Inequalities

Written by Andy Tran for the MaPS Correspondence Program

1 Introduction

In the study of algebra, we are often not just interested in the equality of two expressions but also when one expression will always be greater or less than another. This can often be a challenge when we are dealing with unknown pronumerals. Then we need to use rigorous mathematical techniques to prove that an inequality holds.

We have already used induction to prove some inequalities, but in these notes, we will investigate rigorous proofs for inequalities using the **Rearrangement Inequality**, the fact that **Squares Are Never Negative** and extend this result to important inequalities like the **AM-GM inequality** and extend it to the more general **Power Means Inequality**.

Before we begin, let us review some basic operations in inequalities.

Exercise 1.1 *For each of the following statements, decide if they are true or false. If they are false, provide a counter-example and salvage the statement if possible. Assume that a, b, c and d are real numbers. Be careful to consider all possible values.*

1. If $a > b$, then $a + c > b + c$.
2. If $a > b$, then $ac > bc$.
3. If $a > b$, then $-a < -b$.
4. If $a > b > 0$ and then $\frac{1}{a} < \frac{1}{b}$.
5. If $a > b$ and $c > d$, then $a + c > b + d$.
6. If $a > b$ and $c > d$, then $a - c > b - d$.
7. If $a > b$ and $c > d$, then $ac > bd$.

2 The Rearrangement Inequality

We will begin this section with a motivating example.

Example 2.1 *You are on a TV game show and there are three piles of money in front of you: one consisting of \$5 notes, one consisting of \$20 notes, and one consisting of \$100 notes. You are allowed to take 10 notes from one pile, 20 notes from another pile, and 30 notes from the remaining pile. What is the largest amount of money you can walk away with? What is the smallest amount of money you can walk away with?*

Intuitively, to walk away with the largest amount of money, we would want to take as much of the \$100 notes as possible, then as many as we can remaining from the \$20 and then as little as we can from the \$5. This means we can walk away with at most

$$30 \cdot 100 + 20 \cdot 20 + 10 \cdot 5 = \$3450.$$

Similarly, to walk away with the least amount of money, we would want to take as few of the \$100 notes as possible, then as few as we can remaining from the \$20 and then as many as we can from the \$5. This means that we must walk away with at least

$$10 \cdot 100 + 20 \cdot 20 + 30 \cdot 5 = \$1550.$$

This idea (which has not been formally proven) is the basis of the rearrangement inequality. The formal statement is:

Definition 2.2 ***(The Rearrangement Inequality)***

If $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ are real numbers, and $c_1, c_2, \ldots, c_n$ is a permutation of $b_1, b_2, \ldots, b_n$, then

$$a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1 \leq a_1 c_1 + a_2 c_2 + \cdots + a_n c_n \leq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Intuitively, what this is saying is that if we are **pairing up** each of the a's and the b's and then **summing their products**, then:

- the **largest value** (the right-most term) occurs when you pair the largest terms together, then the next largest, etc. (like taking the most of the largest value note)
- the **smallest value** (the left-most term) occurs when you pair the largest terms with the smallest terms, then the next largest with the next smallest, etc. (like taking the least of the largest value note)
- any other combination of a's and b's will be **between these two values**.

Example 2.3 *Suppose $a, b, c > 0$. Prove that*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3.$$

Notice that the expression $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ is **cyclic**, that is if we cycle the values of the pronumerals (like $a \to b$, $b \to c$, $c \to a$), the expression stays the same. This means that we can cycle the variables such that either $a \geq b \geq c$ or $a \geq c \geq b$. Notice that is also gives us that $\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a}$. or $\frac{1}{b} \geq \frac{1}{c} \geq \frac{1}{a}$ respectively.

In either case, the rearrangement inequality tells us that the pairing of $a \cdot \frac{1}{a} + b \cdot \frac{1}{b} + c \cdot \frac{1}{c}$ is the smallest possible value and any other pairing will be greater than it. Hence,

$$a \cdot \frac{1}{b} + b \cdot \frac{1}{c} + c \cdot \frac{1}{a} \geq a \cdot \frac{1}{a} + b \cdot \frac{1}{b} + c \cdot \frac{1}{c}$$

Which gives us the desired inequality

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3.$$

$\square$

Remark: When we cycled the values of a, b and c so that $a \geq b \geq c$ or $a \geq c \geq b$, this is often mathematically written as "**Without loss of generality** (or **WLOG**) assume that $a \geq b \geq c$ or $a \geq c \geq b$."

