Stage 1:

1. (a)

$y' = (x)'(x-10)^4 + [x(x-10)^4]'$

$= (x-10)^4 + 4x(x-10)^3$

$= (x-10)^3(5x-10)$

$= 5(x-2)(x-10)^3$

let $y' = 0$

$5(x-2)(x-10)^3 = 0$

$x = 2, 10$

(b)

$y' = (x^4)'(x-1)^3 + x^4[(x-1)^3]'$

$= 4x^3(x-1)^3 + 3x^4(x-1)^2$

$= x^3\left[4(x-1)^3 + 3x(x-1)^2\right]$

$= x^3(7x-4)(x-1)^2$

let $y' = 0$

$x^3(7x-4)(x-1)^2 = 0$

$x = 0, \frac{4}{7}, 1$

(c)

$y' = (2x)'(x+5)^3 + 2x[(x+5)^3]'$

$= 2(x+5)^3 + 6x(x+5)^2$

$= 2(4x+5)(x+5)^2$

let $y' = 0$

$2(4x+5)(x+5)^2 = 0$

$x = -\frac{5}{4}, -5$

(d)

$y' = (x^2)'(2x-4)^3 + x^2[(2x-4)^3]'$

$= 2x(2x-4)^3 + x^2 \cdot 3 \cdot 2(2x-4)^2$

$= 2x\left[(2x-4)^3 + 3x(2x-4)^2\right]$

$= 2x(5x-4)(2x-4)^2$

$= 8x(5x-4)(x-2)^2$

let $y' = 0$

$8x(5x-4)(x-2)^2 = 0$

$x = 0, \dfrac{4}{5}, 2$

(e)

$y' = (x^3)'(1-3x^2)^4 + x^3[(1-3x^2)^4]'$

$= 3x^2(1-3x^2)^4 + x^3 \cdot 4 \cdot -6x \cdot (1-3x^2)^3$

$= 3x^2\left[(1-3x^2)^4 - 8x^2(1-3x^2)^3\right]$

$= 3x^2(1-3x^2-8x^2)(1-3x^2)^3$

$= 3x^2(1-11x^2)(1-3x^2)^3$

let $y' = 0$

$3x^2(1-11x^2)(1-3x^2)^3 = 0$

$x = 0, \pm\sqrt{\dfrac{1}{11}}, \pm\sqrt{\dfrac{1}{3}}$

$x = 0, \pm\dfrac{\sqrt{11}}{11}, \pm\dfrac{\sqrt{3}}{3}$

(f)

$y' = (x^4)'(x^2+x)^3 + x^4[(x^2+x)^3]'$

$= 4x^3(x^2+x)^3 + 3x^4(2x+1)(x^2+x)^2$

$= x^3(x^2+x)^2\left[4(x^2+x)+3x(2x+1)\right]$

$= x^3(x^2+x)^2(4x^2+4x+6x^2+3x)$

$= x^6(x+1)^2(10x+7)$

let $y' = 0$

$x^6(x+1)^2(10x+7) = 0$

$x = 0, -1, -\frac{7}{10}$

(g)

$y' = 1(2x+1)^3 + (x-2)\cdot 3\cdot 2\cdot (2x+1)^2$

$= (2x+1)^2[2x+1+6(x-2)]$

$= (2x+1)^2(8x-11)$

let $y' = 0$

$(2x+1)^2(8x-11) = 0$

$x = -\frac{1}{2}, \frac{11}{8}$

(h)

$y' = 2\cdot 3(3x+1)(4x-5)^3 + (3x+1)^2\cdot 3\cdot 4\cdot (4x-5)^2$

$= 6(3x+1)(4x-5)^2[4x-5+2(3x+1)]$

$= 6(3x+1)(10-3)(4x-5)^2$

let $y' = 0$

$6(3x+1)(10-3)(4x-5)^2 = 0$

$x = -\frac{1}{3}, \frac{3}{10}, \frac{5}{4}$

(i)

