Stage 1:

1. Suppose $F(x)$ and $G(x)$ satisfy the following properties:

$$F(3)=2,\ G(3)=4,\ G(0)=3$$

$$F'(3)=-1,\ G'(3)=0,\ G'(0)=-3$$

(a) If $K(x)=F(x)\cdot G(x)$, find $K'(3)$.

(b) If $S(x)=\dfrac{F(x)}{G(x)}$, find $S'(3)$.

(c) If $T(x)=F\left(G(x)\right)$, find $T'(0)$.

2. Given that $y=\dfrac{\sqrt{1+2x}\sqrt[4]{1+4x}\sqrt[6]{1+6x}\cdots\sqrt[100]{1+100x}}{\sqrt[3]{1+3x}\sqrt[5]{1+5x}\sqrt[7]{1+7x}\cdots\sqrt[101]{1+101x}}$, find $\dfrac{dy}{dx}$ at $x=0$.

3. Given that function $y=\cfrac{x}{x+\cfrac{x}{x+\cfrac{x}{x+\cdots}}}$, find $y'(1)$.

Solutions:

1. (a) $K(x)=F(x)\cdot G(x) \;\Rightarrow\; K'(x)=F'(x)\cdot G(x)+F(x)\cdot G'(x)$

$$\Rightarrow\; K'(3)=F'(3)\cdot G(3)+F(3)\cdot G'(3)$$

$$=(-1)\cdot 4+2\cdot 0$$

$$=-4$$

(b) $S(x)=\dfrac{F(x)}{G(x)} \;\Rightarrow\; S'(x)=\dfrac{F'(x)\cdot G(x)-F(x)\cdot G'(x)}{\left[G(x)\right]^2}$

$$\Rightarrow\; S'(3)=\frac{F'(3)\cdot G(3)-F(3)\cdot G'(3)}{\left[G(3)\right]^2}$$

$$=\frac{-1\cdot 4-2\cdot 0}{16}$$

$$=-\frac{1}{4}$$

(c) $T(x)=F\left(G(x)\right) \;\Rightarrow\; T'(x)=F'\left(G(x)\right)\times G'(x)$

$$\Rightarrow\; T'(0)=F'\left(G(0)\right)\times G'(0)$$

$$=F'(3)\times(-3)$$

$$=3$$

2. $$y=\frac{\sqrt{1+2x}}{\sqrt[3]{1+3x}}\cdot\frac{\sqrt[4]{1+4x}}{\sqrt[5]{1+5x}}\cdot\frac{\sqrt[6]{1+6x}}{\sqrt[7]{1+7x}}\cdots\cdots\frac{\sqrt[100]{1+100x}}{\sqrt[101]{1+101x}}$$

$$\Rightarrow\; \frac{dy}{dx}=\left(\frac{\sqrt{1+2x}}{\sqrt[3]{1+3x}}\right)'\cdot\frac{\sqrt[4]{1+4x}}{\sqrt[5]{1+5x}}\cdot\frac{\sqrt[6]{1+6x}}{\sqrt[7]{1+7x}}\cdots\cdots\frac{\sqrt[100]{1+100x}}{\sqrt[101]{1+101x}}$$

$$+\frac{\sqrt{1+2x}}{\sqrt[3]{1+3x}}\cdot\left(\frac{\sqrt[4]{1+4x}}{\sqrt[5]{1+5x}}\right)'\cdot\frac{\sqrt[6]{1+6x}}{\sqrt[7]{1+7x}}\cdots\cdots\frac{\sqrt[100]{1+100x}}{\sqrt[101]{1+101x}}$$

$$+\frac{\sqrt{1+2x}}{\sqrt[3]{1+3x}}\cdot\frac{\sqrt[4]{1+4x}}{\sqrt[5]{1+5x}}\cdot\left(\frac{\sqrt[6]{1+6x}}{\sqrt[7]{1+7x}}\right)'\cdots\cdots\frac{\sqrt[100]{1+100x}}{\sqrt[101]{1+101x}}$$

