21.02

1.

$$\int \frac{x\,dx}{x^2+x+1} = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{1}{x^2+x+1}dx = \underline{\frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$

$$\int \frac{2x+1}{x^2+x+1}dx = \left| \begin{array}{l} u = x^2+x+1 \\ \frac{du}{dx} = 2x+1 \quad dx = \frac{1}{2x+1}du \end{array} \right| = \int \frac{1}{u}du = \ln u = \underline{\ln(x^2+x+1)}$$

$$\int \frac{1}{x^2+x+1}dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx = \left| u = \frac{2x+1}{\sqrt{3}} \right| =$$

$$= \int \frac{2\sqrt{3}}{3(u^2+1)}du = \frac{2\operatorname{arctg}(u)}{\sqrt{3}} = \underline{\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)}$$

$$\frac{du}{dx} = \frac{2}{\sqrt{3}} \quad dx = \sqrt{3}/2\,du$$

2.

$$\int \frac{x\,dx}{\sqrt{5x^2-2x+1}} = \int \frac{\frac{1}{10}(10x-2)+\frac{1}{5}}{\sqrt{5x^2-2x+1}}dx = \frac{1}{10}\int \frac{10x-2}{\sqrt{5x^2-2x+1}}dx + \frac{1}{5}\int \frac{dx}{\sqrt{5x^2-2x+1}}$$

$$= \frac{\sqrt{5x^2-2x+1}}{5} + \frac{\ln\left(\sqrt{\frac{(5x-1)^2}{4}+1} + \frac{5x-1}{2}\right)}{5\sqrt{5}} + C$$

$$\int \frac{10x-2}{\sqrt{5x^2-2x+1}}dx = \left| \begin{array}{l} u = 5x^2-2x+1 \\ \frac{du}{dx} = 10x-2 \quad dx = \frac{1}{10x-2}du \end{array} \right| = \int \frac{1}{\sqrt{u}}du = 2\sqrt{u} = 2\sqrt{5x^2-2x+1}$$

$$\int \frac{dx}{\sqrt{5x^2-2x+1}} = \int \frac{dx}{\left(\sqrt{5}x - \frac{1}{\sqrt{5}}\right)^2 + \frac{4}{5}} = \left| u = \frac{5x-1}{2} \right|$$

$$= \int \frac{2\,du}{5\sqrt{\frac{4}{5}u^2 + \frac{4}{5}}} = \frac{1}{\sqrt{5}} \int \frac{1}{\sqrt{u^2+1}}\, du =$$

$$\frac{du}{dx} = \frac{5}{2} \quad dx = \frac{2}{5} du \Bigg| = \frac{\ln\left(\sqrt{\frac{(5x-1)^2}{4}+1} + \frac{5x-1}{2}\right)}{\sqrt{5}}$$

3. $\int \frac{3x^2+2x-3}{x^3-x}\, dx = \int \frac{3x^2+2x-3}{(x-1)(x+1)x}\, dx = 3\int \frac{dx}{x} +$
$\int \frac{dx}{x-1} - \int \frac{dx}{x+1} = 3\ln|x| + \ln|x-1| -$
$\ln|x+1| + C$

$$\frac{3x^2+2x-3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$3x^2+2x-3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$$3x^2+2x-3 = Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$$

$$3x^2+2x-3 = (A+B+C)x^2 + (B-C)x + (-A)$$

$$\begin{cases} A+B+C = 3 \\ B-C = 2 \\ A = 3 \end{cases} \Leftrightarrow \begin{cases} B = -C \\ 2B = 2 \\ A = 3 \end{cases} \Leftrightarrow \begin{cases} C = -1 \\ B = 1 \\ A = 3 \end{cases}$$

$$4.\ \int \frac{dx}{x^4+1} = \int \frac{dx}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)} =$$

$$= \frac{1}{2\sqrt{2}} \int \frac{x+\sqrt{2}}{(x^2+\sqrt{2}x+1)} - \frac{1}{2\sqrt{2}} \int \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\, dx =$$

