Enrichment stage 1: (2D trig questions)

1. A hill with a uniform slope is inclined at 14° to the horizontal. From the bottom of the hill A, the angle of elevation of T, the top of a tower TB standing on a hill is 25°. On moving 50m up the hill to a point C, the angle of elevation of T is 55°.

   (a) Construct a diagram to represent this information.

   (b) Find the size of ∠ATC.

   ∠TAC = 11°

   ∠ATC = 41° − 11° (Exterior angle of △TAC equals sum of 2 interior opposite angles)

   = 30°

   (c) Find the length of TA correct to one decimal place.

   In △ACT,

   ∠ACT = 139° (Angle sum of a triangle is 180°)

   50/sin30° = TA/sin139°

   TA = 100sin139°

   = 65.6059029

   = 65.6m (1dp)

   (d) Find the height of the tower TB correct to two decimal places.

   ∠ABD = 76° (Angle sum of a triangle is 180°)

   ∠CBT = 104° (Sum of adjacent angles on a straight angle is 180°)

   In △ACT,

   TC/sin11° = 50/sin30°

   TC = 100sin11°

   In △CBT,

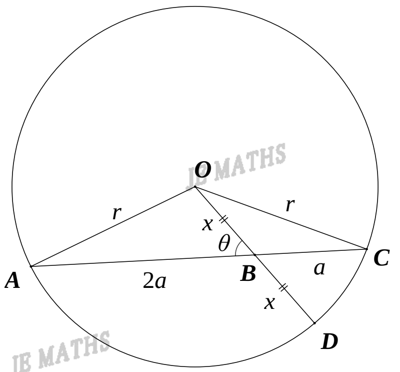
   TB/sin41° = TC/sin104°

   TB = 100sin11°sin41°/sin104°

   = 12.90142385

   ∴ The tower TB is 12.9m high.

2. The following diagram shows a circle with centre O and radius r units. A radius divides a chord in the ratio of 2:1 and is bisected by the chord as shown in the diagram.



(a) Show $r^2 = x^2 + 4a^2 - 4ax\cos\theta$.

In △OAB,

$OA^2 = OB^2 + AB^2 - 2 \times OB \times AB \times \cos\theta$

$r^2 = x^2 + (2a)^2 - 2 \times x \times 2a \times \cos\theta$

$r^2 = x^2 + 4a^2 - 4ax\cos\theta$

(b) Show that the cosine of the angle θ between the chord and the radius is $\dfrac{\sqrt{6}}{4}$.

Similarly in △OBC,

$OC^2 = OB^2 + BC^2 - 2 \times OB \times BC \times \cos(180° - \theta)$

$r^2 = x^2 + a^2 + 2ax\cos\theta$

$(2x)^2 = x^2 + a^2 + 2ax\cos\theta$ (OD is radius)

$3x^2 = a^2 + 2ax\cos\theta$ (1)

From (a),

$r^2 = x^2 + 4a^2 - 4ax\cos\theta$

$(2x)^2 = x^2 + 4a^2 - 4ax\cos\theta$

$3x^2 = 4a^2 - 4ax\cos\theta$ (2)

(1) = (2): $a^2 + 2ax\cos\theta = 4a^2 - 4ax\cos\theta$

$3a^2 = 6ax\cos\theta$

$\cos\theta = a/2x$ (3)

$2 \times (1) + (2)$: $6x^2 + 3x^2 = 2a^2 + 4ax\cos\theta + 4a^2 - 4ax\cos\theta$

