Lengths and Areas

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1 Introduction

The interplay between lengths and areas form the backbone of measurement and geometry. It has been studied for millennia and has allowed civilisation to develop to the point it is at today. This topic will introduce fundamental theorems on **lengths** and **areas** and discuss their applications. Some theorems that will be covered include **Tangent from an external point**, **Pythagoras' Theorem**, the **area of a triangle** and **ratios**.

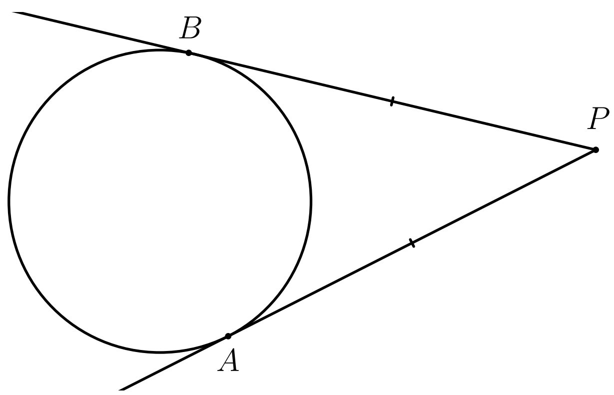
2 Lengths

In your study of mathematics, you would have already seen some theorems about lengths. Here, we shall use these to generate some more interesting results!

2.1 Tangents from an External Point

Recall the following from the *Circles* topic.

Theorem 2.1 (Ice Cream Cone Theorem) *The length of the tangents from an external point to the points of contact on a circle are equal.*



We've already proven this theorem using congruent triangles, so let's apply it to a couple questions!

Example 2.2 *Prove that a quadrilateral $ABCD$ has an incircle if and only if*

$$AB + CD = AD + BC.$$

Solution. First notice that we need to prove an "if and only if" statement, which means we need to show two implications!

The forwards implication ($\Rightarrow$) isn't too hard to show using the Ice Cream Cone Theorem.

Suppose the incircle touches the sides AB, BC, CD, DA at P, Q, R, S respectively. Then by the Ice Cream Cone Theorem, we can let

$$AP = AS = a$$
$$BQ = BP = b$$
$$CR = CQ = c$$
$$DS = DR = d$$

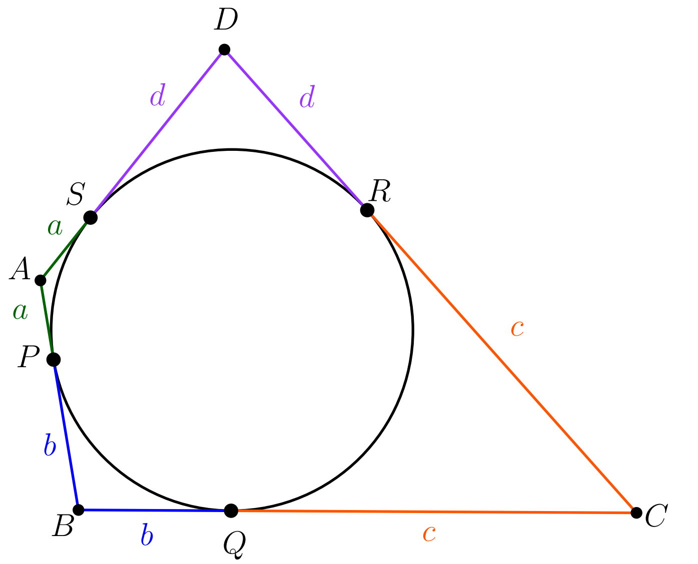
Now we can see that

$$AB + CD = AP + BP + CR + DR$$
$$\implies AB + CD = a + b + c + d$$

and

$$AD + BC = AS + DS + BQ + CQ$$
$$\implies AD + BC = a + d + b + c$$

which implies that $AB + CD = AD + BC$.



