}$$
$$M =_\beta N \Rightarrow N =_\beta M \qquad \text{(sym)}$$
$$M =_\beta N, N =_\beta L \Rightarrow M =_\beta L \qquad \text{(trans)}$$

β-NORMAL FORM

- λ-term is in β-normal form, if it doesn't contain any subterms that are β-redexes
- λ-term M has a β-normal form, if for some N in β- normal form $M =_\beta N$
- What's β-normal form of:
  - $(\lambda x\,.\,x)\,((\lambda yz\,.\,y)\,(\lambda p\,.\,p))$
  - $\Omega = \omega\,\omega = (\lambda x\,.\,x\,x)\,(\lambda x\,.\,x\,x)$

$\lambda xy\,.\,x\,(\lambda z\,.\,z\,x)\,y$ is in β-normal form
$(\lambda x\,.\,x\,x)\,y$ is not in normal form
$(\lambda x\,.\,x\,x)\,y$ has a normal form $y\,y$

LEMMA ABOUT THE REDUCTION OF β-NORMAL FORM

- Let M be in β-NF, then $M \twoheadrightarrow_\beta N \Rightarrow N = M$
- Proof:
  - M doesn't have a redex
  - Then it's impossible to have $M \rightarrow_\beta N$
  - Then $M \twoheadrightarrow_\beta N$ only by reflexivity

$\lambda xy \,.\, x\,(\lambda z \,.\, z\,x)\,y$ is in β-normal form
$(\lambda x \,.\, x\,x)\,y$ is not in normal form
$(\lambda x \,.\, x\,x)\,y$ has a normal form $y\,y$

NOT EVERY TERM HAS A β-NORMAL FORM

- Consider $\Omega = \omega\,\omega = (\lambda x\,.\,x\,x)\,(\lambda x\,.\,x\,x)$
- This is still not a proof
- It may be that there exists M such that
  - $\Omega \twoheadleftarrow M \twoheadrightarrow N$

$$
\begin{aligned}
\Omega &= \omega\omega \\
&= (\lambda x\,.\,x\,x)\,(\lambda x\,.\,x\,x) \\
&\to_\beta (\lambda x\,.\,x\,x)\,(\lambda x\,.\,x\,x) \\
&\to_\beta \cdots
\end{aligned}
$$

EXERCISE

- What are some λ-terms that keep on growing under β rule?

