Methods of Counting

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1 Introduction

Counting the number of ways to choose objects or the number of elements in a given set is a skill required to solve nearly all combinatorics problems. Even problems that aren't explicit counting problems rely on these techniques as well. As such, to succeed in combinatorics it is important to master all the techniques included below.

2 The Multiplication and Addition Principle

These are the most fundamental principles in counting and they come from whether choices are **independent** or **dependent and mutually exclusive**. This may sound technical, but an example will clarify things:

Example 2.1. *Rachel has three pairs of shorts and four T-shirts. What is the total number of outfits that Rachel can wear?*

Clearly, the choice of shorts and T-shirt are independent, and so the total number of choices is $3 \times 4 = 12$. $\square$

Remark: This is called the **Multiplication Principle**.

Example 2.2. *Rachel has four pairs of sneakers and five pairs of thongs. How many choices of footwear does Rachel have?*

Now you can see that these choices are **dependent and mutually exclusive**, in other words, Rachel can only wear one pair of footwear at a time. This means that the total number of choices is $4 + 5 = 9$. $\square$

Remark: This is called the **Addition Principle**.

A useful way to remember whether to use the **multiplication** or **addition** principle is to check if you use "**and**" or "**or**" to describe your choices. That is, in the first example, we could choose from three pairs of shorts **AND** four T-shirts. And in the second example, we could choose from four pairs of sneakers **OR** five pairs of thongs.

Example 2.3. *Jayden is baking a cake and needs to add eggs, flour, butter and sugar to the mixer in any order. How many ways can Jayden add the ingredients to the mixer?*

Notice that Jayden needs to add eggs AND flour AND butter AND sugar, so we will be using the **multiplication principle**. There are 4 possibilities for the first ingredient, which leaves 3 possibilities for the second ingredient, 2 possibilities for the third ingredient and just 1 possibility for the final ingredient. This gives a total of $4! = 4\times3\times2\times1 = 24$ possibilities. □

Remark: We will often write $n!$ as a concise notation to mean $n \times (n-1) \times \cdots \times 3 \times 2 \times 1$.

Exercise 2.4. *Annabelle wants to travel from Sydney to Melbourne but due to the COVID-19 situation, there are no direct flights available. Thus, she must either take a direct train to Melbourne, or she can take a flight via Adelaide. On the day of travelling, there are 4 trains from Sydney to Melbourne, 3 morning flights from Sydney to Adelaide and 5 afternoon flights from Adelaide to Melbourne. How many different travel plans could Annabelle have?*

3 Order and Repetition

When determining the number of possible arrangements in a scenario, the way to count it often depends on whether **order** is important and whether **repetition** is allowed. However, counting problems can be solved without consciously thinking about this!

3.1 Order is important, Repetition is allowed

Example 3.1. *Otto is trying to break into Adam's phone which is locked with a four-digit pass code where each digit is 0-9. What is the total number of codes that Otto could try?*

Here we can see that each of the four digits have 10 choices, and each choice is independent from each other. Using the **multiplication principle**, we see that the total number of choices is $10 \times 10 \times 10 \times 10 = 10000$. Quite a lot of codes to try! □

Remark: Notice that here, the **order** of the digits is **important** and the digits **can be repeated**.

In general, if **order is important** and if **repetition is allowed** then the number of ways to select r objects from n choices is: $\boxed{n^r}$.

3.2 Order is important, Repetition is not allowed

Example 3.2. *Janiru has 10 subjects that he needs to study for. He has decided that he will revise 4 different subjects today, one at 9am, 12pm, 3pm and 6pm respectively. In how many ways can Janiru arrange his study plan for today?*

One way to think about this problem is to see in how many ways we can fill each time slot. At 9am, there are 10 possible subjects to pick from. Then at 12pm, there are 9 subjects to pick from (since he wants to study different subjects). Then there are 8 subjects at 3pm and 7 subjects at 6pm. As each choice is independent from the rest, the total number of choices is $10 \times 9 \times 8 \times 7 = 5040$. □

Remark: Notice that this time, the **order** of the subjects is **important** but the subjects **can not be repeated**.

In general, the number of ways to select r objects from n choices if **order is important** and if **repetition is not allowed** is:

$$\boxed{n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}.}$$

These are known as **permutations.**

3.3 Order is not important, Repetition is not allowed

Example 3.3. *Fiona has 10 pens in her pencil case but she is only allowed to bring 4 of them into the Tournament of Towns. How many combinations of pens could she take?*

Notice that the order that Fiona takes the pens isn't important which makes this a little trickier! We know from the previous example that there are $10 \times 9 \times 8 \times 7 = 5040$ ordered selections, that is if Fiona were to select the 4 pens in a specific order.

But we want to calculate the number of **unordered selections**. So let's find out how many **ordered selections** a single **unordered selection** will correspond to.

Suppose that we have an unordered selection of four pens $\{a, b, c, d\}$. From example 2.3, we know that four objects can be arranged in $4 \times 3 \times 2 \times 1 = 24$ ways. That is, each unordered selection is represented 24 times in the ordered selections. Thus, the total number of ways to select four pens is

$$\frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

$\square$

Remark: notice that in this scenario, the order of the pens is not important and the pens cannot be repeated.

Generally, the number of ways to select r objects from n choices if order is **not important** and if repetition is **not allowed** is:

$$\boxed{\binom{n}{r} = \frac{n(n-1)\dots(n-r+1)}{r(r-1)\dots1} = \frac{n!}{r!(n-r)!}.}$$

This value is so important in combinatorics that it gets its own notation and is often referred to as "n choose r".