However, the backwards direction ($\Leftarrow$) is a little trickier to show! Here, we need to assume that $AB + CD = AD + BC$ and we need to show that the quadrilateral $ABCD$ has an incircle. First we can draw in the circle that is tangent to the sides DA, AB, BC (we know such a circle exists as it can be thought of as the **excircle** of the triangle formed by these lines) and suppose it is tangent at S, P, Q respectively.

From the Ice Cream Cone Theorem, we know that $AP = AS$ and $BP = BQ$. And from the length condition $AB + CD = AD + BC$, we know that there exists some point R on the side CD such that $DR = DS$ and $CR = CQ$. We now need to show that R lies on the same circle as P, Q, S, as this would be the incircle to the quadrilateral $ABCD$.

To do this, we can do some angle chasing, since we have lots of isosceles triangles! Let

$$\angle BAD = 2\alpha,\ \angle CBA = 2\beta,\ \angle DCB = 2\gamma,\ \angle CDA = 2\delta.$$

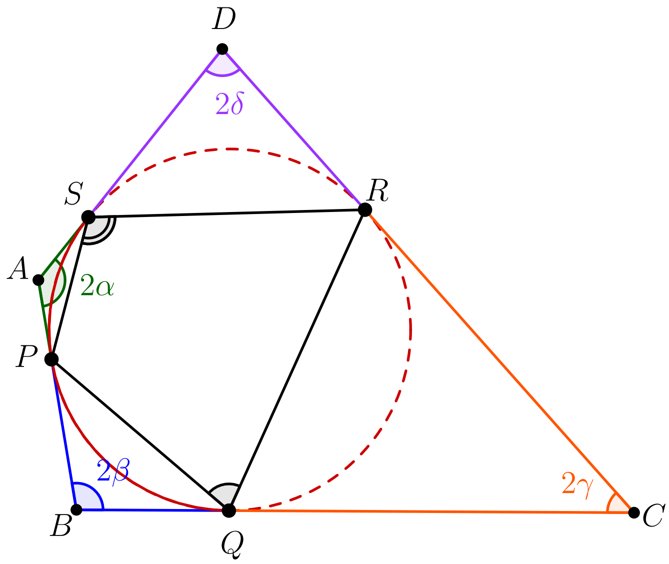
Notice that from the angle sum of the quadrilateral, this implies that $\alpha + \beta + \gamma + \delta = 180°$. Then from base angles of isosceles triangles, we have that

$$\angle ASP = 90 - \alpha,\ \angle DSR = 90 - \delta,\ \angle BQP = 90 - \beta,\ \angle CQR = 90 - \gamma.$$

Then angles on a straight line gives us that

$$\angle PSR = \alpha + \delta,\ \angle PQR = \beta + \gamma$$

Which means that $\angle PSR + \angle PQR = \alpha + \delta + \beta + \gamma = 180^\circ$, implying that $PQRS$ is cyclic and an incircle of $ABCD$. □



Exercise 2.3 *Consider $\triangle ABC$ with its incircle tangent to BC, CA, AB at D, E, F respectively. Let the lengths $AE = AF = x$, $BD = BF = y$ and $CD = CE = z$. Let the excircle opposite A of $\triangle ABC$ be tangent to BC, CA, AB at X, Y, Z respectively. Then prove that $AY = AZ = x + y + z$, $BX = BZ = z$, and $CX = CY = y$.*

Remark: Here we are rewriting the side lengths of the triangle in terms of lengths formed from the incircle, that is, we can let $AB = x + y$, $BC = y + z$ and $CA = z + x$. This process is called the **incircle substitution** and notice how it enforces the **triangle inequality** on the side lengths.

2.2 Pythagoras' Theorem

Perhaps one of the most fundamental theorems about lengths is none other than Pythagoras' Theorem!