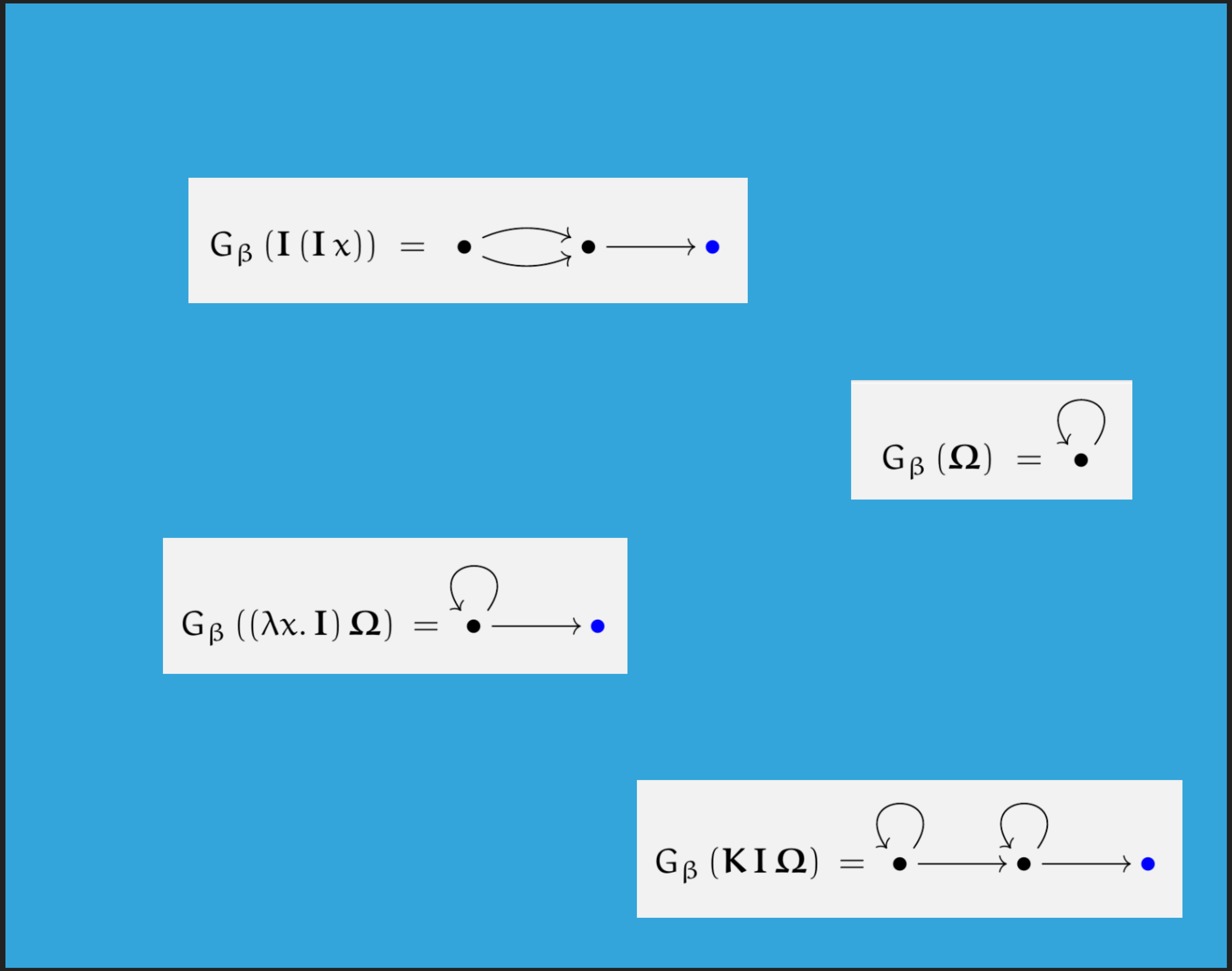
NOT EVERY SEQUENCE OF REDUCTIONS LEADS TO β-NORMAL FORM

$$
\begin{aligned}
\boldsymbol{K}\,\boldsymbol{I}\,\boldsymbol{\Omega} &= \boldsymbol{K}\,\boldsymbol{I}\,((\lambda x\,.\,x\,x)\,(\lambda x\,.\,x\,x)) \\
&\to_\beta \boldsymbol{K}\,\boldsymbol{I}\,((\lambda x\,.\,x\,x)\,(\lambda x\,.\,x\,x)) \\
&\to_\beta \;\cdots
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{K}\,\boldsymbol{I}\,\boldsymbol{\Omega} &= (\lambda xy\,.\,x)\,\boldsymbol{I}\,\boldsymbol{\Omega} \\
&\to_\beta (\lambda y\,.\,\boldsymbol{I})\,\boldsymbol{\Omega} \\
&\to_\beta \boldsymbol{I}
\end{aligned}
$$

GRAPH OF REDUCTIONS OF A TERM M

- Directed multigraph with
  - Vertices: $\{N \mid M \twoheadrightarrow_\beta N\}$
  - Edges: $\to_\beta$

