

Optimization of the CMB-galaxy cross-correlation signal for studying the Integrated Sachs-Wolfe effect

Arthur Diniz Meirelles

Advisor: Edivaldo Moura Santos

Coadvisor: Ronaldo Carloto Batista

Institute of Physics
Universidade de São Paulo

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1 Theoretical Aspects

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Introduction

- The integrated Sachs-Wolfe (ISW) effect occurs due to time changes in gravitational potentials;
- During periods of transition from or to matter-dominated eras of the Universe, potentials change more rapidly;
- Naturally, a statistical correlation between CMB temperature maps and gravitational potentials is expected.

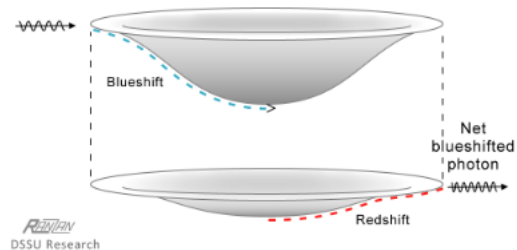


Figure 2: Illustration of the ISW effect. In this case, the gravity wells flattened, leaving a net blueshift.

The Λ CDM Model

To build the theoretical framework used, it is assumed

- The Universe is homogeneous and isotropic at large scales;
- Initial conditions are obtained by assuming an inflationary period when the Universe was very compact and energetic;
- The Universe is composed of baryonic matter, radiation, neutrinos, cold dark matter and a dark energy component described by a cosmological constant Λ ;

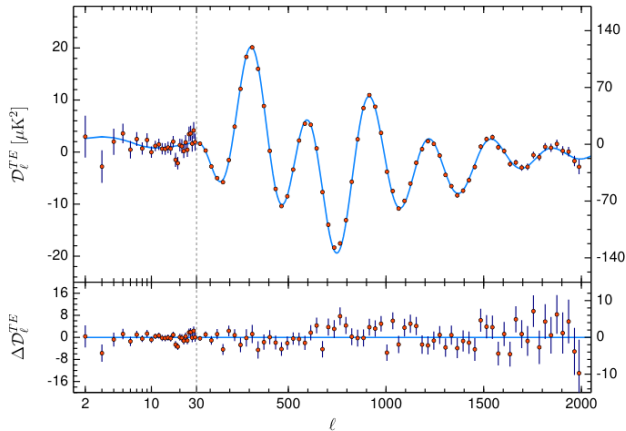
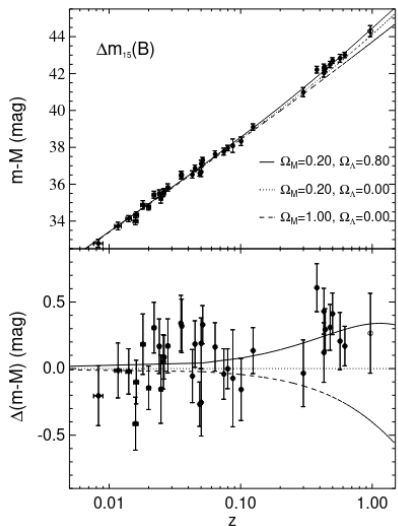
The homogeneous and isotropic background is first set up, and linear perturbations are added to it.

The Perturbed Metric

We express the perturbed metric with two functions $\Psi(\mathbf{x}, t)$ and $\Phi(\mathbf{x}, t)$

$$\begin{cases} g_{00} = -1 - 2\Psi(\mathbf{x}, t), \\ g_{0i} = g_{i0} = 0, \\ g_{ij} = a^2(t)\delta_{ij}[1 + 2\Phi(\mathbf{x}, t)]. \end{cases} \quad (1)$$

The Λ CDM Model



The Cosmic Microwave Background

The temperature of the Cosmic Microwave Background (CMB) can be expressed by

$$T(\mathbf{x}, \hat{\mathbf{p}}, t) = \bar{T}(t)[1 + \Theta(\mathbf{x}, \hat{\mathbf{p}}, t)]. \quad (2)$$

The temperature perturbation Θ can be expressed in Fourier space according to

$$\Theta(\hat{\mathbf{k}}, \mu) = \sum_{\ell=0}^{\infty} (2\ell + 1)(-i)^{\ell} \Theta_{\ell}(\hat{\mathbf{k}}) \mathcal{P}_{\ell}(\mu), \quad (3)$$

where $\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{p}}$ and \mathcal{P}_{ℓ} are Legendre polynomials.

Analytical Approximation

Working with first-order equations and the tightly coupled limit, and assuming recombination to be an instantaneous process happening at conformal time $\eta = \eta_*$, we can obtain

$$\begin{aligned} \Theta_\ell(k, \eta_0) \approx & [\Theta_0(k, \eta_*) + \Psi(k, \eta_*)] j_\ell[k(\eta_0 - \eta_*)] \\ & + i v_b(k, \eta_*) \left\{ j_\ell[k(\eta_0 - \eta_*)] - (\ell + 1) \frac{j_\ell[k(\eta_0 - \eta_*)]}{k(\eta_0 - \eta_*)} \right\} \\ & + \int_0^{\eta_0} d\eta e^{-\tau} [\Psi'(k, \eta) - \Phi'(k, \eta)] j_\ell[k(\eta_0 - \eta)]. \end{aligned} \quad (4)$$

Here, the third term in the equation describes the Integrated Sachs-Wolfe effect.