Theorem 2.4 *(Pythagoras' Theorem) In a right-angled triangle with side lengths a, b and c, where c is the hypotenuse, we have*

$$a^2 + b^2 = c^2.$$

Many students learn about this theorem very early on in their maths education, but often you are not taught how to prove it! In fact, there are at least 367 known proofs of Pythagoras' Theorem (if you are interested, you can google *"The Pythagorean Proposition"*), but here, we will cover one such innovative proof.

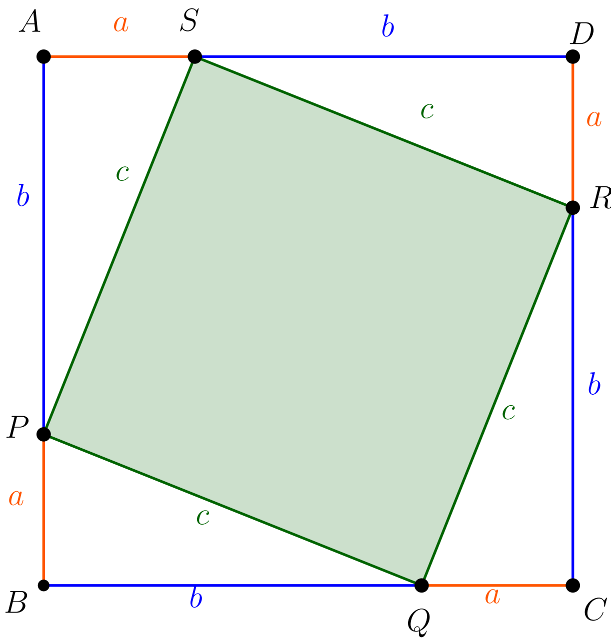
Example 2.5 *Prove Pythagoras' Theorem.*

Here, we construct a square $ABCD$ with side length $a + b$. But since c is the length of the hypotenuse of a right angled triangle with side lengths a and b, we can define points P, Q, R and S on the square as in the diagram below so that $PB = QC = RD = SA = a$ and $PA = SD = RC = QB = b$. This means that $PQ = QR = RS = SP = c$.

Now we can write the area of the square $ABCD$ as $(a+b)^2$, that is the square of its side length. But we can also write it as $c^2 + 4 \cdot \frac{ab}{2}$, that is the green square plus the four smaller triangles. This gives us

$$
\begin{aligned}
(a+b)^2 &= c^2 + 4 \cdot \frac{ab}{2} \\
\implies a^2 + 2ab + b^2 &= c^2 + 2ab \\
\implies a^2 + b^2 &= c^2
\end{aligned}
$$

Which proves Pythagoras' Theorem. □



Exercise 2.6 *Consider four points A, B, C and D on the plane. Prove that $AC \perp BD$ if and only if*

$$AB^2 + CD^2 = AD^2 + BC^2.$$

Recall that you need to prove both directions!

Exercise 2.7 *Let a, b, and c be the lengths of the sides of a triangle. Let d be the length of a cevian to the side of length a. If the cevian divides the side of length a into two segments of length m and n, with m adjacent to c and n adjacent to b, then prove that*

$$b^2 m + c^2 n = a(d^2 + mn).$$

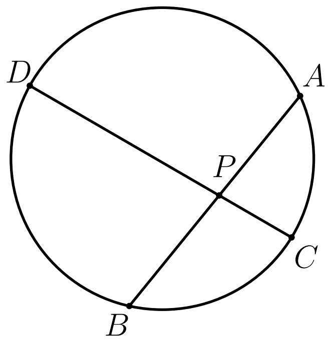
Remark: This result is called **Stewart's Theorem**. In the case where the cevian is a median, this result simplifies to $b^2 + c^2 = 2(a^2 + d^2)$, which we call **Apollonius' Theorem**.