Exercise 2.4 *Prove the rearrangement inequality in the case of three terms. To start this, you can suppose that* $a_1 \le a_2 \le a_3$ *and* $b_1 \le b_2 \le b_3$ *and prove that*

$$a_1b_2 + a_2b_3 + a_3b_1 \le a_1b_1 + a_2b_2 + a_3b_3.$$

Think about how this idea can generalise to all possible arrangements as well as the smallest value for the rearrangement inequality. Also think about how this idea can generalise to any number of terms.

Exercise 2.5 *If* a, b *and* c *are positive real numbers, prove that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}.$$

3 Squares are never negative

When we are dealing with arbitrary pronumerals, it is often very difficult to determine if one side of an equation is greater than or less than another side. This is where the power of "**Squares are never negative**" becomes very useful. That is, for any real number x, we will have that

$$x^2 \ge 0.$$

This is a rather obvious statement, but it has many incredible applications.

Exercise 3.1 *Why is the statement "Squares are always positive" incorrect?*

You can see that another useful fact that we can deduce from this is that the **equality** only occurs when $x = 0$.

Now let's apply this to some more interesting inequalities:

Example 3.2 *Prove that for any real numbers* a *and* b,

$$a^2 + b^2 \ge 2ab$$

and determine when equality holds.

To prove that an inequality is true, we'd want to rewrite it as a square and then use the fact that squares are never negative. You may also find the terms a^2, b^2 and $2ab$ a little bit familiar! In fact we can say:

$$\begin{aligned} (a-b)^2 &\ge 0 \text{ (Squares are never negative)} \\ \implies a^2 - 2ab + b^2 &\ge 0 \\ \implies a^2 + b^2 &\ge 2ab \end{aligned}$$

And we also know that the only time equality would occur is when $a - b = 0$ in other words, $a = b$. $\square$

Example 3.3 *Prove that if* x *and* y *are positive real numbers, then* $\frac{1}{x} + \frac{1}{y} \ge \frac{4}{x+y}$ *and find when equality occurs.*

Again, we will eventually want to rewrite this as a square to say that the inequality holds for all possible values of x and y. One way to start this problem is to simplify it by removing the fractions.

$$\begin{aligned}
\frac{1}{x}+\frac{1}{y} &\geq \frac{4}{x+y} \\
\iff \frac{x+y}{xy} &\geq \frac{4}{x+y} \\
\iff (x+y)^2 &\geq 4xy \\
\iff x^2+2xy+y^2 &\geq 4xy \\
\iff x^2-2xy+y^2 &\geq 0 \\
\iff (x-y)^2 &\geq 0.
\end{aligned}$$

And we know that the last statement is true, so that means that our original inequality is true. Also we note that equality occurs if and only if $x=y$. □

Remark: Notice that in our proof, we had to multiply both sides of the inequality by the denominators xy and $x+y$. We must be very careful that we know that they are **positive** which is why the inequality still holds. If we did not know the sign, we would not be able to multiply across.

Exercise 3.4 *Prove that the sum of a positive number and its reciprocal is always greater than or equal to 2.*

Exercise 3.5 *Prove that for all real numbers a, b, c: $a^2+b^2+c^2 \geq ab+bc+ca$ and when does equality occur?*

4 The AM-GM Inequality

The AM-GM inequality is a very powerful statement that says that the **Arithmetic Mean** of a set of positive numbers is always greater than or equal to its **Geometric Mean**.

The **Arithmetic Mean** of a set of *positive* real numbers $\{x_1, x_2, \ldots, x_n\}$ is the mean that we are most familiar with, and is defined by:

$$\frac{x_1+x_2+\ldots+x_n}{n}$$

The **Geometric Mean** of $\{x_1, x_2, \ldots, x_n\}$ however, is defined by:

$$\sqrt[n]{x_1 x_2 \cdots x_n}$$

Thus, the AM-GM inequality states that:

Definition 4.1 ***(The AM-GM Inequality)***

Given any set of *positive* real numbers $x_1, x_2, \ldots x_n$,

$$\frac{x_1+x_2+\ldots+x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

And equality occurs if and only if $x_1 = x_2 = \cdots = x_n$.