$9x^2 = 6a^2$

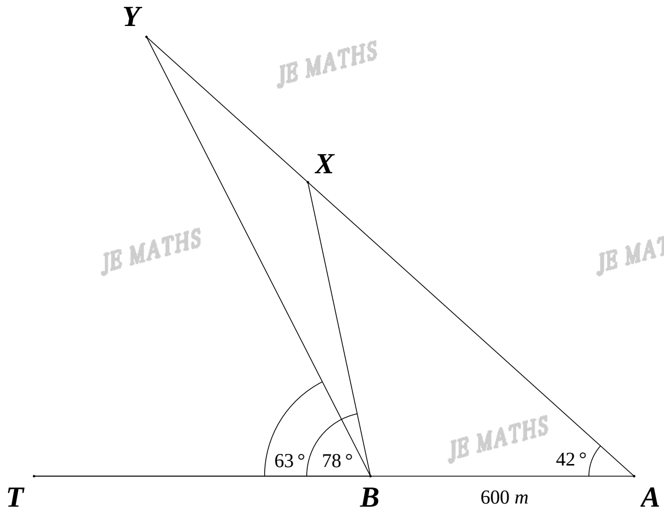
$a^2/x^2 = 3/2$

$a/x = \sqrt{3/2} > 0$ as $a, x > 0$ (4)

Sub (4) into (3): $\cos\theta = \sqrt{3}/(2\sqrt{2})$

$\therefore \cos\theta = \sqrt{6}/4$

3. A man is walking along a straight road towards T when he notices, from A, that two trees X and Y are in line at an angle $42°$ from his line of travel. Having walked a further 600 metres to B, the tree X is at an angle $78°$ and Y at an angle of $63°$ from his line of travel. What is the distance between the tree X and Y, to the nearest metre?



Let XY be x and XB be y.

$\angle ABX = 102°$

$\angle BXA = 36°$ (Angle sum of $\triangle ABX$)

In $\triangle ABX$,

$\frac{y}{\sin 42°} = \frac{600}{\sin 36°}$

$y = \frac{600\sin 42°}{\sin 36°}$

$\angle XBY = 78° - 63°$

$= 15°$

$\angle BYX = 63° - 42°$ (Exterior angle $\angle YBT$ is equal to the sum of $\angle YAB$ and $\angle BYX$)

$= 21°$

In $\triangle XBY$,

$\frac{x}{\sin 15°} = \frac{y}{\sin 21°}$

$x = \frac{600\sin 42° \sin 15°}{\sin 36° \sin 21°}$

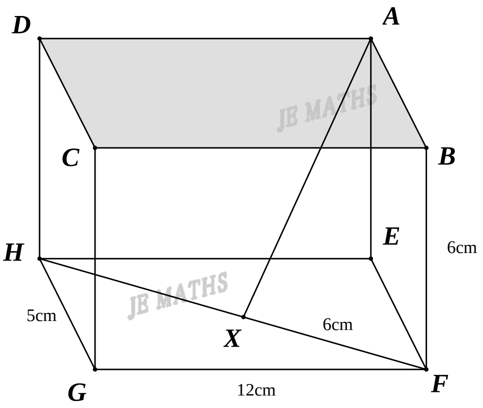
$= 493.30$

$\therefore XY = 493\text{m}$ (nearest metre)

Enrichment stage 2: (3D trig questions)

1. A pencil case open at the top ABCD in the shape of a rectangular prism has dimensions 5cm by 6cm by 12cm. A pencil AX is placed in the case such that it rests at points A and X where X is 6cm along the diagonal FH.

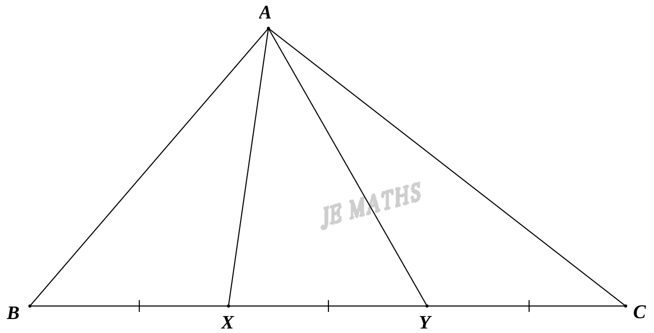
   (a) Find the length of EX correct to 2 decimal places.



   (b) Find the angle that the pencil AX makes with the base plane (EFGH) to the nearest minute?

   (c) An ant stands at point D. It walks on the outside of the pencil case to F. What is the shortest distance from D to F? Justify your answer.

2. ABC is a triangle. BC is trisected at points X and Y.



(a) Express cos∠AXB in terms of the lengths of AB, AX and BC.