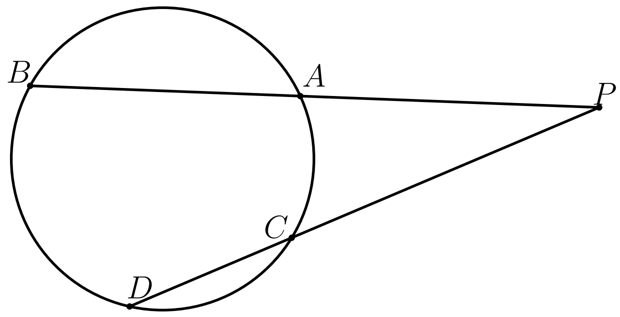
2.3 Length Ratios

In some cases, we might not just be interested in one length, but rather the ratio of two lengths! This will often manifest itself in congruent or similiar triangles, and indeed we have seen an example before with the following theorem in the *Circles unit*.

Theorem 2.8 (Power of a Point) *Given a circle and any point P not on the circle, let a line passing through P intersect the circle at A and B, and let another line passing through P intersect the circle at C and D. Then*

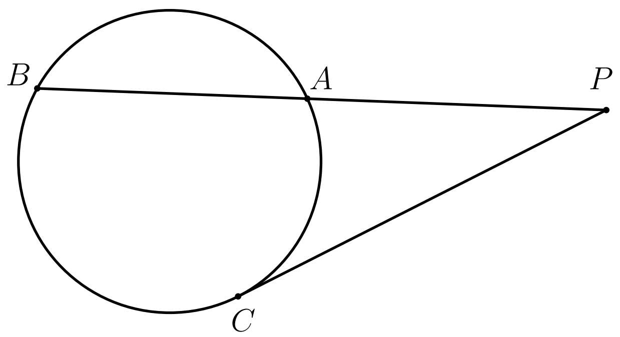
$$PA \cdot PB = PC \cdot PD$$





Remark: Note that P can lie inside or outside the circle as in the two diagrams above. In a special case, the two points of intersection can coincide, so that the secant becomes a tangent. As in the diagram below, we would have

$$PA \cdot PB = PC^2.$$



We can try apply this with the following exercise!

Exercise 2.9 *Circles $\Gamma_1$and Γ_2 intersect at A and B. Let a common tangent to Γ_1 and Γ_2 touch the circles at C and D. Prove that AB bisects CD.*

Another interesting result comes from looking at length ratios involving an angle bisector!

Theorem 2.10 (Angle Bisector Theorem) *In a triangle ABC, let the angle bisector of $\angle BAC$ meet BC at D. Then we have*

$$\frac{AB}{BD} = \frac{AC}{CD}.$$

There are a few ways to prove this theorem and we will explore a couple of them here!

Example 2.11 *In a triangle ABC, let the angle bisector of $\angle BAC$ meet BC at D. Let the line passing through C and parallel to AD meet AB at P.*

(a) Prove that $AP = AC$.

(b) Prove that $\frac{AB}{BD} = \frac{AC}{CD}$.

Solution. Since $AD||PC$, we have that by corresponding angles $\angle BAD = \angle APC$ and by alternating angles $\angle DAC = \angle ACP$. But since AD is an angle bisector, we have that $\angle BAD = \angle DAC$, and so $\angle APC = \angle ACP$. But this implies that APC is an isosceles triangle, and so $AP = AC$, answering part (a).

But we also have that $\triangle ABD \sim \triangle PBC$ as they are equal angles from the parallel lines. This gives us the side ratios

