Circles

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1 Introduction

The circle is a fundamental shape that appears everywhere in nature and has been used to inspire the study of geometry, astronomy and calculus. Being such a basic and symmetric shape, circles have many unique properties that have been studied for millennia. This topic will introduce many fascinating results that can be proven about circles and their applications to geometrical problems.

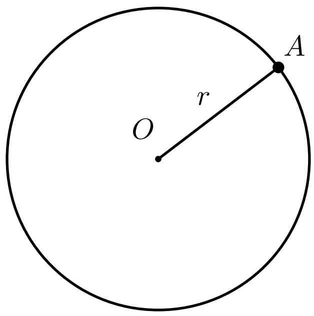
We will begin by introducing the **key theorems** involved with circles and learn how to **prove** them. We will then look at an **example** of problems that use these theorems.

2 Definition

Before we can proceed, we must address the important question: what *is* a **circle**? How do we define *what* a **circle** is? This might seem very obvious at first, but in these notes, we will be **proving** properties of circles and so it is <u>imperative</u> that we have a **rigorous**, **mathematical definition** for a **circle**.

There are a few ways we can define what a circle is, but for the purposes of these notes, we will use the following:

Definition 2.1 *A circle is a set of points in the plane that are a fixed* ***distance*** *(called the* ***radius****) away from a fixed* ***point*** *(called the* ***centre****).*

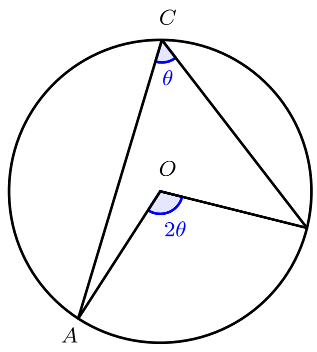


This means that if we have a circle with centre O and radius r, then for any point on the circle A, we have that $OA = r$. This will be a crucial fact for all the proofs in these notes.

3 Angles

Now we can look at a few important theorems in circle geometry!

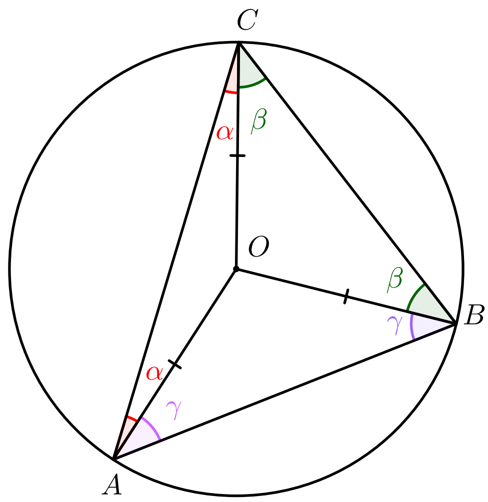
Theorem 3.1 *The angle that an arc subtends at the centre of a circle is twice the angle subtended at the circumference.*



So in this case, O is the centre of the circle, and AB can be any arc on the circle. Then for any point C on the circumference (on the same side as O), we will have that $\angle AOB = 2\angle ACB$. But why should this be the case? Let us prove it!

Example 3.2 *Prove Theorem 3.1*

As we want to prove this property about circles, we must use the definition of a circle! So let's fill in some missing lines and see what we get!



Since O is the centre of the circle, we must have that

$$OA = OB = OC.$$

Notice that this gives us lots of isosceles triangles! We know that the base angles of isosceles triangles are equal so

$$\angle OCA = \angle OAC = \alpha$$
$$\angle OBC = \angle OCB = \beta$$
$$\angle OAB = \angle OBA = \gamma$$

Now we can find our angles of interest in terms of α, β and γ! Clearly

$$\angle ACB = \alpha + \beta. \tag{1}$$

And using the angle sum of a triangle, we see that

$$\angle AOB = 180^\circ - 2\gamma. \tag{2}$$

But notice that the angle sum of $\triangle ABC$ is 180° so that

$$2\alpha + 2\beta + 2\gamma = 180^\circ.$$

Or in other words,

$$180^\circ - 2\gamma = 2\alpha + 2\beta.$$

Combining this with (2) gives us that

$$\begin{aligned}
\angle AOB &= 2\alpha + 2\beta \\
\angle AOB &= 2(\alpha + \beta) \\
\angle AOB &= 2\angle ACB \quad \text{from (1)}
\end{aligned}$$

