Dynamics

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September 10, 2020

[0] This lecture will be recorded. By attending the live sessions, you agree to the recording, and you understand that your image, voice, and name may be disclosed to classmates.

Welcome

Lecture Recordings This lecture will be recorded. By attending the live sessions, you agree to the recording, and you understand that your image, voice, and name may be disclosed to classmates

During the lecture
I will mute the audience
Ask questions through 'Chat'

Kinematics
Describe state and geometric path

What is Dynamics

Kinematics
Describe state and geometric path

Dynamics
How much torques to apply so we can achieve the desired task?
How does a system move as a reaction of torques?

Why do we care about Dynamics

Forward Dynamics (for forward simulation)

- Given
  - $\mathbf{q}, \dot{\mathbf{q}}$: current positions and velocities
  - τ: torques applied at each motor
- Find
  - $\ddot{\mathbf{q}}$: resulting accelerations

$$\ddot{\mathbf{q}} = f(\mathbf{q}, \dot{\mathbf{q}}, \tau)$$

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Inverse Dynamics (for control)

- Given
  - $\mathbf{q}, \dot{\mathbf{q}}$: positions, velocities
  - $\ddot{\mathbf{q}}^*$: accelerations to achieve the desired task
- Find
  - τ: torques needed to achieve the desired task

$$\tau = g(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}^*)$$

Outline

Rigid-Body Dynamics

Invers Dynamics Controller

Task-space Dynamics

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Rigid-Body Dynamics

Invers Dynamics Controller

Task-space Dynamics

Dynamics

- $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}} \in \mathbb{R}^{\mathcal{N}}$: positions, velocities, accelerations
- $\boldsymbol{\tau} \in \mathbb{R}^{\mathcal{N}}$: torques

Rigid-body Dynamics

Dynamics

- $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}} \in \mathbb{R}^{\mathcal{N}}$: positions, velocities, accelerations
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Equation of Motion

Dynamics

- $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}} \in \mathbb{R}^{\mathcal{N}}$: positions, velocities, accelerations
- $\tau \in \mathbb{R}^{\mathcal{N}}$: torques

Equation of Motion

$$\tau(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q})$$

Rigid-body Dynamics

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where $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$: masss matrix

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$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$: centrifugal and Coriolis force

Rigid-body Dynamics

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$\mathbf{g}(\mathbf{q}) \in \mathbb{R}^{\mathcal{N}}$: gravity vector

Dynamics

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$\mathbf{g}(\mathbf{q}) \in \mathbb{R}^{\mathcal{N}}$: gravity vector

(shorter version) $\tau = \mathbf{M}\ddot{\mathbf{q}} + \mathbf{h}$

where $\mathbf{h} \in \mathbb{R}^{\mathcal{N}} = \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}$

What are $\mathbf{M}(\mathbf{q}), \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}), \mathbf{g}(\mathbf{q})$?
Again, most robotics libraries provide functions for these values

Inverse Dynamics (for control)

- Given
  - $\mathbf{q}, \dot{\mathbf{q}}$: current positions, velocities
  - $\ddot{\mathbf{q}}^*$: desired accelerations to achieve the desired task
- Find
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$$\tau = \mathbf{M}\ddot{\mathbf{q}}^* + \mathbf{h}$$

Forward Dynamics (for forward simulation)

- Given
  - $\mathbf{q}, \dot{\mathbf{q}}$: current positions and velocities
  - τ: torques applied on the system
- Find
  - $\ddot{\mathbf{q}}$: expected accelerations after applying τ

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\tau - \mathbf{h})$$

Kinetic Energy

The work needed to move a body with mass m from rest to its stated linear velocity $\dot{\mathbf{p}}$ and angular velocity ω

$$\mathsf{K}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m \dot{\mathbf{p}}^2 + \frac{1}{2} \omega^\top I \omega$$

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(Gravitational) Potential Energy
Energy held by an object becuase of its position relative to other object (e.g., gravitational field)

$$\mathsf{P}(\mathbf{q}) = mgh$$

Total energy

Lagrangian function

$$\mathsf{L}(\mathbf{q},\dot{\mathbf{q}}) = \mathsf{K}(\mathbf{q},\dot{\mathbf{q}}) - \mathsf{P}(\mathbf{q},\dot{\mathbf{q}})$$