In △ABX,

$\cos\angle AXB = (AX^2 + BX^2 - AB^2)/(2AX \times BX)$

$= [AX^2 + (BC/3)^2 - AB^2]/(2AX \times BC/3)$

(b) Hence, or otherwise show that $AB^2 - AC^2 = 3(AX^2 - AY^2)$, giving reasons.

In △AXY,

$\cos(180° - \angle AXB) = (AX^2 + XY^2 - AY^2)/(2AX \times XY)$

$-\cos\angle AXB = [AX^2 + (BC/3)^2 - AY^2]/(2AX \times BC/3)$

$\therefore [AX^2 + (BC/3)^2 - AY^2]/(2AX \times BC/3) = -[AX^2 + (BC/3)^2 - AB^2]/(2AX \times BC/3)$, from (a)

$AX^2 + (BC/3)^2 - AY^2 = -AX^2 - (BC/3)^2 + AB^2$

$2AX^2 + 2BC^2/9 = AB^2 + AY^2$ (1)

Similarly in △AXY and △AYC,

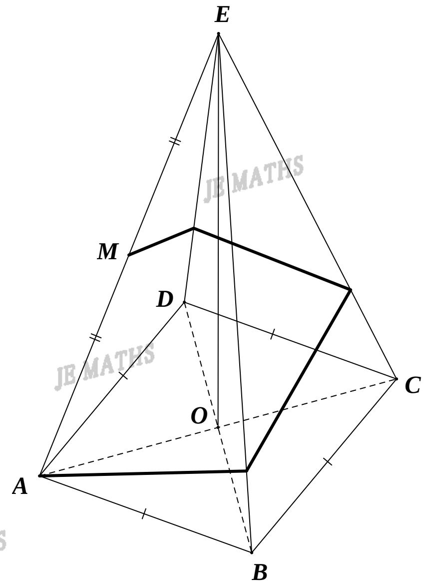
$2AY^2 + 2BC^2/9 = AC^2 + AX^2$ (2)

(1) − (2): $2AX^2 - 2AY^2 = AY^2 - AX^2 + AB^2 - AC^2$

$\therefore AB^2 - AC^2 = 3(AX^2 - AY^2)$

3. ABCDE is a pyramid with a square base of side length 10cm and height 12cm, as shown in the diagram below.

(a) Find the length of edge AE.



$AC^2 = AB^2 + BC^2$

$= 10^2 + 10^2$

$= 200$

$AC = 10\sqrt{2}$ cm

$OA = AC/2 = 5\sqrt{2}$ cm

In $\triangle AOE$,

$AE^2 = AO^2 + OE^2$

$= (5\sqrt{2})^2 + 12^2$

$= 194$

$AE = \sqrt{194}$ cm

(b) Calculate the size of ∠AEB to the nearest minute.

In $\triangle ABE$,

$\cos\angle AEB = (AE^2 + EB^2 - AB^2) / (2 \times AE \times EB)$

$= [(\sqrt{194})^2 + (\sqrt{194})^2 - 10^2] / (2 \times \sqrt{194} \times \sqrt{194})$

$= 72/97$

$\angle AEB = \cos^{-1}(72/97)$

$= 42.07502205$

$= 42°5'$ (nearest minute)

(c) A piece of string is attached to point A and wound around the pyramid to connect to the midpoint M, of edge AE. Find the length of the shortest possible piece of string. Give your answer to 2 decimal places.

$\angle AEM = 4 \times \angle AEB$

$= 4 \times 42.07502205°$

$= 168.3000882°$

$= 168°18'$

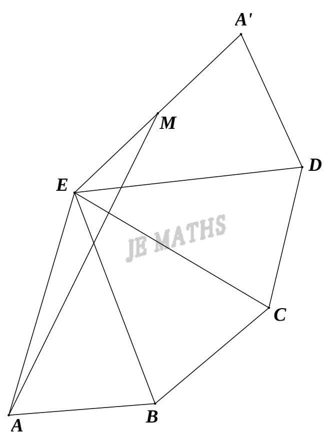
$EM = EA'/2 = \sqrt{194}/2$

$AM^2 = (\sqrt{194})^2 + (\sqrt{194}/2)^2 - 2\sqrt{194} \times \sqrt{194}/2 \times \cos 168°18'$

