Problem Set 7

Written by James Bang and Andy Tran for the MaPS Correspondence Program

Instructions

- Some of these problems are based off the notes "*Inequalities*". Some other are revision problems for the previous notes.
- They are in roughly difficulty order and get quite difficult, so you are **not** expected to be able to solve every problem.
- However, please attempt as many questions as you can and submit your solutions to your mentor for marking and feedback.
- You may (and encouraged to) submit incomplete solutions if you can not solve a problem completely.
- You may type your solutions or submit a pdf of a **clear** scan/photo of **legible** written solutions.
- Feel free to discuss these problems with your peers and on the forum but the solutions you submit must be written by yourself.

Problems

1. Suppose that a rectangle has a perimeter of 20m. What is the largest possible area of such a rectangle?

   *Note: You must **prove** that your answer is the maximal area.*

2. Prove that for all positive integers n,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

3. Prove that for all positive reals a, b, c, we have

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geqslant a + b + c.$$

4. (a) Prove that if p is a prime number, then each of $1, 2, 3, \ldots, p-1$ has a unique inverse in $(\bmod\, p)$.

   (b) Prove that 1 and $p-1$ are the only numbers that are their own inverse.

   (c) Prove Wilson's Theorem.

5. Let x, y be non-negative reals with sum 2. Prove that

$$x^2y^2(x^2+y^2) \leqslant 2.$$