Problem Set 2

MaPS Correspondence Program

Instructions

- This problem set is based off the notes "*Divisibility*".
- They are in roughly difficulty order and get quite difficult, so you are **not** expected to be able to solve every problem.
- However, please attempt as many questions as you can and submit your solutions to your mentor for marking and feedback.
- You may (and encouraged to) submit incomplete solutions if you can not solve a problem completely.
- You may type your solutions or submit a pdf of a **clear** scan/photo of **legible** written solutions.
- Feel free to discuss these problems with your peers and on the forum but the solutions you submit must be written by yourself.

Problems

1. (a) Is it true that if a, b and n are integers, $n|ab \implies n|a$ or $n|b$? If yes, prove it. If not, find a counter-example and salvage the statement.

   (b) Is it true that if a, b and n are integers, $a|n$ and $b|n \implies ab|n$? If yes, prove it. If not, find a counter-example and salvage the statement.

2. Prove that there are infinitely many primes of the form $6n + 5$ where n is a positive integer.

3. Prove that the sequence

   $$1, 61, 661, 6661, 66661, \ldots$$

   has infinitely many numbers divisible by 7.

4. Prove that the fraction

   $$\frac{21n + 4}{14n + 3}$$

   is irreducible for every positive integer n. *[Note: this problem appeared in the 1959 International Mathematical Olympiad!]*

5. (a) Prove that if $2^n - 1$ is a prime number, then n must be a prime. *[Note: primes of this form are called Mersenne Primes]*

   (b) Prove that if $2^n + 1$ is a prime number, then n must be a power of 2. *[Note: primes of this form are called Fermat Primes]*