1. Consider the curve $f(x) = x^3$. Let $y = mx + b$ be a tangent.

(a) Expand $(x-A)(x-B)^2$.

(b) Find the x-coordinate of the point of contact in terms of m.

(c) Hence, find the derivative of $f(x) = x^3$.

1. (a) $(x-A)(x-B)^2=(x-A)(x^2-2Bx+B^2)$

$$=x^3-2Bx^2+B^2x-Ax^2+2ABx-AB^2$$
$$=x^3+(-A-2B)x^2+(2AB+B^2)x-AB^2$$

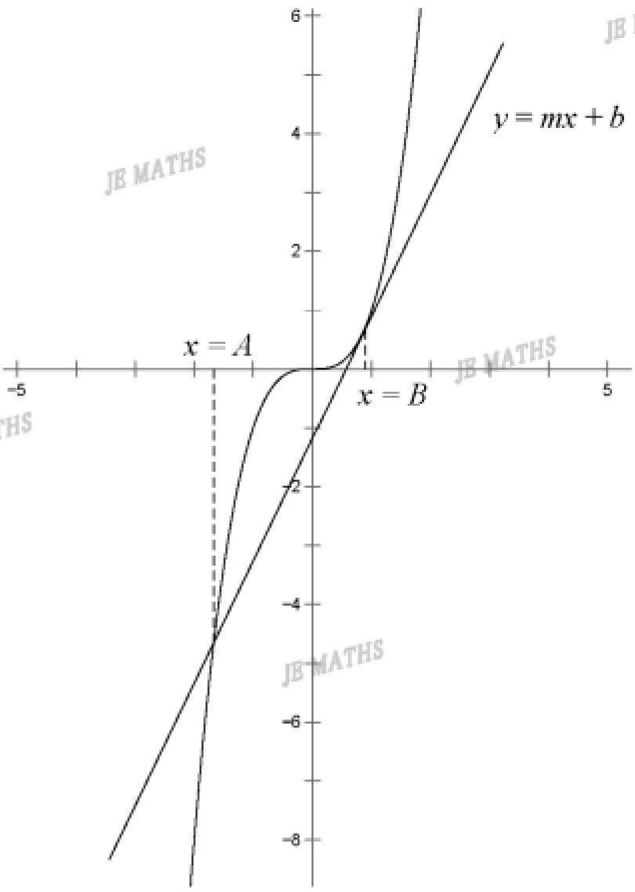
(b) Solve $\begin{cases} y=x^3 \\ y=mx+b \end{cases}$ for the x-coordinate of point of contact of the tangent and parabola.

$x^3=mx+b \Rightarrow x^3-mx-b=0$

The solution of the equation $x^3-mx-b=0$ gives the x-coordinate of point of contact.

The equation $x^3-mx-b=0$ can be expressed in the form of $(x-A)(x-B)^2=0$ as $x^3-mx-b=x^3+(-A-2B)x^2+(2AB+B^2)x-AB^2$ by finding proper values of coefficients A and B.

Hence, the solution of $x^3-mx-b=0$, which representing the x-coordinate of point of contact, can be obtained from the solution of the equation $(x-A)(x-B)^2=0$, where $x=B$ is the x-coordinate of point of contact and $x=A$ is the x-coordinate of point of intersection between the tangent and parabola.



Let $x^3 - mx - b = x^3 + \left(-A - 2B\right)x^2 + \left(2AB + B^2\right)x - AB^2$.

By matching coefficients,

$-A - 2B = 0$ (1)

$2AB + B^2 = -m$ (2)

$-AB^2 = -b$ (3)

From (1),

$A = -2B$

Sub $A = -2B$ into (2):

$-4B^2 + B^2 = -m$

$B^2 = \dfrac{m}{3}$

$x = B = \pm\sqrt{\dfrac{m}{3}}$

(c) $x = \pm\sqrt{\dfrac{m}{3}} \Rightarrow x^2 = \dfrac{m}{3}$

$\Rightarrow m = 3x^2$

$\Rightarrow f'(x) = 3x^2$