Deriving Rigid-body Dynamics

Total energy

Lagrangian function

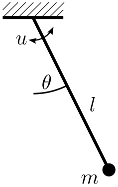
$$\mathsf{L}(\mathbf{q},\dot{\mathbf{q}}) = \mathsf{K}(\mathbf{q},\dot{\mathbf{q}}) - \mathsf{P}(\mathbf{q},\dot{\mathbf{q}})$$

The euler-lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = \tau$$

Deriving Rigid-body Dynamics

Example (2D Pendulum)

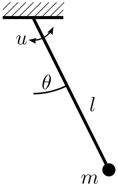


configuration θ

mass m

length l

Example (2D Pendulum)



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Position and Velocity

$$\mathbf{x} = \begin{bmatrix} l\sin(\theta) \\ l\cos(\theta) \end{bmatrix} \qquad \dot{\mathbf{x}} = \begin{bmatrix} l\dot{\theta}\cos(\theta) \\ l\dot{\theta}\sin(\theta) \end{bmatrix}$$

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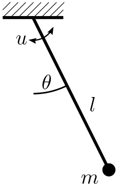

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Kinetic energy $\mathsf{K}(\theta, \dot{\theta}) = \frac{1}{2}m\dot{\mathbf{x}}^2 = \frac{1}{2}ml^2\dot{\theta}^2$

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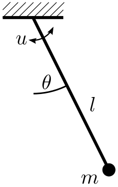

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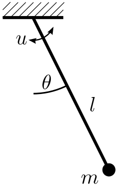

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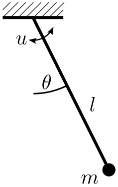

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$$\frac{\partial \mathbf{L}(\theta, \dot{\theta})}{\partial \dot{\theta}} = m l^2 \dot{\theta},$$

Deriving Rigid-body Dynamics

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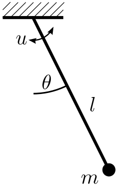

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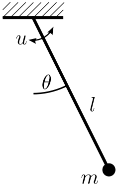

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$$\tau = \frac{d}{dt}\frac{\partial \mathbf{L}(\theta,\dot{\theta})}{\partial \dot{\theta}} - \frac{\partial \mathbf{L}(\theta,\dot{\theta})}{\partial \theta} = ml^2\ddot{\theta} + mgl\sin(\theta)$$

Outline

Rigid-Body Dynamics

Invers Dynamics Controller

Task-space Dynamics

Inverse Kinematic Control

- Inputs: desired end-effector velocity
- Outputs: joint velocity

Kinematic v.s. Dynamic Control

Inverse Kinematic Control

- Inputs: desired end-effector velocity
- Outputs: joint velocity

Inverse Dynamic Control

- Inputs: current position, current velocity, desired acceleration
- Outputs: joint torque

Inverse Dynamic Control

Question: How could we find the desired accelerations?

Assuming that we have a nice and smooth trajectory $\mathbf{q}_1^{ref}, \mathbf{q}_2^{ref}, \ldots, \mathbf{q}_T^{ref}$

At each time step t, we can find desired accelerations $\ddot{\mathbf{q}}_t^{ref}$

$$\dot{\mathbf{q}}_t^{ref} = \frac{\mathbf{q}_t^{ref} - \mathbf{q}_{t-1}^{ref}}{\triangle t} \text{ my desired velocity}$$

$$\ddot{\mathbf{q}}_t^{ref} = \frac{\dot{\mathbf{q}}_t^{ref} - \dot{\mathbf{q}}_{t-1}^{ref}}{\triangle t} \text{ my desired accelerations}$$

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No, this is unstable

No, in practice, errors will accumulate over time.

We need to have to some **feedback**

Desired Accelerations with Feedback

$$\ddot{\mathbf{q}}_t^* = \ddot{\mathbf{q}}_t^{ref} + \mathbf{D}(\dot{\mathbf{q}}^{ref} - \dot{\mathbf{q}}) + \mathbf{K}(\mathbf{q}^{ref} - \mathbf{q})$$

$\mathbf{K} \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$: stiffness matrix

$\mathbf{D} \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$: damping matrix

Inverse Dynamic Control

Desired Accelerations with Feedback

$$\ddot{\mathbf{q}}_t^* = \ddot{\mathbf{q}}_t^{ref} + \mathbf{D}(\dot{\mathbf{q}}^{ref} - \dot{\mathbf{q}}) + \mathbf{K}(\mathbf{q}^{ref} - \mathbf{q})$$

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$\mathbf{D} \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$: damping matrix

This is a standard!

