Graph Theory

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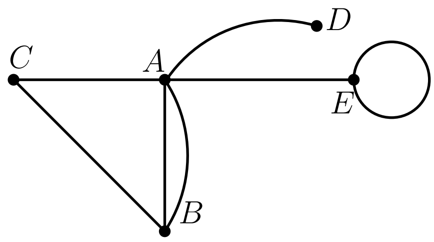
1 Introduction

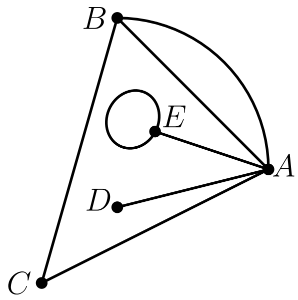
In mathematics, **Graph Theory** can be thought of as the study of points and the way that they are connected. Given their simple definition, they can encode a lot of information and can be applied to many different scenarios.

Graph theory has countless applications in today's society, for example in computer science, physics, ecology, genetics, economics, sociology and many more areas. This unit will introduce the terminology to study graphs and some key properties such as the Euler Characteristic.

2 Graphs

In mathematics, a **graph** is defined to be a set of points (called **vertices**), some of those are connected (by **edges**). Often, we are only interested in which vertices are connected (or how they are connected), rather than the shape of the edges, or position of the vertices. Consider the following two graphs, which are **mathematically identical**.



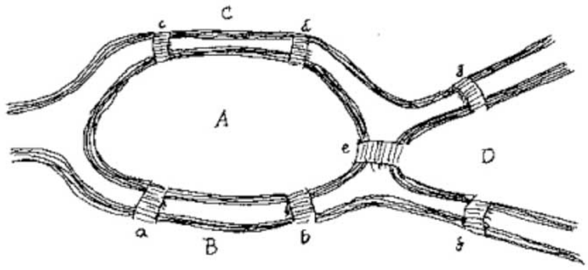


These are the same because in both graphs, there are two edges between A and B, one between A and C, one between A and D, one between A and E, one between B and C, and one from E to itself.

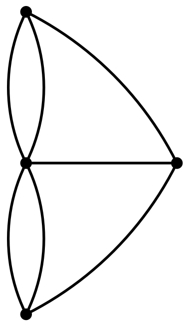
In its simplest form, we can treat all edges to be identical and all vertices to be identical. But we can also make more sophisticated graphs by **labelling**, **colouring**, **weighting** or **directing** edges or vertices.

Many questions can be interpreted as graph theory problems, reducing it to a much simpler form, and allowing us to use well established tools in graph theory. To illustrate this, consider the following exercise.

Exercise 2.1 ***(The Seven Bridges of Königsberg)*** *Suppose that a river flows through the middle of a city, dividing it into four land masses. As seen in the diagram, there are 7 bridges that connect these four pieces of land. Is it possible to start at any one of these islands and cross every bridge exactly once where each island can only be reached from the bridges?*



Hint: You may quickly notice that this scenario can be simplified into a graph! We are essentially interested in the landmasses (vertices) and how the bridges connect them (edges). This means that we can reinterpret the problem to the diagram below.



Can you start from any vertex and trace a path across all the edges, using each one exactly once?

Remark: such a path is called an *Eulerian Trail*, named after the famous mathematician Leonard Euler who first solved this problem, pioneering the field of Graph Theory!

3 Core Definitions and Properties

In this section, we will learn the basic vocabulary in graph theory and will prove some key properties.

Definition 3.1 *The **degree** of a vertex is the number of edges entering the vertex.*

Example 3.2 *Prove that the sum of degrees of all vertices in a graph is always an even number.*

We are interested in the total of the degrees in the graph. Instead of counting the total degree through each **vertex**, we can count the total degree through each **edge**. Since each edge connects two vertices, each edge contributes 2 to the sum of all degrees. This means that if there are E edges, the total degree is equal to $2E$ which is always an even number. $\square$

Exercise 3.3 *Prove that there is always an even number of vertices that have an odd degree.*

Remark: this is often called the **Hand-shaking Lemma**.

Definition 3.4 *A **connected graph** is a graph such that it is possible to travel between any pair of vertices along the edges.*

Definition 3.5 *A **cycle** is a closed path along edges which doesn't pass through the same edge twice.*

Definition 3.6 *A **tree** is a connected graph that contains no cycles. A **forest** is a graph that contains no cycles (ie. it only has trees). However, note that there are many equivalent ways to define what a tree is!*

Example 3.7 *A connected graph G with n vertices must have at least $n-1$ edges.*

We prove this statement by induction. For $n = 1$ and $n = 2$ it is obvious. Assume that it is true for n and prove itfor $n+1$. Consider a connected graph G with $n+1$ vertices.

Case 1. All vertices of G have degree at least 2. Then the total degree of G (the sum of degrees of all vertices) is at least $2(n+1)$. But we know from the previous example, that this number equals $2E$ where E is the number of edges. Hence the graph has at least $n+1$ edges.

