Extra Enrichment:

1. Evaluate: $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$.

2. Find the value of k for which $\lim_{x\to 3}\dfrac{4x^2+kx+7k-6}{2x^2-5x-3}$ exists and hence evaluate the limit.

3. Evaluate: $\lim_{x \to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$.

4. Evaluate: $\lim_{x \to \infty} \frac{\left(\sqrt{x^2+4}+x\right)^2}{\sqrt[3]{x^6+4}}$.

Solutions:

1. $$\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)=\lim_{n\to\infty}\frac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n}$$

$$=\lim_{n\to\infty}\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}$$

$$=\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}+n}$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$=\frac{1}{\sqrt{1+0}+1}$$

$$=\frac{1}{2}$$

2. $$\lim_{x\to 3}\frac{4x^2+kx+7k-6}{2x^2-5x-3}=\lim_{x\to 3}\frac{4x^2+kx+7k-6}{(x-3)(2x+1)}$$

When $x\to 3$, $(x-3)\to 0$ in the denominator.

If $4x^2+kx+7k-6$ do not have a factor $(x-3)$,

then $4x^2+kx+7k-6\to C$ as $x\to 3$, where C is a non-zero value,

hence $\lim_{x\to 3}\frac{4x^2+kx+7k-6}{(x-3)(2x+1)}=\frac{C}{0}=\infty$, the limit does not exist.

The limit only exists when $4x^2+kx+7k-6$ has a factor $(x-3)$.

$4x^2+kx+7k-6$ must be divisible by $x-3$:

$$\begin{array}{r l}
 & 4x \quad +(k+12) \\
(x-3) & \overline{)\,4x^2+kx \quad +7k-6} \\
 & \underline{4x^2-12x} \\
 & (k+12)x+7k-6 \\
 & \underline{(k+12)x-3(k+12)} \\
 & 10k+30
\end{array}$$

The remainder must be zero: $10k+30=0$

$$\Rightarrow\quad k=-3$$

Let $k=-3$: $\lim\limits_{x\to 3}\dfrac{4x^2+kx+7k-6}{2x^2-5x-3}=\lim\limits_{x\to 3}\dfrac{4x^2-3x-27}{2x^2-5x-3}$

$$=\lim_{x\to 3}\frac{(x-3)(4x+9)}{(x-3)(2x+1)}$$

$$=\lim_{x\to 3}\frac{4x+9}{2x+1}$$

$$=\frac{12+9}{6+1}$$

$$=3$$

3. $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1}=\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1}\cdot\dfrac{\sqrt{x}+1}{\sqrt{x}+1}\cdot\dfrac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}$ (rationalize both numerator and denominator)

$$=\lim_{x\to 1}\frac{\left(\sqrt[3]{x}-1\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)}{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}\cdot\frac{\sqrt{x}+1}{1}\cdot\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}$$

$$=\lim_{x\to 1}\frac{x-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}$$

$$=\lim_{x\to 1}\frac{\sqrt{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}$$

$$=\frac{1+1}{1+1+1}$$

$$=\frac{2}{3}$$

4. $\lim\limits_{x\to\infty}\dfrac{\left(\sqrt{x^2+4}+x\right)^2}{\sqrt[3]{x^6+4}}=\lim\limits_{x\to\infty}\dfrac{x^2+4+2x\sqrt{x^2+4}+x^2}{\sqrt[3]{x^6+4}}$

$$=\lim_{x\to\infty}\frac{2x^2+4+2\sqrt{x^4+4x^2}}{\sqrt[3]{x^6+4}}$$

$$=\lim_{x\to\infty}\frac{\dfrac{\left(2x^2+4+2\sqrt{x^4+4x^2}\right)}{x^2}}{\dfrac{\left(\sqrt[3]{x^6+4}\right)}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2+\frac{4}{x^2}+\frac{2\sqrt{x^4+4x^2}}{\sqrt{x^4}}}{\frac{\sqrt[3]{x^6+4}}{\sqrt[3]{x^6}}}$$

$$= \lim_{x\to\infty} \frac{2+\frac{4}{x^2}+2\sqrt{1+\frac{4}{x^2}}}{\sqrt[3]{1+\frac{4}{x^6}}}$$

$$= \frac{2+0+2\sqrt{1+0}}{\sqrt[3]{1+0}}$$

$$= \frac{2+2}{1}$$

$$= 4$$