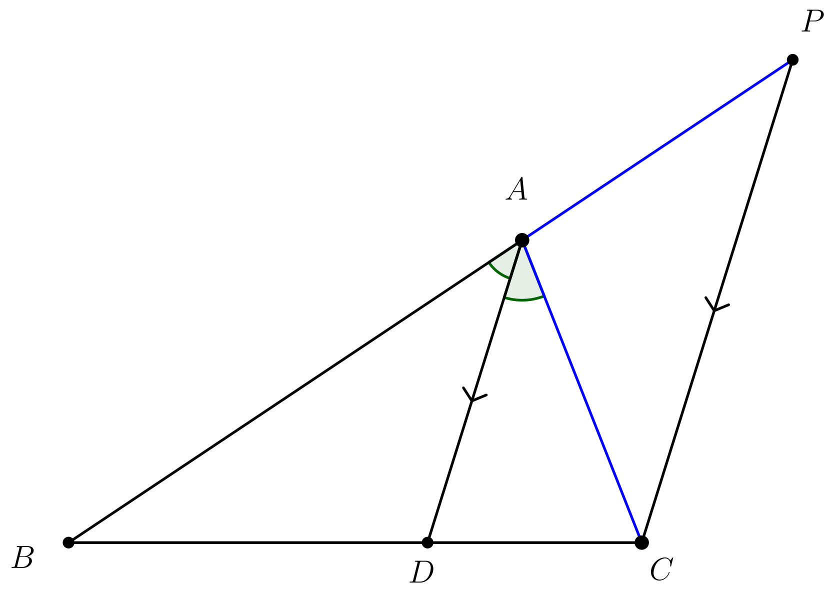
$$\frac{BA}{AP} = \frac{BD}{DC}$$

but using the result from (a), we have

$$\implies \frac{BA}{AC} = \frac{BD}{DC}$$

$$\implies \frac{BA}{BD} = \frac{AC}{DC}$$

which completes the proof. $\square$



Now we present another proof of the Angle Bisector Theorem!

Exercise 2.12 *In a triangle ABC, let the angle bisector of $\angle BAC$ meet BC at D. Let the line AD extended meet the circumcircle of $\triangle ABC$ again at P.*

(a) Prove that $BP = CP$.

(b) Prove that $\triangle ABD \sim \triangle CPD$ and $\triangle ACD \sim \triangle BPD$.

(c) Prove that $\frac{AB}{BD} = \frac{AC}{CD}$.

3 Areas

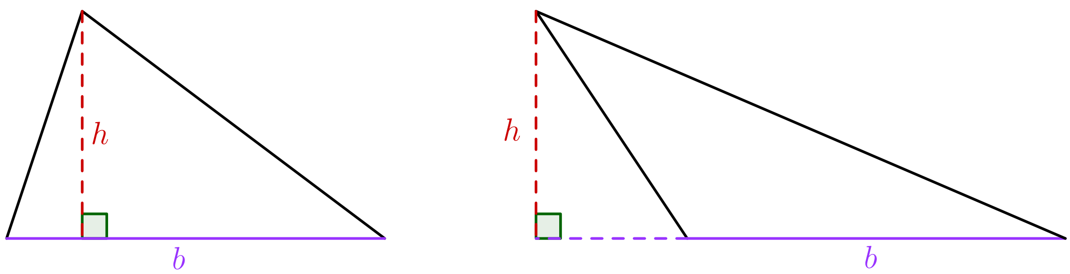
We now look at some key results involving areas, which is essentially the product of lengths!

3.1 Triangles

Since any polygon can be decomposed into triangles, finding the area of a triangle is an important problem in geometry, which has led to many different formulae which we will explore here. The most basic formula is

Theorem 3.1 (Area of a triangle) *For a triangle with base b and perpendicular height h, the area of the triangle A is given by*

$$A = \frac{1}{2}bh.$$



Notice that the height can lie outside of the triangle!

This simple formula is usually the starting point for many other possible formulae! For example, from Problem Set 4, we saw that:

Theorem 3.2 (Area of a triangle) *For a triangle ABC, if r is its inradius, and s is its semiperimeter, then the area A of the triangle is given by*

$$A = rs.$$

If you are familiar with trigonometry, you can also rewrite the height h in terms of the sides and angles of the triangle!

