Mathematical Induction

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1 Introduction

Last term, we were introduced to the importance of **proofs** in mathematics and saw a few methods of proof. In this topic, we will investigate another method of proof: **Mathematical Induction**. This is a very useful tool that is used in many scenarios to prove that a statement is **true** for **infinitely** many integers (but this idea can be extended to more than just integers).

This means that if you ever stumble across a question where you are required to prove a statement for **infinitely** many positive integers, **mathematical induction** should come to mind! In this set of notes, we will explore a few different types of problems that can be solved using induction

2 Structure of Mathematical Induction

Every proof by induction has two steps:

- **The Base Case:** prove that the statement is true for the **smallest** required number.
- **The Inductive Step:** prove that if the statement is true for some particular integer, say k, then it is true for the next integer, that is $k + 1$.

An analogy for this is to prove that a chain of infinitely many dominoes will fall over. If we prove that:

- **The Base Case:** the **first** domino will fall over.
- **The Inductive Step:** if **any** domino falls over, the **next** one will fall over as well

Then we can combine these statements to conclude that all infinitely many dominoes will fall over.

3 Applications

Induction can be used to solve a diverse range of problems. Here we explore a few possible applications.

3.1 Series

These are expressions that contain n terms and we wish to prove that some equality holds where n can be any positive integer.

Example 3.1 *Prove that*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{1}$$

for all positive integers n.

Base Case: We prove that (1) holds for $n = 1$. $LHS = 1$ and $RHS = \frac{1 \cdot 2}{2} = 1$. Thus $LHS = RHS$ and so we have verified (1) for the base case.

Inductive Step: We now assume that (1) is true for an arbitrary integer $n = k$, so we are assuming that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

We now try to prove that (1) is true for $n = k + 1$.

From our assumption, we can then deduce that

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ \implies 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1) + 2(k+1)}{2} \\ \implies 1 + 2 + \cdots + k + (k+1) &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Which is (1) for $n = k + 1$. Thus, we have proven that if (1) is true for some arbitrary integer k, it will also be true for $k + 1$. Since it is true for $n = 1$, it will also be true for $n = 1 + 1 = 2$, and so it is also true for $n = 2 + 1 = 3$ and so on. Thus, $1 + 2 + \cdots n = \frac{n(n+1)}{2}$ for all positive integers n. □

Exercise 3.2 *Prove that for all positive integers n,*

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

3.2 Inequalities

Let's look at another example where we apply induction to prove an inequality.

Example 3.3 *Prove that $2^n > n^2$ for $n \geq 5$*

Base Case: If $n = 5$,

$$LHS = 2^5 = 32$$

and

$$RHS = 5^2 = 25.$$

So $LHS > RHS$. Hence, $2^n > n^2$ holds for the base case $n = 5$.

Inductive Step: We now assume that the statement is true for some integer $n = k \geq 5$, ie. assume that $2^k > k^2$. We now try to prove that it is true for $n = k + 1$, ie. we try to prove that $2^{k+1} > (k+1)^2$. From our assumption we have:

$$
\begin{aligned}
& 2^k > k^2 \\
\implies & 2 \cdot 2^k > 2 \cdot k^2 \\
\implies & 2^{k+1} > k^2 + k^2 \\
\implies & 2^{k+1} > k^2 + 4k \quad \text{Since } k > 4 \\
\implies & 2^{k+1} > k^2 + 2k + 2k \\
\implies & 2^{k+1} > k^2 + 2k + 1 \quad \text{Since } k \geq 1 \\
\implies & 2^{k+1} > (k+1)^2
\end{aligned}
$$

Hence, if $2^n > n^2$ for some integer $n = k$, then it is also true for $n = k + 1$. Since it is true for $n = 5$, then it is also true for $n = 5 + 1 = 6$, and also for $n = 6 + 1 = 7$ and so on. Thus, $2^n > n^2$ for all integers $n \geq 5$. □

Exercise 3.4 *Show that for all positive integers $n > 1$:*

$$12^n > 7^n + 5^n.$$

Exercise 3.5 *Prove that for all positive integers n:*

$$(2n)! \geq 2^n (n!)^2.$$

3.3 Recurrences

Induction can also be a useful tool to prove **recurrences**, that is, sequences where each term is defined from the previous terms.

