Stage 1:

1. (a) $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{2t-4}{2}=t-2$

(b) $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{\frac{-1}{t^2}}{2t}=\dfrac{-1}{2t^3}$

(c) $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{-3t^{-4}}{1}=\dfrac{-3}{t^4}$

(d) $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{6t^2}{\frac{1}{2\sqrt{t}}}=12t^{\frac{5}{2}}$

(e) $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{-\frac{1}{2}t^{-\frac{3}{2}}}{-2t^{-3}}=\dfrac{1}{4}t^{\frac{3}{2}}$

2. (i) $\dfrac{dx}{dt}=\dfrac{dx}{du}\cdot\dfrac{du}{dt}$ (let $u=t+1,\ x=\dfrac{u^2}{2}$)

$=2\cdot\dfrac{1}{2}u\cdot 1$

$=t+1$

$\dfrac{dy}{dt}=\dfrac{dy}{du}\cdot\dfrac{du}{dt}$ (let $u=t+1,\ y=\dfrac{u^3}{3}$)

$=3\cdot\dfrac{1}{3}u^2\cdot 1$

$=(t+1)^2$

$\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{(t+1)^2}{t+1}=t+1$

(ii)

(a)

$$\left.\frac{dy}{dx}\right|_{t=-3} = -2$$

$$x = \frac{(-3+1)^2}{2} = 2$$

$$y = \frac{(-3+1)^2}{3} = \frac{-8}{3}$$

ans: the tangent has a gradient of -2 at $\left(-2, \frac{-8}{3}\right)$

(b)

$$\left.\frac{dy}{dx}\right|_{t=0} = 1$$

$$x = \frac{(0+1)^2}{2} = \frac{1}{2}$$

$$y = \frac{(0+1)^2}{3} = \frac{1}{3}$$

ans: the tangent has a gradient of 1 at $\left(\frac{1}{2}, \frac{1}{3}\right)$

(c)

$$\left.\frac{dy}{dx}\right|_{t=\frac{1}{2}} = \frac{3}{2}$$

$$x = \frac{(\frac{1}{2}+1)^2}{2} = \frac{9}{8}$$

$$y = \frac{(\frac{1}{2}+1)^2}{3} = \frac{27}{24} = \frac{9}{8}$$

ans: the tangent has a gradient of 1 at $\left(\frac{9}{8}, \frac{9}{8}\right)$

(iii)

(a) let $\dfrac{dy}{dx} = 5$

$t+1=5$

$t=4$

$$x = \frac{(4+1)^2}{2} = \frac{25}{2}$$

$$y = \frac{(4+1)^2}{3} = \frac{125}{3}$$

ans: the tangent has a gradient of 5 at $\left(\dfrac{25}{2}, \dfrac{125}{3}\right)$

(b) let $\dfrac{dy}{dx} = \tan 135°$

$t+1=-1$

$t=-2$

$$x = \frac{(-2+1)^2}{2} = \frac{1}{2}$$

$$y = \frac{(-2+1)^2}{3} = \frac{-1}{3}$$

ans: the tangent has a gradient of 5 at $\left(\dfrac{1}{2}, \dfrac{-1}{3}\right)$

(iv)

$$y - \frac{(t+1)^3}{3} = (t+1)\left(x - \frac{(t+1)^2}{2}\right)$$

$$y = (t+1)x - \frac{(t+1)^3}{2} + \frac{(t+1)^3}{3}$$

$$y = (t+1)x - \frac{(t+1)^3}{6}$$

(v)

x-int: $0=(t+1)x-\frac{(t+1)^3}{6}$

$(t+1)x=\frac{(t+1)^3}{6}$

$x=\frac{(t+1)^2}{6}$

y-int: $y=(t+1)\cdot 0-\frac{(t+1)^3}{6}$

$y=-\frac{(t+1)^3}{6}$

$$area=\frac{1}{2}\left|\frac{(t+1)^3}{6}\right|\cdot\frac{(t+1)^2}{6}$$

$$=\left|\frac{(t+1)^5}{72}\right|u^2$$

3. (i)

$t=\frac{x}{c}$, $y=\frac{c}{\frac{x}{c}}=\frac{c^2}{x}$

$\therefore xy=c^2$

(ii)

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{-c}{t^2}}{c}=-\frac{1}{t^2}$$

$y-\frac{c}{t}=-\frac{1}{t^2}(x-ct)$

$y=-\frac{1}{t^2}(x-ct)$

$y=-\frac{1}{t^2}x+\frac{c}{t}+\frac{c}{t}$

$y=-\frac{1}{t^2}x+\frac{2c}{t}$

(iii)