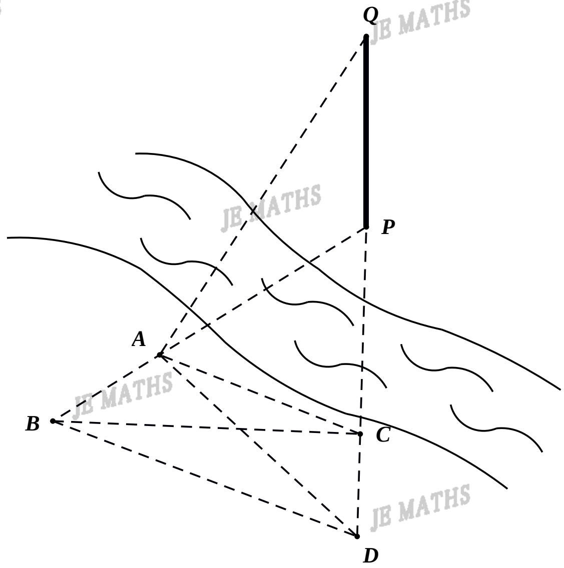
$= 432.4692859$

$AM = 20.79589589$

$= 20.80$ cm (2dp)



4. In the figure, QP is a **vertical tower** on one side of a river. A, B, C and D are points on the other side of the river. P, A, B, D and C are on the same horizontal plane.
Given that $AB = CD = 20$ m, $AC = 35$ m, $AD = BC = 45$ m
and the angle of elevation of Q from A is 30^o.
(a) Find $\angle PAC$, correct to the nearest 0.1^o.



(b) Find the length of AP. (2dps)

(c) Hence, find the height of the tower *QP*. (1dp)

In ΔAPQ,

$\tan 30° = QP/AP$

$QP = AP\tan 30°$

$= 61.26 \times \tan 30°$

$= 35.368$

$= 35.4$ m

(d) Suppose *G* is a point in the region *ABCD* including the boundary, what is the *greatest angle of elevation* of the top of the tower (*Q*) from *G*, to the nearest degree?
When G is at the midpoint on AC, it will be the shortest distance to point P,
ie, the greatest angle of elevation of Q from AC appears at the midpoint of AC.