Correlation Functions

To calculate the CMB autocorrelation function C_ℓ^{tt} , we expand Θ in spherical harmonics

$$\Theta(\mathbf{x}, \hat{\mathbf{p}}, t) = \sum_{\ell=1}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} a_{\ell m}(\mathbf{x}, t) Y_{\ell m}(\hat{\mathbf{p}}), \quad (5)$$

and calculate the autocorrelation between the $a_{\ell m}$ terms:

$$\langle a_{\ell m} a_{\ell' m'}^* \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{tt} \quad (6)$$

Other autocorrelation spectra C_ℓ^{xx} follow the same process. It is common to use

$D_\ell^{XX} = \frac{\ell(\ell+1)}{2\pi} C_\ell^{xx}$ for some spectra for better visualization. We can also write

$$C_\ell^{tt} = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} a_{\ell m} a_{\ell' m'}^* \quad (7)$$

Cosmic Variance

The low number of $a_{\ell m}$ coefficients for lower multipoles ℓ leads to a high uncertainty in this region called cosmic variance

$$\left(\frac{\Delta C_{\ell}^{XX}}{C_{\ell}^{XX}} \right)_{CV} = \sqrt{\frac{2}{2\ell + 1}} \quad (8)$$



Table 1: Best-fit values of cosmological parameters reported in Planck 2018 results using TT+TE+EE+lowE and gravitational lensing data.

The Matter Power Spectrum

To calculate a cross-correlation function, we need the 3D matter power spectrum $P(k, z)$, defined by

$$\langle \delta(\mathbf{k}, z) \delta^*(\mathbf{k}', z) \rangle = (2\pi)^3 P(k, z) \delta_D(\mathbf{k} - \mathbf{k}')$$

We have used CAMB's implementation of the HALOFIT model to calculate the matter power spectrum.

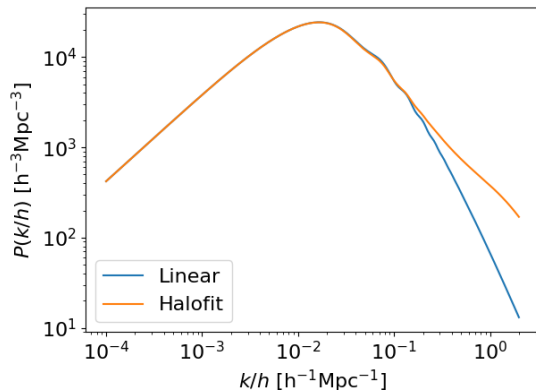


Figure 4: Matter power spectrum calculated using both a linear approximation and the HALOFIT model.

The Cross-correlation Spectrum

To trace the matter density anisotropies, we used galaxy contrast maps, which can be calculated using

$$\delta_g(\mathbf{n}) = \frac{N_g(\mathbf{n}) - \bar{N}_g}{\bar{N}_g} \quad (9)$$

It is assumed that $\delta_g = b_g \delta$, where b_g is the bias factor, which we assume to be a slowly varying function of redshift.

The galaxy autocorrelation C_ℓ^{gg} and cross-correlation function C_ℓ^{tg} can then be calculated with

$$\langle a_{\ell m}^t a_{\ell' m'}^g \rangle = C_\ell^{tg} \delta_{\ell \ell'} \delta_{m m'} \quad \langle a_{\ell m}^g a_{\ell' m'}^g \rangle = C_\ell^{gg} \delta_{\ell \ell'} \delta_{m m'} \quad (10)$$

Analytical Formula

Given fields x and y , representing either the ISW contribution to the CMB temperature ($x, y = t$) or the galaxy contrast ($x, y = g$), we can calculate the associated auto- or cross-correlation spectra using [1]

$$C_{\ell}^{xy} = \frac{2}{\pi} \int dk k^2 W_{\ell}^x(k) W_{\ell}^y(k) P(k), \quad (11)$$

with

$$W_{\ell}^t = -3\Omega_m \left(\frac{H_0}{k} \right)^2 \int dz \frac{d[(1+z)D(z)]}{dz} j_{\ell}[k\chi(z)] \quad (12)$$

$$W_{\ell}^g = \int dz b_g(z) \frac{dN}{dz} D(z) j_{\ell}[k\chi(z)] \quad (13)$$

ISW Contribution: Temperature Autocorrelation

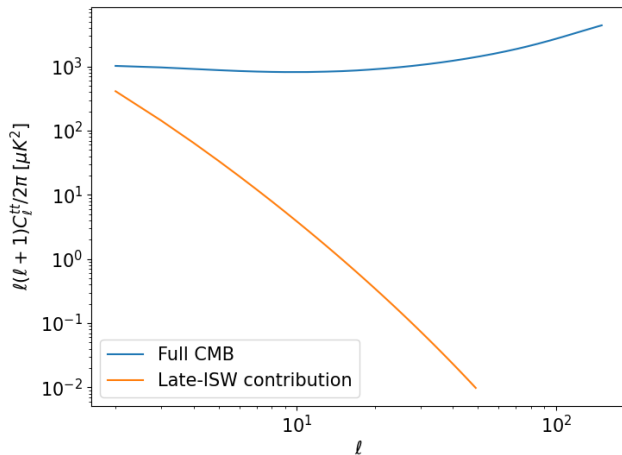


Figure 5: CMB autocorrelation comparison between full spectrum and ISW contribution.