$y - \text{int}: \dfrac{2c}{t}$

$0 = \dfrac{-1}{t^2}x + \dfrac{2c}{t}$

$x = \dfrac{2c}{t} \cdot t^2$

$x = 2ct$

$x - \text{int}: 2ct$

$area = \dfrac{1}{2} \cdot 2ct \cdot \dfrac{2c}{t} = 2c^2$ square units

$\therefore$ area of ΔOAB is constant as T varies

(iv)

$$M_{AB} = \left(\frac{0+2ct}{2}, \frac{\frac{2c}{t}+0}{2} \right) = (ct, \frac{c}{t}) = T$$

$$d_{AT} = \sqrt{(ct-0)^2 + (\frac{2c}{t} - \frac{c}{t})^2}$$

$$= \sqrt{c^2t^2 + \frac{c^2}{t^2}}$$

$d_{AT} = d_{BT}$ (T is midpoint of AB)

$$d_{OT} = \sqrt{c^2t^2 + \frac{c^2}{t^2}}$$

$\therefore d_{AT} = d_{OT} = d_{BT}$

$d_{AB} = d_{AT} + d_{BT} = 2d_{OT}$

$\therefore AB = 2OT$

Stage 2:

1. (a)

$$x = 4 - y^2$$

$$\frac{dx}{dy} = -2y$$

$$\frac{dy}{dx} = \frac{1}{-2y} = \frac{-1}{2\sqrt{4-x}}$$

(b)

$$\frac{1}{y} = x^{\frac{1}{5}}$$

$$x = \frac{1}{y^5} = y^{-5}$$

$$\frac{dx}{dy} = -5y^{-6}$$

$$\frac{dy}{dx} = \frac{1}{-5y^{-6}} = \frac{1}{-5\sqrt[5]{x^6}}$$

(c)

$$\frac{1}{y} = \sqrt[3]{x+1}$$

$$x = \frac{1}{y^3} - 1$$

$$\frac{dx}{dy} = -3y^{-4}$$

$$\frac{dy}{dx} = \frac{1}{-3y^{-4}} = \frac{-1}{3\sqrt[3]{(x+1)^4}}$$

2. (i) $y^k = x$

$$\frac{dx}{dy} = ky^{k-1}$$

$$\frac{dy}{dx} = \frac{1}{ky^{k-1}} = \frac{1}{k(x^{\frac{1}{k}})^{k-1}} = \frac{1}{ky^{1-\frac{1}{k}}} = \frac{1}{k}x^{\frac{1}{k}-1}$$

(ii)

let $y = (x^{\frac{1}{k}})^n$, $u = x^{\frac{1}{k}}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= n(x^{\frac{1}{k}})^{n-1} \cdot \frac{1}{k}x^{\frac{1}{k}-1} = \frac{n}{k}x^{\frac{n}{k}-\frac{1}{k}}x^{\frac{1}{k}-1} = \frac{n}{k}x^{\frac{n}{k}-1}$$

Stage 3:

1. (a) $y=-3(x^4-2x^2-1)^{-1}$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=3(x^4-2x^2-1)^{-2}(4x^3-4x)$$

$$=\frac{12x^3-12x}{(x^4-2x^2-1)^2}$$

let $\frac{dy}{dx}=0$

$12x^3-12x=0$

$x(x^2-1)=0$

$x=0,1,-1$

when $x=0$, $y=\frac{-3}{0^4-2\cdot0^2-1}=3$

when $x=1$, $y=\frac{-3}{1^4-2\cdot1^2-1}=\frac{3}{2}$

when $x=-1$, $y=\frac{-3}{(-1)^4-2\cdot(-1)^2-1}=\frac{3}{2}$

ans: $(0,3),(1,\frac{3}{2}),(-1,\frac{3}{2})$

(b) $y=(x^2-4x)^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=\frac{1}{2}(x^2-4x)^{-\frac{1}{2}}(2x-4)$$

$$=\frac{x-2}{\sqrt{x^2-4x}}$$

let $\frac{dy}{dx}=0$

$x-2=0$

$x=2$

when $x=2$, $y=\sqrt{2^2-4\cdot2}=\sqrt{-4}$

ans: no point on the curve

2. (a)

$$\frac{dy}{dx} = -(x^2+1)^{-2} \cdot 2x = \frac{-2x}{(x^2+1)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{-2 \cdot 1}{(1^2+1)^2} = \frac{-1}{2}, \quad y = \frac{1}{2}$$

$$l_T : y - \frac{1}{2} = 1(x-1) \cdot \frac{-1}{2}$$

$$y = -\frac{1}{2}x - 1$$

$$l_N : y - \frac{1}{2} = 2(x-1)$$

$$y = 2x - \frac{3}{2}$$

(b)