In ΔPGC,

$\tan 73.4° = PG/GC$

$PG = GC\tan 73.4°$

$= 17.5 \times \tan 73.4°$

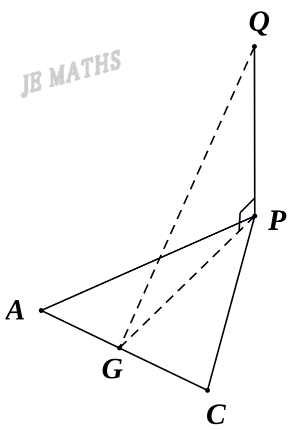
In ΔQGP,

$\tan\angle QGP = QP/PG$

$= 35.4/(17.5\tan 73.4°)$

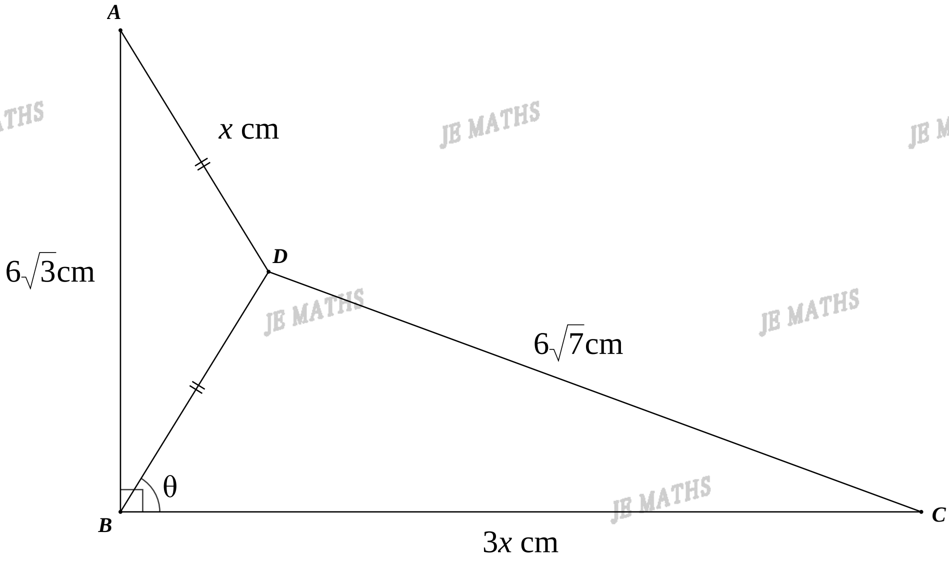
$\angle QGP = 31.09065372$

$= 31°$



Enrichment stage 3: (trig showing questions)

1. Given that AD = DB = x cm, BC = $3x$ cm, AB = $6\sqrt{3}$ cm, CD = $6\sqrt{7}$ cm and $\angle DBC = \theta°$, where $\theta > 70$.



(a) By using the cosine rule in △DBC, show that $\cos\theta = \dfrac{5x^2 - 126}{3x^2}$.

(b) By considering △ADB, show that $\sin\theta = \dfrac{3\sqrt{3}}{x}$.

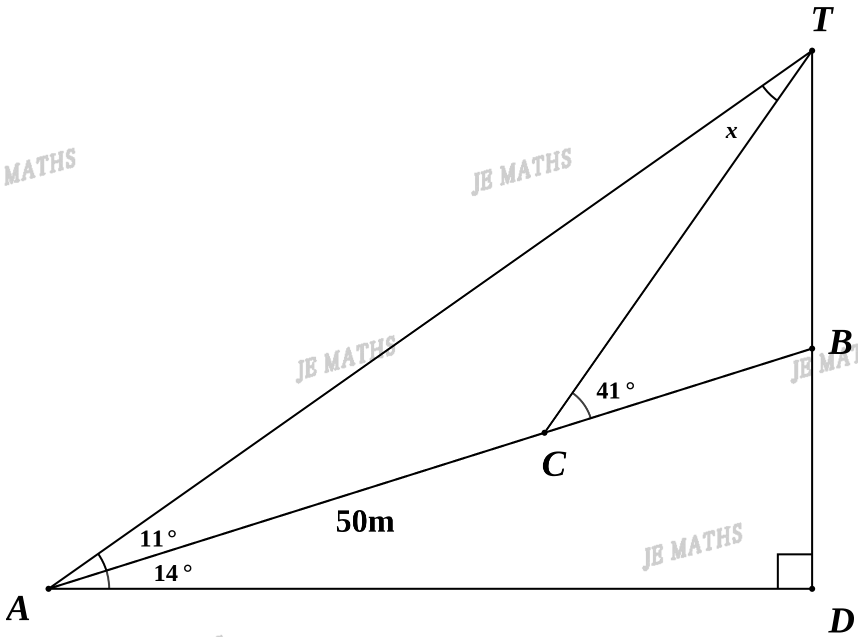
(c) Hence, using parts (a) and (b) and the identity $\sin^2\theta + \cos^2\theta = 1$, show that $16x^4 - 1017x^2 + 15876 = 0$.

(d) Find all possible values of *x*.

(e) Find the value of θ, to the nearest degree.

Enrichment stage 1:

1. (a)



(b)

∠TAC = 11°

∠ATC = 41° - 11° (Exterior angle of △TAC equals sum of 2 interior opposite angles)

= 30°

(c) In △ACT,

∠ACT = 139° (Angle sum of a triangle is 180°)

50/sin30° = TA/sin139°

TA = 100sin139°

= 65.6059029

= 65.6m (1dp)

(d)

∠ABD = 76° (Angle sum of a triangle is 180°)

∠CBT = 104° (Sum of adjacent angles on a straight angle is 180°)

In △ACT,

TC/sin11° = 50/sin30°

TC = 100sin11°

In △CBT,

TB/sin41° = TC/sin104°

TB = 100sin11° sin41°/sin104°

= 12.90142385

∴The tower TB is 12.9m high.