$$+\cdots\cdots$$

$$+\frac{\sqrt{1+2x}}{\sqrt[3]{1+3x}}\cdot\frac{\sqrt[4]{1+4x}}{\sqrt[5]{1+5x}}\cdot\frac{\sqrt[6]{1+6x}}{\sqrt[7]{1+7x}}\cdots\cdots\left(\frac{\sqrt[100]{1+100x}}{\sqrt[101]{1+101x}}\right)' \qquad \text{(by product rule)}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=0}=\left(\frac{\sqrt{1+2x}}{\sqrt[3]{1+3x}}\right)'_{x=0}+\left(\frac{\sqrt[4]{1+4x}}{\sqrt[5]{1+5x}}\right)'_{x=0}+\left(\frac{\sqrt[6]{1+6x}}{\sqrt[7]{1+7x}}\right)'_{x=0}+\cdots+\left(\frac{\sqrt[100]{1+100x}}{\sqrt[101]{1+101x}}\right)'_{x=0}$$

$$=0+0+0+\cdots+0$$
$$=0$$

Check the derivative of the general term $\left(\frac{\sqrt[k]{1+kx}}{\sqrt[k+1]{1+(k+1)x}}\right)'$, where $k=2,3,4,\dots 100$.

$$\left(\frac{\sqrt[k]{1+kx}}{\sqrt[k+1]{1+(k+1)x}}\right)'=\left((1+kx)^{\frac{1}{k}}\times\left[1+(k+1)x\right]^{-\frac{1}{k+1}}\right)'$$

$$=\frac{1}{k}(1+kx)^{\frac{1}{k}-1}\cdot(1+kx)'\times\left[1+(k+1)x\right]^{-\frac{1}{k+1}}$$

$$+(1+kx)^{\frac{1}{k}}\times\left(-\frac{1}{k+1}\right)\left[1+(k+1)x\right]^{-\frac{1}{k+1}-1}\cdot\left[1+(k+1)x\right]'$$

(by product rule and chain rule)

$$=\frac{1}{k}(1+kx)^{\frac{1-k}{k}}\cdot k\times\left[1+(k+1)x\right]^{-\frac{1}{k+1}}$$

$$+(1+kx)^{\frac{1}{k}}\times\left(-\frac{1}{k+1}\right)\left[1+(k+1)x\right]^{-\frac{k+2}{k+1}}\cdot(k+1)$$

$$=(1+kx)^{\frac{1-k}{k}}\times\left[1+(k+1)x\right]^{-\frac{1}{k+1}}-(1+kx)^{\frac{1}{k}}\times\left[1+(k+1)x\right]^{-\frac{k+2}{k+1}}$$

$$\therefore\ \left(\frac{\sqrt[k]{1+kx}}{\sqrt[k+1]{1+(k+1)x}}\right)'=(1+kx)^{\frac{1-k}{k}}\times\left[1+(k+1)x\right]^{-\frac{1}{k+1}}-(1+kx)^{\frac{1}{k}}\times\left[1+(k+1)x\right]^{-\frac{k+2}{k+1}}$$

$$\therefore\ \left(\frac{\sqrt[k]{1+kx}}{\sqrt[k+1]{1+(k+1)x}}\right)'_{x=0}=1-1=0$$

3. $y=\cfrac{x}{x+\cfrac{x}{x+\cfrac{x}{x+\cdots}}} \Rightarrow y=\dfrac{x}{x+y}$

$$\Rightarrow xy+y^2=x$$

$$\Rightarrow \left(xy+y^2\right)'=(x)'$$

$$\Rightarrow y+x\times\frac{dy}{dx}+2y\times\frac{dy}{dx}=1$$

$$\Rightarrow (x+2y)\times\frac{dy}{dx}=1-y$$

$$\Rightarrow \frac{dy}{dx}=\frac{1-y}{x+2y}$$

$$y=\cfrac{x}{x+\cfrac{x}{x+\cfrac{x}{x+\cdots}}} \Rightarrow y=\frac{x}{x+y}$$

$$\Rightarrow y=\frac{1}{1+y} \qquad (\text{Let } x=1)$$

$$\Rightarrow y^2+y-1=0$$

$$\Rightarrow y_{1,2}=\frac{-1\pm\sqrt{5}}{2}$$

$$\Rightarrow y=\frac{-1+\sqrt{5}}{2} \quad \left(\text{omit } y=\frac{-1-\sqrt{5}}{2}<0,\text{ as when } x=1,\ y>0\right)$$

$$\frac{dy}{dx}=\frac{1-y}{x+2y} \Rightarrow y'(1)=\frac{1-\frac{-1+\sqrt{5}}{2}}{1+(-1+\sqrt{5})}=\frac{3\sqrt{5}-5}{10}$$

$$\therefore\ y'(1)=\frac{3\sqrt{5}-5}{10}$$