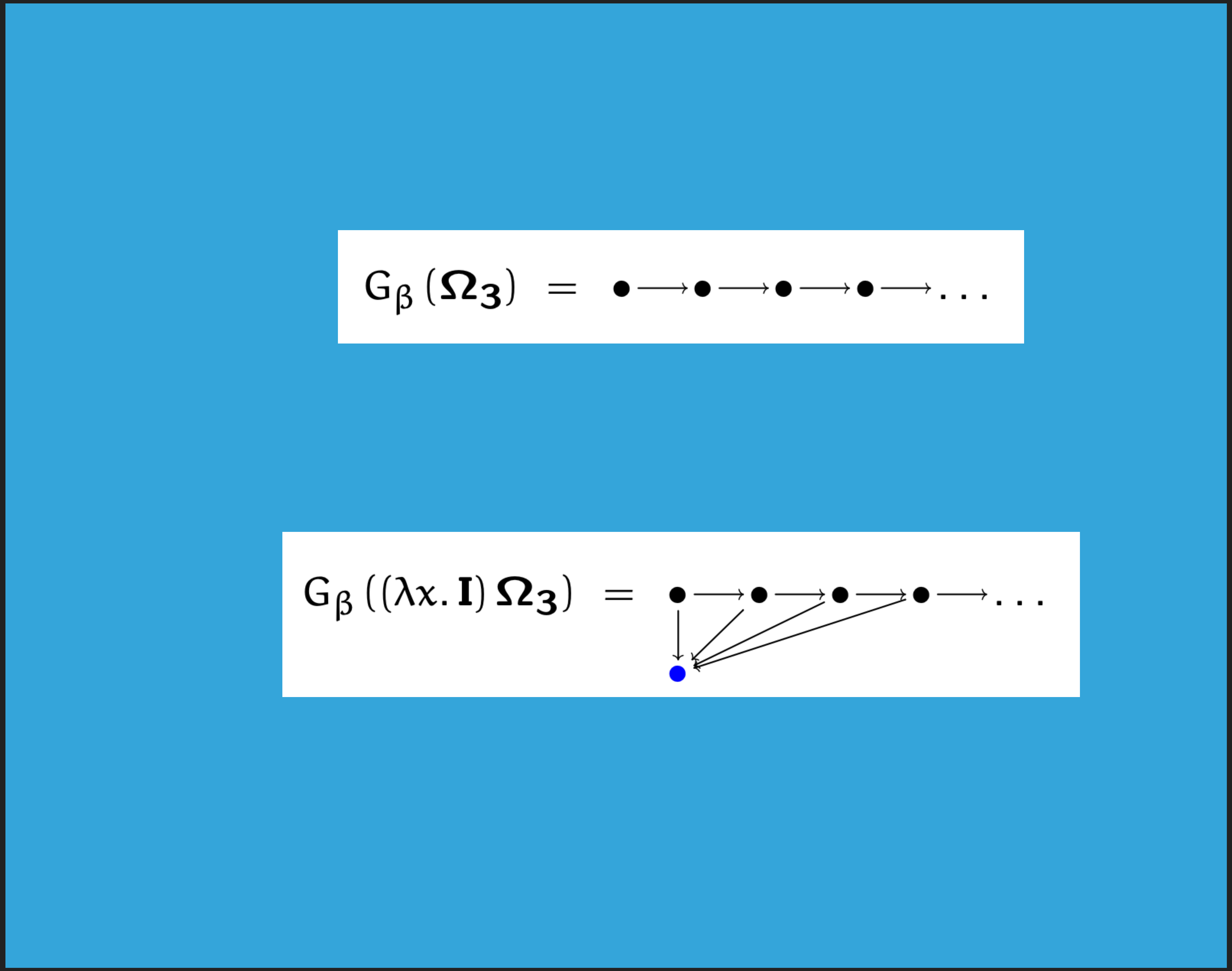


EXERCISE

- What are the graphs of:
  - $\Omega_3 = (\lambda x\,.\,x\,x\,x)\,(\lambda x\,.\,x\,x\,x)$
  - $(\lambda x\,.\,I)\,\Omega_3$

EXERCISE

- What are the graphs of:
  - $\Omega_3 = (\lambda x\,.\,x\,x\,x)\,(\lambda x\,.\,x\,x\,x)$
  - $(\lambda x\,.\,I)\,\Omega_3$
- Note
  - Not every graph is finite
  - Some infinite graphs have β-normal form



- Fixed Point
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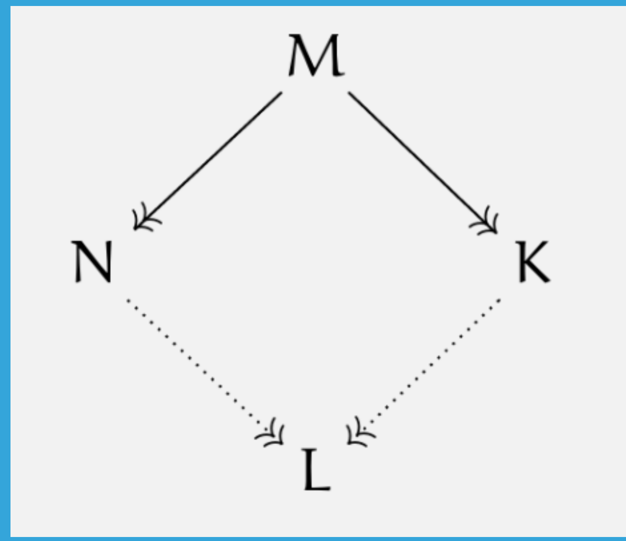
THEOREM

