Simple Linear Regression

Prof Wells

STA 295: Stat Learning

February 6th, 2024

Outline

In today's class, we will. . .

- Discuss theoretical foundation for linear regression
- Perform inference for simple linear models
- Implement simple linear regression in R

Section 1

Foundations

Linear Regression

- Suppose we have one or more predictors $(X_1, X_2, \ldots, X_p)$ and a *quantitative* response variable Y, and that

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- We'll use **Simple Linear Regression** (SLR) to build intuition about all linear models

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- So we are **estimating** an **approximation** to a relationship between response and predictors.

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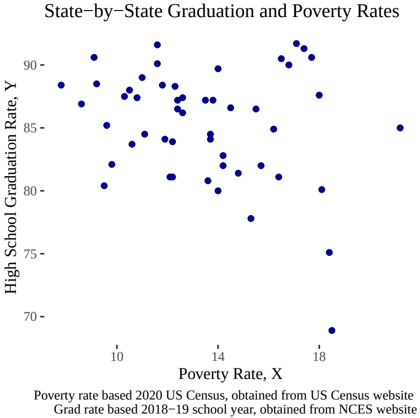

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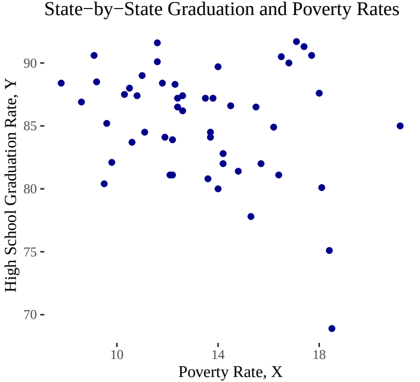

Poverty rate based 2020 US Census, obtained from US Census website
Grad rate based 2018−19 school year, obtained from NCES website

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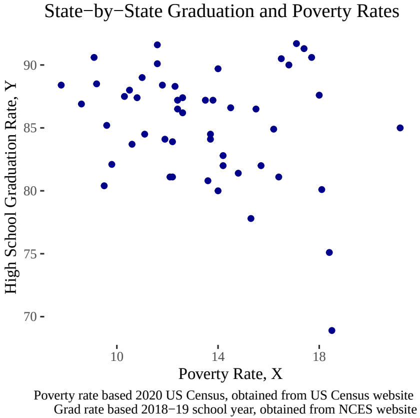

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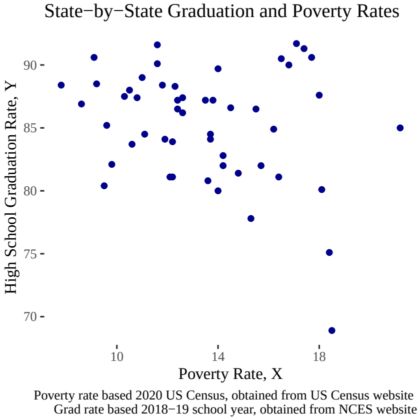

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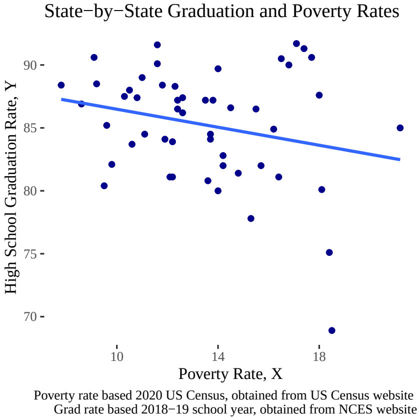

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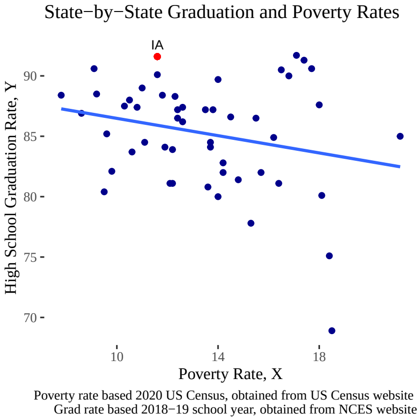

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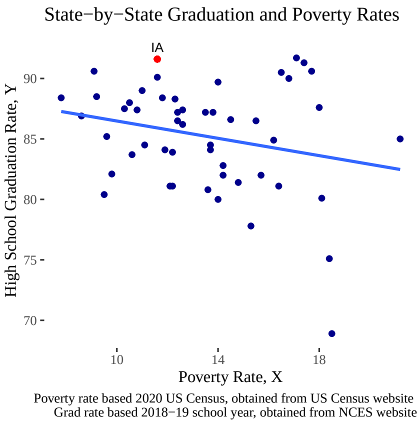