3.4 Order is not important, Repetition is allowed

Example 3.4. *Amanda is at her local supermarket which sells 10 types of fruits. If she wants to buy 4 pieces of fruit for her family, how many fruit combinations could Amanda buy?*

Firstly notice that in the fruit combinations, the **order of the fruit doesn't matter**, and the **fruits can be repeated**. For example, Amanda could simply buy 4 oranges, or perhaps she could buy 2 apples, a banana and a pineapple.

To count the number of ways this can be done, we will make an analogy to placing the fruit on a checkout counter with dividers. Using 9 dividers, we can uniquely split the fruit into individual sections, for example:

apples | bananas | grapes | peaches | pineapples | dates | cherries | pears | oranges | figs

And then a selection of 4 oranges would look like

$$| \; | \; | \; | \; | \; | \; | \; | \circ \circ \circ \circ |$$

and a selection of 2 apples, a banana and a pineapple would look like

$$\circ \circ \; | \circ | \; | \; | \circ | \; | \; | \; | \; |$$

This gives us a way of representing any selection of 4 fruits **uniquely** as a combination of four circles and 9 dividers in a line. Also notice that any combination of four circles and 9 dividers in a line gives you a **unique** selection of 4 fruits.

Thus, the number of ways the fruits can be bought is equal to the number of ways 4 identical circles and 9 identical dividers can be arranged in a line. This can also be thought of as having 13 objects in a line, and choosing 4 of them to be the circles (and the rest are forced to be dividers). This can be done is $\binom{13}{4} = 715$ ways. $\square$

Remark: To be precise, what we have done is find a **bijection** (ie. a one-to-one correspondence) between the selections of 4 fruits from 10 choices to the arrangements of 4 circles and 9 dividers in a line. This means that they both have the same number of arrangements. For some combinatorial problems, finding a proper bijection is a crucial part of the soution.

Remark: This technique is often called the Supermarket Shopping Principle, or the Stars and Bars method.

In general, the number of ways to make a selection of r objects from n categories where order is **not important** and repetition **is allowed** is

$$\boxed{\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}.}$$

4 Arrangements with Restrictions

When we are counting the number of arrangements of something, we will often have extra restrictions that we will need to account for. However, the basics of the addition and multiplication principles, and the importance of order and repetition stay the same.

Example 4.1. *In a group with* 5 *boys and* 6 *girls, how many subgroups are there with three boys and three girls?*

To solve this problem, we can see that the boys and girls are independent so we can consider them separately. The number of ways to choose 3 boys out of a group of 5 is $\binom{5}{3} = 10$ since the group is unordered and repetition is not allowed. Similarly, the number of ways of choosing 3 girls from a group of 6 is $\binom{6}{3} = 20$.

As selecting the boys and girls are independent choices (notice the word **AND**), we multiply these together to get a total of 200 choices. □

Exercise 4.2. *How many subgroups are there with the same number of boys as girls?*

Example 4.3. *In example 3.1, Otto learnt that Adam's four-digit pass code had 10000 possibilities if the digits could be anything between 0-9. Otto overheard that the digits in Adam's pass code were in a strictly increasing order. How many pass codes does Otto have to try now?*

We now have the restriction that the four-digits are in strictly increasing order. What we need to do is convert this into a framework that we already know how to count.

Notice that any selection of 4 distinct digits has a unique way to be arranged in a strictly increasing order. For example, the four digits $\{4, 0, 9, 6\}$ can only be arranged in increasing order as $\{0, 4, 6, 9\}$. That is, there is a **one-to-one correspondence** between the number of selections of 4 distinct digits to the possible pass codes.

The number of ways to pick 4 distinct digits from 10 choices, where order is not important and repetition is not allowed is $\binom{10}{4}$. Hence, there are $\binom{10}{4} = 210$ possible pass codes.
□

Exercise 4.4. *How many possible 4-digit codes are there such that the digits are weakly increasing?*

5 Extra Practice Problems

1. (a) How many ways can 20 different gifts be given out to 180 students if each student may receive more than one gift?
   (b) How many ways can 20 different gifts be given out to 180 students if each student may receive at most one gift?
   (c) How many ways can 20 identical gifts be given out to 180 students if each student may receive at most one gift?
   (d) How many ways can 20 identical gifts be given out to 180 students if each student may receive more than one gift?
2. (a) How many distinct strings of letters can be formed using the letters APRICOT?
   (b) How many distinct strings of letters can be formed using the letters APPLE
   (c) How many distinct strings of letters can be formed using the letters BANANA?
3. In how many ways can the letters TRANDY be rearranged to form a distinct string of letters?
   (a) What if the string of letters must begin with A?
   (b) What if the T must come before the Y?
   (c) What if the letters A, N, and D must appear in that order, but not necessarily together?
   (d) What if the letters A, N and D must appear together, but not necessarily in that order?
   (e) What if the letters A, N and D must appear together and in that order?
4. (a) How many numbers below 2022 are divisible by all of 2, 3 and 5?
   (b) How many numbers below 2022 are divisible by exactly two of 2, 3 or 5?
   (c) How many numbers below 2022 are divisible by exactly one of 2, 3 or 5?
   (d) How many numbers below 2022 are divisible by at least one of 2, 3 or 5?
   (e) How many numbers below 2022 are divisible by none of 2, 3 or 5?
5. (a) How many different arrangements of 52 distinct cards can we have in a deck?
   (b) It is estimated that 107 billion humans have ever lived on this planet, and that the universe is approximately 14 billion years old. If every person who has ever lived shuffled a deck of 52 cards each second ever since the creation of the universe until now, how many deck configurations could we create?
   (c) Express this as a percentage of the total possible number of deck configurations.