2. (a)

In $\triangle$OAB,

$OA^2 = OB^2 + AB^2 - 2 \times OB \times AB \times \cos\theta$

$r^2 = x^2 + (2a)^2 - 2 \times x \times 2a \times \cos\theta$

$r^2 = x^2 + 4a^2 - 4ax\cos\theta$

(b)

Similarly in $\triangle$OBC,

$OC^2 = OB^2 + BC^2 - 2 \times OB \times BC \times \cos(180° - \theta)$

$r^2 = x^2 + a^2 + 2ax\cos\theta$

$(2x)^2 = x^2 + a^2 + 2ax\cos\theta$ (OD is radius)

$3x^2 = a^2 + 2ax\cos\theta \quad (1)$

From (a),

$r^2 = x^2 + 4a^2 - 4ax\cos\theta$

$(2x)^2 = x^2 + 4a^2 - 4ax\cos\theta$

$3x^2 = 4a^2 - 4ax\cos\theta \quad (2)$

(1) = (2): $a^2 + 2ax\cos\theta = 4a^2 - 4ax\cos\theta$

$3a^2 = 6ax\cos\theta$

$\cos\theta = a/2x \quad (3)$

$2 \times (1) + (2)$: $6x^2 + 3x^2 = 2a^2 + 4ax\cos\theta + 4a^2 - 4ax\cos\theta$

$9x^2 = 6a^2$

$a^2/x^2 = 3/2$

$a/x = \sqrt{(3/2)} > 0$ as $a, x > 0 \quad (4)$

Sub (4) into (3): $\cos\theta = \sqrt{3}/(2\sqrt{2})$

$\therefore \cos\theta = \sqrt{6}/4$

3. Let XY be x and XB be y.

$\angle ABX = 102°$

$\angle BXA = 36°$ (Angle sum of $\triangle$ABX)

In $\triangle$ABX,

$y/\sin42° = 600/\sin36°$

$y = 600\sin42°/\sin36°$

$\angle XBY = 78° - 63°$

$= 15°$

$\angle BYX = 63° - 42°$ (Exterior angle $\angle$YBT is equal to the sum of $\angle$YAB and $\angle$BYX)

$= 21°$

In $\triangle$XBY,

$x/\sin15° = y/\sin21°$

$x = 600\sin42°\sin15°/\sin36°\sin21°$

$= 493.30$

$\therefore$ XY = 493m (nearest metre)

Enrichment stage 2:

1. (a)

Let $\angle HFE$ be θ.

$\tan\theta = 12/5$

$\theta = 67°23'$

In $\triangle EXF$,

$EX^2 = EF^2 + XF^2 - 2 \times EF \times XF \times \cos 67°23'$

$= 37.926...$

$EX = \sqrt{37.926...}$

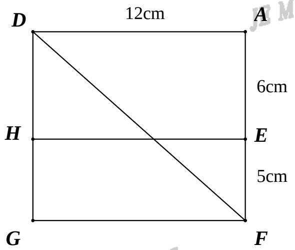
$= 6.16$ cm (2dp)

(b) In $\triangle AXE$,

$\angle AXE = \tan^{-1}(6/6.16)$

$= 44°15'$ (nearest minute)

(c)

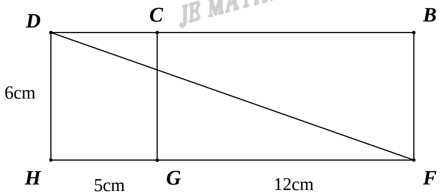


$DF^2 = DA^2 + (AE + EF)^2$

$= 12^2 + (6 + 5)^2$

$= 265$

$DF = \sqrt{265}$



$DF^2 = DH^2 + HF^2$

$= 6^2 + (5 + 12)^2$

$= 325$

$DF = \sqrt{325}$

$\therefore$Quicker to walk along the back ADHE than the base HEFG, i.e. first method

Shortest distance is $\sqrt{265}$ cm.