What is Stiffness?

Stiffness K
A matrix of coefficient that specifies how hard the robot should react to position errors

Low Stiffness

High Stiffness

What is Damping?

Damping D
A matrix of coefficient that specifies how hard the robot should react to velocity errors

Low Stiffness High Stiffness

What are the values of Stiffness and Damping?

Assuming that we have only 1 joint

What are K and D such that

$$\mathbf{M}(\ddot{\mathbf{q}}^{ref} - \ddot{\mathbf{q}}) + \mathbf{D}(\dot{\mathbf{q}}^{ref} - \dot{\mathbf{q}}) + \mathbf{K}(\mathbf{q}^{ref} - \mathbf{q}) = \mathbf{0}$$

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Choose your stiffness K, by tuning

What are the values of Stiffness and Damping?

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Choose your stiffness $\mathbf{K}$, by tuning
Choose your damping $\mathbf{D}$ by

$$\mathbf{D} = 2\sqrt{\mathbf{K}\mathbf{M}}$$

Natural frequency of a system

What are the values of Stiffness and Damping?

Assuming that we have only 1 joint

What are $\mathbf{K}$ and $\mathbf{D}$ such that

$$\mathbf{M}(\ddot{\mathbf{q}}^{ref} - \ddot{\mathbf{q}}) + \mathbf{D}(\dot{\mathbf{q}}^{ref} - \dot{\mathbf{q}}) + \mathbf{K}(\mathbf{q}^{ref} - \mathbf{q}) = \mathbf{0}$$

Choose your stiffness $\mathbf{K}$, by tuning
Choose your damping $\mathbf{D}$ by

$$\mathbf{D} = 2\sqrt{\mathbf{KM}}$$

Natural frequency of a system

In practice, it is a <u>trial and error</u> process

Let’s try

Outline

Rigid-Body Dynamics

Invers Dynamics Controller

Task-space Dynamics

Question: Given Task-space trajectory, how to find the torques?
Can we describe the equation of motion in terms of the task-space positions, velocities, and accelerations

Task-space Dynamics

Task space

$$\mathbf{x} \in \mathbb{R}^6 = \begin{bmatrix} \boldsymbol{\theta} \\ \mathbf{p} \end{bmatrix}, \quad \dot{\mathbf{x}} \in \mathbb{R}^6 = \begin{bmatrix} \boldsymbol{\omega} \\ \dot{\mathbf{p}} \end{bmatrix}, \quad \ddot{\mathbf{x}} \in \mathbb{R}^6 = \begin{bmatrix} \dot{\boldsymbol{\omega}} \\ \ddot{\mathbf{p}} \end{bmatrix}$$

Task space

$$\mathbf{x} \in \mathbb{R}^6 = \begin{bmatrix} \boldsymbol{\theta} \\ \mathbf{p} \end{bmatrix}, \quad \dot{\mathbf{x}} \in \mathbb{R}^6 = \begin{bmatrix} \boldsymbol{\omega} \\ \dot{\mathbf{p}} \end{bmatrix}, \quad \ddot{\mathbf{x}} \in \mathbb{R}^6 = \begin{bmatrix} \dot{\boldsymbol{\omega}} \\ \ddot{\mathbf{p}} \end{bmatrix}$$

Spatial Force/ Wrench

$$\mathbf{F} \in \mathbb{R}^6 = \begin{bmatrix} \mathbf{m} \\ \mathbf{f} \end{bmatrix}$$

$\mathbf{m} \in \mathbb{R}^3$ angular force / moment

$\mathbf{f} \in \mathbb{R}^3$ linear force

Relationship with τ

$$\begin{cases} \tau = \mathbf{J}^\top \mathbf{F} \\ \mathbf{F} = (\mathbf{J}^\top)^\dagger \tau \end{cases}$$

Task-space Dynamics

Question: Can we describe the dynamics in terms of f and $\ddot{x}$?

