Enrichment stage 1: (Probability without drawing tree diagrams)

1. Urn A contains 3 red and 5 black marbles. Urn B contains 2 red and 8 black marbles. From urn A, one marble is chosen at random and placed into urn B.

   (a) What is the probability that this marble is red?

   P(R in A) = 3/8

   (b) Now from urn B (with the extra marble included), two marbles are drawn at random in succession without replacement. Find the probability that they are both the same colour.

   P(same) = P(RR in B) + P(BB in B)

   = P(R in A) × [P(RR in B) + P(BB in B)] + P(B in A) × [P(RR in B) + P(BB in B)]

   = 3/8 × (3/11 × 2/10 + 8/11 × 7/10) + 5/8 × (2/11 × 1/10 + 9/11 × 8/10)

   = 139/220

2. In a best of five sets tennis match (where the first player to win three sets wins the match), Chris has a probability of $\frac{2}{3}$ of winning each set. What is the probability of him winning this particular match?

   Case 1: Chris wins in 3 sets

   P(WWW) = (2/3)^3

   = 8/27

   Case 2: Chris wins in 4 sets

   P(WWLW) + P(WLWW) + P(LWWW) = 3 × (2/3)^3 × (1/3)

   = 8/27

   Case 3: Chris wins in 5 sets

   P(WWLLW) + P(WLWLW) + P(LWWLW) + P(WLLWW) + P(LWLWW) + P(LLWWW)

   = 6 × (2/3)^3 × (1/3)^2

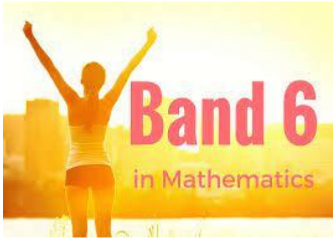
   = 16/81

   P(Chris wins) = 8/27 + 8/27 + 16/81

   = 64/81

Enrichment stage 2: (Condition probability)

1. The probabilities that two students A and B can be promoted to **Band 6** are 0.8 and 0.7 respectively. The probabilities that students A and B can be promoted to Band 6 and then to the university are 0.56 and 0.35 respectively.



Let M be the event that student A can be promoted to Band 6.

N be the event that student B can be promoted to Band 6.

X be the event that student A can be promoted to the university.

Y be the event that student B can be promoted to the university.

(Give your answers correct to 3 significant figures if necessary.)

(a) Find the probability that both students can be promoted to Band 6 and then to the university.

(b) If both students can be promoted to Band 6, find the probability that:

(i) student A can be promoted to the university.

(ii) student B can be promoted to the university.

(iii) both of them can be promoted to the university.

(iv) only one of them can be promoted to the university.

2. A box contains 10 balls as follows:

	Red balls (R)	White balls (W)
Ball contains stars	1	3
Ball contains dots	2	4

Two balls are drawn at random from the box, in succession and **without replacement**.
Event R and S are defined as follows:

R: exactly one of the balls drawn is red.

S: exactly one of the balls drawn contains stars.

(a) Find the probability that exactly one of the balls drawn is red.

(b) Find the probability that exactly one of the balls drawn contains stars.

(c) Find the probability that exactly one of the balls drawn is red **and** exactly one of the balls drawn contains stars.

(d) Find the probability that exactly one of the balls drawn is red **or** exactly one of the balls drawn contains stars.

Enrichment stage 3: (probability and algebra)

1. Two standard 6 sided dice are rolled. The number on the upper face of the first die determines the coefficient b and the number on the upper face of the second die determines the value of c in the quadratic equation $x^2 + bx + c = 0$. With the aid of a diagram or otherwise determine the probability that the first quadratic equation will have two different real roots.

2. A bag contains 180 marbles, some of which are red, some blue and the rest are yellow. The probability of drawing a blue marble is $\frac{1}{12}$ and the probability of drawing a red is $\frac{2}{5}$. Find:

(a) The number of yellow marbles in the bag.

(b) the number of yellow marbles that need to be added to the bag to change the probability of obtaining a yellow marble to $\frac{4}{5}$.

3. A race consists of only three horses A, B and C. A is twice as likely to win the race as B and B is twice as likely to win as C.

(a) What is the probability that C wins the race?

Let p be the probability that C wins.

P(C wins) = p

P(B wins) = 2 × P(C wins) = p

P(A wins) = 2 × P(B wins) = 4p

Total probability is 1

4p + 2p + p = 1

p = 1/7

(b) What is the probability that A does not win the race?