2. (a)

In $\triangle ABX$,

$\cos\angle AXB = (AX^2 + BX^2 - AB^2)/(2AX \times BX)$

$= [AX^2 + (BC/3)^2 - AB^2]/(2AX \times BC/3)$

(b)

In $\triangle AXY$,

$\cos(180° - \angle AXB) = (AX^2 + XY^2 - AY^2)/(2AX \times XY)$

$-\cos\angle AXB = [AX^2 + (BC/3)^2 - AY^2]/(2AX \times BC/3)$

$\therefore [AX^2 + (BC/3)^2 - AY^2]/(2AX \times BC/3) = -[AX^2 + (BC/3)^2 - AB^2]/(2AX \times BC/3)$, from (a)

$AX^2 + (BC/3)^2 - AY^2 = -AX^2 - (BC/3)^2 + AB^2$

$2AX^2 + 2BC^2/9 = AB^2 + AY^2$ (1)

Similarly in $\triangle AXY$ and $\angle AYC$,

$2AY^2 + 2BC^2/9 = AC^2 + AX^2$ (2)

(1) - (2): $2AX^2 - 2AY^2 = AY^2 - AX^2 + AB^2 - AC^2$

$\therefore AB^2 - AC^2 = 3(AX^2 - AY^2)$

3. (a)

$AC^2 = AB^2 + BC^2$

$= 10^2 + 10^2$

$= 200$

$AC = 10\sqrt{2}$ cm

$OA = AC/2 = 5\sqrt{2}$ cm

In $\triangle AOE$,

$AE^2 = AO^2 + OE^2$

$= (5\sqrt{2})^2 + 12^2$

$= 194$

$AE = \sqrt{194}$ cm

(b)

In $\triangle ABE$,

$\cos\angle AEB = (AE^2 + EB^2 - AB^2) / (2 \times AE \times EB)$

$= [(\sqrt{194})^2 + (\sqrt{194})^2 - 10^2] / (2 \times \sqrt{194} \times \sqrt{194})$

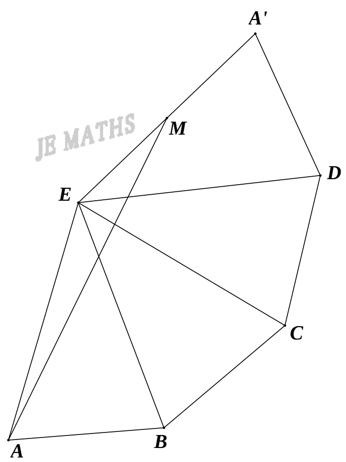
$= 72/97$

$\angle AEB = \cos^{-1}(72/97)$

$= 42.07502205$

$= 42°5'$ (nearest minute)

(c)



$\angle AEM = 4 \times \angle AEB$

$= 4 \times 42.07502205°$

$= 168.3000882°$

$= 168°18'$

$EM = EA' / 2 = \sqrt{194}/2$

$AM^2 = (\sqrt{194})^2 + (\sqrt{194}/2)^2 - 2\sqrt{194} \times \sqrt{194}/2 \times \cos 168°18'$

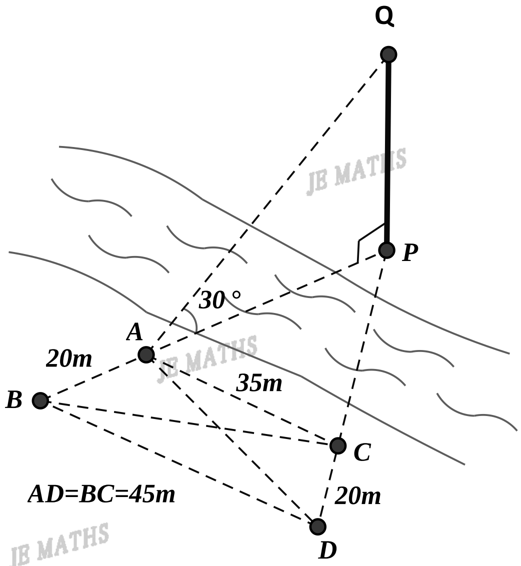
$= 432.4692859$

$AM = 20.79589589$

$= 20.80$ cm (2dp)

4.

