Colourings and Invariants

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1 Introduction

As the name implies, an *invariant* is something which **doesn't change**. They are a powerful, simple and rigorous method mainly used to show that some configuration is **impossible**. Remember that to show that a configuration is **possible**, we just need to provide **one example**, but to show that something is **impossible**, we must show that **every single case will not produce a valid solution**.

Invariants can come up in numerous shapes and forms, and so they often require a bit of exploring before they can be found. This set of notes will explore some common types of invariants using some examples to illustrate such ideas.

2 Numerical Invariants (Whiteboard Questions)

Problem 2.1

On a whiteboard are the numbers $1, 2, \ldots, 100$. In each second, Ellen selects two numbers a, b already on the board, removes them and writes the number $a + b$ on the board.

(a) How many moves does it take for exactly one number to remain on the board?

(b) Prove that no matter how Ellen does her moves, the final number is always the same (and find this number).

Proof. Intuitively speaking, every time we do an operation, the number of numbers on the board **decreases by exactly one** since we delete two numbers and add one. This means by the time we get to only 1 number remaining, we must have done $100 - 1 = 99$ operations. □

If we operate a and b, then we get $a + b$. Then, if we operate the numbers $a + b$ and c, then we get $a + b + c$; in short, it is obvious that the **sum** of all numbers on the board never changes (is **invariant**) regardless of the operation. It follows that the final number will always be equal to $1 + 2 + \cdots + 100 = 5050$. □

Problem 2.2

In a whiteboard, the numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2022}$ are written. At any stage, James may pick two numbers a, b on the board, delete them and write the number $ab + a + b$. He repeats this until there is only one number remaining. What is this number?

Solution. This doesn't look as simple as the previous example, since the invariant isn't immediately obvious to us. However, notice that $ab+a+b$ looks very much like $(a+1)(b+1)$, except it's missing a -1. Hence the operation is $a, b \longrightarrow (a+1)(b+1) - 1$, or

$$(a+1) - 1, (b+1) - 1 \longrightarrow (a+1)(b+1) - 1.$$

It now becomes clear that operating any three numbers a, b, c in any order then gives $(a+1)(b+1)(c+1) - 1$, and so on. In other words, if the numbers $a_1, a_2, \ldots, a_n$ are written on the board at the moment, the following expression

$$(a_1 + 1)(a_2 + 1) \cdots (a_n + 1) - 1$$

remains the same i.e. is **invariant**, whatever operation we choose. Hence, the final number on the board will always be equal to

$$(1+1)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{2022}\right) - 1 = \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \cdots \times \frac{2023}{2022} - 1 = 2023 - 1 = 2022. \square$$

Problem 2.3. Same question as **Problem 2.1**, except that Ellen writes the number ab on the board instead of $a + b$ on every move.

Problem 2.4. Dmitry the Magician has a set of 24 magical rods of length $1, 2, \ldots, 24$. Every night, he picks two rods of lengths a, b and holds them at right angles to each other. He then exclaims "hypotenate!", which makes the two rods instantly disappear and are replaced by a third rod that would have completed the triangle in the air. After many nights, Dmitry has only one rod remaining. Find the length of this rod.

For clarity, the operation is taking two rods length a, b and replacing them with $\sqrt{a^2 + b^2}$.

Problem 2.5. Vaishnavi has the numbers $1, 2, \ldots, 2022$ written on the whiteboard. In each move, she takes two numbers a, b, and replaces them by the number $ab - a - b + 2$. She repeats this until only one number remains. What is this final number?

Problem 2.6. Ellen is back! On the whiteboard are the numbers $1, 2, 3, \ldots, 10$. This time, she selects two numbers a and b and replaces them with $\dfrac{4a+3b}{5}$ and $\dfrac{4b-3a}{5}$. If at some point all the numbers are equal, Ellen will buy everyone chocolates. Will everyone get chocolates?

3 Parity/Modulo Invariance

Problem 3.1

On a sheet of paper are the numbers $1, 2, \ldots, 100$. Isaac picks two numbers a, b and replaces them by $|a-b|$. He repeats this until there is only one number remaining. Prove that the final number must be even.

Unfortunately, this question isn't quite as simple as the $a+b$ variant since depending on how Isaac chooses his moves, the final answer may not be the same (although despite that, it's always even). How do we address this?

Proof. We consider the number **modulo 2** (which makes sense because we want to show the final number is even). It is trivial to check that $|a-b| \equiv a+b \pmod{2}$, regardless of whether $a > b$ or $a < b$.

The operation of the problem then becomes the following. Isaac has the numbers $1, 2, \ldots, 100$ on the sheet of paper, and on each turn he takes a, b and replaces them with a number with the same parity as $a+b$. It becomes obvious that the final number has the same parity as $1+2+\cdots+100 = \frac{100\cdot 101}{2}$, which is an even number. It follows that the final number must always be even. $\square$

Problem 3.2

In the planet of Dmitros, there are $3n$ chameleons for some $n \geq 1$: n reds, n blues and n yellows. In each day, it is possible for exactly one pair of chameleons of different colours to merge and turn into a chameleon of the third colour (so for example, a red and blue may combine to become a single yellow). This operation continues until no more such moves can be made. Is it possible that exactly one chameleon remains?

Solution. Let's consider the number of chameleons of each colour modulo 2. It is quite easy to show that in any of the three types of possible moves, the number of red, blue and yellow chameleons at any step must have the same parity. This cannot occur for the final configuration, since the numbers in (1,0,0) cannot all have the same parity. $\square$

Problem 3.3. The numbers $1, 2, \ldots, 100$ are written on the whiteboard. Rachel may choose three numbers, say a, b, c, and replace them with the number $a+b-2c$. She keeps doing this until exactly two numbers remain. Is it possible that the sum of the two final numbers is divisible by 3?