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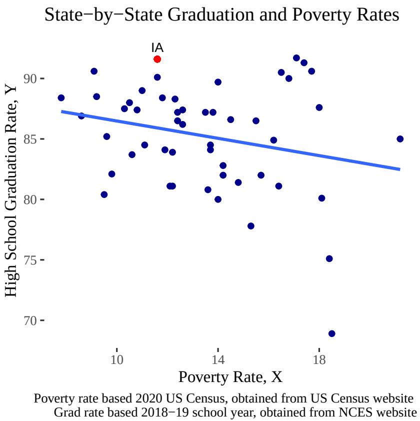

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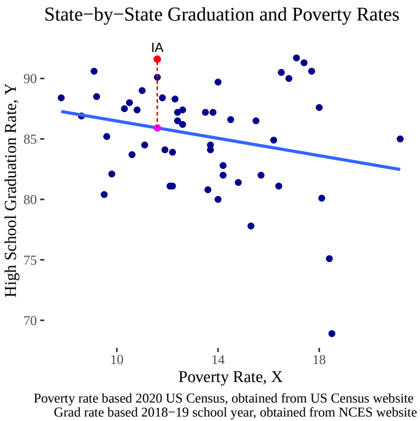

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But Iowa's actual graduation rate is 91.6

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- **Residuals** are the leftover variation in the data after accounting for model fit.
- Each observation (X_i, Y_i) has its own residual e_i, which is the difference between the observed (Y_i) and predicted $(\hat{Y}_i)$ value:

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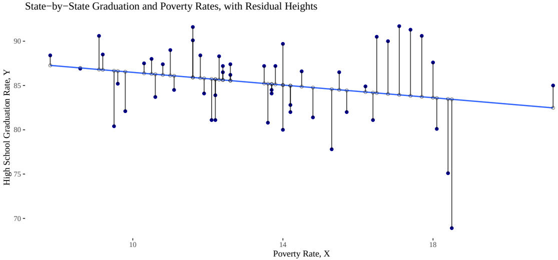

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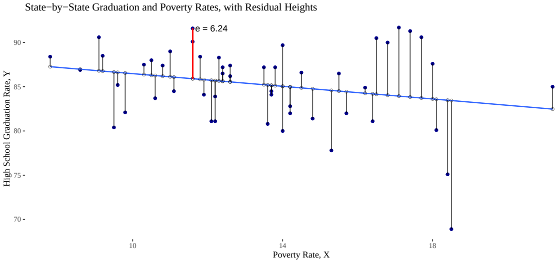
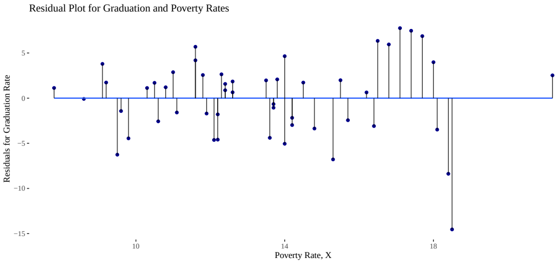


- Iowa's residual is

$$e = Y - \hat{Y} = 91.6 - 85.36 = 6.24$$

Residual Plot

- To visualize the degree of accuracy of a linear model, we use residual plots:



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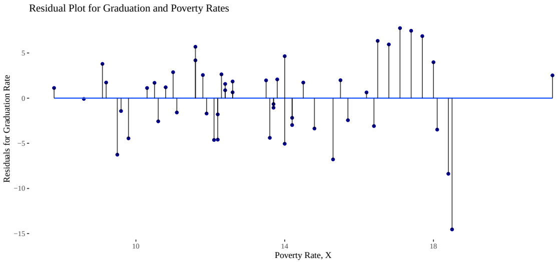

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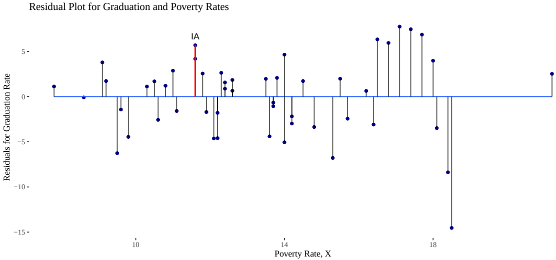

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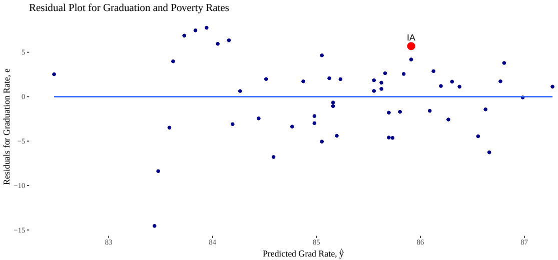

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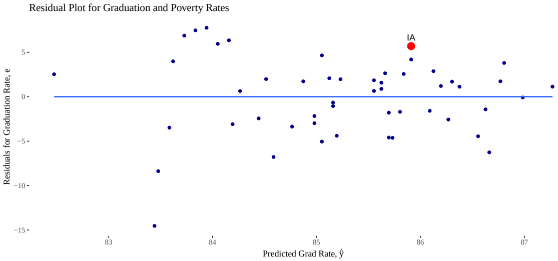

- This residual plot can still be used to determine accuracy of model, but can be used when we have more than 1 predictor.