(a) InΔABC,



cos$\angle$BAC = $(20^2+35^2-45^2)/(2\times20\times35)$

= -2/7

$\angle$BAC = cos^(-1)(-2/3)

$\angle$PAC = 180°- $\angle$BAC

= 180°- cos^(-1)(-2/3)

= 73.3984504

= 73.4°

(b) ΔACB≡ΔACAD (SSS)

$\angle$BAC=$\angle$DCA

(corresponding angles in the congruent triangles)

$\angle$PAC =$\angle$PCA = 73.4°

In the isosceles ΔAPC,

$\angle$APC = 180°- 2$\angle$PAC

= 180°-2×73.4°=33.2°

AP/sin73.4°=35/sin33.2°

AP=35sin73.4°/sin33.2°

= 61.2555674

= 61.26 m

(c) InΔAPQ,

tan 30° = QP/AP

QP = APtan30°

= 61.26×tan30°

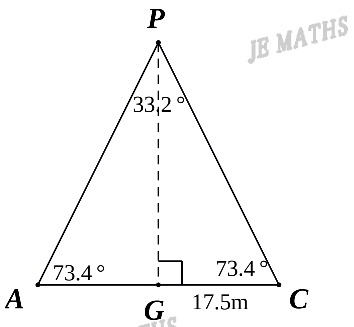
= 35.368…

= 35.4 m

(d) When G is at the midpoint on AC, it will be the shortest distance to point P,

ie, the greatest angle of elevation of Q from AC appears at the midpoint of AC.

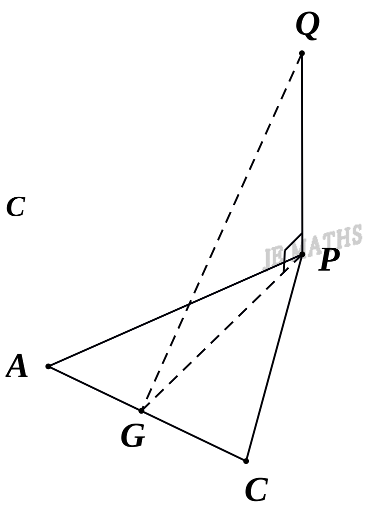
In ΔPGC,



tan73.4° = PG/GC

PG = GCtan73.4°

= 17.5×tan73.4°



In ΔQGP,

tan$\angle$QGP=QP/PG

= 35.4/(17.5tan73.4°)

$\angle$QGP = 31.09165372

= 31°

Enrichment stage 3:

1. (a)

$\cos\theta = [x^2 + (3x)^2 - (6\sqrt{7})^2]/(2 \times x \times 3x)$

$= (x^2 + 9x^2 - 252)/6x^2$

$= (10x^2 - 252)/6x^2$

$= (5x^2 - 126)/3x^2$

(b) In △ADB,

$\angle ABD = 90° - \theta$

$\cos(90° - \theta) = [x^2 + (6\sqrt{3})^2 - x^2]/(2 \times x \times 6\sqrt{3})$

$= 108/12x\sqrt{3}$

$= 9/x\sqrt{3}$

$= 3\sqrt{3}/x$

$\therefore \sin\theta = 3\sqrt{3}/x$

(c)

$\sin^2\theta + \cos^2\theta = 1$

$(3\sqrt{3}/x)^2 + [(5x^2 - 126)/3x^2]^2 = 1$

$27/x^2 + (25x^4 - 1260x^2 + 15876)/9x^4 = 1$

$243x^2 + 25x^4 - 1260x^2 + 15876 = 9x^4$

$16x^4 - 1017x^2 + 15876 = 0$

(d)

$x^2 = [1017 \pm \sqrt{(1017^2 - 4 \times 16 \times 15876)}]/32$

$= (1017 \pm 135) / 32$

$x^2 = 36$ or $441/16$

$\therefore x = 6$ or 5.25 only ($x > 0$)

(e)

When $x = 6$,

$\sin\theta = 3\sqrt{3}/6 = \sqrt{3}/2$

$\theta = 60°$

When $x = 5.25$,

$\sin\theta = 3\sqrt{3}/5.25$

$\theta = 82°$

$\therefore \theta = 82°$ only ($\theta > 70°$)