$$y' = -4\pi x(x^2-3x+1)^3 + 3(2x-3)(x^2-3x+1)^2 \cdot -2\pi x^2$$

$$= -2\pi x(x^2-3x+1)^2[2(x^2-3x+1)+3x(2x-3)]$$

$$= -2\pi x(x^2-3x+1)^2(8x^2-15x+2)$$

let $y' = 0$

$$-2\pi x(x^2-3x+1)^2(8x^2-15x+2) = 0$$

for $x^2-3x+1=0$

$$\Delta = (-3)^2 - 4\cdot 1\cdot 1 = 5$$

$$x = \frac{3\pm\sqrt{5}}{2}$$

for $8x^2-15x+2=0$

$$\Lambda = 15^2 - 4\cdot 8\cdot 2 = 161$$

$$x = \frac{15\pm\sqrt{161}}{2\cdot 8} = \frac{15\pm\sqrt{161}}{16}$$

$$x = 0, \frac{3\pm\sqrt{5}}{2}, \frac{15\pm\sqrt{161}}{16}$$

(j)

$$y' = 2\sqrt{x+2} + 2x\cdot\frac{1}{2}(x+2)^{-\frac{1}{2}}$$

$$= 2\sqrt{x+2} + \frac{x}{\sqrt{x+2}}$$

$$= \frac{3x+4}{\sqrt{x+2}}$$

let $y' = 0$

$$\frac{3x+4}{\sqrt{x+2}} = 0$$

$$x = -\frac{4}{3}$$

(k)

$$y' = -6x\sqrt{3-x} - 3x^2 \cdot \frac{1}{2} \cdot -1(3-x)^{-\frac{1}{2}}$$

$$= -6x\sqrt{3-x} + \frac{3x^2}{2}\frac{1}{\sqrt{3-x}}$$

$$= \frac{-12x(3-x)+3x^2}{2\sqrt{3-x}}$$

$$= \frac{15x^2-36x}{2\sqrt{3-x}}$$

$$= \frac{3x(5x-12)}{2\sqrt{3-x}}$$

let $y' = 0$

$$x = 0, \frac{12}{5}$$

(l)

$$y' = 2x\sqrt{1-x^2} + x^2 \cdot \frac{1}{2} \cdot -2x(1-x^2)^{-\frac{1}{2}}$$

$$= 2x\sqrt{1-x^2} + \frac{x^3}{\sqrt{1-x^2}}$$

$$= \frac{2x-2x^3-x^3}{\sqrt{1-x^2}}$$

$$= \frac{x(2-3x^2)}{\sqrt{1-x^2}}$$

let $y' = 0$

$$\frac{x(2-3x^2)}{\sqrt{1-x^2}} = 0$$

$$x = 0, \pm\sqrt{\frac{2}{3}} = 0, \pm\frac{\sqrt{6}}{3}$$

(m)

$$y' = -4x\sqrt{4-x^2} + (3-2x^2)\cdot\frac{1}{2}\cdot -2x(4-x^2)^{-\frac{1}{2}}$$

$$= -4x\sqrt{4-x^2} - \frac{x(3-2x^2)}{\sqrt{4-x^2}}$$

$$= \frac{-4x(4-x^2)-3x+3x^3}{\sqrt{4-x^2}}$$

$$= \frac{-19x+6x^3}{\sqrt{4-x^2}}$$

$$= \frac{x(6x^2-19)}{\sqrt{4-x^2}}$$

let $y' = 0$

$$\frac{x(6x^2-19)}{\sqrt{4-x^2}} = 0$$

$$x = 0, \pm\sqrt{\frac{19}{6}} = 0, \pm\frac{\sqrt{114}}{6}$$

2. (i)

$$y' = mx^{m-1}(a-x)^n + x^m \cdot -1\cdot n(a-x)^{n-1}$$

$$= mx^{m-1}(a-x)^n - nx^m(a-x)^{n-1}$$

$$= (a-x)^{n-1}[mx^{m-1}(a-x) - nx^m]$$

$$= (a-x)^{n-1}(amx^{m-1} - mx^m - nx^m)$$

$$= (a-x)^{n-1}\cdot x^{m-1}\cdot(am-mx-nx)$$

let $y' = 0$

$$(a-x)^{n-1}\cdot x^{m-1}\cdot(am-mx-nx) = 0$$

For $am - mx - nx = 0$

$mx + nx = am$

$$x = \frac{am}{m+n}$$

since $\dfrac{am}{m+n} = \dfrac{a}{1+\dfrac{m}{n}} \to 0 < \dfrac{a}{1+\dfrac{m}{n}} < a$

ie $x = \dfrac{am}{m+n}$ is the x-coordinate of the point T which lies between 0 and a.