□

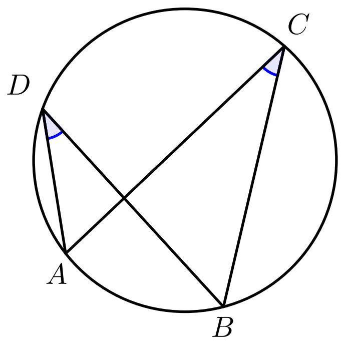
Remark: Notice how every step of the proof either uses known **definitions** or **theorems**. To prove something rigorously in mathematics, it is important that we only use definitions and theorems in each step.

That being said, now that we have proven Theorem 3.1, we can use it in future proofs!

Exercise 3.3 *What is the significance of this theorem when AB is a diameter?*

Let's look at some other theorems.

Theorem 3.4 *An arc subtends a constant angle at the circumference of the circle.*

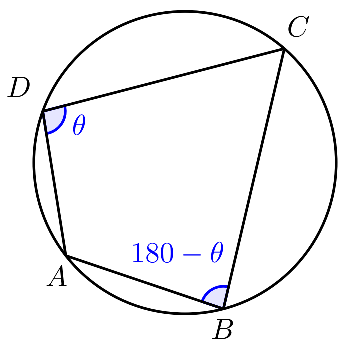


Here, AB is any arc on the circle and C and D are any points on the circumference on the same side of A and B. Then we will have

$$\angle ACB = \angle ADB.$$

Exercise 3.5 *Prove Theorem 3.4*

Theorem 3.6 *Opposite angles of a cyclic quadrilateral are supplementary.*



So if we take any four points on a circle A, B, C, D in that order, then we will always have that

$$\angle ADC + \angle ABC = 180^\circ$$

Exercise 3.7 *Prove Theorem 3.6*

Extension: If you have seen circle geometry before, or if you are looking for a challenge, do some research on **directed angles** and explain why Theorem 3.4 and Theorem 3.6 are actually the same thing!

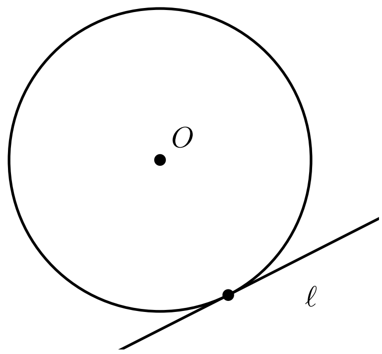
4 Tangents

In geometry, we are interested in studying the relationship between geometric objects and as **circles** and **lines** are the some of the simplest geometric objects, we can investigate the possible relationships between them.

It is not too hard to see that a line can intersect a circle either 0, 1 or 2 times. They intersect 0 times if the line simply doesn't pass the circle, or they intersect 2 times if the passes through the circle. But there is the very interesting case where a line intersects the circle exactly once (or when the line "touches" the circle)!

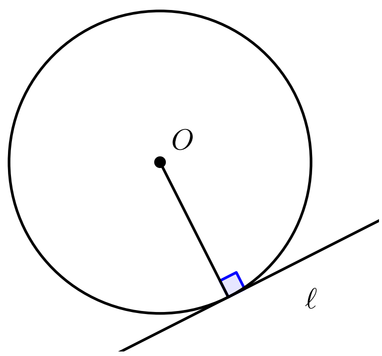
Let us formally define what a tangent is.

Definition 4.1 *A **tangent** is a line that intersects a circle at **exactly one** point.*



Now we shall look at a few key theorems involving tangents!