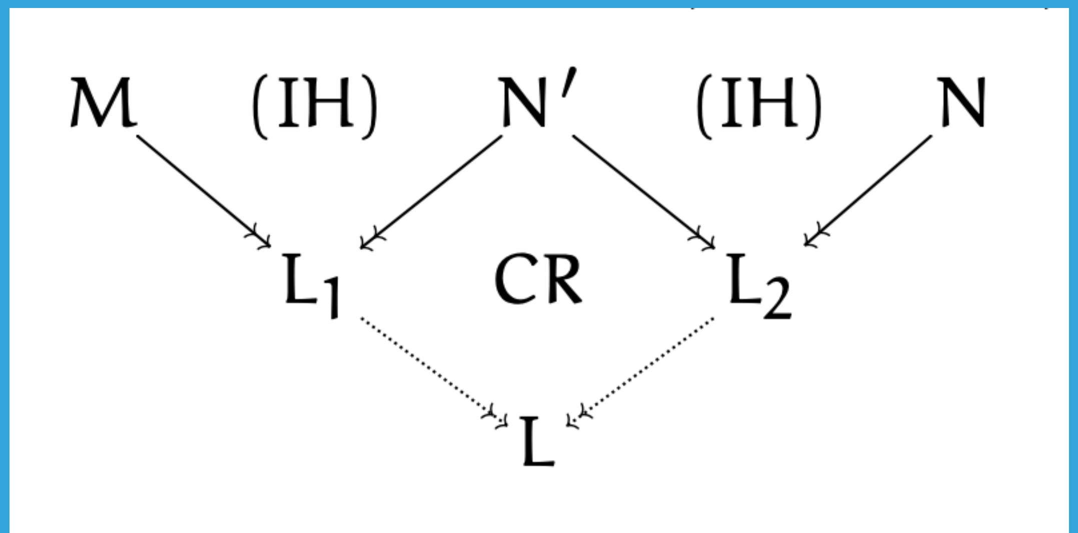
- If $M \twoheadrightarrow_\beta N, M \twoheadrightarrow_\beta K,$ then $\exists L : N \twoheadrightarrow_\beta L, K \twoheadrightarrow_\beta L$
  - Diamond property
  - Confluence



CONSEQUENCE-1

- If $M =_\beta N,$ then $\exists L : N \twoheadrightarrow_\beta L, K \twoheadrightarrow_\beta L$

- Proof: induction by $=_\beta$
  - $M =_\beta N,$ since $M \twoheadrightarrow_\beta N$. Take $L = N$
  - $M =_\beta N,$ since $N \twoheadrightarrow_\beta M$. By the hypothesis, $\exists L_1 : N \twoheadrightarrow_\beta L_1, M \twoheadrightarrow_\beta L_1$. Take $L = L_1$
  - $M =_\beta N,$ since $M =_\beta N', N' =_\beta N$. Then $\longrightarrow$



CONSEQUENCE-2

- If M has N as β-normal form, then $M \twoheadrightarrow_\beta N$
- Proof:
- Let $M =_\beta N, N$ is in β-normal form
- By Consequence-1, $\exists L : M \twoheadrightarrow_\beta L, N \twoheadrightarrow_\beta L$
- By Lemma, $N = L$

Ω doesn't have a normal form:

Let $\Omega \twoheadrightarrow_\beta N, N$ in β-normal form

But $\Omega \twoheadrightarrow_\beta \Omega, \Omega$ is not in β-normal form

CONSEQUENCE-3

- Any λ-term has **at most** one β-normal form
- Proof:
  - Let M have 2 β-normal forms N, N'
  - By consequence-1, $N \twoheadrightarrow_\beta L, N' \twoheadrightarrow_\beta L$
  - By Lemma, $N = L = N'$

Now we can prove inequalitites

$$\lambda xy\,.\,x = true \neq false = \lambda xy\,.\,y$$

They are two different β-normal forms

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HOW TO DO REDUCTIONS?

- Variable x – reduction is done
- Abstraction $\lambda x\,.\,M$ – reduce M
- Application $M\,N$ – reduce M or reduce N
  - $(\ldots((x\,N_1)\,N_2)\ldots N_k)$ – reduce each N_i left-to-right
  - $(\ldots(((\lambda x\,.\,M)\,N_1)\,N_2)\ldots N_k)$ – what to reduce?