Case 2. There is a vertice v in G which has degree 1 (sometimes it is called a **leaf**). Remove the vertice v and its adjacent edge from the graph. The remaining graph has n vertices and still must be connected. But by the inductional assumption, such a graph has at least $n-1$ vertices and therefore G has at least n edges. □

Example 3.8 *For a **connected** graph G with n vertices, prove that the following statements are equivalent*

(1) G does not contain any cycles.

(2) G contains exactly $n-1$ edges.

(3) For any two vertices, there exists exactly one path joining the two vertices.

(4) The removal of any edge disconnects the graph.

To show (2) $\implies$ (1), we can prove it by the contrapositive. If G has a cycle, you can remove an edge of that cycle and still have a connected graph. From Example 3.7 we know that it will have $\geq n-1$ edges, so it started with at least n edges, a contradiction.

To show (1) $\implies$ (2), start with the graph on n vertices and 0 edges. This has exactly n "connected components". At every second, let's add in some edge to reconstruct G. Then since there are no cycles, the number of connected components decreases by 1 every time. Hence, the first time G is connected, there is exactly one connected component so we have added in $n-1$ edges.

(1) $\iff$ (3) is also true since if there is a cycle, then there exist two distinct paths between two vertices within that cycle. And vice versa, if there are two distinct paths between two vertices, we can join them to make a cycle.

(3) $\implies$ (4) is true since if there is exactly one path between any two vertices, then deleting some edge will disconnect the two vertices on either side of that edge.

Finally, (4) $\implies$ (1) is true because if G contains a cycle than the removal one of theedges of this cycle will not disconnect the graph. □

Definition 3.9 *A **directed graph** is one in which the edges have a direction (from one vertex to another)*

Definition 3.10 *Two vertices are **adjacent** if there is an edge between them. Two adjacent vertices are called **neighbours**.*

Definition 3.11 *The **complement** of a graph G, often denoted by G^C, has the same vertices as G, but whose edges are between pairs of vertices that were not adjacent in G. In other words, between each pair of vertices, if there is an edge, remove it, and if there is no edge, add one in.*

Example 3.12 *Prove that the **complement** of an **unconnected graph** is connected. And what about the converse?*

To prove a property about graphs (or any mathematical object), we must only use what we know from the **definitions**.

Suppose that we have an **unconnected** graph G. By the definition of being **unconnected**, this means that the vertices of G can be split into two non-empty groups, say $\mathcal{A}$ and $\mathcal{B}$ such that there are no edges between the two groups.

The **complement** graph, let's say G^C, by definition only has the edges which weren't in the original graph G. But in G, we know that there are no edges between the vertices in $\mathcal{A}$ to the vertices in $\mathcal{B}$. This means that in G^C, there is an edge from each vertex in $\mathcal{A}$ to each vertex in $\mathcal{B}$.

We wish to prove that G^C is **connected**. This means that we need to prove that for any two vertices in G^C, there exists a path between them, travelling along the edges. There are 2 ways to choose two vertices:

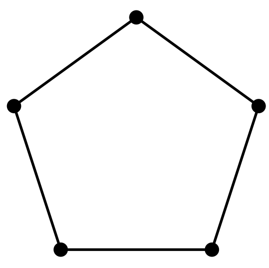
- **One vertex in $\mathcal{A}$ and one vertex in $\mathcal{B}$.**
  We know that in G^C, there exists an edge between each vertex in $\mathcal{A}$ to each vertex in $\mathcal{B}$, so there exists a path between these two vertices.

- **Two vertices in the same group** ($\mathcal{A}$ **or** $\mathcal{B}$).
  WLOG assume that the two vertices are in $\mathcal{A}$ and let's call them p and q. As mentioned before, in G^C, p and q are connected to every vertex in $\mathcal{B}$. Since $\mathcal{B}$ is non-empty, we can pick any vertex in $\mathcal{B}$, say k. Then we know that there is a path from p to k and then from k to q, so there exists a path between the two vertices p and q.

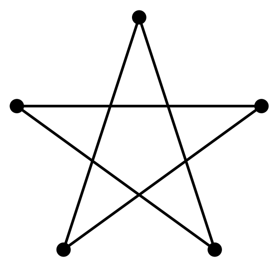
Hence, G^C is **connected** as there exists a path between each pair of vertices. $\square$

However, notice that the converse (the **complement** of a **connected** graph is unconnected) is false, as we can consider the following graph:

G



G^C



We can see that both G and G^C are connected, so the converse is false. $\square$

Now you can test your understanding with these exercises.

