Enrichment stage 1: (log expression and equation)

1. Simplify the following expressions:

 (a) $\log_b \frac{x^2}{\sqrt{y}} - 2\log_b xy$

 (b) $\frac{\log_a x^3}{2\log_a x}$

2. Given that $\log_a\left(x\sqrt{y}\right)=1$ and $\log_a\left(x^2y^2\right)=1$, calculate the value of $\log_a\left(y\sqrt{x}\right)$.

3. If $\log_{10}(1+y)-\log_{10}(1-y)=x$, prove that $y=\frac{10^x-1}{10^x+1}$.

4. Given that $2^{2x} - 2y(2^x) - 1 = 0$, show that $x = \log_2[y + \sqrt{(y^2 + 1)}]$.

5. Evaluate $\dfrac{1}{1+\log_a bc} + \dfrac{1}{1+\log_b ac} + \dfrac{1}{1+\log_c ab}$, where a, b, c > 0 and not equal to 1.

6. Solve for x: $(x^2 - 3x + 1)^{x+1} = 1$. Start by considering cases, e.g. $a^0 = 1$

Enrichment stage 2: (log function)

1. Show that $f(x) = \log_{10}\left(1+10^x\right) - \dfrac{x}{2}$ is an even function.

f (-x) = log10 (1 + 10-x) + x/2

= log10 [(10x + 1)/10x] + x/2

= log10 (10x + 1) - log10 (10x) + x/2

= log10 (10x + 1) - x + x/2

= log10 (1 + 10x) - x/2

= f (x)

2. If $f(x) = \log_{10}(1+x) - \log_{10}(1-x)$, for -1 < x < 1.

   Find the value of $f\left(\dfrac{2x}{1+x^2}\right)$ in terms of $f(x)$.

f [2x/(1 + x²)] = log10 [1 + 2x/(1 + x²)] - log10 [1 - 2x/(1 + x²)]

= log10 [(x² + 2x + 1)/(1 + x²)] - log10 [(x² - 2x + 1)/(1 + x²)]

= log10 [(x² + 2x + 1)/(x² - 2x + 1)]

= log10 [(x + 1)²/(x - 1)²]

= 2 log10 [(x + 1)/(x - 1)]

= 2[log10 (x + 1) - log10 (x - 1)]

= 2f (x)

Competition stage 1:

1. Given that $3^{x+4} = 4^{x+3}$, find x.

2. Given that $2^{x-2} = 3^{x+2}$, find x.

3. Given that $a^{x+c} = b^{x+d}$, find x.

Competition stage 2:

1. (a) It is given that $a = k_1n + r_1$ and $b = k_2n + r_2$, where a, b, k_1, k_2, r_1 and r_2 are integers. Show that $ab = k_3n + r_1r_2$, where k_3 is an integer.

$ab = (k_1n + r_1)(k_2n + r_2)$

$= k_1k_2n^2 + k_1r_2n + k_2r_1n + r_1r_2$

$= (k_1k_2n + k_1r_2 + k_2r_1)n + r_1r_2$

$= k_3n + r_1r_2$, where $k_3 = k_1k_2n + k_1r_2 + k_2r_1$ is an integer

(b) Hence, show that $a^k = cn + r_1^k$, where c is an integer.

$a^k = a \times a \times a \times \ldots \times a$ (k terms)

$= (m_1n + r_1^2) \times a \times \ldots \times a$, from (a)

$= (m_2n + r_1^3) \times \ldots \times a$, from (a)

$= \ldots$

$= (m_{k-2}n + r_1^{k-1}) \times a$, from (a)

$= cn + r_1^k$, from (a)

(c) Deduce that $4 \times 27^k + 22$ is divisible by 13 for all positive integers k.

$27 = 2 \times 13 + 1$

$4 \times 27^k + 22 = 4 \times (13c + 1^k) + 22$, from (b)

$= 4 \times (13c + 1) + 22$

$= 52c + 4 + 22$

$= 52c + 26$

$= 13(4c + 2)$

$= 13N$, where $N = 4c + 2$ is an integer

Therefore, $4 \times 27^k + 22$ is divisible by 13.

Enrichment stage 1:

1. (a)

$\log_b(x^2/\sqrt{y}) - 2\log_b(xy) = 2\log_b x - 1/2 \times \log_b y - 2\log_b x - 2\log_b y$

$= -3/2 \times \log_b y$

(b)