ISW Contribution: Galaxy Autocorrelation and Cross-correlation

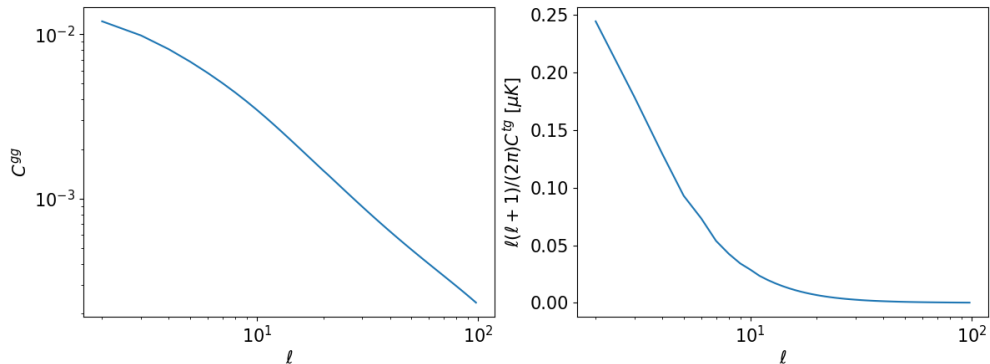


Figure 6: Galaxy autocorrelation spectrum (left) and late-ISW contribution to the galaxy-CMB cross-correlation (right).

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Selection Function Parametrization

The function $\frac{dN}{dz}$ in equation (13) is called the selection function. We are assuming its parametrization to be [2]

$$\frac{dN}{dz}(z|\lambda, \beta, z_0) dz = \frac{\beta}{\Gamma(\lambda)} \left(\frac{z}{z_0}\right)^{\beta\lambda-1} \exp\left[-\left(\frac{z}{z_0}\right)^\beta\right] d\left(\frac{z}{z_0}\right) \quad (14)$$

We have explored how to maximize the cross-correlation signal using an idealized selection function.

2MASS Bands Comparison

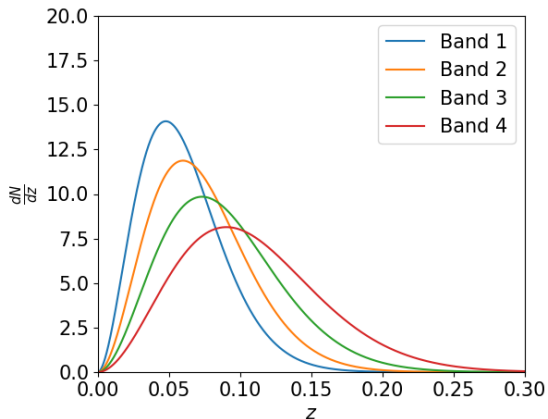


Figure 7: Selection function calculated for the 4 bands of the 2MASS catalog.

Band	z_0	β	λ
1	0.043	1.825	1.524
2	0.054	1.800	1.600
3	0.067	1.765	1.636
4	0.084	1.723	1.684

Table 2: Parameter values for the 4 bands of the 2MASS catalog.

Exploring the Parameter Space

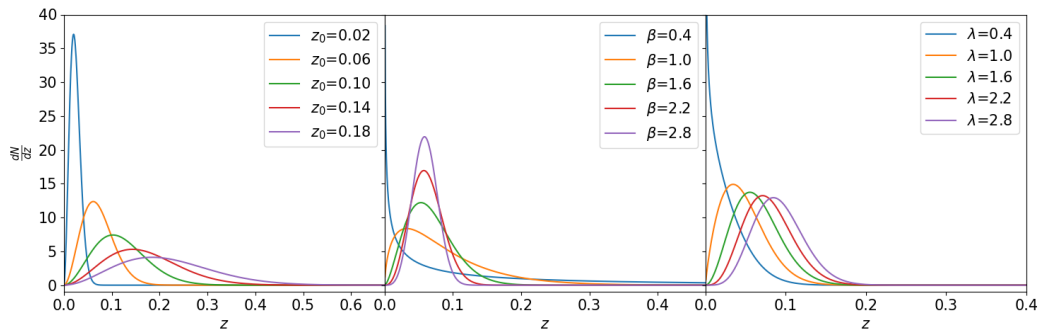


Figure 8: Selection function calculated for various parameter values.