Exercise 4.2 *Show that the AM-GM inequality when $n = 2$ is equivalent to "Squares are Never Negative"*

Let's apply this to a couple examples.

Example 4.3 *For positive real numbers $x_1, x_2, \ldots, x_n$, prove that* $\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n$.

You can probably see that for an inequality like this, it would be a lot harder to rewrite the *LHS* as a square or sum of squares. This is where the AM-GM can show its true power!

Considering the set of numbers $\{\frac{x_1}{x_2}, \frac{x_2}{x_3}, \cdots, \frac{x_{n-1}}{x_n}, \frac{x_n}{x_1}\}$, we know that their arithmetic mean is greater than or equal to their geometric mean:

$$\frac{\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1}}{n} \geq \sqrt[n]{\frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdot \cdots \cdot \frac{x_{n-1}}{x_n} \cdot \frac{x_n}{x_1}}.$$

Conveniently, all the fractions on the RHS cancel out with each other!

$$\begin{aligned}
\implies \frac{\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1}}{n} &\geq \sqrt[n]{1} \\
\implies \frac{\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1}}{n} &\geq 1 \\
\implies \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} &\geq n
\end{aligned}$$

which was what needed to be proven. □

Example 4.4 *For all positive real numbers a, b, c prove that $(a + b)(b + c)(c + a) \geq 8abc$.*

A first instinct should probably be to expand the brackets and simplify. So that

$$\begin{aligned}
(a + b)(b + c)(c + a) \geq 8abc &\iff abc + a^2b + ac^2 + a^2c + b^2c + b^2a + bc^2 + abc \geq 8abc \\
&\iff a^2b + ac^2 + a^2c + b^2c + b^2a + bc^2 \geq 6abc \qquad (1)
\end{aligned}$$

From here, there are a couple ways to prove that (1) is true!

Approach 1. A simple way to prove that it is true, is to use the **AM-GM inequality**! We can apply it to the 6 terms on the LHS to find that

$$\begin{aligned}
\frac{a^2b + ac^2 + a^2c + b^2c + b^2a + bc^2}{6} &\geq \sqrt[6]{a^2b \cdot ac^2 \cdot a^2c \cdot b^2c \cdot b^2a \cdot bc^2} \\
\implies \frac{a^2b + ac^2 + a^2c + b^2c + b^2a + bc^2}{6} &\geq \sqrt[6]{a^6b^6c^6} \\
\implies \frac{a^2b + ac^2 + a^2c + b^2c + b^2a + bc^2}{6} &\geq abc \\
\implies a^2b + ac^2 + a^2c + b^2c + b^2a + bc^2 &\geq 6abc
\end{aligned}$$

which concludes the proof. □

Approach 2. Alternatively, we just use the fact that **squares are never negative** to prove (1), but in a sneaky way!

$$\begin{aligned}
a^2b + ac^2 + a^2c + b^2c + b^2a + bc^2 &\geq 6abc \\
\iff a^2b + ac^2 + a^2c + b^2c + b^2a + bc^2 - 6abc &\geq 0 \\
\iff (a^2c - 2abc + b^2c) + (a^2b - 2abc + bc^2) + (ab^2 - 2abc + ac^2) &\geq 0 \\
\iff c(a^2 - 2ab + b^2) + b(a^2 - 2ac + c^2) + a(b^2 - 2bc + c^2) &\geq 0 \\
\iff c(a - b)^2 + b(a - c)^2 + a(b - c)^2 &\geq 0
\end{aligned}$$

And the last statement is true because **the sum of squares is never negative**. □

Exercise 4.5 *In an earlier example, we proved that if* $a, b, c > 0$ *then*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3.$$

Prove this again using the AM-GM inequality.

Exercise 4.6 *Find the maximum value of* xyz *given that* x*,* y *and* z *are positive real numbers satisfying* $x + 2y + 3z = 3$.

Exercise 4.7 *Let* $a_1, a_2, \ldots, a_n$ *be positive real numbers such that* $a_1 a_2 \cdots a_n = 1$ *Prove that*

$$(a_1 + 1)(a_2 + 1) \cdots (a_n + 1) \geq 2^n.$$