# Polynomials

Written by Andy Tran, James Bang for the MaPS Correspondence Program

## 1 Introduction

**Polynomials** are a special type of mathematical expression with far reaching applications in fields like chemistry, physics, economics and many other areas in mathematics. Their simple definition leads to their versatility in many mathematical problems as well as the development of much theory about them. We will explore the definitions of polynomials and extend this to the **Remainder Theorem**, **Factor Theorem**, **Fundamental Theorem of Algebra** and **Vieta's Formulae**.

## 2 Definitions

A **polynomial** $P(x)$ is any expression that can be created by additions, subtractions and multiplications of real numbers and the pronumeral $x$. More rigorously, a polynomial $P = P(x)$ of degree $n$ is any expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are any real numbers. Notice that this just looks like the sum of non-negative integer powers of $x$. So for example, the following are all polynomials:

- $3x^2 + 2x - 1$
- $2x^5 - \sqrt{5}x^{1000} - \pi$
- $-\frac{5}{2}$

But these are not:

- $\frac{1}{x}$ (the pronumeral cannot have a negative degree)
- $2x^{\frac{5}{2}} - 5$ (the pronumeral cannot have a fractional degree)

Now that we know what a polynomial is, let's define some important terminology which we will be referring to later on! For the polynomial

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

- $a_0, a_1, \ldots, a_n$ are called the **coefficients**.
- $a_0$ is called the **constant term**, as it is the part of the polynomial that doesn't change as $x$ varies.
- $a_n$ is called the **leading coefficient** and $a_nx^n$ is called the **leading term**.
- The polynomial is called **monic** if the **leading coefficient** is 1.
- $n$ is called the **degree** of the polynomial. In general, it is the highest nonzero power of $x$ that exists within its expansion. We denote the degree as $\deg(P(x)) = \deg(P)$. For example, $\deg(x^3) = 3$ and $\deg(0x^2 + x + 1) = 1$.
- A polynomial is called:
  - **constant** if $n = 0$;
  - **linear** if $n = 1$;
  - **quadratic** if $n = 2$;
  - **cubic** if $n = 3$;
  - **quartic** if $n = 4$;
  - and so on.

Let's check our understanding with the following exercise:

**Exercise 2.1.** *For each of the following expressions, determine if it is a polynomial or not. If it is a polynomial, state the constant term, leading coefficient, whether or not it is monic, and the degree. If it is not a polynomial, explain why.*

(a) $3x^4 - \frac{1}{2}x - 9$

(b) $2x^5 + x^6$

(c) $\sqrt{2}$

(d) $-2x^2 - 3.5\sqrt{x}$

# 3 Operations on Polynomials

No, we are not performing surgery on polynomials, but since polynomials are just expressions, we can try to apply the familiar **mathematical** operations of **addition**, **multiplication** and **division**. (Note that subtraction is essentially addition of negative numbers).

## 3.1 Addition and Multiplication

It's not too hard to see that the sum or product of two polynomials will also be a polynomial, since all the powers of $x$ will remain as non-negative integers. To demonstrate this, lets look at a couple of examples. If we take

$$P(x) = x^2 + 3x + 2 \text{ and } Q(x) = 2x + 1$$

then we would have that

$$\begin{aligned} P(x) + Q(x) &= (x^2 + 3x + 2) + (2x + 1) \\ &= x^2 + 5x + 3 \end{aligned}$$

and

$$\begin{aligned} P(x) \cdot Q(x) &= (x^2 + 3x + 2) \cdot (2x + 1) \\ &= 2x^3 + x^2 + 6x^2 + 3x + 4x + 2 \\ &= 2x^3 + 7x^2 + 7x + 2. \end{aligned}$$

So far so good. Now let's prove some properties about sums and products of polynomials!