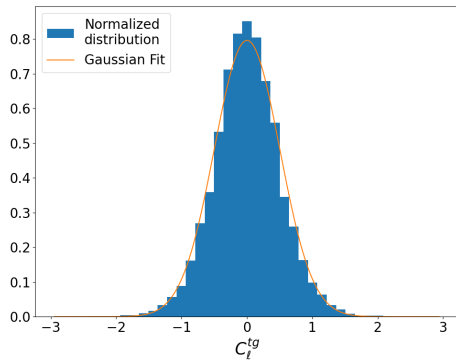
Null Hypothesis

For the process of finding a galaxy survey with an idealized selection function, we first needed an estimation for the probability of C_ℓ^{tg} being 0, which would be our null hypothesis. The following process was used:

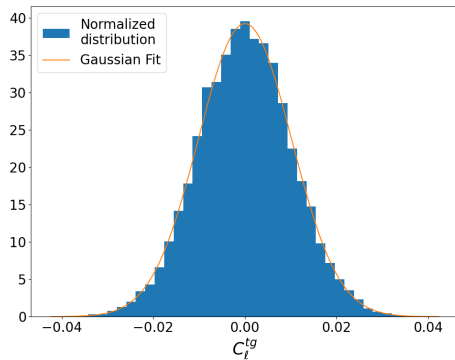
- Synthesize multiple uncorrelated CMB temperature and galaxy contrast maps using HEALPix;
- Calculate the cross-correlation C_ℓ^{tg} for each pair of uncorrelated maps at each multipole;
- For each multipole ℓ , a histogram of C_ℓ^{tg} was produced;
- Each histogram was fit with Gaussian distributions with average $\mu = \mu_\ell \approx 0$ and $\sigma^2 = \sigma_\ell^2$.

The null hypothesis is compatible with $\Omega_m = 1$.

Synthesized Maps' Histograms



(a) $\ell = 4$



(b) $\ell = 30$

Figure 9: Distribution of cross-correlation values on different multipoles for 10^4 maps synthesized with null cross-correlation.

Null Hypothesis

For f_ℓ corresponding to the Gaussian fit made for the multipole ℓ

$$f_\ell(C_\ell^{tg}) = \frac{1}{\sqrt{2\pi\sigma_\ell^2}} \exp \left[-\frac{1}{2} \left(\frac{C_\ell^{tg} - \mu_\ell}{\sigma_\ell} \right)^2 \right], \quad (15)$$

we have defined the null hypothesis probability distribution to be

$$P_{\text{null}} = \prod_{\ell=2}^{\ell_{\text{max}}} f_\ell(C_\ell^{tg}) \quad (16)$$

Exploring the Parameter Space

For various points (β, z_0, λ) in the parameter space, we have calculated the ratio $P_{\text{null}}(\beta, z_0, \lambda) / P_{\text{null}}^{\text{2MASS}}$ and produced the heat maps in Figure 10.

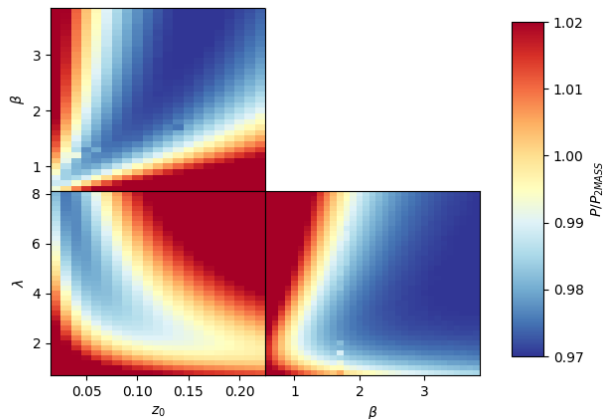


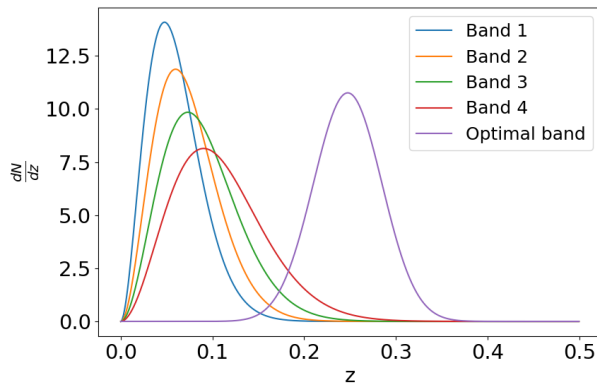
Figure 10: Heat maps used to explore the parameter space.

Minimizer

The heat maps provided initial points to run an algorithm that minimizes $P_{\text{null}}(z_0, \beta, \lambda)$. The point found that minimizes the null hypothesis probability in that region of the parameter space was

$$(\beta, z_0, \lambda) = (3.088, 0.1508, 4.9401) \quad (17)$$

Properties of the Minimum



The selection function found is deeper than that of 2MASS band 1. It does not favor galaxies at $z^* = 0.63$, the estimated redshift at which the accelerated expansion started.