NORMAL-ORDER REDUCTION STRATEGY

- $(\ldots(((\lambda x\,.\,M)\,N_1)\,N_2)\ldots N_k)$ – what to reduce?
  - Reduce the redex $(\lambda x\,.\,M)\,N_1$

EXERCISE

- Apply normal-order reduction strategy to the term:
  - $(\lambda xy\,.\,x)((\lambda z\,.\,z)\,5)\,6$
  - Ω

APPLICATIVE-ORDER REDUCTION STRATEGY

- $(\ldots(((\lambda x\,.\,M)\,N_1)\,N_2)\ldots N_k)$ – what to reduce?
  - Reduce N_1 to N_1' – β-normal form
  - Reduce N_2 to N_2' – β-normal form
  - …
  - Reduce N_k to N_k' – β-normal form
  - Reduce the redex $(\lambda x\,.\,M)\,N_1'$

EXERCISE

- Apply applicative-order reduction strategy to the term
  - $(\lambda x\,.\,x)((\lambda z\,.\,z)\,y)$
  - $(\lambda x\,.\,(\lambda y\,.\,y)x)\,((\lambda z\,.\,z)\,(\lambda w\,.\,w))$

HEAD NORMAL FORM

- Any λ-term has either of 2 forms:
  - $\lambda x_0, x_1, \ldots, x_n \,.\, y\, N_0\, N_1 \ldots N_k, k \geq 0, n \geq 0$
  - $\lambda x_0, x_1, \ldots, x_n \,.\, (\lambda z \,.\, M)\, N_0\, N_1 \ldots N_k, k > 0, n \geq 0$
- $\lambda x_0, x_1, \ldots, x_n \,.\, y\, N_0\, N_1 \ldots N_k, k \geq 0, n \geq 0$ – Head Normal Form
- y – Head Variable
- $(\lambda z \,.\, M)\, N_0$ – Head Redex

WEAK HEAD NORMAL FORM

- A λ-term is in WHNF if it's either:
  - $\lambda x_0, x_1, \ldots, x_n \,.\, y\, N_0\, N_1 \ldots N_k,\ k \geq 0, n \geq 0$
  - $\lambda x_0, x_1, \ldots, x_n \,.\, M$ – no redexes
- In Haskell, a constructor is allowed in head position

NORMALIZATION THEOREM

- If a term has a β-normal form, you can get it by reducing the leftmost outermost redex
  - Normal-order reduction always gets to a normal form
  - Applicative-order reduction strategy may not get to the normal form

$$
\begin{aligned}
\boldsymbol{K}\,\boldsymbol{I}\,\boldsymbol{\Omega} &= \boldsymbol{K}\,\boldsymbol{I}\,((\lambda x\,.\,x\,x)\,(\lambda x\,.\,x\,x)) \\
&\rightarrow_\beta \boldsymbol{K}\,\boldsymbol{I}\,((\lambda x\,.\,x\,x)\,(\lambda x\,.\,x\,x)) \\
&\rightarrow_\beta \;\cdots
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{K}\,\boldsymbol{I}\,\boldsymbol{\Omega} &= (\lambda xy\,.\,x)\,\boldsymbol{I}\,\boldsymbol{\Omega} \\
&\rightarrow_\beta (\lambda y\,.\,\boldsymbol{I})\,\boldsymbol{\Omega} \\
&\rightarrow_\beta \boldsymbol{I}
\end{aligned}
$$

PROPERTIES

- Normal-order reduction is effective, but may be not efficient
  - $(\lambda x\,.\,F\,x\,(G\,x)\,x)\,N \rightarrow_\beta F\,N\,(G\,N)\,N$
    - Computes N three times
  - $(\lambda xy\,.\,y)\,N \rightarrow_\beta \lambda y\,.\,y$
    - Computes N zero times
- Applicative-order reduction strategy computes N once in both cases

CALL-BY-...

- Call-by-value: applicative-order reduction (Eager)
- Call-by-name: normal-order reduction
- Call-by-need: NO + let* (Lazy)
  - $square(13+42) \to let\, x = 13+42\, in\, x * x \to let\, x = 55\, in\, x * x \to 55 * 55 \to 3025$