P(A doesn't win) = 1 - P(A wins)

= 1- 4/7

= 3/7

4. A jar contains 4 blue marbles and M red marbles, where M > 1. Jack takes two marbles out simultaneously. If the probability that they are both the same colour is 0.6, find all possible values of M.

P(BB) + P(RR) = 3/5

4/(4+M)×3/(3+M) + M/(4+M)×(M-1)/(3+M) = 3/5

12/[(4+M)(3+M)] + M(M-1)/[(4+M)(3+M)] = 3/5

5(12 + M² - M) = 3(4+M)(3+M)

60+5M²-5M = 3M²+21M+36

2M² - 26M + 24 = 0

M² - 13M + 12 = 0

(M - 1)(M - 12) = 0

M = 1 or M = 12

M = 12 only as M > 1

5. A hat contains m red and n white marbles. One marble is randomly selected and its colour noted. It is then returned to the hat along with k other marbles of the same colour. A second marble is chosen at random.

(a) Find the probability that the second marble is red.

$P(RR) = m/(m+n) \times (m+k)/(m+n+k)$

$= m(m+k)/(m+n)(m+n+k)$

$P(WR) = n/(m+n) \times m/(m+n+k)$

$= mn/(m+n)(m+n+k)$

P(Second marble is red) = P(RR) + P(WR)

$= m(m+k)/(m+n)(m+n+k) + mn/(m+n)(m+n+k)$

$= m(m+n+k)/(m+n)(m+n+k)$

$= m/(m+n)$

(b) What can you conclude about the probability found in (a)?

The probability of the second marble being red is independent of the value of k.

In other words, regardless of how many marbles are added before the second marble is drawn,

the probability will be the same as the probability of drawing a red marble with replacement.

Competition stage 1: (birthday paradox)

1. What is the probability that the two of them have a birthday on the same day of the week, if there are

(a) 7 people in a room.

Let's find the probability that the birthdays of all 7 people fall on 7 **different** days of the week

The 1st person has a 100% chance of a unique day of the week(of course) = 1 = 7/7

The 2nd has a (1 − 1/7) chance = 6/7

The 3rd has a (1 − 2/7) chance (all but 2 days) = 5/7

The 4th has a (1 − 3/7) (all but 3 days) = 4/7

The 5th has a (1 − 4/7) chance = 3/7

The 6th has a (1 − 5/7) chance = 2/7

The 7th has a (1 − 6/7) = 1/7

Probability (all have birthdays of different days)

= (7/7)*(6/7)*(5/7)*(4/7)*(3/7)*(2/7)*(1/7)

= (7×6×5×4×3×2×1) / (7^7)

= 5040/7^7

= 720/ 117649

= 0.006119899 approx.

So, an approximately 0.612% chance that all 7 have birthdays on 7 different days of the week

The probability that 2 or more have birthdays on the **same** day

= (100 − 0.612)% approx.

= 99.39% approx.

(b) 23 people in a room. (1 year = 365 days)

Let's find the probability that the birthdays of all 23 people fall on 23 different days of the year:

The 1st person has a 100% chance of a unique day of a year (of course) = 1 = 365/365

The 2nd has a (1 − 1/365) chance = 364/365

The 3rd has a (1 − 2/365) chance (all but 2 days) = 363/365

The 4th has a (1 − 3/365) (all but 3 days) = 362/365

….

The 23rd has a (1 − 22/365) (all but 22 days) = 343/365

Probability (all have birthdays of different days) =

(365/365)×(364/365)×(363/365)×(362/365)×….×(343/365)

= (365×364×….×343) / (365^23)

= 0.4927 approx.

So there is an approximately 49.27% chance that all 23 have birthdays on 23 different days of the year.

So, the probability that 2 or more have birthdays on the same day

= (100 −49.27)% approx.

= 50.73% approx.