Figure 11: Selection functions comparison.

Properties of the Minimum

The selection function found reduces the maximum value of the cross-correlation function, but its peak is at higher redshifts ($\ell \approx 10$), reducing the influence of cosmic variance in the signal.

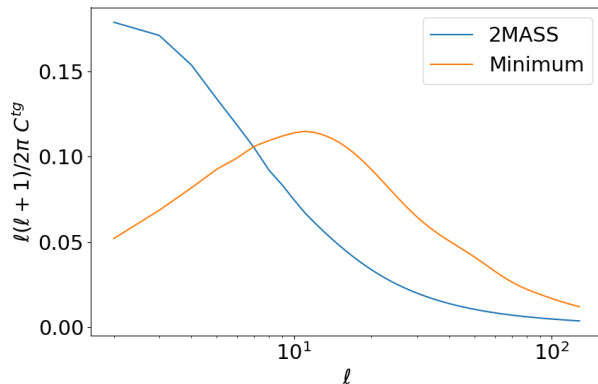


Figure 12: Theoretical cross-correlation spectrum comparison.

Discussion

- The ratio $P_{\text{null}}(z_0, \beta, \lambda) / P_{\text{null}}^{2\text{MASS}} = 0.971$ for the minimum;
- Despite the small statistical gain, this optimal selection function yields reasonably better results for constraints on Ω_m , as will be discussed;
- A similar work has reported a small preference towards the non-null signal with changes to the depth of the survey and fainter magnitude limits [3].

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WMAP Data

The CMB temperature maps used in this project are the ones provided by WMAP9 [4].

- Three frequency bands were used, which are named bands Q (40GHz), V (60GHz) and W (90GHz);
- The temperature intensity maps' noise power can be well modeled as uncorrelated Gaussian fluctuations;
- Despite having a lower resolution (0.3°) compared to Planck's (0.083°), this project focuses on lower multipoles;
- We have combined both maps' masks and used it for analysing each one, resulting in a fraction $f_{\text{sky}} = 0.7$.

WMAP Data

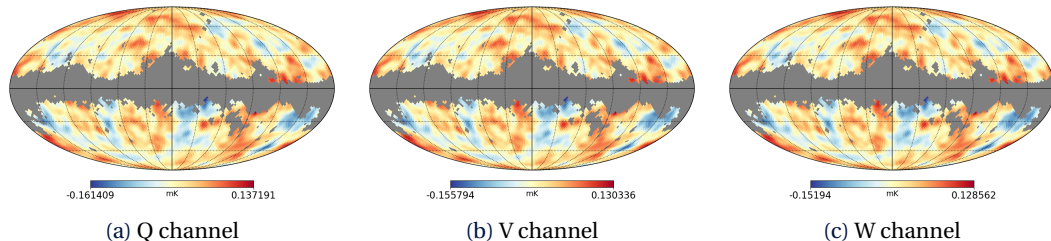


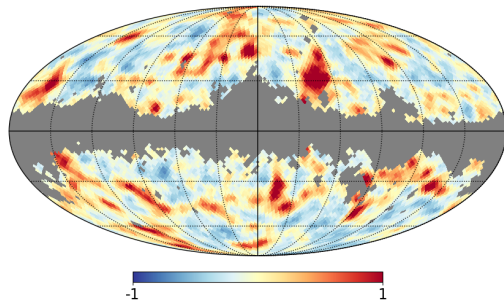
Figure 13: Mollweide projection of three (Q, V, W) WMAP9 CMB temperature (in mK) maps in galactic coordinates with an $f_{\text{sky}} = 0.70$ mask applied

2MASS Catalog

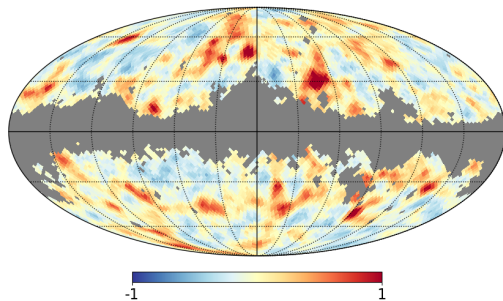
Wide sky coverage is a very influential aspect of studying cross-correlation spectra, and the 2MASS catalog contains raw imaging data covering 99.998% of the sky [5], which was the dataset used.

- We have used the K_s ($2.16\mu\text{m}$) band of the Extended Source Catalog (XSC);
- The data of the K_s band obtained from a 20 mag aperture (K_{20}) were corrected for galactic extinction, and the remaining data was further divided into 4 bands: Bands 1 ($12.0 < K'_{20} < 12.5$), 2 ($12.5 < K'_{20} < 13.0$), 3 ($13.0 < K'_{20} < 13.5$) and 4 ($13.5 < K'_{20} < 14.0$);
- Each band has a different selection function, as shown in Figure 7.