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  - No, as we will see later with *penalized regression* (Ch 6, ISLR)

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- R^2 is the proportion of variation in the response explained by the model.

Section 2

Inference for Linear Models

Statistical Inference

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- **Tools**: confidence intervals, hypothesis tests
- **The Problems**: Our model will change if built using a different random sample. So in addition to estimates, we need to know about variability

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  - i.e. if $\hat{\theta}$ is approximately Normally distributed and $C = .95$, then $t_C^* \approx 2$.
- The value $\mathrm{SE}(\hat{\theta})$ is the standard error of $\hat{\theta}$, or the standard deviation of the sampling distribution

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If one or more of these conditions do not hold, our predictions may not be accurate and we should be skeptical of inferential claims.

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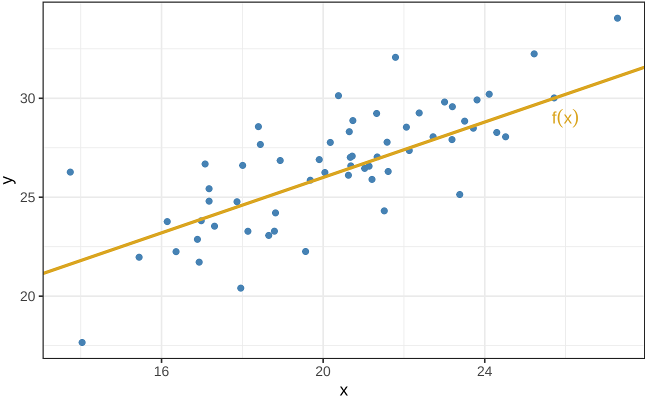
Simulated Data from true model



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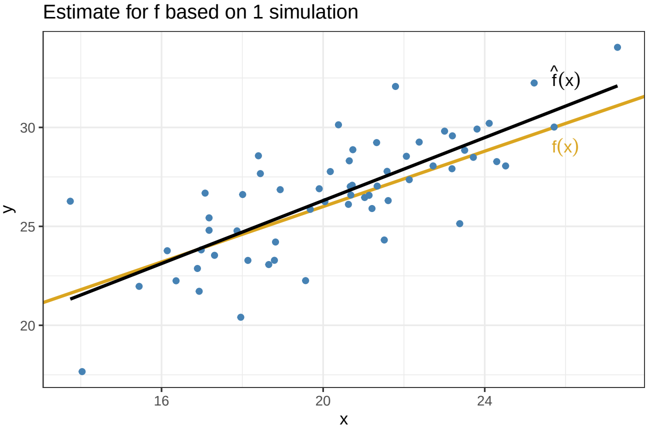

The Sampling Distribution of $\hat{\beta}_1$

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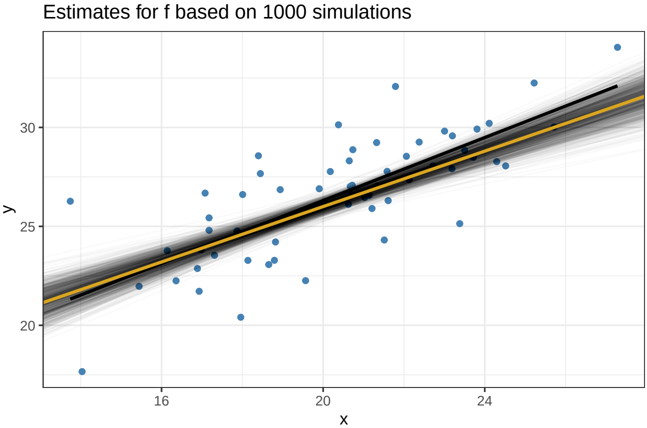

