Kinematics (Continued)

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During the lecture
I will mute the audience
Ask questions through ’Chat’

Last time

Representations of Rigid Bodies
Configuration and task
Homogeneous Transformation Matrix

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Forward Kinematics
Given configuration position/velocity, find task position/velocity

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Configuration and task
Homogeneous Transformation Matrix

Forward Kinematics
Given configuration position/velocity, find task position/velocity

Inverse Kinematics
Given task velocity
Use inverse kinematics to find configuration velocity

Outline

Redundancy Resolution

Orientations

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Redundancy Resolution

Orientations

What is Redundancy?

[0] © Youtube: Ministry of Silly Walks

What is Redundancy?

Redundancy
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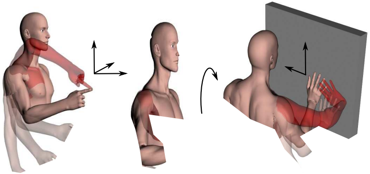
There are more than one solutions to achieve the desired task

Redundancy
More degree-of-freedom than what is required for your task
There are more than one solutions to achieve the desired task

EXAMPLE

- keep your finger tip and the same position and move your elbows



- joint-space velocity but no task-space velocity

Redundancy

Mathematically....

Redundancy

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- There are many solutions to $\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$

Redundancy

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- There are many solutions to $\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$
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- There exist a nullspace $\mathcal{N}(\mathbf{J}) \neq 0$

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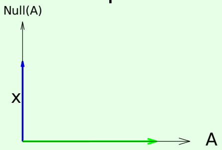
Nullspace

The nullspace of a matrix **A** consists of all the vectors **x** that satisfy $\mathbf{Ax} = \mathbf{0}$



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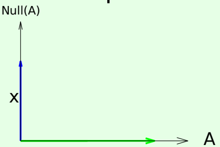

so... our elbows generate some joint velocity $\dot{\mathbf{q}}^0$, and $\mathbf{J}\dot{\mathbf{q}}^0 = \mathbf{0}$

Inverse Kinemaitcs Controller with Redudnacy Resolution

$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}\mathbf{x} + \mathbf{N}\dot{\mathbf{q}}^0$ where

$\mathbf{N} = \mathbf{I} - \mathbf{J}^{\dagger}\mathbf{J}$ is the nullspace projection matrix of $\mathbf{J}$

Redundancy Resolution

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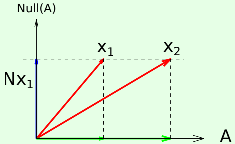
Nullspace Projection Matrix

A nullspace projection of $\mathbf{A}$ projects a vector onto $\mathcal{N}(A)$



Inverse Kinemaitcs Controller with Redudnacy Resolution

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- $\mathbf{N}$ projects $\dot{\mathbf{q}}^0$ onto the nullspace of the jacobian.
- $\dot{\mathbf{q}}^0$ can be **ANY** arbitrary vector.
  $\mathbf{N}\dot{\mathbf{q}}^0$ has **NO** effect on the $\dot{\mathbf{x}}$

$$\mathbf{J}\mathbf{N}\dot{\mathbf{q}}^0 = \mathbf{J}(\mathbf{I} - \mathbf{J}^{\dagger}\mathbf{J})\dot{\mathbf{q}}^0 = (\mathbf{J} - \mathbf{J}\mathbf{J}^{\dagger}\mathbf{J})\dot{\mathbf{q}}^0 = (\mathbf{J} - \mathbf{J})\dot{\mathbf{q}}^0 = \mathbf{0}$$

Question: How do we choose $\dot{q}^0$?
Ideally, we like to move our arms in a comfortable way, or reduce the amount of energy we consume for a given task.

Redundancy

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Move the redundant dimension to some default positions

$$\dot{\mathbf{q}}^0 = \mathbf{q}^0 - \mathbf{q}$$

$\mathbf{q}^0$ is the home configuration

$\mathbf{q}$ is the current configuration

Matlab Example

Without Redundancy Resolution

With Redundancy Resolution

Outline

Redundancy Resolution

Orientations

Orientations

What about orientations?
Previously, we only control the positions and ignore the orientations.
How do we control orientations?

