# Problem Set 12

Written by Andy Tran for the MaPS Correspondence Program

Instructions

- Some of these problems are based off the notes *"Graph Theory"*. Some other are revision problems for the previous notes.

- They are in roughly difficulty order and get quite difficult, so you are **not** expected to be able to solve every problem.

- However, please attempt as many questions as you can and submit your solutions to your mentor for marking and feedback.

- You may (and encouraged to) submit incomplete solutions if you can not solve a problem completely.

- You may type your solutions or submit a pdf of a **clear** scan/photo of **legible** written solutions.

- Feel free to discuss these problems with your peers and on the forum but the solutions you submit must be written by yourself.

Problems

1. In a country, every two cities are connected either by a road, or by an air route. Prove that one of these methods of travel allows one to get from every city to every other city using only that method of travel.

2. Let $x \neq y$ be integers. Show, by factorisation or otherwise, that $x - y \mid x^n - y^n$. Hence show that $x - y \mid P(x) - P(y)$ whenever $P(x)$ is a polynomial with integer coefficients.

3. Suppose there are three houses on a two-dimensional world and each of them needs to be connected to the electricity, water, and gas utilities. However, since this is a 2-dimensional world, these connections cannot not overlap. Is it possible to connect the three houses to the three utilities?
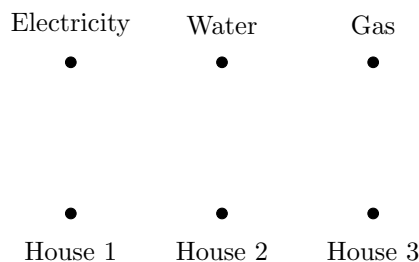
4. A square $ABCD$ has side length 10. X and Y are midpoints of sides AD and CD, respectively. What is the total area of triangles XPA and YBP, where P is the point of intersection of AY and XB?

5. Tony the truck driver lives in a country with n cities where each city is connected to at least $\frac{n}{2}$ other cities by road. Prove that there exists a city which Tony can live in, where he can start driving from his home and visit every other city exactly once and return to his home at the end.