Problem Set 11

Written by Andy Tran for the MaPS Correspondence Program

Instructions

- Some of these problems are based off the notes "*Lengths and Areas*".
- They are in roughly difficulty order and get quite difficult, so you are **not** expected to be able to solve every problem.
- However, please attempt as many questions as you can and submit your solutions to your mentor for marking and feedback.
- You may (and encouraged to) submit incomplete solutions if you can not solve a problem completely.
- You may type your solutions or submit a pdf of a **clear** scan/photo of **legible** written solutions.
- Feel free to discuss these problems with your peers and on the forum but the solutions you submit must be written by yourself.

Problems

1. For any rectangle $ABCD$, show that $PA^2 + PC^2 = PB^2 + PD^2$ for all points P.
2. Five boys and five girls are standing in a line. How many possible arrangements of the line are there if:
   (a) there are no restrictions?
   (b) one boy insists on standing at the front of the line?
   (c) one girl must be closer to the front of the line than her sister?
   (d) a group of three friends insist upon not being separated?
   (e) no boy is standing next to another boy?
   (f) there are five pairs that insist upon not being separated?
3. Let P be a point inside the equilateral $\triangle ABC$. Let the perpendiculars from P to the sides BC, CA, AB be D, E, F respectively. Prove that $PD + PE + PF$ is equal to the length of an altitude of $\triangle ABC$.

   The altitude of $\triangle ABC$ is the line segment from one of the vertices to the opposite side.
4. Let $P(x)$ be a **cubic** polynomials with roots r_1, r_2, r_3. It is given that
   $$\frac{P(\frac{1}{2}) + P(-\frac{1}{2})}{P(0)} = 1013.$$
   Find the value of $\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}$.
5. Let $ABCD$ be a parallelogram, and let P be a point on the side CD. Let the line through P that is parallel to AD intersect the diagonal AC at Q. Prove that
   $$(\text{Area of} \triangle\, BCP)^2 = (\text{Area of} \triangle\, QBP) \;\times\; (\text{Area of} \triangle\, ABP).$$