Task-space Dynamics

Configuration $\rightarrow$ Task-space

$$\dot{\mathsf{x}} = \mathsf{J}\dot{\mathsf{q}}$$

$$\ddot{\mathsf{x}} = \mathsf{J}\ddot{\mathsf{q}} + \dot{\mathsf{J}}\dot{\mathsf{q}}$$

Task-space Dynamics

Configuration $\rightarrow$ Task-space

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$$

$$\ddot{\mathbf{x}} = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}}$$

Task-space $\rightarrow$ Configuration

$$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}\dot{\mathbf{x}}$$

$$\ddot{\mathbf{q}} = \mathbf{J}^{\dagger}\ddot{\mathbf{x}} - \mathbf{J}^{\dagger}\dot{\mathbf{J}}\dot{\mathbf{q}}$$

Task-space Dynamics

Configuration $\rightarrow$ Task-space

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$$

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Task-space $\rightarrow$ Configuration

$$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}\dot{\mathbf{x}}$$

$$\ddot{\mathbf{q}} = \mathbf{J}^{\dagger}\ddot{\mathbf{x}} - \mathbf{J}^{\dagger}\dot{\mathbf{J}}\dot{\mathbf{q}}$$

Substitute $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ into the equation of motion $\tau = \mathbf{M}\ddot{\mathbf{q}} + \mathbf{h}$

$$\tau = \mathbf{M}(\mathbf{J}^{\dagger}\ddot{\mathbf{x}} - \mathbf{J}^{\dagger}\dot{\mathbf{J}}\dot{\mathbf{q}}) + \mathbf{h}$$

Configuration → Task-space

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$$

$$\ddot{\mathbf{x}} = \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}}$$

Task-space → Configuration

$$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}\dot{\mathbf{x}}$$

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Substitute $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ into the equation of motion $\tau = \mathbf{M}\ddot{\mathbf{q}} + \mathbf{h}$

$$\tau = \mathbf{M}(\mathbf{J}^{\dagger}\ddot{\mathbf{x}} - \mathbf{J}^{\dagger}\dot{\mathbf{J}}\dot{\mathbf{q}}) + \mathbf{h}$$

Premultiply both sides by $(\mathbf{J}^{\top})^{\dagger}$, we get

$$(\mathbf{J}^{\top})^{\dagger}\tau = (\mathbf{J}^{\top})^{\dagger}\mathbf{M}\mathbf{J}^{\dagger}\ddot{\mathbf{x}} - (\mathbf{J}^{\top})^{\dagger}\mathbf{M}\mathbf{J}^{\dagger}\dot{\mathbf{J}}\dot{\mathbf{q}} + (\mathbf{J}^{\top})^{\dagger}\mathbf{h}$$

Task-space Dynamics

$$(\mathbf{J}^\top)^\dagger \tau = (\mathbf{J}^\top)^\dagger \mathbf{M}\mathbf{J}^\dagger \ddot{\mathbf{x}} - (\mathbf{J}^\top)^\dagger \mathbf{M}\mathbf{J}^\dagger \dot{\mathbf{J}}\dot{\mathbf{q}} + (\mathbf{J}^\top)^\dagger \mathbf{h}$$

Task-space Dynamics

$$(\mathbf{J}^\top)^\dagger \tau = (\mathbf{J}^\top)^\dagger \mathbf{M}\mathbf{J}^\dagger \ddot{\mathbf{x}} - (\mathbf{J}^\top)^\dagger \mathbf{M}\mathbf{J}^\dagger \dot{\mathbf{J}}\dot{\mathbf{q}} + (\mathbf{J}^\top)^\dagger \mathbf{h}$$

Expressing $\mathbf{F} = (\mathbf{J}^\top)^\dagger \tau$

Task-space Dynamics

$$(\mathbf{J}^\top)^\dagger \tau = (\mathbf{J}^\top)^\dagger \mathbf{M}\mathbf{J}^\dagger \ddot{\mathbf{x}} - (\mathbf{J}^\top)^\dagger \mathbf{M}\mathbf{J}^\dagger \dot{\mathbf{J}}\dot{\mathbf{q}} + (\mathbf{J}^\top)^\dagger \mathbf{h}$$

