Class Activity 12-5

1. Multivariate Normal distributions have a number of theoretical properties, which is one reason they are frequently used as models for multivariate phenomena. The joint density function for the bivariate Normal distribution where X and Y have $N(0,1)$ marginal distribution and correlation ρ is given by:

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right)$$

where $\exp(z) = e^z$ is used to improve readability.

   (a) Show **using the formula for the joint PDF** that if X and Y are uncorrelated, then X and Y are independent.

   (b) Show that indeed the marginal distribution of X and of Y is $N(0,1)$. To do so...

      i. Verify the following equality (completing the square in terms of x):

$$x^2 - 2\rho xy + y^2 = (x - \rho y)^2 + (1-\rho^2)y^2$$

      ii. Make an appropriate substitution in the exponential, perform an appropriate integration and then simplify.

   (c) Show that for **every** value of x, the conditional density of Y given $X = x$ is Normally distributed. (A similar fact true for X given $Y = y$, and the proof is analogous).

   (d) Use the preceding result to find formula for $E[Y|X=x]$ and $\text{Var}(Y|X=x)$. Does the conditional expectation of Y depend on the value of X? Does the conditional variance of Y depend on the value of X?

   (e) Using the results from the previous part, show that $\text{Var}(Y) = 1$ using Law of Total Variance.