- By Cauchy stress tensor, find trial stresses tensor.
- Find principal trial stresses from the trial stresses tensor.
- Find the Coulomb Yield (Fmc) and cutoff (Ftc) values based on the principal trial stresses.

$$Fmc(\boldsymbol{\sigma}) = \frac{\sigma_1 - \sigma_3}{2} + \frac{\sigma_1 + \sigma_3}{2}\sin(\phi) - C\cos(\phi) = 0$$
$$Ftc(\boldsymbol{\sigma}) = \sigma_1 - t_c = 0$$

Elastic region:

Condition: $Fmc \leq 0$ *and* $Ftc \leq 0$

The stress vector stays unchanged,

mStressVector = trailStressVector;

No need any return mapping here

Zone of axial return

First need to find whether there is intersection between the Yield and cutoff functions.

Find the root of *Fmc* .

$$\frac{\sigma_1 - \sigma_3}{2} = 0$$

$$fme => \frac{\sigma_1 + \sigma_3}{2}\sin(\phi) - C\cos(\phi) = 0$$

$$\frac{\sigma_1 + \sigma_3}{2} = \frac{C}{\tan(\phi)}$$

at the root point (along the horizontal axis), $\sigma_1 = \sigma_3$ then:

$$\sigma_1 = \frac{C}{\tan(\phi)}$$

In order to guaranty that cutoff function to cross the Yield function,

$$t_c < \frac{C}{\tan(\phi)}$$

Then we need to find the shear at the intersection point. Setting $\sigma_1 = t_c$ in the yield function:

$$\frac{t_c - \sigma_3}{2} + \frac{t_c + \sigma_3}{2}\sin(\phi) - C\cos(\phi) = 0$$

It leads

$$\sigma_3 = \frac{t_c(1 + \sin\phi\ ) - 2C\cos\phi}{1 - \sin\phi}$$

Then

$$\tau_{corner} = \frac{\sigma_1 - \sigma_3}{2} = \frac{C\cos\phi - t_c\sin\phi}{1 - \sin\phi}$$

We need

$$\tau^{trial} < \tau_{corner}$$

Moreover, this region is associated with $Ftc \geq 0$.

If these three conditions are met, we need to map the trial stresses to the tension-cutoff curve. It means:

$$\sigma_1^{map} = t_c$$

Now need to find σ_3. We move parallel to the horizontal axis, it means the shear value stays constant.

$$\frac{t_c - \sigma_3}{2} = \frac{\sigma_1^{trial} - \sigma_3^{trial}}{2}$$

$$\sigma_3 = t_c - \sigma_1^{trial} + \sigma_3^{trial}$$

They are the corrected principal stresses. They need to be rotated back to the element axes system. We need to use the rotation matrix.

$$\boldsymbol{\sigma} = \boldsymbol{R}\boldsymbol{\sigma_p}\boldsymbol{R}^{-1}$$

The rotation matrix is orthogonal

$$\boldsymbol{R}^{\boldsymbol{T}} = \boldsymbol{R}^{-1} => \ \boldsymbol{\sigma} = \boldsymbol{R}\boldsymbol{\sigma_p}\boldsymbol{R}^{-1}$$

Where

$$\boldsymbol{\sigma_p} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Zone of tensile corner return

This zone is located under the line which is perpendicular to the flow function and passes through the intersection point of yield function and tension cutoff.

First we find the intersection point. The intersection point is related whether the tension-cutoff crosses the yield function.

The tension-cut-off crosses the yield function:

In this case of crossing the shear can be found like the previous region:

$$\tau_{corner} = \frac{\sigma_1 - \sigma_3}{2} = \frac{C \cos\phi - t_c \sin\phi}{1 - \sin\phi}$$

And the normal stress can be found by

$$\sigma_{corner} = \frac{\sigma_1 + \sigma_3}{2} = \frac{t_c - C \cos\phi}{1 - \sin\phi}$$

The tension-cutoff is located outside:

Then this point is equal to the root, which was derived in the previous section:

$$\tau_{corner} = \frac{\sigma_1 - \sigma_3}{2} = 0$$

$$\sigma_{corner} = \frac{\sigma_1 + \sigma_3}{2} = \frac{C}{\tan(\phi)}$$

Flow function:

The flow function is:

$$Gmc(\boldsymbol{\sigma}) = \frac{\sigma_1 - \sigma_3}{2} + \frac{\sigma_1 + \sigma_3}{2}\sin(\psi) = 0$$

$$\frac{\sigma_1 - \sigma_3}{2} = -\frac{\sigma_1 + \sigma_3}{2}\sin(\psi)$$

The slope of this line is $-\sin(\psi)$. The slope of a line perpendicular is then

$$m = \frac{1}{\sin(\psi)}$$

If we consider a line as

$$\tau = m\sigma + B$$

And substituting the intersection point

$$\tau_{corner} = m\,\sigma_{corner} + B$$

$$B = \tau_{corner} - m\,\sigma_{corner}$$

Then

$$g = (\tau - \tau_{corner}) - \frac{1}{\sin(\psi)}(\sigma - \sigma_{corner}) = 0$$

This zone is defined in the region below this function and above the axial region (above the shear at intersection).

$$\geq 0 \ \ and \ \ \tau_{trial} > \tau_{corner}$$

Then

$$\frac{\sigma_1 - \sigma_3}{2} = \tau_{corner}$$

$$\frac{\sigma_1 + \sigma_3}{2} = \sigma_{corner}$$

By solving this system:

$$\sigma_1 = \sigma_{corner} + \tau_{corner}$$

$$\sigma_3 = \sigma_{corner} - \tau_{corner}$$

They are the corrected principal stresses. They need to be rotated back to the element axes system. We need to use the rotation matrix.

$$\boldsymbol{\sigma} = \boldsymbol{R}\boldsymbol{\sigma_p}\boldsymbol{R^{-1}}$$

The rotation matrix is orthogonal

$$\boldsymbol{R^T} = \boldsymbol{R^{-1}} => \ \boldsymbol{\sigma} = \boldsymbol{R}\boldsymbol{\sigma_p}\boldsymbol{R^{-1}}$$

Where

$$\boldsymbol{\sigma_p} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Zone of regular failure

This zone is associated with the region above the yield function and above the function g derived in the previous section.

$$Fmc > 0 \ and \ g < 0$$

We can use the derivative of the flow function to define the direction and find the return point on the yield surface.

$$\frac{\partial G}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} \frac{1}{2}(1 + \sin\psi) \\ 0 \\ -\frac{1}{2}(1 - \sin\psi) \end{bmatrix} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Then a parametrized line can be defined by:

$$\sigma_1 = \sigma_1^{trial} + \dot{\lambda} n_1$$

$$\sigma_3 = \sigma_3^{trial} + \dot{\lambda} n_3$$

At yield function,

$$Fmc = \frac{\sigma_1 - \sigma_3}{2} + \frac{\sigma_1 + \sigma_3}{2} \sin(\phi) - C\cos(\phi) = 0$$

$$\frac{1 + \sin\phi}{1 - \sin\phi}\sigma_1 - \sigma_3 = \frac{2C\cos\phi}{1 - \sin\phi}$$

Solving this 3 equations:

$$\dot{\lambda} = \frac{C_2 - \sigma_1^{trial} C_1 + \sigma_3^{trial}}{n_1 C_1 - n_3}$$

Where

$$C_1 = \frac{1 + \sin\phi}{1 - \sin\phi}$$

$$C_2 = \frac{2C\cos\phi}{1 - \sin\phi}$$

Then

$$\boldsymbol{\sigma}_1 = \dot{\lambda}\frac{\partial G}{\partial \boldsymbol{\sigma}} + \boldsymbol{\sigma}_1^{trial}$$

They are the corrected principal stresses. They need to be rotated back to the element axes system. We need to use the rotation matrix.

$$\boldsymbol{\sigma} = \boldsymbol{R}\boldsymbol{\sigma}_{\boldsymbol{p}}\boldsymbol{R}^{-1}$$

The rotation matrix is orthogonal

$$\boldsymbol{R}^{\boldsymbol{T}} = \boldsymbol{R}^{-1} => \ \boldsymbol{\sigma} = \boldsymbol{R}\boldsymbol{\sigma}_{\boldsymbol{p}}\boldsymbol{R}^{-1}$$

Where

$$\boldsymbol{\sigma}_{\boldsymbol{p}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Rotation matrix

The rotation matrix is derived from the eigenvectors of the Cauchy stresses. Having three eigenvectors related to the principal stresses

$$[\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \boldsymbol{v}_3]$$

Normalizing the vectors, it results in rotation matrix

$$\boldsymbol{R} = \left[\frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} \quad \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} \quad \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|}\right]$$