$$y^2 = 8 - x$$

$$x = 8 - y^2$$

$$\frac{dy}{dx} = -\frac{1}{2y} = -\frac{1}{2\sqrt{8-x}}$$

$$\left.\frac{dy}{dx}\right|_{x=4} = -\frac{1}{2\sqrt{4}} = -\frac{1}{4}$$

$$y = \sqrt{8-4} = 2$$

$$l_T : y - 2 = \frac{-1}{4}(x-4)$$

$$y = -\frac{1}{4}x + 3$$

$$l_N : y - 2 = 4(x-4)$$

$$y = 4x - 14$$

3. (a)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -(x^2 + k)^{-2} \cdot 2x$$

$$= -\frac{2x}{(x^2 + k)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = -\frac{2}{(k+1)^2}$$

$$-\frac{1}{2} = -\frac{2}{(k+1)^2}$$

$$(k+1)^2 = 4$$

$$k + 1 = \pm 2$$

$$k = 1, -3$$

(b)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2}(x + k)^{-\frac{1}{2}} \cdot 1$$

$$= \frac{1}{2\sqrt{x + k}}$$

$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{1}{2\sqrt{4 + k}}$$

$$-m = 2\sqrt{4 + k}$$

$$-2\sqrt{4 + k} = -6$$

$$\sqrt{4 + k} = 3$$

$$4 + k = 9$$
$$k = 5$$

4. (i)

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=-(8-x)^{-2}\cdot -1$$

$$=\frac{1}{(8-x)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=p}=\frac{1}{(8-p)^2}$$

$$y=\frac{1}{8-p}$$

$$l_T: y-\frac{1}{8-p}=\frac{1}{(8-p)^2}(x-p)$$

$$y=\frac{1}{(8-p)^2}x+\frac{1}{8-p}-\frac{p}{(8-p)^2}$$

$$y=\frac{1}{(8-p)^2}x+\frac{8-2p}{(8-p)^2}$$

(ii)

(a) $0=\frac{1}{(8-p)^2}\cdot 0+\frac{8-2p}{(8-p)^2}$

$$0=8-2p$$

$$p=4$$

$$y=\frac{1}{(8-4)^2}x+\frac{8-8}{(8-4)^2}$$

$$y=\frac{x}{16}$$

(b) $0=\frac{1}{(8-p)^2}\cdot 10+\frac{8-2p}{(8-p)^2}$

$\frac{-10}{(8-p)^2}=\frac{8-2p}{(8-p)^2}$

$-10=8-2p$

$-10=8-2p$

$2p=18$

$p=9$

$y=\frac{1}{(8-9)^2}x+\frac{8-18}{(8-9)^2}$

$y=x-10$

Stage 4:

1. (i)

semicircle:

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx}=\frac{1}{2}(16-x^2)^{-\frac{1}{2}}\cdot -2x$$

$$=\frac{-x}{\sqrt{16-x^2}}$$

$$y-\sqrt{16-t^2}=\frac{-t}{\sqrt{16-t^2}}(x-t)$$

$$y=\frac{-t}{\sqrt{16-t^2}}(x-t)+\sqrt{16-t^2}$$

semi ellipse:

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=\frac{1}{2}k(16-x^2)^{-\frac{1}{2}}\cdot -2x$$

$$=\frac{-xk}{\sqrt{16-x^2}}$$

$$y-k\sqrt{16-t^2}=\frac{-tk}{\sqrt{16-t^2}}(x-t)$$

$$y=\frac{-tk}{\sqrt{16-t^2}}(x-t)+k\sqrt{16-t^2}$$

(ii)
semicircle:

$$0=\frac{-t}{\sqrt{16-t^2}}(x-t)+\sqrt{16-t^2}$$

$$t(x-t)=16-t^2$$

$$tx-t^2=16-t^2$$

$$tx=16$$

$x=\dfrac{16}{t}$, x-int: $\dfrac{16}{t}$

semi ellipse:

$$0=\frac{-tk}{\sqrt{16-t^2}}(x-t)+k\sqrt{16-t^2}$$

$$tk(x-t)=k(16-t^2)$$

$$tx-t^2=16-t^2$$

$$tx=16$$

$x=\dfrac{16}{t}$, x-int: $\dfrac{16}{t}$

ans: the two tangents meet at the x-axis (same x-intercept)

(iii)

$A(\dfrac{16}{t},0)$, $OA=\dfrac{16}{t}$

$B(4,0)$, $OB=4$

$C(t,0)$, $OC=t$

$$LHS=OB^2=4^2=16=\frac{16}{t}\cdot t=OA\cdot OC=RHS$$

2. (i)