**Example 3.1.** *For two polynomials $P(x)$ and $Q(x)$, prove that $\deg(PQ) = \deg(P) + \deg(Q)$. Note that $PQ$ is the polynomial obtained by multiplying $P(x)$ and $Q(x)$, that is $P(x)Q(x)$*

Solution: Suppose that $\deg(P) = m$ and $\deg(Q) = n$, in other words,

$$P(x) = a_m x^m + a_{m-1}x^{m-1} + \dots$$

$$Q(x) = b_n x^n + b_{n-1}x^{n-1} + \dots$$

This means that starting with the highest degree term,

$$P(x) \cdot Q(x) = a_m x^m \cdot b_n x^n + \dots = a_m b_m x^{m+n} + \dots$$

So $P(x) \cdot Q(x)$ has degree $m + n$. Thus, $\deg(PQ) = \deg(P) + \deg(Q)$. □

**Remark:** Notice that the function of taking the degree of a polynomial looks like $f(xy) = f(x) + f(y)$. Can you think of any other functions that satisfy this property?

**Exercise 3.2.** *For two polynomials $P(x)$ and $Q(x)$ with degrees $m$ and $n$ respectively, what what can we say about the possible degree of the polynomial $P(x) + Q(x)$?*

**Exercise 3.3.** *Given that the degree of a polynomial needs to satisfy the conditions in Example 3.1 and Exercise 3.2, what do you think should be the degree of the zero polynomial $P(x) = 0$?*

### 3.2 Division

We've seen that addition and multiplication work quite intuitively with no issues. However, dividing two polynomials **will not necessarily give a polynomial.** For instance, if we have two polynomials $x^2 - 1$ and $x$, then the quotient

$$\frac{x^2 - 1}{x} = x - \frac{1}{x}$$

is **not** a polynomial due to the $\frac{1}{x}$ term.

However, back when we first learnt about integers, we also had the issue where dividing two integers doesn't necessarily give you another integer! You may recall that we actually dealt with this problem in topic 2, where we used the division algorithm!

**Theorem 3.4.** ***(The Division Algorithm)*** *For any two given integers $n$ and $d > 0$, there are unique numbers $q$ and $r$ such that*

$$n = dq + r$$

*and $0 \leq r < d$.*

Or in other words, for any given **dividend** ($n$) and a **divisor** ($d$), there will always be a *unique* **quotient** ($q$) and **remainder** ($r$). And if $r = 0$, then we say that $d$ divides $n$, and write it as $d|n$.

So it turns out that we can still divide polynomials together, but we may sometimes get a remainder. We can slightly adjust the statement of the theorem to suit polynomials.

**Theorem 3.5.** ***(The Division Algorithm (Polynomials))*** *For any two given polynomials $P(x)$ and $D(x)$ with $\deg(P) \geq \deg(D)$, there are unique polynomials $Q(x)$ and $R(x)$ such that*

$$P(x) = D(x)Q(x) + R(x)$$

*and $\deg(R) < \deg(D)$.*

In words, we are doing a similar thing to the integers, that is for any given **dividend** ($P(x)$) and a **divisor** ($D(x)$), there will always be a *unique* **quotient** ($Q(x)$) and **remainder** ($R(x)$). And in the same sense as with integers, if $R(x)$ is 0, then we say that $D(x)$ divides $P(x)$ or $D(x)|P(x)$.

For example, if we wanted to divide $x^3 - 2x + 2$ by $x + 1$, we could rewrite it as

$$x^3 - 2x + 2 = (x + 1)(x^2 - x - 1) + 3.$$

Where we would say that the remainder is 3. Notice that the degree of the divisor $x + 1$ is 1, and the degree of the remainder 3 is 0.

You can see that the **degree** of the polynomial is treated as a "size" for the polynomial.

And notice that we are reusing a lot of the same vocabulary and notations for both integers and polynomials. This is because integers and polynomials act remarkably similar in ways that you wouldn't expect! In fact, in **algebra**, polynomials and integers are examples of the object called *Euclidean domain*!
So now we know that polynomials can be divided, but how do we actually do it? It turns out that just like with integers, we can use the familiar technique of long division!