Expressing $\mathbf{F} = (\mathbf{J}^\top)^\dagger \tau$

Dynamics equation expressed in taskspace

$$\mathbf{F} = \mathbf{\Lambda}\ddot{\mathbf{x}} + \eta$$

$$\text{where } \mathbf{\Lambda} = (\mathbf{J}^\top)^\dagger \mathbf{M}\mathbf{J}^\dagger$$

$$\eta = (\mathbf{J}^\top)^\dagger \mathbf{h} - \mathbf{\Lambda}\dot{\mathbf{J}}\dot{\mathbf{q}}$$

Task-space Inverse Dynamics Controller

from our plannar: $\mathbf{x}^{ref}, \dot{\mathbf{x}}^{ref}, \ddot{\mathbf{x}}^{ref}$

Task-space Inverse Dynamics Controller

from our plannar: $\mathbf{x}^{ref}, \dot{\mathbf{x}}^{ref}, \ddot{\mathbf{x}}^{ref}$
from forward kineamtics: $\mathbf{x}, \dot{\mathbf{x}}$

Task-space Inverse Dynamics Controller

from our plannar: $\mathbf{x}^{ref}, \dot{\mathbf{x}}^{ref}, \ddot{\mathbf{x}}^{ref}$
from forward kineamtics: $\mathbf{x}, \dot{\mathbf{x}}$

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$$\ddot{\mathbf{x}}^* = \ddot{\mathbf{x}}^{ref} + \mathbf{D}(\dot{\mathbf{x}}^{ref} - \dot{\mathbf{x}}) + \mathbf{K}(\mathbf{x}^{ref} - \mathbf{x})$$

with feedback

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$$\mathbf{\Lambda} = (\mathbf{J}^\top)^\dagger \mathbf{M} \mathbf{J}^\dagger$$ task-space mass matrix

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$$\ddot{\mathbf{x}}^* = \ddot{\mathbf{x}}^{ref} + \mathbf{D}(\dot{\mathbf{x}}^{ref} - \dot{\mathbf{x}}) + \mathbf{K}(\mathbf{x}^{ref} - \mathbf{x}) \quad \text{with feedback}$$

$$\boldsymbol{\Lambda} = (\mathbf{J}^\top)^\dagger \mathbf{M} \mathbf{J}^\dagger \quad \text{task-space mass matrix}$$

$$\eta = (\mathbf{J}^\top)^\dagger \mathbf{h} - \boldsymbol{\Lambda} \dot{\mathbf{J}} \dot{\mathbf{q}} \quad \text{task-space } \mathbf{h}$$

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$$\mathbf{\Lambda} = (\mathbf{J}^\top)^\dagger \mathbf{M} \mathbf{J}^\dagger \qquad \text{task-space mass matrix}$$

$$\eta = (\mathbf{J}^\top)^\dagger \mathbf{h} - \mathbf{\Lambda} \dot{\mathbf{J}} \dot{\mathbf{q}} \qquad \text{task-space } \mathbf{h}$$

$$\mathbf{F} = \mathbf{\Lambda} \ddot{\mathbf{x}}^* + \eta \qquad \text{task-space force to achieve } \ddot{\mathbf{x}}^*$$

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$\ddot{\mathbf{x}}^* = \ddot{\mathbf{x}}^{ref} + \mathbf{D}(\dot{\mathbf{x}}^{ref} - \dot{\mathbf{x}}) + \mathbf{K}(\mathbf{x}^{ref} - \mathbf{x})$	with feedback
$\boldsymbol{\Lambda} = (\mathbf{J}^\top)^\dagger \mathbf{M} \mathbf{J}^\dagger$	task-space mass matrix
$\eta = (\mathbf{J}^\top)^\dagger \mathbf{h} - \boldsymbol{\Lambda} \dot{\mathbf{J}} \dot{\mathbf{q}}$	task-space **h**
$\mathbf{F} = \boldsymbol{\Lambda} \ddot{\mathbf{x}}^* + \eta$	task-space force to achieve $\ddot{\mathbf{x}}^*$
$\tau = \mathbf{J}^\top \mathbf{F}$	torque to achieve $\ddot{\mathbf{x}}^*$

Let's try