Enrichment stage 1:

1. (a) P(R in A) = 3/8

   (b) P(same) = P(RR in B) + P(BB in B)

   = P(R in A) $\times$ [P(RR in B) + P(BB in B)] + P(B in A) $\times$ [P(RR in B) + P(BB in B)]

   = 3/8 $\times$ (3/11 $\times$ 2/10 + 8/11 $\times$ 7/10) + 5/8 $\times$ (2/11 $\times$ 1/10 + 9/11 $\times$ 8/10)

   = 139/220

2. Case 1: Chris wins in 3 sets

   P(WWW) = $(2/3)^3$

   = 8/27

   Case 2: Chris wins in 4 sets

   P(WWLW) + P(WLWW) + P(LWWW) = 3 $\times (2/3)^3 \times (1/3)$

   = 8/27

   Case 3: Chris wins in 5 sets

   P(WWLLW) + P(WLWLW) + P(LWWLW) + P(WLLWW) + P(LWLWW) + P(LLWWW)

   = 6 $\times (2/3)^3 \times (1/3)^2$

   = 16/81

   P(Chris wins) = 8/27 + 8/27 + 16/81

   = 64/81

Enrichment stage 2:

1. (a) $P(E) = P(M\cap X)\times P(N\cap Y)$
$= 0.56\times 0.35$
$= 0.196$

(b)
(i) $P(X|M) = P(X\cap M)/P(M)$
$= 0.56/0.8$
$= 0.7$

(ii) student B can be promoted to the university.
$P(Y|N) = P(Y\cap N)/P(N)$
$= 0.35/0.7$
$= 0.5$

(iii) $P(E) = P(X|M)\times P(Y|N)$
$= 0.7\times 0.5$
$= 0.35$

(iv) $P(E) = P(X|M)\times(1-P(Y|N))+(1-P(X|M))\times P(Y|N)$
$= 0.7\times(1-0.5)+(1-0.7)\times 0.5$
$= 0.5$

2. (a) P(R) = P(red, not red)+P(not red, red)
$= 3/10\times 7/9+7/10\times 3/9$
$= 7/15$

(b) P(S) = P(star, not star)+P(not star, star)
$= 4/10\times 6/9+6/10\times 4/9$
$= 8/15$

(c) $P(R\cap S) = P(S\cap R)$
= P(red star, white not star) +P(white not star, red star)+P(red not star, white star) +P(white star, red not star)
$= 1/10\times 4/9+4/10\times 1/9+2/10\times 3/9+3/10\times 2/9$
$= 2/9$

(d) $P(R|S) = P(S\cap R)/P(S)$
$= (2/9)/(8/15)$
$= 5/12$

Enrichment stage 2:

1. For $x^2 + bx + c = 0$ to have two different real roots,

Second Die

First Die	1	2	3	4	5	6
1						
2						
3	×	×				
4	×	×	×			
5	×	×	×	×	×	×
6	×	×	×	×	×	×

$\triangle = b^2 - 4c > 0$

$b^2 > 4c$

$\therefore$P($x^2 + bx + c = 0$ has two different real roots) = 17/36

2. (a)

P(B) = 1/12

P(R) = 2/5

Number of blue marbles = 1/12 × 180 = 15

Number of red marbles = 2/5 × 180 = 72

$\therefore$ Number of yellow marbles = 180 - 15 - 72 = 93

(b)

$(93 + x)/(180 + x) = 4/5$

$465 + 5x = 4x + 720$

$x = 720 - 465$

= 255 yellow marbles

3. (a) Let p be the probability that C wins.

P(C wins) = p

P(B wins) = 2 × P(C wins) = p

P(A wins) = 2 × P(B wins) = 4p

Total probability is 1

4p + 2p + p = 1

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(b) What is the probability that A does not win the race?

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4. $P(BB) + P(RR) = 3/5$

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$(M - 1)(M - 12) = 0$

$M = 1$ or $M = 12$

$M = 12$ only as $M > 1$

5. (a)

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$= m(m + k)/(m + n)(m + n + k)$

$P(WR) = n/(m + n) \times m/(m + n + k)$

$= mn/(m + n)(m + n + k)$

P(Second marble is red) = P(RR) + P(WR)

$= m(m + k)/(m + n)(m + n + k) + mn/(m + n)(m + n + k)$

$= m(m + n + k)/(m + n)(m + n + k)$

$= m/(m + n)$

(b)

The probability of the second marble being red is independent of the value of k.

In other words, regardless of how many marbles are added before the second marble is drawn, the probability will be the same as the probability of drawing a red marble with replacement.

Competition stage 3: (birthday paradox)

1. (a)

Let's find the probability that the birthdays of all 7 people fall on 7 **different** days of the week:
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Let's find the probability that the birthdays of all 23 people fall on 23 different days of the year:
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Probability (all have birthdays of different days) =
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