Problem 3.4. Many handshakes are exchanged on the first day of class. To attempt to make things interesting, the following rule is proposed. If a person has finished exchanging an odd number of handshakes, he or she must be wearing a party hat; otherwise, they must not be wearing a hat. Prove that at any moment there is an even number of party hats being worn in the room.

Problem 3.5. Clive has a single pile of 100 chocolates in front of him. In any move, he may select any pile with more than 2 chocolates, eat one chocolate and then split the remaining pile into two smaller piles. Would it be possible for Clive to reach a situation where every pile has exactly three chocolates?

Problem 3.6. In the planet of Jamos are 10 red, 15 blue and 20 yellow chameleons. Whenever two chameleons of different colours meet, they both change to the third colour. Is it possible that at some point in time all the chameleons are of the same colour?

4 Chessboard/Colouring Invariance

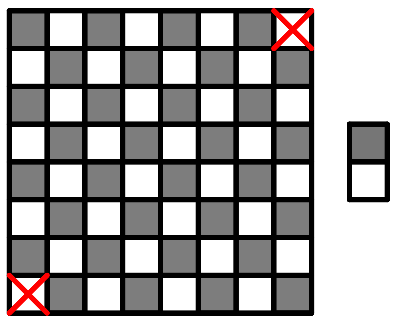
Problem 4.1

Consider an 8×8 chessboard with the top-right and bottom-left corners removed. Is it possible to tile the resultant chessboard with 1×2 dominoes?

Solution. No, it is not possible to do so.

Consider the following chessboard colouring (on the following page) with the two missing squares on white squares: it is easy to see that any domino must cover exactly one black square and one white square (as the blacks and whites are adjacent).

However, simply counting the number of black and white squares gives that there are 32 black squares and 30 white squares. Since they are not equal, it follows that the remaining board cannot be tiled with dominoes with no spaces. ☐



Problem 4.2. Consider the following table:

1	1	1	1
1	1	1	1
1	1	1	1
1	-1	1	1

In one move, we may switch the signs of all the numbers in a row, column or a parallel to one of the diagonals (including the corner squares). Show that there will always be at least one -1.

Problem 4.3. Arch-rivals Vicky and Vickie are playing a game on a board with an infinitely long line of squares. They take turns placing their symbol, either an Y or E in the squares and the first person to have three of their symbols in a row wins. If Vicky goes first, does either player have a winning strategy?

Problem 4.4. Suppose we have an $m \times n$ chessboard coloured in the usual alternating way black and white. A move consists of switching the colour of every square in the same row, column or 2×2 square (so black goes to white and vice versa). For what m, n is it possible to end up with exactly one black square?

Problem 4.5. We are tiling a 8×8 chessboard with 1×3 tiles, such that exactly one square is left untiled. Find all possible positions of this square.

Problem 4.6. On every square of a 4×4 grid is a coin. Precisely one coin is heads up. A move consists of flipping all coins in a row, column or line parallel to the diagonal (in particular, this means we can flip any corner square by itself). Find all positions of this heads coin if after a finite number of moves it is possible for all coins to be oriented the same way.

5 Geometric Invariance

Problem 5.1

A biologist has a 10×10 array of cells, 9 of which are infected. If one healthy cell has 2 infected neighbours, then it too will become infected. If this process repeats itself, can the infection spread to every cell?

Solution. For any configuration, we consider the **perimeter** of all the infected squares, as the number of unit edges for which an infected square touches a non-infected square. Let P be the perimeter: we show that P never increases.

For a square to become infected, it must be adjacent to at least two other infected squares. However, when this square becomes infected, the total perimeter must either stay the same or decrease, since at most two other edges of the perimeter form, while exactly two edges are destroyed with the new infected cell.

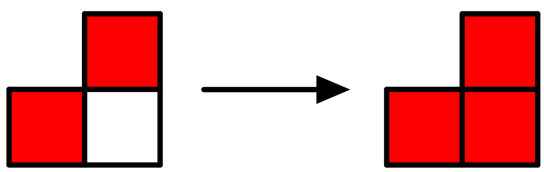


Figure 1: The perimeter stays the same in this case.

Initially, the perimeter of the shape is at most $4 \times 9 = 36$, since each of the four edges of the square contribute at most one edge to the perimeter. If it were possible for the entire grid to be infected, the perimeter must be exactly 40. However, this is impossible since the perimeter must never increase. It follows that the infection cannot spread to every cell. $\square$

Problem 5.2. Given a polygon $\mathcal{P}$, a new polygon $\mathcal{Q}$ can be created using the following procedure. The polygon $\mathcal{P}$ is divided into two polygons by a straight cut, then one of the resulting polygons is flipped over and joined back to the other polygon along the edge created by the cut. This procedure is only allowed if the two polygons do not overlap after being joined. Is it possible to transform $\mathcal{P} = \square$ into $\mathcal{Q} = \triangle$ by applying the procedure a finite amount of times?

Problem 5.3. An $n \times n$ square is tiled with 1×1 tiles, some of which are coloured. Thanom is allowed to colour in any uncoloured tile that shares edges with at least three coloured tiles. She discovers that by repeating this process all tiles will eventually be coloured. Show that initially there must have been at least $\frac{n^2+2n}{3}$ coloured tiles.