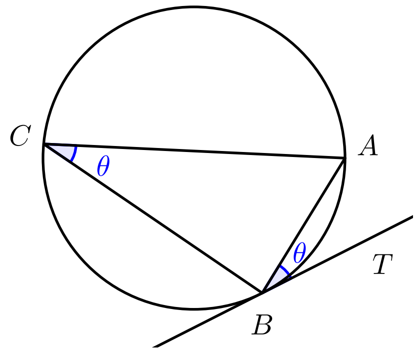
Theorem 4.2 *The radius from the centre to the point of contact of a tangent to a circle is perpendicular to the tangent*



Remark: This theorem arises from the fact that the radius is a line of symmetry of the diagram, so the radius must be perpendicular to the tangent, however this is not a proof!

Exercise 4.3 *Prove Theorem 4.2. Hint: you may want to try a proof by contradiction! If the angle is not* $90°$*, then on one side it is smaller than* $90°$*...*

Theorem 4.4 ***(Alternate Segment Theorem)*** *The angle that a tangent makes with a chord of the circle is equal to the angle subtended by the chord in the alternate segment.*



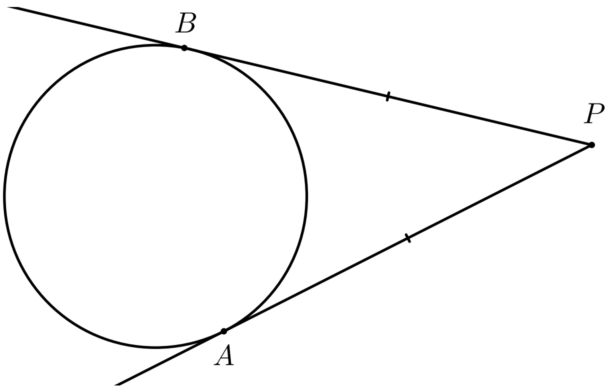
What this means is that if AB is any chord of the circle, let T be a point on the tangent to the circle at B. Let C be any point on the circumference of the circle in the segment excluding $\angle ABT$. Then

$$\angle ABT = \angle ACB.$$

Exercise 4.5 *Prove the Alternate Segment Theorem.*

Hint: Remember that we will have to use the definition of a circle, and we can use any result we have already proven!

Theorem 4.6 *The length of the tangents from an external point to the points of contact are equal.*



This theorem tells us that if P is any external point to the circle, then we can draw the two tangents from P to the circle, meeting it at A and B, and we will have

$$PA = PB.$$

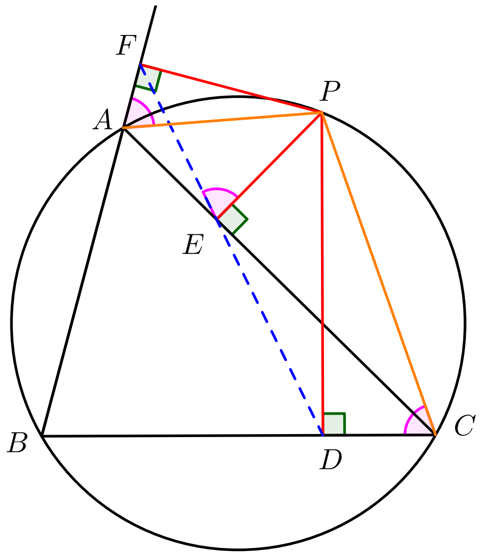
Remark: This theorem is often colloquially referred to as the *"ice-cream cone theorem"*. Why do you think that's the case?

Exercise 4.7 *Prove Theorem 4.6*

5 Applications

The theorems that we have proven above can be used to prove many fascinating properties about circles. Below is one example and there are several more in the problem set.

Example 5.1 ***(Simson Line)*** *Consider a triangle ABC and a circle $\mathcal{C}$ that passes through the three vertices. Let P be a point of $\mathcal{C}$ different from A, B and C. Let D, E, F be the feet of the perpeniduclars from P to BC, CA and AB respectively (possibly extended). Prove that D, E and F are collinear.*