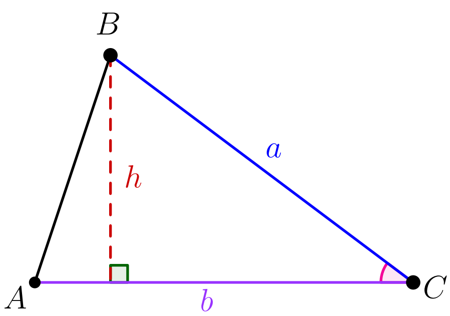
Theorem 3.3 (Area of a triangle) *For a triangle ABC, if we denote $BC = a$ and $CA = b$, then its area A is given by*

$$A = \frac{1}{2}ab\sin C$$

Proof. We know that the area can already be represented as $A = \frac{1}{2}bh$, so perhaps we can try rewrite h in terms of side lengths and angles. In particular, we have $\sin C = \frac{h}{a}$, so then $h = a\sin C$. Substituting this into our original formula gives us

$$A = \frac{1}{2}ab\sin C$$

which is our desired result. □



If you are very comfortable with trigonometry, you can have a go at extending this result a little further!

Theorem 3.4 (Area of a triangle) *For a triangle ABC with side lengths a, b, c and circumradius R (that is, radius of circle passing through the three vertices), then its area A is given by*

$$A = \frac{abc}{4R}.$$

Exercise 3.5 *Prove the above theorem.*

Hint: it might be a good idea to start with the formula $A = \frac{1}{2}ab\sin C$ and try to rewrite $\sin C$ in terms of c and R.

Another interesting formula writes the area of the triangle purely in terms of the side lengths!

Theorem 3.6 (Heron's Formula) *For a triangle ABC with side lengths a, b, c, let s be its semiperimeter (that is, $s = \frac{a+b+c}{2}$). Prove that its area A is given by*

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Exercise 3.7 *Prove the above theorem.*

3.2 Area Ratios

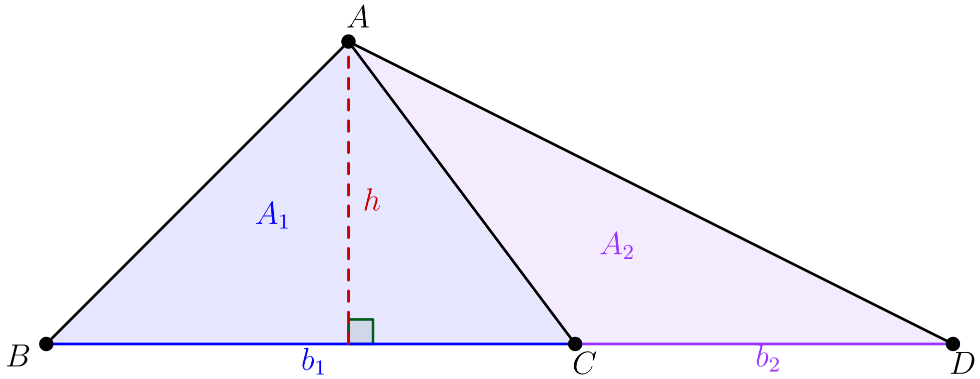
Just like with length ratios, we might not be interested in a single area, but rather the ratio between two areas. This can often be connected with length ratios, since areas are simply the product of lengths! A common instance where this comes up is the following.

Lemma 3.8 *If two triangles have the same height, then the ratio of their areas is equal to the ratio of their bases.*

This is true because we can write their areas using the formula $A = \frac{1}{2}bh$ and then since their heights are equal, they cancel out!

$$\frac{A_1}{A_2} = \frac{\frac{1}{2}b_1 h}{\frac{1}{2}b_2 h}$$
$$\implies \frac{A_1}{A_2} = \frac{b_1}{b_2}$$

This situation occurs quite often when we have two triangles who bases lie on the same line and they have the same other vertex. So in the diagram below, the two triangles $\triangle ABC$ and $\triangle ADC$ have their bases on the line BD and share the vertex A.