Example 3.6 *Consider a sequence of numbers defined by $a_0 = 0$ and $a_i = 2a_{i-1} + 1$ for $i = 1, 2, 3, \ldots$. Prove that $a_n = 2^n - 1$ for $n \in \mathbb{Z}_0^+$ (that is, all non-negative integers).*

To understand what this recurrence means, each term is **double** the previous term **plus one**, and the starting term is 0. So the first few terms looks like 0, 1, 3, 7, 15, ... You can probably see the pattern in these numbers, but we need to use induction to prove that it will always hold.

Base Case: For the smallest value of n, we are given that $a_0 = 0$, but $2^0 - 1 = 0$. So the equation $a_n = 2^n - 1$ holds for the base case where $n = 0$

Inductive Step: Now we assume that $a_k = 2^k - 1$ for some integer k.

We have that

$$
\begin{aligned}
a_{k+1} &= 2a_k + 1 \text{ by the definition of the sequence} \\
a_{k+1} &= 2(2^k - 1) + 1 \text{ by the assumption} \\
a_{k+1} &= 2^{k+1} - 2 + 1 \\
a_{k+1} &= 2^{k+1} - 1
\end{aligned}
$$

Hence, if $a_k = 2^k - 1$, then $a_{k+1} = 2^{k+1} - 1$. Thus, by induction, $a_n = 2^n - 1$ for all non-negative integers n.

□

Exercise 3.7 *Prove that the recurrence defined by $x_{n+1} = 3x_n - 2$ where $x_0 = 0$ is $x_n = -(3^n - 1)$.*

3.4 Combinatorics

Induction also has many applications to problems outside of number theory! Here we will look at an example from combinatorics.

Example 3.8 *Prove that the number of ways to choose a (possible empty) subset of n elements is 2^n where $n \geq 1$.*

For example, if we have **two elements** in a set, say $\{A, B\}$ then there are **4 possible subsets**:

$$\{\}, \{A\}, \{B\}, \{A, B\}.$$

Or if we have **three elements** in a set, say $\{A, B, C\}$, there are **8 possible subsets**:

$$\{\}, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}$$

Notice that the **order** of the elements is **not important**. Now let's use induction to prove that this always holds!

Base Case: If we only have 1 element, then there are $2 = 2^1$ possible subsets: one with the element and one without. Thus the statement is true in the base case.

Inductive Step: Assume that there are 2^k possible subsets from a set of k elements. If we have $k + 1$ elements, let's count how many subsets we can form by considering the final element:

- If we **include the final element**, then any subset of the remaining k elements can be included. By the assumption there were 2^k subsets from a set of k elements, so there are a total of 2^k subsets **including the final element**.
- In a similar way, if we do **not include the final element**, we can still form a subset with any subset of the remaining k elements. Again by the assumption, there are 2^k such subsets, so there are 2^k subsets **not including the final element**.

Hence in total, among $k + 1$ elements, there are $2^k + 2^k = 2^{k+1}$ subsets, and so the statement is true by induction. □

Remark: we could have actually done the proof starting from $n \geq 0$, because there is only $1 = 2^0$ ways of selecting a subset from 0 elements: by taking the empty set!

Exercise 3.9 *Suppose that n chords are drawn in a circle such that every pair of chords intersect exactly once and no three chords are concurrent. Prove that the chords cut the circle into $\frac{n^2+n+2}{2}$ regions.*

4 Strong Induction

A variant of induction, called **strong induction**, follows the same principle as induction but uses a more powerful assumption in the inductive step, that is:

- **The Inductive Step:** prove that if the statement is true for *all integers up to a particular number*, say $1, 2, 3, \ldots, k$, then it is true for the next number $k + 1$.