2MASS Catalog



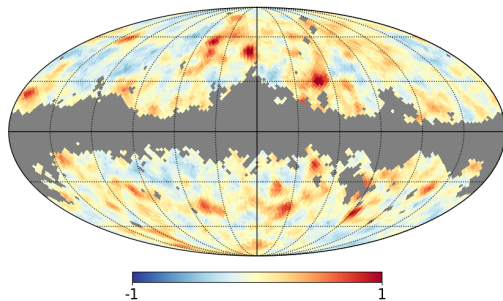
(a) Band 1



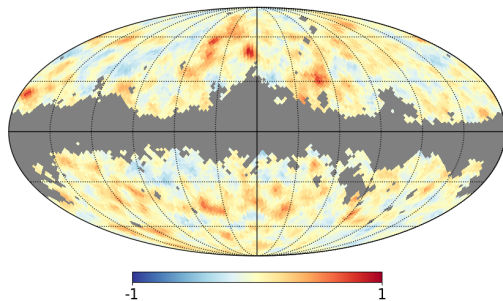
(b) Band 2

Figure 14: Mollweide projection of the 2MASS-XSC galaxy contrast maps in galactic coordinates with the combined 2MASS+WMAP mask applied ($f_{sky} = 0.70$).

2MASS Catalog



(a) Band 3



(b) Band 4

Figure 15: Mollweide projection of the 2MASS-XSC galaxy contrast maps in galactic coordinates with the combined 2MASS+WMAP mask applied ($f_{sky} = 0.70$).

Correlation Spectra Estimator

To estimate the correlation spectra that describe the pixelized maps d with primordial signal s and noise n (with \mathbf{S} and \mathbf{N} being the corresponding covariance matrices), we could use the following likelihood function.

$$\mathcal{L} = P(\mathbf{d}|\mathbf{C}) = \frac{1}{(2\pi)^{n_{\text{dim}}/2} |\mathbf{C}|^{1/2}} \exp\left(-\frac{1}{2} \mathbf{d}^T \mathbf{C}^{-1} \mathbf{d}\right), \quad \mathbf{C} = \mathbf{S} + \mathbf{N} \quad (18)$$

With a prior $\pi(\mathbf{S})$ we can use Bayes' Theorem to obtain

$$P(\mathbf{C}|\mathbf{d}) \propto \pi(\mathbf{S}) P(\mathbf{d}|\mathbf{C}) \quad (19)$$

Correlation Spectra Estimator

If we can sample from $P(\mathbf{C}|\mathbf{s}, d)$ and $P(\mathbf{s}|\mathbf{C}, d)$. Then we iterate

$$\mathbf{s}^{i+1} \leftarrow P(\mathbf{s}|\mathbf{C}^i, d) \quad (20)$$

$$\mathbf{C}^{i+1} \leftarrow P(\mathbf{C}|\mathbf{s}^{i+1}, d), \quad (21)$$

to obtain a sample of $\{(\mathbf{s}^i, \mathbf{C}^i)\}$.

The Blackwell-Rao Estimator

We can then use the Blackwell-Rao estimator to obtain an approximation for $P(C_\ell|\mathbf{d})$

$$P(C_\ell|\mathbf{d}) \approx \frac{1}{N_G} \sum_{i=1}^{N_G} P(C_\ell|\sigma_\ell^i), \quad (22)$$

where

$$\sigma_\ell^i = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \mathbf{s}_{\ell m} \mathbf{s}_{\ell m}^\dagger. \quad (23)$$

We then maximize the probability $P(C_\ell|\mathbf{d})$ to obtain the best-fit C_ℓ .

Matrices Used

In this work, \mathbf{s} and \mathbf{S} are defined by

$$\mathbf{s}^T = (s_{00}^{tg}, s_{01}^{tg}, s_{11}^{tg}, \dots, s_{\ell_{\max}0}^{tg}, \dots, s_{\ell_{\max}\ell_{\max}}^{tg}), \quad (24)$$

$$\mathbf{S} = \text{diag}(S_0^{tg}, S_1^{tg}, S_1^{tg}, \dots, S_{\ell_{\max}}^{tg}, \dots, S_{\ell_{\max}}^{tg}), \quad (25)$$

where

$$\mathbf{s}_{\ell m}^{tg} = \begin{pmatrix} a_{\ell m}^t \\ a_{\ell m}^g \end{pmatrix}, \quad \mathbf{S}_{\ell}^{tg} = \begin{pmatrix} C_{\ell}^{tt} & C_{\ell}^{tg} \\ C_{\ell}^{tg} & C_{\ell}^{gg} \end{pmatrix}. \quad (26)$$

Correlation Spectra Obtained

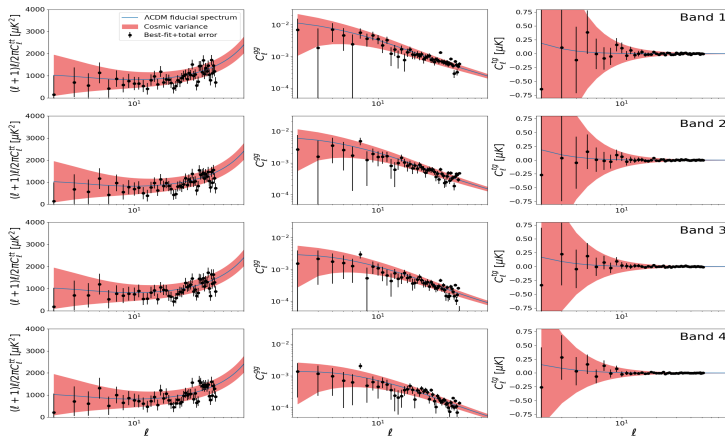


Figure 16: Comparison between theoretical correlation spectra and the ones estimated from WMAP and 2MASS.