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=2a(x-h)\cdot 1$$

$$=2ax-2ah$$

let $\frac{dy}{dx}=m$

$$2ax-2ah=m$$
$$2ax=2ah+m$$
$$x=h+\frac{m}{2a}$$
$$y=a(h+\frac{m}{2a}-h)^2+k$$
$$=\frac{m^2}{4a}+k$$

$$P(h+\frac{m}{2a},\frac{m^2}{4a}+k)$$

let $\frac{dy}{dx}=-m$

$$2ax-2ah=-m$$
$$2ah=2ax-m$$
$$x=h-\frac{m}{2a}$$
$$y=a(h-\frac{m}{2a}-h)^2+k$$
$$=\frac{m^2}{4a}+k$$

$$Q(h-\frac{m}{2a},\frac{m^2}{4a}+k)$$

(ii)

tangent at $P(h+\frac{m}{2a},\frac{m^2}{4a}+k)$:

$$y-\frac{m^2}{4a}-k=m(x-h-\frac{m}{2a})$$

$$y=mx-mh-\frac{m^2}{2a}+\frac{m^2}{4a}+k \quad 1)$$

tangent at $Q(h-\frac{m}{2a},\frac{m^2}{4a}+k)$:

$$y-\frac{m^2}{4a}-k=m(x-h+\frac{m}{2a})$$

$$y=-mx+mh-\frac{m^2}{2a}+\frac{m^2}{4a}+k \quad 2)$$

1) and 2): intersection of tangents

$$2y=-\frac{m^2}{a}+\frac{m^2}{2a}+2k$$

$$y=\frac{-m^2}{4a}+k$$

normal at $P(h+\frac{m}{2a},\frac{m^2}{4a}+k)$:

$$y-\frac{m^2}{4a}-k=-\frac{1}{m}(x-h-\frac{m}{2a})$$

$$y=-\frac{1}{m}x+\frac{h}{m}+\frac{1}{2a}+\frac{m^2}{4a}+k \quad 3)$$

normal at $Q(h-\frac{m}{2a},\frac{m^2}{4a}+k)$:

$$y-\frac{m^2}{4a}-k=\frac{1}{m}(x-h+\frac{m}{2a})$$

$$y=\frac{1}{m}x-\frac{h}{m}+\frac{1}{2a}+\frac{m^2}{4a}+k \quad 4)$$

3) and 4): intersection of tangents

$$2y=\frac{1}{a}+\frac{m^2}{2a}+2k$$

$$y=\frac{1}{2a}+\frac{m^2}{4a}+k$$

$$d_{PQ}=h+\frac{m}{2a}-h+\frac{m}{2a}=\frac{m}{a}$$

length of vertical diagonal:

$$\frac{1}{2a}+\frac{m^2}{4a}+k+\frac{m^2}{4a}-k=\frac{1}{2a}+\frac{m^2}{2a}=\frac{m^2+1}{2a}$$

$$area=\frac{1}{2}\cdot\frac{m}{a}\cdot\frac{m^2+1}{2a}=\frac{m(1+m^2)}{4a^2}$$

3. (i)

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=2a(x-h)\cdot 1$$

$$=2a(x-h)$$

$$y-\left[a(t-h)^2+k\right]=2a(t-h)(x-k)$$

$$y=2a(t-h)x-2at^2+2ath+a(t-h)^2+k$$

$$y=2a(t-h)-2at^2+2ath+at^2-2ath+ah^2+k$$

$$y=2a(t-h)-at^2+ah^2+k$$

(ii)

$$y=2a(t-h)x-at^2+ah^2+k$$

$$0=2a(t-h)\cdot 0-at^2+ah^2+k$$

$$at^2=ah^2+k$$

$$t^2=h^2+\frac{k}{a}$$

$$t=\pm\sqrt{h^2+\frac{k}{a}}$$

(iii)

y-coordinate of P:

$$y=2a(t-h)\cdot h-at^2+ah^2+k$$

$$y=2ath-2ah^2-at^2+ah^2+k$$

$$y=-at^2+2ath-ah^2+k$$

$$y=-a(t-h)^2+k$$

$$d_{VP}=\left|k-\left[-a(t-h)^2+k\right]\right|$$

$$=|a|(t-h)^2 \quad 1)$$

$$y=ax^2-2axh+h^2a+k$$

$$AOS: x=\frac{-(-2ah)}{2a}=h$$

$$d_T=|t-h|$$

$$(d_T)^2=(t-h)^2 \quad 2)$$

1) and 2): $\dfrac{d_{VP}}{(d_T)^2}=\dfrac{|a|(t-h)^2}{(t-h)^2}=|a|$

$\therefore VP$ is proportional.