# 4 Polynomial Division

Let's do a quick revision of long division. Say perhaps we want to divide 2798 by 13. Then what we would do is at each line, subtract off the largest multiple of 13 and then "drop" the next value of the dividend 2798 and repeatedly do this until the remainder is less than 13.

$$
\begin{array}{r}
215 \\
13\,\overline{)\,2798} \\
\underline{26\phantom{98}} \\
19\phantom{8} \\
\underline{13\phantom{8}} \\
68 \\
\underline{65} \\
3
\end{array}
$$

Now let's say that we want to divide $2x^3 + 7x^2 + 9x + 8$ by $x + 3$. We can do a similar thing where we can place the polynomials in the division box, and at each line, subtract the "largest" multiple of $x + 3$. But recall that the "size" of a polynomial can be though of as its **degree**. This means that at each line, we want to subtract off the multiple of $x + 3$ that will cancel out the **leading term**.

$$
\begin{array}{r}
2x^2 + 1x + 6 \\
x + 3\,\overline{)\,2x^3 + 7x^2 + 9x + 8} \\
\underline{2x^3 + 6x^2\phantom{+ 9x + 8}} \\
x^2 + 9x\phantom{+ 8} \\
\underline{x^2 + 3x\phantom{+ 8}} \\
6x + 8 \\
\underline{6x + 18} \\
-10
\end{array}
$$

Hence, we have $2x^3 + 7x^2 + 9x + 8 = (x + 3)(2x^2 + x + 6) - 10$.

**Remark:** Notice how similar the procedure is for long division in integers and polynomials.

Note that if the polynomial $Q(x)$ is missing some coefficients (that is, some of the coefficients of $x^k$ are zero), then in the division process, we write it out in full form, for example, to divide $x^3 - 1$ by $x + 2$, we may need to express $x^3 - 1$ as $x^3 + 0x^2 + 0x - 1$.

Also note that many polynomials will have negative terms as well. You can still apply the above procedure but you will need to be very careful with the signs. This also means that you can have a remainder that appears negative (like $-5$) but this is okay since the size of the remainder is measured by its degree (here, it is 0).

**Exercise 4.1.** *Use long division to divide* $Q(x)$ *by* $P(x)$ *and express* $Q(x) = A(x)P(x) + B(x)$*, where:*

(a) $P(x) = x + 4$ and $Q(x) = x^3 + 3x^2 - 2x + 5$,

(b) $P(x) = x^2 + x + 1$ and $Q(x) = x^6 - 1$.

## 4.1 The Remainder Theorem

**Example 4.2.** *Find the remainder when a polynomial* $x^3 - 5x^2 + 2x + 6$ *is divided by* $x - 2$.

As we are trying to find the remainder, one option is to perform the **long division** and see what we get at the end. However, this seems like a lot of work, so maybe we can try to exploit the **division algorithm**.
We know that there exists some quotient $Q(x)$ and a remainder $R(x)$ such that

$$x^3 - 5x^2 + 2x + 6 = (x - 2)Q(x) + R(x).$$

But we know that $R(x)$ needs to have a *lower degree than the divisor*. But the divisor $(x-2)$ has degree 1, so $R(x)$ must have degree 0, so it must be a constant, say $r$. This means that

$$x^3 - 5x^2 + 2x + 6 = (x - 2)Q(x) + r$$

and this equation must hold for **all values** of $x$. Lets see what happens when we substitute $x = 2$:

$$\begin{aligned} 2^3 - 5 \cdot 2^2 + 2 \cdot 2 + 6 &= (2 - 2) \cdot Q(x) + r \\ -2 &= r. \end{aligned}$$

So you can see that without having to work out the quotient, we can still deduce that the remainder is $-2$. □

In general, if we want to find the remainder of $P(x)$ divided by $x - a$, the division algorithm means we can rewrite the polynomial as

$$P(x) = (x - a)Q(x) + r$$

where $r$ is remainder, and then substituting $x$ for $a$ gives us that[1]

$$P(a) = r.$$

Which is a very fast and easy way of working out the remainder! This result is so useful we give it a special (and not surprising) name:

[1]Note that the notation $P(a)$ means taking the polynomial $P(x)$ and replacing all the $x$'s with $a$'s.

**Theorem 4.3.** ***(The Remainder Theorem)*** *The remainder when a polynomial $P(x)$ is divided by $x - a$ is $P(a)$.*

In general, division is particularly interesting when the remainder is equal to 0. In the integers, this means that the divisor is a **factor** of the dividend. In a similar spirit, polynomials can have factors, that is another polynomial that divides it with **no remainder**.