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Monte Carlo Markov Chains

By defining the likelihood $\mathcal{L}(C_\ell|\theta)$ and the prior $p(\theta)$, one can use Markov chains to obtain samples of each parameter in θ following the posterior

$$P(\theta|C_\ell) = \mathcal{L}(C_\ell|\theta)p(\theta) \quad (27)$$

Joint posteriors can be easily obtained from each parameter sample, strongly simplifying data analysis for multiple parameter models.

Planck Likelihoods

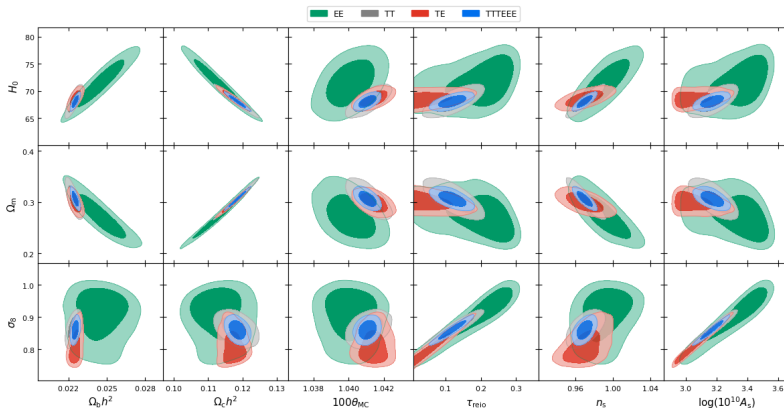


Figure 17: Joint posterior distributions of cosmological parameters using Planck's CMB temperature (T) and polarization (E) data.

Likelihood Profiling

The likelihood of each point in the spectrum was assumed to be Gaussian

$$\mathcal{L}(C_{\ell, \text{theo}}^{xy} | C_{\ell, \text{data}}^{xy}) = \frac{1}{\sigma_{\ell} \sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{C_{\ell, \text{data}}^{xy} - C_{\ell, \text{theo}}^{xy}}{\sigma_{\ell}} \right)^2 \right]. \quad (28)$$

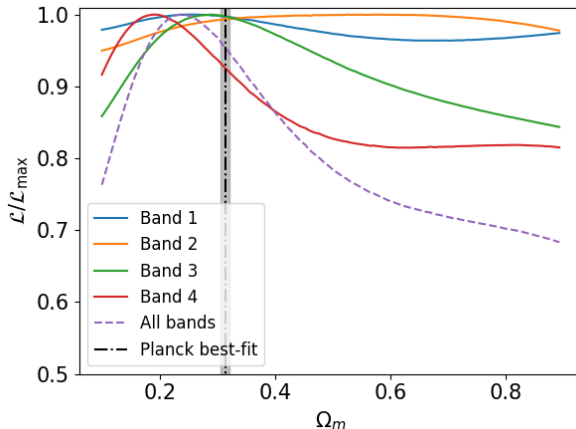
We are varying only Ω_m , so $C_{\ell, \text{theo}}^{xy}$ is a functions of only Ω_m . The likelihood of a full spectrum is

$$\mathcal{L}(\Omega_m | C_{\text{data}}^{xy}) = \prod_{\ell=2}^{\ell_{\max}} \mathcal{L}(C_{\ell, \text{theo}}^{xy} | C_{\ell, \text{data}}^{xy}). \quad (29)$$

The $C^{tg} + C^{gg}$ joint-likelihood is

$$\mathcal{L}(\Omega_m | C_{\text{data}}^{tg}, C_{\text{data}}^{gg}) = \mathcal{L}(\Omega_m | C_{\text{data}}^{tg}) \mathcal{L}(\Omega_m | C_{\text{data}}^{gg}). \quad (30)$$

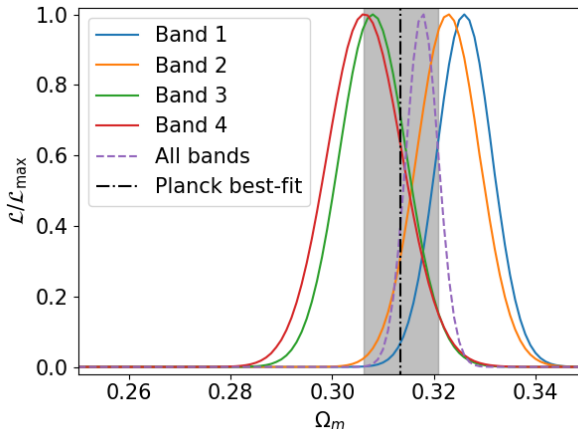
Results for 2MASS



- All curves are compatible with Planck;
- Not much constraining power on Ω_m ;
- Bands 3 and 4 – which have the deepest selection functions – have the most constraining power amongst all 4.