$\log_a x^3 - 2\log_b x = 3\log_a x - 2\log_a x$

$= \log_a x$

2. $\log_a(x\sqrt{y}) = 1 \quad \Rightarrow \quad x\sqrt{y} = a \quad (1)$

$\log_a(x^2y^2) = 1 \quad \Rightarrow \quad x^2y^2 = a \quad (2)$

$(2)/(1) : y = 1/a \quad (3)$

Sub (3) into (1): $x = a\sqrt{a} \quad (4)$

$\log_a(y\sqrt{x}) = \log_a(1/a \times \sqrt{(a\sqrt{a})})$

$= \log_a(a^{-1} \times a^{3/4})$

$= \log_a(a^{-1/4})$

$= -1/4$

3. $\log_{10}(1 + y) - \log_{10}(1 - y) = x$

$\log_{10}[(1 + y)/(1 - y)] = x$

$10^x = (1 + y)/(1 - y)$

$10^x - y10^x = 1 + y$

$10^x - 1 = y(10^x + 1)$

$y = (10^x - 1)/(10^x + 1)$

4. Let $u = 2^x$.

$u^2 - 2yu - 1 = 0$

$u = \{2y \pm \sqrt{[(-2y)^2 - 4 \times 1 \times (-1)]}\}/2$

$= [2y \pm \sqrt{(4y^2 + 4)}]/2$

$= [2y \pm 2\sqrt{(y^2 + 1)}]/2$

$= y \pm \sqrt{(y^2 + 1)}$

$\because u = 2^x > 0$

$u = y + \sqrt{(y^2 + 1)} > 0$

$2^x = y + \sqrt{(y^2 + 1)}$

$x = \log_2[y + \sqrt{(y^2 + 1)}]$

5. $1/(1 + \log_a bc) + 1/(1 + \log_b ac) + 1/(1 + \log_c ab)$
$= 1/(1 + \log bc/\log a) + 1/(1 + \log ac/\log b) + 1/(1 + \log ab/\log c)$
$= \log a/(\log a + \log bc) + \log b/(\log b + \log ac) + \log c/(\log c + \log ab)$
$= \log a/\log abc + \log b/\log abc + \log c/\log abc$
$= (\log a + \log b + \log c)/\log abc$
$= \log abc/\log abc$
$= 1$

6. Case 1: Exponent is 0
$x + 1 = 0$
$x = -1$
Case 2: Base is 1
$x^2 - 3x + 1 = 1$
$x^2 - 3x = 0$
$x(x - 3) = 0$
$x = 0, 3$
Case 3: Base is -1 and exponent is even
$x^2 - 3x + 1 = -1$
$x^2 - 3x + 2 = 0$
$(x - 1)(x - 2) = 0$
$x = 1, 2$
Test $x = 1$ to check if exponent is even.
$x + 1 = 2$ is even
Test $x = 2$ to check if exponent is even.
$x + 1 = 3$ is odd
$\therefore$ Only $x = 1$
$\therefore x = -1, 0, 1, 3$

Enrichment stage 2:

1. $f(-x) = \log_{10}(1 + 10^{-x}) + x/2$

   $= \log_{10}[(10^x + 1)/10^x] + x/2$

   $= \log_{10}(10^x + 1) - \log_{10}(10^x) + x/2$

   $= \log_{10}(10^x + 1) - x + x/2$

   $= \log_{10}(1 + 10^x) - x/2$

   $= f(x)$

2. $f[2x/(1 + x^2)] = \log_{10}[1 + 2x/(1 + x^2)] - \log_{10}[1 - 2x/(1 + x^2)]$

   $= \log_{10}[(x^2 + 2x + 1)/(1 + x^2)] - \log_{10}[(x^2 - 2x + 1)/(1 + x^2)]$

   $= \log_{10}[(x^2 + 2x + 1)/(x^2 - 2x + 1)]$

   $= \log_{10}[(x + 1)^2/(x - 1)^2]$

   $= 2\log_{10}[(x + 1)/(x - 1)]$

   $= 2[\log_{10}(x + 1) - \log_{10}(x - 1)]$

   $= 2f(x)$

Competition stage 1:

1. $3^x 3^4 = 4^x 4^3$
   $3^x/4^x = 4^3/3^4$
   $(3/4)^x = 4^3/3^4$
   $\log(3/4)^x = \log(4^3/3^4)$
   $x\log(3/4) = \log(4^3/3^4)$
   x(log3-log4)=3log4-4log3
   x=(3log4-4log3)/(log3-log4)

2. $2^x 2^{-2} = 3^x 3^2$
   $2^x/3^x = 3^2/2^{-2}$
   $(2/3)^x = 3^2 2^2$
   $\log(2/3)^x = \log(3^2 2^2)$
   $x\log(2/3) = \log(3^2 2^2)$
   x(log2-log3)=2log3+2log2
   x=(2log3+2log2)/(log2-log3)

3. $a^x a^c = b^x b^d$
   $a^x/b^x = b^d/a^c$
   $(a/b)^x = b^d/a^c$
   $\log(a/b)^x = \log(b^d/a^c)$
   $x\log(a/b) = \log(b^d/a^c)$
   x(loga-logb)=dlogb-cloga
   x=(dlogb-cloga)/(loga-logb)

Competition stage 2:

1. (a)

$ab = (k_1n + r_1)(k_2n + r_2)$

$= k_1k_2n^2 + k_1r_2n + k_2r_1n + r_1r_2$

$= (k_1k_2n + k_1r_2 + k_2r_1)n + r_1r_2$

$= k_3n + r_1r_2$, where $k_3 = k_1k_2n + k_1r_2 + k_2r_1$ is an integer

(b)

ak = a × a × a × ... × a (k terms)

= (m1n + r1 ^{2}) × a × ... × a, from (a)

= (m2n + r1 ^{3}) × ... × a, from (a)

= ...

= (mk-2n + r1k-1) × a, from (a)

= cn + r1k, from (a)

(c)

$27 = 2 \times 13 + 1$

$4 \times 27k + 22 = 4 \times (13c + 1k) + 22$, from (b)

$= 4 \times (13c + 1) + 22$

$= 52c + 4 + 22$

$= 52c + 26$

$= 13(4c + 2)$

$= 13N$, where $N = 4c + 2$ is an integer

Therefore, $4 \times 27k + 22$ is divisible by 13.