Rigid-Body Motion

Position

$$\mathbf{x} \in \mathbb{R}^6 = \begin{bmatrix} \boldsymbol{\theta} \\ \mathbf{p} \end{bmatrix}$$

$\boldsymbol{\theta} \in \mathbb{R}^3$ orientation

$\mathbf{p} \in \mathbb{R}^3$ translation

Rigid-Body Motion

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(Spatial) Velocity/Twist

$$\dot{\mathbf{x}} \in \mathbb{R}^6 = \begin{bmatrix} \boldsymbol{\omega} \\ \dot{\mathbf{p}} \end{bmatrix}$$

$\boldsymbol{\omega} \in \mathbb{R}^3$ angular velocity

$\dot{\mathbf{p}} \in \mathbb{R}^3$ linear velocity

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Move to robot from point $\mathbf{p}$ to $\mathbf{p}^*$
Find $\dot{\mathbf{p}}^* = \mathbf{p}^* - \mathbf{p}$, and then inverse kinematics to find $\dot{\mathbf{q}}$

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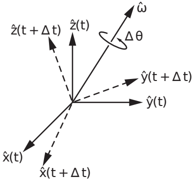
Move to robot from point p to p^*
Find $\dot{\mathbf{p}}^* = \mathbf{p}^* - \mathbf{p}$, and then inverse kinematics to find $\dot{\mathbf{q}}$

Move to robot from point θ to θ^*
Find $\omega = ?$

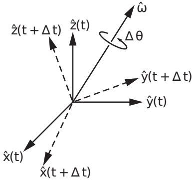
Angular Velocity



current rotation: $\mathbf{R} = [\hat{x}(t), \hat{y}(t), \hat{z}(t)]$
desired rotation: $\mathbf{R}^* = [\hat{x}(t+\triangle t), \hat{y}(t+\triangle t), \hat{z}(t+\triangle t)]$
angular velocity $\omega = ?$

[0] More details in Modern Robotics, Section 3.3

Angular Velocity



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Skew Interpertation of a Twist

$$
\begin{aligned}
\dot{\mathsf{R}} &= \omega \times \mathsf{R} \\
&= \mathbb{S}(\omega)\mathsf{R}
\end{aligned}
$$

$$
\mathbb{S}(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}
$$

$\mathbb{S}(\omega)$: skew-symmetric matrix representation of ω

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Angular Velocity

Desired position $\mathbf{R}^* \in \mathbb{SO}(3)$
provided by the planner

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Approximation of Angular Velocity

$$\begin{aligned}\mathbb{S}(\omega) &= \dot{\mathbf{R}}\mathbf{R}^{-1} \\ &= \dot{\mathbf{R}}\mathbf{R}^{\top} \qquad \text{(since } \mathbf{R}^{\top} = \mathbf{R}^{-1}\text{)}\end{aligned}$$

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&\Downarrow
\end{aligned}
$$

$$
\dot{\mathbf{R}}\mathbf{R}^{\top} = (\mathbf{R}^* - \mathbf{R})\mathbf{R}^{\top} = \mathbf{R}^*\mathbf{R}^{\top} - \mathbf{I}
$$

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$$\Downarrow$$

$$\dot{\mathbf{R}}\mathbf{R}^{\top} = (\mathbf{R}^* - \mathbf{R})\mathbf{R}^{\top} = \mathbf{R}^*\mathbf{R}^{\top} - \mathbf{I}$$

$$\Downarrow$$

Convert skew symmetric matrix back to angles

$$\omega = \frac{1}{2}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Matlab Example

Example (Move orientations)

$\mathbf{q} \in \mathbb{R}^7$: joints
$\mathbf{x} \in \mathbb{R}^3$: hand orientations
$\mathbf{J} \in \mathbb{R}^{3\times 7}$: Jacobian