$$= \frac{1}{2\sqrt{2}} \left(\frac{1}{2} \ln(x^2+x\sqrt{2}+1) + \operatorname{arctg}(x\sqrt{2}+1) - \frac{1}{2} \ln(x^2-x\sqrt{2}+1) + \right.$$

$$\left. + \operatorname{arctg}(x\sqrt{2}-1) + \mathbb{C} \right.$$

$$\int \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} dx = \frac{1}{2} \int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} dx + \frac{1}{\sqrt{2}} \int \frac{dx}{x^2+\sqrt{2}x+1} =$$

$$= \frac{1}{2} \ln(x^2+x\sqrt{2}+1) + \operatorname{arctg}(x\sqrt{2}+1)$$

$$\int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} dx = |u = x^2+\sqrt{2}x+1| = \int \frac{1}{u} du = \ln(u)$$

$$= \ln(x^2+\sqrt{2}x+1)$$

$$\int \frac{dx}{x^2+\sqrt{2}x+1} = \int \frac{dx}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}} = |u = \sqrt{2}x+1| =$$

$$= \sqrt{2} \int \frac{du}{u^2+1} = \sqrt{2} \operatorname{arctg}(\sqrt{2}x+1)$$

$$\int \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1} dx = \frac{1}{2} \int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1} dx - \frac{1}{\sqrt{2}} \int \frac{dx}{x^2-\sqrt{2}x+1} =$$

$$= \frac{\ln(x^2-x\sqrt{2}+1)}{2} - \operatorname{arctg}(x\sqrt{2}-1)$$

$$\int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1} dx = |u = x^2-\sqrt{2}x+1| = \int \frac{1}{u} du = \ln(x^2-\sqrt{2}x+1)$$

$$\int \frac{dx}{x^2-\sqrt{2}x+1} = \int \frac{dx}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}} = |u = \sqrt{2}x-1| =$$

$$= \sqrt{2} \int \frac{1}{u^2+1} du = \sqrt{2} \operatorname{arctg}(\sqrt{2}x-1)$$

5. $\int \frac{x^3+x-1}{(x^2+2)^2}\,dx = \int\left(\frac{-x-1}{(x^2+2)^2}+\frac{x}{x^2+2}\right)dx = \int \frac{x}{x^2+2}\,dx -$

$-\int \frac{x+1}{(x^2+2)^2}\,dx = \frac{\ln|x^2+2|}{2} + \frac{x-2}{4(x^2+2)} + \frac{\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right)}{4\sqrt{2}} + C$

$$\frac{Ax+B}{(x^2+2)^2}+\frac{Cx+D}{x^2+2}=\frac{x^3+x-1}{(x^2+2)^2}$$

$$Ax+B+Cx^3+2Cx+Dx^2+2D=x^3+x-1$$

$$Cx^3+Dx^2+(A+2C)x+(B+2D)=x^3+x-1$$

$$\begin{cases} C=1 \\ D=0 \\ A+2C=1 \\ B+2D=-1 \end{cases} \Longleftrightarrow \begin{cases} C=1 \\ D=0 \\ A=-1 \\ B=-1 \end{cases}$$

$$\int \frac{x}{x^2+2}\,dx = |u=x^2+2| = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|x^2+2|$$

$$\int \frac{x+1}{(x^2+2)^2}\,dx = \int \frac{x}{(x^2+2)^2}\,dx + \int \frac{dx}{(x^2+2)^2} = \frac{x-2}{4(x^2+2)} + \frac{\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right)}{4\sqrt{2}}$$

$$\int \frac{x}{(x^2+2)^2} = |u=x^2+2| = \frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2u} = -\frac{1}{2(x^2+2)}$$

$$\int \frac{dx}{(x^2+2)^2} = \frac{x}{4(x^2+2)} + \frac{1}{4}\int \frac{dx}{x^2+2} = \frac{x}{4(x^2+2)} + \frac{\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right)}{4\sqrt{2}}$$