Example 4.1 *Define the Fibonacci Sequence by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Define $\phi = \dfrac{1 + \sqrt{5}}{2}$ and prove that $F_n \geq \phi^{n-2}$ for $n \geq 2$. Note that $\phi^2 = \phi + 1$.*

Base Case: $F_2 = F_1 + F_0 = 1 + 0 = 1$ and $\phi^{2-2} = \phi^0 = 1$, and so $F_2 \geq \phi^0$, thus the base case holds.

Inductive Step: Assume that $F_n \geq \phi^{n-2}$ for all integers less than or equal to some integer k (and greater than or equal to 2), ie. assume that $F_k \geq \phi^{k-2}$, $F_{k-1} \geq \phi^{k-3}$, $F_{k-2} \geq \phi^{k-4}, \ldots$.

We now wish to prove that the statement holds for $n = k + 1$, ie. $F_{k+1} \geq \phi^{k-1}$. Thus, we have:

$$\begin{aligned}
F_{k+1} =& F_k + F_{k-1} \\
\geq& \phi^{k-2} + \phi^{k-3} \quad \text{From our inductive assumption} \\
=& \phi^{k-3}(\phi + 1) \\
=& \phi^{k-3} \cdot \phi^2 \quad \text{Using the fact that } \phi^2 = \phi + 1 \\
=& \phi^{k-1}
\end{aligned}$$

Hence, we have shown that $F_{k+1} \geq \phi^{k-1}$ which completes our induction. $\square$

Remark: Notice how this implies that the Fibonacci Sequence grows **exponentially**, which is not immediately obvious but is a trait shared by most recurrences.

Exercise 4.2 *Prove that any positive integer can be uniquely expressed as the sum of distinct powers of 2.*

5 Induction Trap!

When we use induction, we need to be careful to make sure that all of our assumptions hold! Consider this "proof".

Proposition 5.1 *All cats are black.*

To prove this, we shall instead prove that "For any group of n cats, if at least one of them is black, then they are all black." for all positive integers n.

Base Case: If there is a group of one cat, and at least one of them is black, then all the cats in this group are clearly black.

Inductive Step: Now assume that among any group of 1, 2, ... , k cats, if at least one of them is black, then they are all black.

In a group of $k + 1$ cats, say $\{c_1, c_2, c_3, \ldots, c_k, c_{k+1}\}$ assume that one of them, say c_1 is black. We can now consider two different subsets of these cats: Group 1 $\{c_1, c_2\}$ and Group 2 $\{c_1, c_3, c_4, \ldots, c_{k+1}\}$. These groups contain 2 and k cats respectively, and both contain at least one black cat, and so by the inductive assumption, all of those cats must be black.

Hence, all $k + 1$ cats are black, and so by induction, for any group of n cats, if there is at least one black cat, then they are all black. We can then take the set of all cats in the world, which contains at least one black cat, and so all cats are black. $\square$.

What has gone wrong here? Clearly not all cats in the world are black!

What has actually happened is that our inductive step only works if $k \geq 3$, and our base case only addresses the case where $k = 2$. That is, the statements are **individually true**, but combining them together **does not** prove that all cats are black.

6 Harder Practice Problems

These are a few harder problems you can try for practice.

1. Prove that for all positive integers n, $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

   Notice how this implies that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$

2. Prove that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} < 1$ for all positive integers n.

3. Prove that for all positive integers $(\sqrt{n})^n \leq n! \leq \left(\frac{n+1}{2}\right)^n$

4. The nth triangular number is defined as the sum of the first n positive integers. Let T_n denote the sum of the first n triangular numbers. Derive a formula for T_n and hence or otherwise prove that $T_n + 4T_{n-1} + T_{n-2} = n^3$