Exercise 3.13 *A house contains a finite number of rooms, with doors connecting the rooms. The only way to access a room is by walking through a door, except for the living room with a main entrance (which leads to the outside). If there is a painting in all rooms with an odd number of doors, prove that there must be at least one painting in this house.*

Exercise 3.14 *Among a group of 6 people, each pair of people are either both friends or both enemies. Prove that there exists a group of 3 people such that they are all friends with each other, or all enemies with each other.*

4 Euler's Characteristic

Euler's Characteristic is often taught in the context of **polyhedra**. Let's explore this with an exercise.

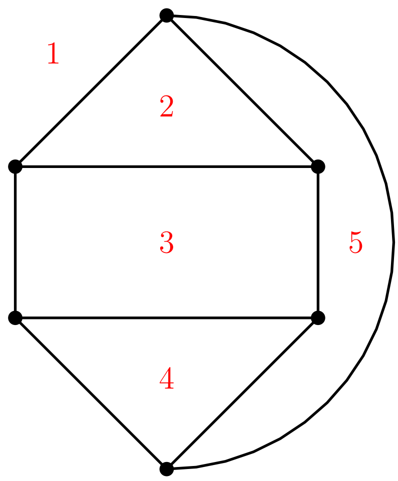
Exercise 4.1 *Fill out the table below of the number of vertices, edges and faces (and vertices − edges + faces) for each polyhedron. Also calculate this for two extra polyhedra of your choosing. Do you see any interesting patterns?*

Polyhedron	Vertices	Edges	Faces	$V - E + F$
Triangular Prism	6	9	5	2
Square Pyramid				
Tetrahedron				
Cube				
Octahedron				
Dodecahedron				
Icosahedron				

Remark: The value of $V - E + F$ is called the *Euler Characteristic.*

You may have noticed that the language we use for **graphs** is quite similar to the language we use with **polyhedra** (vertices and edges)! So maybe we can calculate an Euler Characteristic for graphs as well!

The **vertices** and **edges** will be the same as we have defined them previously, but we can consider a **face** as a region of the plane that is enclosed by edges (including the **exterior face**). Let's look at an example.

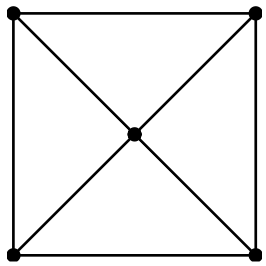


In this graph, the faces have been labelled in red. Notice that 1 represents the exterior face.

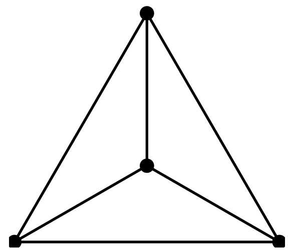
Vertices	Edges	Faces	$V - E + F$
6	9	5	2

Remark: Not every graph can be plotted in such a way that none of its edges cross each other. Graphs which can be plotted like that are called **planar**. For example, a graph with 5 vertices such that any pair of its vertices is connected by an edge, is not planar.

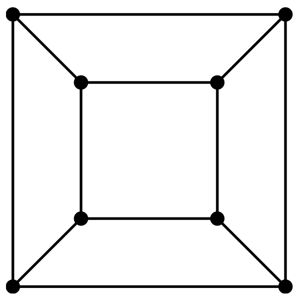
Exercise 4.2 *Calculate the Euler Characteristic of the following graphs. Also calculate this for two* ***connected*** *graphs of your own choosing.*



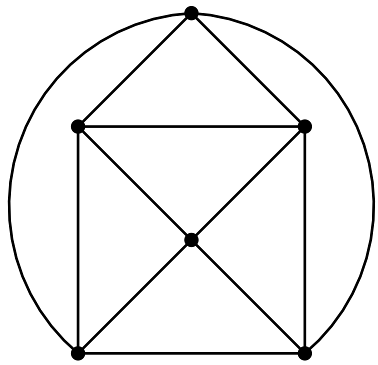
Vertices	Edges	Faces	$V - E + F$



Vertices	Edges	Faces	$V - E + F$



Vertices	Edges	Faces	$V - E + F$



Vertices	Edges	Faces	$V - E + F$

Remark: you may notice some interesting connections between the calculations of the past two exercises! In particular, you may notice that **polyhedra** can be represented as **graphs** on a plane!

Another pattern is that the values for V and F are swapped for the **cube** and **octahedron**, and the same for the **dodecahedron** and **icosahedron**! This is because these shapes are **duals** of each other, which means that's what we get when we replace all **vertices** with **faces**, and vice versa. In a similar sense, graphs can have **duals** as well! And notice that the **tetrahedron** is the dual of itself.

These five solids are called **platonic solids** and are the only equivalents of a "regular shape" in 3 dimensions!

Euler's Characterstic can be a useful tool in solving some problems!

Exercise 4.3 *Suppose there are 2022 lines in the plane such that no two are parallel and no three concurrent. Find the number of regions that these 2022 lines split the plane into.*

Exercise 4.4 *To wrap up, research some real-world applications of graph theory. You may want to focus on a subject/field that you are interested in, there will almost certainly be applications of graph theory!*