Example 3.9 (2003 AIMO Q4) *Side AB of $\triangle ABC$ is produced to P, so that $PB = AB$; BC is produced to Q, so that $QC = CB$; and CA is produced to R, so that $RA = AC$. The area of $\triangle ABC$ is 51. What is the area of $\triangle PQR$?*

Solution. Here, we can make use of the previous lemma a few times! Notice that $\triangle ABC$ and $\triangle PBC$ have their bases on the line AP and share the vertex C, so their heights are the same! But since their bases are equal, their areas must also be equal. In a similar way, $\triangle PBC$ and $\triangle PQC$ have their bases on the line BQ and share the vertex P, so their areas are also equal. This gives us that

$$|\triangle ABC| = |\triangle PBC| = |\triangle PQC|.$$

In particular, this means

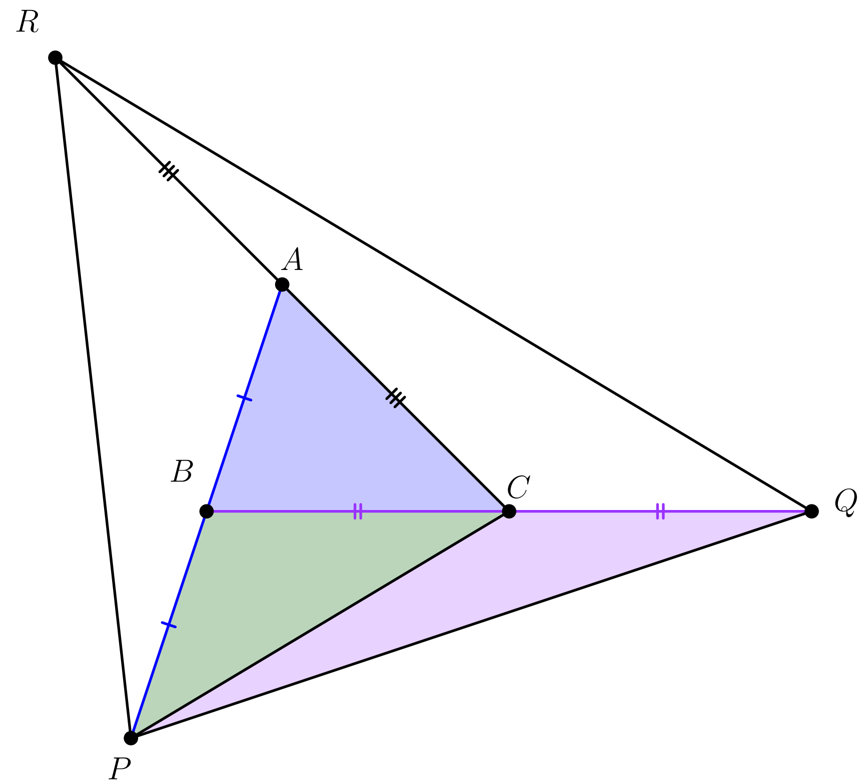
$$|\triangle PBQ| = 2|\triangle ABC|$$

And repeating this argument on other vertices gives us

$$|\triangle QCR| = 2|\triangle ABC|$$
$$|\triangle RAP| = 2|\triangle ABC|$$

Summing over all smaller triangles, we can conclude that

$$|\triangle PQR| = 7|\triangle ABC|$$
$$\implies |\triangle PQR| = 357$$



Exercise 3.10 (2010 AIMO Q7) *Suppose $\triangle ABC$ has area 2010, and let X, Y and z be points on the sides AB, BC and CA, respectively, such that:*

$$\frac{AX}{XB} = \frac{2}{3}, \ \frac{BY}{YC} = \frac{32}{35}, \ \frac{CZ}{ZA} = \frac{1}{5}$$

Determine the area of $\triangle XYZ$.

Exercise 3.11 (2012 AIMO Q8) *$ABCD$ is a trapezium with $AD \parallel BC$ and AC intersecting BD at P. The area of $ABCD$ is 225. The area of $\triangle BPC$ is 49. What is the area of $\triangle APD$?*