Immediately, we can see a lot of cyclic quadrilaterals! $ABCP$ is clearly cyclic as all its vertices lie on $\mathcal{C}$. This gives us that

$$\angle PAB + \angle PCB = 180°. \tag{3}$$

Also notice that $AFPE$ is cyclic quadrilateral, since $\angle AFP + \angle AEP = 90° + 90° = 180°$. This gives us that

$$\angle FAP = \angle FEP. \tag{4}$$

Simialrly, $DEPC$ is cyclic quadrilateral, since $\angle PDC = \angle PEC = 90°$. This gives us that

$$\angle PED + \angle PCD = 180°. \tag{5}$$

Now consider $\angle FED$:

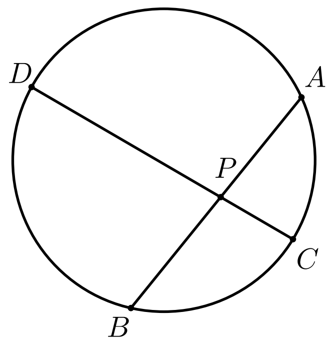
$$\begin{aligned} \angle FED &= \angle FEP + \angle PED \\ &= \angle FAP + 180° - \angle PCD \quad \text{from (4) and (5)} \\ &= \angle FAP + 180° - \angle PCB \\ &= \angle FAP + \angle PAB \quad \text{from (3)} \\ &= 180° \end{aligned}$$

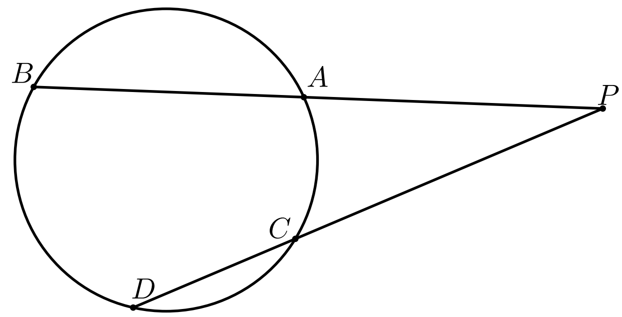
Since $\angle FED = 180°$, we see that D, E and F are collinear. □

6 Power of a Point (Extension)

Theorem 6.1 ***(Power of a Point)*** *Given a circle and any point P not on the circle, let a line passing through P intersect the circle at A and B, and let another line passing through P intersect the circle at C and D. Then*

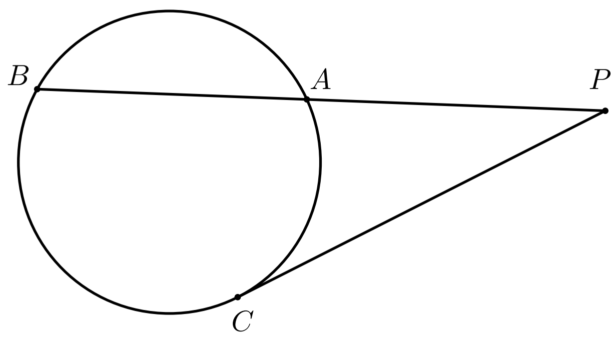
$$PA \cdot PB = PC \cdot PD$$





Remark: Note that P can lie inside or outside the circle as in the two diagrams above. In a special case, the two points of intersection can coincide, so that the secant becomes a tangent. As in the diagram below, we would have

$$PA \cdot PB = PC^2.$$



Exercise 6.2 *Prove the three cases of the Power of a Point Theorem.*

Hopefully you have been wondering why this theorem is called "Power of a Point"! This has to do with the following definition.

Definition 6.3 *Given a circle $\mathcal{C}$ and a point P on the plane, let d be the distance from P to the centre of the circle and let r be the radius of the circle. Then define the* ***power*** *of P to be*

$$h = d^2 - r^2$$

Exercise 6.4 *Prove that this quantity h is equal to the quantities in the Power of a Point Theorem 5.1.*

This definition leads to some interesting properties about circles!

Exercise 6.5 *Consider two circles in the plane. What is the locus of points which have equal power to both circles?*

Remark: This is called the **radical axis** of the two circles.

Exercise 6.6 *Now consider three circles in the plane. For each pair of circles, construct the radical axis. Prove that these three radical axes are concurrent.*

Remark: The point of intersection is called the **radical centre** of the three circles.