Enrichment stage 1: (Function substitution)

1. Suppose f is a function whose domain is the interval [-5, 5], which means $-5 \le x \le 5$ and $f(x) = \dfrac{x}{x+3}$ for every x in the interval [0, 5], which means $0 \le x \le 5$.

   (a) If f is an even function, then evaluate $f(-2)$ and $f(-3)$.

   f is an even function, which means $f(-x) = f(x)$

   Because 2 and 3 are both in the interval [0, 5],

   $f(2) = 2/5$, $f(3) = 3/5$

   $f(-2) = f(2) = 2/5$

   $f(-3) = f(3) = 3/5$

   (b) If f is an odd function, then evaluate $f(-2)$ and $f(-3)$.

   f is an odd function, which means $f(-x) = -f(x)$

   $f(-2) = -f(2) = -2/5$

   $f(-3) = -f(3) = -3/5$

2. Given that a quadratic function: $f(x) = x^2 + bx + 36$.
   When $f(x) = 0$, the solutions of it are integers, what is the number of integer values b can have?

   $x^2 - bx + 36 = 0$ can be rewritten as $(x - m)(x - n)$, where m and n are both integers.

   $\therefore b = m + n$ and $mn = 36$

   m and n are either both positive or both negative.

   Possible positive values of (m, n) are (1, 36), (2, 18), (3, 12), (4, 9), (6, 6), (9, 4), (12, 3), (18, 2), (36, 1).

   $\therefore b$ can take values 37, 20, 15, 13 and 12.

   Possible negative values of (m, n) are (−1, −36), (−2, −18), (−3, −12), (−4, −9), (−6, −6), (−9, −4),

   (−12, −3), (−18, −2), (−36, −1).

   $\therefore b$ can take values −37, −20, −15, −13 and −12.

   $\therefore$ There are 10 possible values for b.

Enrichment stage 2: (Even and odd)

1. For the function $f(x)$,

   (a) Show that $g(x) = f(x) + f(-x)$ is even, and $h(x) = f(x) - f(-x)$ is odd.

   (b) Deduce that any function can be written as the sum of even functions, odd functions, or even and odd functions.

2. For all positive integers n, $f(n) = 2n$ if n is even and $f(n) = 3n$ if n is odd. If p is a prime number greater than 2, what is the value of $f\left[f(p-1)-p\right]$.

Enrichment stage 2: (Absolute value function)

1. Given $f(x)=\begin{cases}2x+9 & \text{if } x<0\\ 3x-10 & \text{if } x\geq 0\end{cases}$.

   (a) Evaluate $f(1)$.

   (b) Evaluate $f(-3)$.

   (c) Evaluate $f(|x|+1)$.

2. Solve for x:

   (a) $|3x-2|<|x-3|$

(b) $|3x - 2| < x - 3$ by using discussing method.

(i) For $x \le 2/3$,

$-3x + 2 < x - 3$

$4x > 5$

$x > 5/4$

∴No solution as the intervals don't overlap.

(ii) For $x > 2/3$,

$3x - 2 < x - 3$

$2x < -1$

$x < -1/2$

∴No solution as the intervals don't overlap.

(c) $|3x - 2| < -3$

$|3x - 2| > 0$ for all x

∴No solutions for x

3. (a) Simplify $\sqrt{(x+1)^2}$

$\sqrt{(x + 1)^2} = |x + 1|$

(b) Simplify $\sqrt{(x^2+6x+11)^2}$

$x^2 + 6x + 11 = (x + 3)^2 + 2 > 0$ for all real x

$\sqrt{(x^2 + 6x + 11)^2} = |x^2 + 6x + 11|$

$= x^2 + 6x + 11$ as $x^2 + 6x + 11 > 0$

Enrichment stage 1:

1. (a)

f is an even function, which means $f(-x) = f(x)$

Because 2 and 3 are both in the interval [0, 5],

$f(2) = 2/5$, $f(3) = 3/5$

$f(-2) = f(2) = 2/5$

$f(-3) = f(3) = 3/5$

(b)

f is an odd function, which means $f(-x) = -f(x)$

$f(-2) = -f(2) = -2/5$

$f(-3) = -f(3) = -3/5$

2. $\therefore b = m + n$ and $mn = 36$

m and n are either both positive or both negative.

Possible positive values of (m, n) are (1, 36), (2, 18), (3, 12), (4, 9), (6, 6), (9, 4), (12, 3), (18, 2), (36, 1).

$\therefore b$ can take values 37, 20, 15, 13 and 12.

Possible negative values of (m, n) are (-1, -36), (-2, -18), (-3, -12), (-4, -9), (-6, -6), (-9, -4), (-12, -3), (-18, -2), (-36, -1).

$\therefore b$ can take values -37, -20, -15, -13 and -12.

$\therefore$ There are 10 possible values for b.

Enrichment stage 2:

1. (a)

$g(-x) = f(-x) + f(-(-x))$
$= f(-x) + f(x)$
$= g(x)$

$\therefore g(x)$ is even.

$h(-x) = f(-x) - f(-(-x))$
$= f(-x) - f(x)$
$= -[f(x) - f(-x)]$
$= -h(x)$

$\therefore h(x)$ is odd.

(b)

$g(x) + h(x) = f(x) + f(-x) + f(x) - f(-x)$
$= 2f(x)$

$f(x) = [g(x) + h(x)]/2$

$\therefore$Any function $f(x)$ can be expressed as the sum of an odd and even function.

2. p is odd because the only prime number that is even is 2 and p is larger than 2

$\therefore$ $p - 1$ is even and $p - 2$ is odd

$[f(p - 1) - p] = f[2(p - 1) - p]$
$= f(p - 2)$
$= 3(p - 2)$

Enrichment stage 2:

1. (a)

Because $1 \geq 0$,
$f(1) = 2(1) + 9$
$= 11$

(b)
Because $-3 < 0$,
$f(-3) = 3(-3) - 10$
$= -19$

(c)
Because $|x| + 1 \geq 1 > 0$,
$f(|x| + 1) = 3(|x| + 1) - 10$
$= 3|x| - 7$

2. (a)

$|3x - 2|^2 < |x - 3|^2$
$9x^2 - 12x + 4 < x^2 - 6x + 9$
$8x^2 - 6x - 5 < 0$
$(4x - 5)(2x + 1) < 0$
$-1/2 < x < 5/4$

(b)
(i) For $x \leq 2/3$,
$-3x + 2 < x - 3$
$4x > 5$
$x > 5/4$
∴No solution as the intervals don't overlap.

(ii) For $x > 2/3$,
$3x - 2 < x - 3$
$2x < -1$
$x < -1/2$
∴No solution as the intervals don't overlap.

(c) $|3x - 2| > 0$ for all x
∴No solutions for x

3. (a) $\sqrt{(x + 1)^2} = |x + 1|$

(b) $x^2 + 6x + 11 = (x + 3)^2 + 2 > 0$ for all real x
$\sqrt{(x^2 + 6x + 11)^2} = |x^2 + 6x + 11|$
$= x^2 + 6x + 11$ as $x^2 + 6x + 11 > 0$