Problem Set 10

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Instructions

- Some of these problems are based off the notes "*Polynomials*". Some other are revision problems for the previous notes.
- They are in roughly difficulty order and get quite difficult, so you are **not** expected to be able to solve every problem.
- However, please attempt as many questions as you can and submit your solutions to your mentor for marking and feedback.
- You may (and encouraged to) submit incomplete solutions if you can not solve a problem completely.
- You may type your solutions or submit a pdf of a **clear** scan/photo of **legible** written solutions.
- Feel free to discuss these problems with your peers and on the forum but the solutions you submit must be written by yourself.

Problems

1. If $P(x)$ is a polynomial with degree m and $Q(x)$ is a polynomial of degree n, what is the degree of the polynomial $P(Q(x))$?

   Note: $P(Q(x))$ is the polynomial obtained when each x in $P(x)$ is replaced by $Q(x)$. For example, if $P(x) = x^2 + 2x - 5$ and $Q(x) = x + 1$, then $P(Q(x)) = (x+1)^2 + 2(x+1) - 5$.

2. On a cube, there are seven vertices marked 0 and one marked 1. It is permitted to add 1 to any two neighbouring vertices (that is, two vertices connected by an edge). Is it possible that all the numbers are divisible by 3 after a finite number of steps?

3. Consider the polynomial
   $$P(x) = x^{2022} - 2x^{2021} + 1$$
   (a) What is the remainder when $P(x)$ is divided by $x - 2$?
   (b) What is the remainder when $P(x)$ is divided by $x + 1$?
   (c) What is the remainder when $P(x)$ is divided by $x^2 - x - 2$?

4. Inez's swimming team consists of n people and they each have their own locker labelled with their names. Inez is feeling a little sneaky and wants to rearrange the name labels in such a way that no locker has the correct name label. Suppose that the number of ways Inez can do this is D_n.
   (a) Find D_1, D_2, D_3.
   (b) Prove that $D_n = (n-1)(D_{n-1} + D_{n-2})$.
   (c) Find D_7.

5. Find all polynomials $P(x)$ with real coefficients such that
   $$(x - 27)P(3x) = 27(x - 1)P(x).$$