Figure 18: C^{tg} only likelihood profiles.

Results for 2MASS



- All likelihoods are compatible with Planck;
- Very high constraining power on Ω_m when C^{gg} is introduced, comparable to Planck's;
- No significant difference in constraining power between each band.

Figure 19: Joint likelihood profiles

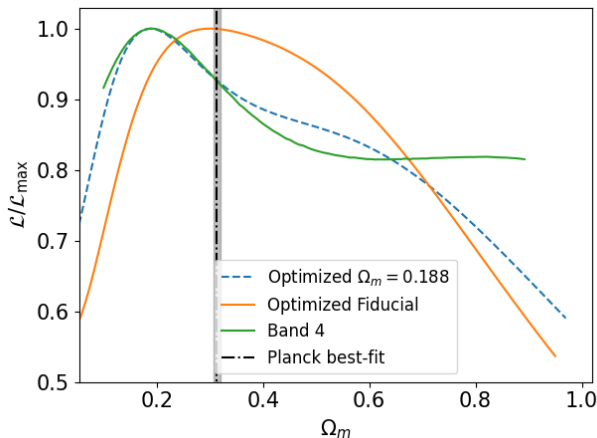
Forecast for the Optimized Band

To estimate the behavior of a survey that follows our idealized selection function in this analysis, we have produced two synthetic cross-correlation spectra using that selection function. Both use the Λ CDM model with Planck's best-fit parameters, only differing in Ω_m :

- One of the datasets was produced using Planck's best-fit of $\Omega_m = 0.3153$;
- The other dataset was produced using $\Omega_m = 0.188$, the value that maximizes the likelihood of band 4 for the C^{tg} only likelihood.

Band 4's errors were used as estimated for both synthetic datasets.

Forecast for the Optimized Band



- Constraining power still low;
- Reasonable improvement in the constraining power.

Figure 20: Optimized band C^{tg} only profile.

Discussion

- The optimized band being deeper than 2MASS means errors on the spectra should be lower, meaning the errors used are overestimated, which indicates the constraints could be better for a real survey following our optimized selection function;
- The optimized band was not found by optimizing the Ω_m constraints, but rejecting the null cross-correlation hypothesis. $\Omega_m = 1$ leads to $C_\ell^{tg} = 0$, and the fast decrease in likelihood for higher Ω_m is noticeable.

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Conclusions

- We have obtained the likelihood profiles for Ω_m obtained from cross-correlation data by studying the ISW effect;
- The constraints obtained using only C^{tg} were not strong, but can be used as a complement to other datasets for a combined analysis;
- In the 2MASS catalog, the 2 deepest bands – bands 3 and 4 – yielded better constraining power;
- The Planck collaboration also studied the ISW effect using the cross-correlation spectrum[6], obtaining significantly better results with deeper surveys (SDSS[7] and Planck's Kappa map).

Conclusions

- An optimized band capable of maximizing the ISW signal was obtained, prioritizing galaxies at higher redshifts and increasing the cross-correlation signal in a region less affected by cosmic variance;
- The likelihoods obtained for artificial data calculated using the optimized band improved the constraints on Ω_m ;
- Combining different matter tracers leads to improved results overall.

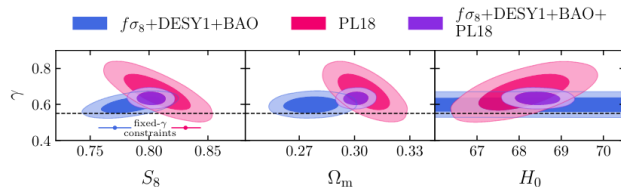
Thank You

Future Prospects

Exploring the growth function parametrization [8]

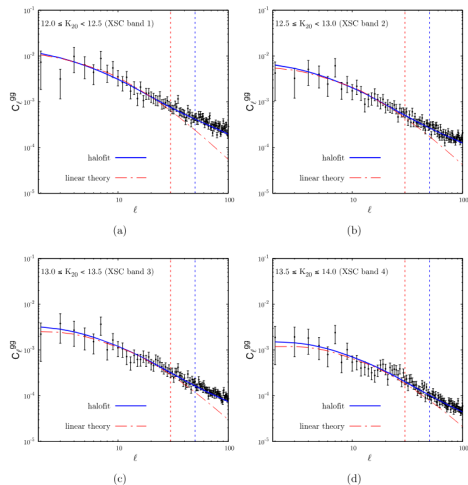
$$\frac{d \ln D(a)}{d \ln a} = \Omega_m^\gamma(a), \quad (31)$$

with γ as a free parameter [9] might be worth to explore.



Bias Factor Fit

The bias factor was fitted to the data using the galaxy autocorrelation function with limiting multipoles for the linear model and HALOFIT.



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