Problem Set 6

MaPS Correspondence Program

Instructions

- Some of these problems are based off the notes "*Modular Arithmetic*". Some other are revision problems for the previous notes.
- They are in roughly difficulty order and get quite difficult, so you are **not** expected to be able to solve every problem.
- However, please attempt as many questions as you can and submit your solutions to your mentor for marking and feedback.
- You may (and encouraged to) submit incomplete solutions if you can not solve a problem completely.
- You may type your solutions or submit a pdf of a **clear** scan/photo of **legible** written solutions.
- Feel free to discuss these problems with your peers and on the forum but the solutions you submit must be written by yourself.

Problems

1. Prove that every prime number greater than 3 leaves a remainder of either 1 or 5 when divided by 6.
2. In triangle ABC, let P, Q, R be points on sides BC, CA, AB respectively. Let $\mathcal{C}_1$ be the circle passing through A, Q and R and let $\mathcal{C}_2$ be the circle passing through B, P and R. Let the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$ other than R be X. Prove that $CPXQ$ is a cyclic quadrilateral.
3. (a) Find the residues of $2^{340} \pmod{11}$ and $2^{340} \pmod{31}$. The answer should be between 0 and 10 for the first congruence and between 0 and 30 for the second one.
   (b) Find the residue of $2^{340} \pmod{341}$.
   (c) What does this say about the converse of Fermat's Little Theorem?
4. The Fibonacci numbers are defined by the recurrence $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Prove that every positive integer can be written as the sum of distinct Fibonacci numbers, no two of which are consecutive.
5. Let n be an integer such that $n - 3$ is a multiple of 8. Let k be an integer greater than 3. Prove that

$$n^{2^{k-3}} - 2^{k-1} - 1$$

is a multiple of 2^k.
Note: this problem appeared as Q3 in the 2013 AMO