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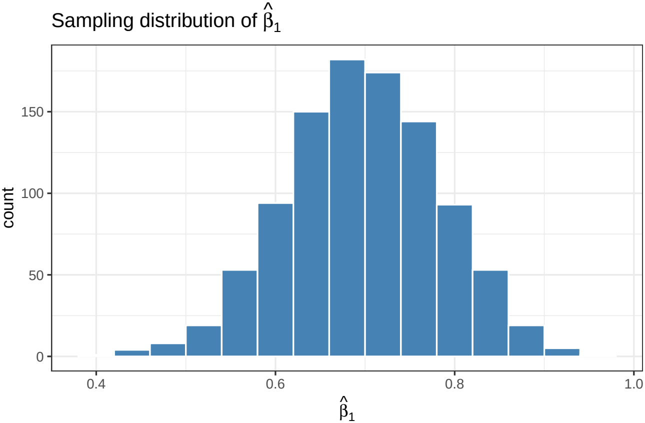

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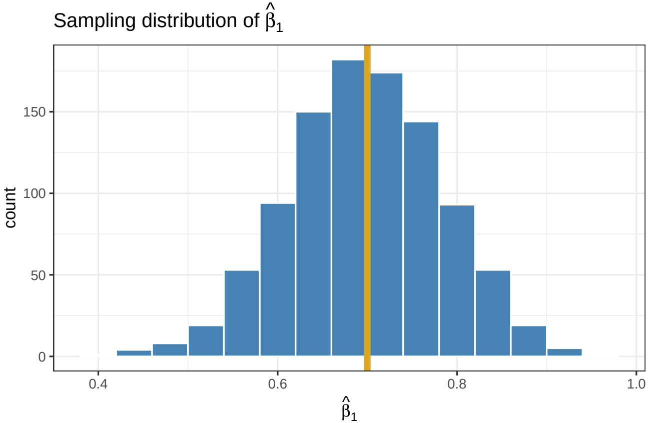

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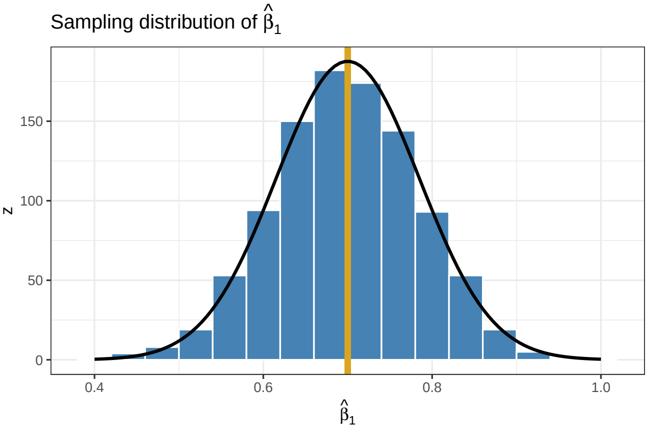

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3. $\hat{\beta}_1 | X \sim N(\beta_1, \frac{\sigma^2}{S_{XX}})$.

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Interpretation We are *95% confident* that the true slope relating x and y lies between lower and upper bound of this interval.

Hypothesis test for $\hat{\beta}_1$

Suppose we are interested in testing the claim that the slope is zero.

$$H_0 : \beta_1^0 = 0 \qquad \text{vs} \quad H_A : \beta_1^0 \neq 0$$

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- The p-value for an observed test statistic t is the probability that a randomly chosen value from the t-dist is larger in absolute value than $|t|$.
- An observed t with p-value less than a desired significance level (often $\alpha = 0.05$) gives good evidence against the null-hypothesis.

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  - For details, see DeGroot and Schervish "Probability and Statistics" (or take STA 336)

Section 3

Linear Models in R

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```
## # A tibble: 6 x 3
##   state      Graduates Poverty
##   <chr>          <dbl>   <dbl>
## 1 Alabama         91.7    17.1
## 2 Alaska          80.4     9.5
## 3 Arizona         77.8    15.3
## 4 Arkansas        87.6    18
## 5 California      84.5    13.7
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```

- We fit a linear model using the `lm` function in R:

```
poverty_mod <- lm(Graduates ~ Poverty, data = poverty)
```

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  - We can obtain a high-level summary of the model using summary()

```
summary(poverty_mod)
```

```
##
## Call:
## lm(formula = Graduates ~ Poverty, data = poverty)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -14.541  -2.774   0.876   2.543   7.758
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  90.0615     2.8347  31.772   <2e-16 ***
## Poverty      -0.3579     0.2056  -1.741    0.088 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 4.396 on 49 degrees of freedom
## Multiple R-squared:  0.05823,    Adjusted R-squared:  0.03901
## F-statistic:  3.03 on 1 and 49 DF,  p-value: 0.08802
```

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mod_summary <- summary(poverty_mod)
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```

```
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```

```
mod_summary$sigma
```

```
## [1] 4.395734
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```

```
poverty_mod$residuals
```

```
##          1          2          3          4          5          6
##  7.7579486 -6.2618192 -6.7862069  3.9800264 -0.6587896 -4.5955859
```