Problem Set 4

MaPS Correspondence Program

Instructions

- Some of these problems are based off the notes "*Circles*".
- They are in roughly difficulty order and get quite difficult, so you are **not** expected to be able to solve every problem.
- However, please attempt as many questions as you can and submit your solutions to your mentor for marking and feedback.
- You may (and encouraged to) submit incomplete solutions if you can not solve a problem completely.
- You may type your solutions or submit a pdf of a **clear** scan/photo of **legible** written solutions.
- Feel free to discuss these problems with your peers and on the forum but the solutions you submit must be written by yourself.

Problems

1. Suppose that $ABCD$ is a quadrilateral with a circle inscribed in it. In other words, there is a circle tangent to all four sides of the quadrilateral. Prove that
$$AB + CD = AD + BC.$$

2. Find the last 2 digits of
$$1! + 2! + 3! + 4! + \cdots + 2022!$$
Note that $n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$.

3. For a triangle ABC, define its *incircle* as the circle that is tangent to the three sides of the triangle. Define its *inradius* to be the radius of its incircle, and denote it by r. Define the triangle's *semiperimeter* to be half of its perimeter, and denote it by s. Prove that the area A of the triangle is given by
$$A = rs.$$

4. Let n be a positive integer and consider the set of numbers $\{1, 2, 3, \ldots, 2n\}$. Show that if you select any $n+1$ of these numbers:
   (a) There exist two of which are relatively prime (that is, they share no common factors other than 1).
   (b) There exist two of which one divides the other.

5. Let $ABCD$ be a convex quadrilateral such that no pairs of opposite sides are parallel. Let AB and CD meet at E and let AD and BC meet at F. Define the following four circles:
   - $\mathcal{C}_1$ passing through A, E and D.
   - $\mathcal{C}_2$ passing through A, F and B.
   - $\mathcal{C}_3$ passing through B, C and E.
   - $\mathcal{C}_4$ passing through D, C and F.

   Prove that the four circles pass through a common point.