But if we are dividing by a polynomial of the form $x - a$, we know how to find the remainder very efficiently. So this means we can also work out whether or not $x-a$ is a factor of the polynomial by checking if the remainder is 0.

**Theorem 4.4.** ***(The Factor Theorem)*** *$x-a$ is a factor of a polynomial $P(x)$ if and only if $P(a) = 0$.*

If $x - a$ is a factor of $P(x)$ (or equivalently if $P(a) = 0$), we will often say that $a$ is a **root** or a **zero** of the polynomial $P(x)$.

**Exercise 4.5.** *Use the factor theorem to find roots to the polynomial $x^3 - 2x^2 - 5x + 6$ and hence factorise it. You may need to do some trial and error.*

# 5 Roots of Polynomials

Expressions of the form $x - a$ are really important for polynomials as they can be thought of as the "building blocks" of polynomials. Hopefully this will ring a bell, as you may have been taught that **primes** can be thought of as the "building blocks" of the integers! In particular, we had the following theorem:

**Theorem 5.1.** *(**The Fundamental Theorem of Arithmetic**) Every integer $n > 1$ can be uniquely represented as a product of primes. That is, it has a unique representation*

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$$

*where $p_1, p_2, p_3, \dots p_k$ represent the 1st, 2nd, 3rd prime number and so on, and $a_1, a_2, a_3, \dots, a_k, k$ are non-negative integers.*

By now you probably get the idea that there is an equivalent statement for **polynomials**!

**Theorem 5.2.** *(**The Fundamental Theorem of Algebra**) Every polynomial $P(x)$ with degree $n$ has exactly $n$ roots. That is, it has a unique representation*

$$P(x) = A(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_n)$$

*where $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n$ are the roots to $P(x)$ and $A$ is the leading coefficient.*

Note that there may be repeated roots like

$$x^2 - 2x + 1 = (x - 1)(x - 1)$$

and be warned that the roots are not necessarily **real numbers**! For example, the polynomial $x^2 + 1$ cannot be factorised over the real numbers, but if we allow $i = \sqrt{-1}$, then we have that

$$x^2 + 1 = (x + i)(x - i).$$

We do not provide the proof of this theorem here as it is rather long for these notes. However if you are interested, we encourage you to search for it, as the proof is accessible for bright high school students.

**Exercise 5.3.** *Prove that a polynomial of degree $n$ cannot have more than $n$ roots (including repeated roots).*

# 6 Vieta's Formulas

The **Fundamental Theorem of Algebra** gives us another way to express polynomials (just like a **prime factorisation** for integers)! For the sake of simplicity, let's consider a monic quadratic (a polynomial of degree 2 and leading coefficient 1). In general, this would look like $x^2 + ax + b$ which would factorise to something like

$$x^2 + ax + b = (x - \alpha)(x - \beta)$$

where $\alpha, \beta$ are the roots to the polynomial. But we can expand the RHS to see that

$$\begin{aligned} x^2 + ax + b &= x^2 - \alpha x - \beta x + \alpha\beta \\ x^2 + ax + b &= x^2 - (\alpha + \beta)x + \alpha\beta \end{aligned}$$

Since we have the same polynomial on both sides of the equation, the **coefficients** must be the same, so we have that $a = -(\alpha + \beta)$ and $b = \alpha\beta$. However, this will often be written as

$$\alpha + \beta = -a \quad \text{and} \quad \alpha\beta = b.$$

In particular, this gives us a way to write the **sum** and the **product** of the roots in terms of the polynomial's **coefficients**. This is especially interesting for polynomials with real coefficients, but complex roots, as it means that its roots still sum and multiply to a real number!

These formulas are called **Vieta's formulas** and can be generalised to polynomials of any degree.

**Exercise 6.1.** *Suppose now that we have a non-monic quadratic, say $ax^2 + bx + c$. Find an expression for $\alpha + \beta$ and $\alpha\beta$ in terms of $a, b, c$.*

**Exercise 6.2.** *If the polynomial $ax^3 + bx^2 + cx + d$ has roots $\alpha, \beta, \gamma$, find Vieta's formulas by equating the coefficients in a similar way to above.*