$$f\left(\frac{am}{m+n}\right) = \left(\frac{am}{m+n}\right)^m \left(a - \frac{am}{m+n}\right)^n$$

$$= \frac{a^m m^m}{(m+n)^m}\left(\frac{am+an-am}{m+n}\right)^n$$

$$= \frac{a^m a^n m^m n^n}{(m+n)^m (m+n)^n}$$

$$= \frac{a^{m+n} m^m n^n}{(m+n)^{m+n}}$$

ans: $T\left(\dfrac{am}{m+n}, \dfrac{a^{m+n} m^m n^n}{(m+n)^{m+n}}\right)$

(ii)

$x = \dfrac{am}{m+n}$ (when $m = n$)

$= \dfrac{am}{2m}$

$= \dfrac{a}{2}$

$$f\left(\frac{a}{2}\right) = \left(\frac{a}{2}\right)^m - \left(a - \frac{a}{2}\right)^n$$

$$= \left(\frac{a}{2}\right)^m \cdot \left(\frac{a}{2}\right)^n$$

$= \left(\dfrac{a}{2}\right)^{m+n}$ (when $m = n$)

$$= \left(\frac{a}{2}\right)^{2m}$$

ans: $T\left(\dfrac{a}{2}, \left(\dfrac{a}{2}\right)^{2m}\right)$

3. (a)

$y' = (uv)'w + (uv)w'$

$= w(u'v + uv') + uvw'$

$= u'vw + uv'w + uvw'$

(b)

$y' = 1(x-1)^4\sqrt{2x+1} + x\cdot 4(x-1)^3\sqrt{2x+1} + x(x-1)^4\cdot\frac{1}{2}\cdot 2(2x+1)^{-\frac{1}{2}}$

$= (x-1)^4\sqrt{2x+1} + 4x(x-1)^3 + \frac{x(x-1)^4}{\sqrt{2x+1}}$

$= \frac{(x-1)^4(2x+1) + 4x(x-1)^3(2x+1) + x(x-1)^4}{\sqrt{2x+1}}$

$= \frac{(x-1)^3\left[(x-1)(2x+1) + 4x(2x+1) + x(x-1)\right]}{\sqrt{2x+1}}$

$= \frac{(x-1)^3\left(10x^2 - 2x + 5x - 1 + x^2 - x\right)}{\sqrt{2x+1}}$

$= \frac{(x-1)^3\left(11x^2 + 2x - 1\right)}{\sqrt{2x+1}}$

let $y' = 0$

$11x^2 + 2x - 1 = 0$

$\Delta = 2^2 - 4\cdot 11\cdot -1 = 48$

$x = \frac{-2\pm 4\sqrt{3}}{2\cdot 11} = \frac{-1\pm 2\sqrt{3}}{11}$

ans: $x = 1, \frac{-1\pm 2\sqrt{3}}{11}$

Stage 2:

1. (a)

$$y' = \frac{4(2x+1) - 2(4x-1)}{(2x+1)^2}$$

$$= \frac{6}{(2x+1)^2}$$

let $y' = 0$

$$\frac{6}{(2x+1)^2} = 0$$

no solution for x

(b)

$$y' = \frac{2(x^2-2) - 2x(2x)}{(x^2-2)^2}$$

$$= \frac{-2x^2-4}{(x^2-2)^2}$$

let $y' = 0$

$$\frac{-2x^2-4}{(x^2-2)^2} = 0$$

$$-2x^2 - 4 = 0$$

$$-2x^2 = 4$$

$$x^2 = -2$$

no real solution for x

(c)

$$y' = \frac{2x(x^2+1) - 2x(x^2-4)}{(x^2+1)^2}$$

$$= \frac{10x}{(x^2+1)^2}$$

let $y' = 0$

$$\frac{10x}{(x^2+1)^2} = 0$$

$$x = 0$$

(d)

$$y' = \frac{6x(x^3-2) - 3x^2(3x^2)}{(x^3-2)^2}$$

$$= \frac{6x^4 - 12x - 9x^4}{(x^3-2)^2}$$

$$= \frac{-3x^4 - 12x}{(x^3-2)^2}$$

let $y' = 0$

$$\frac{-3x^4 - 12x}{(x^3-2)^2} = 0$$

$$-3x^4 - 12x = 0$$

$$-3x(x^3+4) = 0$$

$$x = 0, \sqrt[3]{-4}$$

(e)

$$y' = \frac{2(x-1)(x^2+2x)-(2x+2)(x-1)^2}{(x^2+2x)^2}$$

$$= \frac{2(x-1)\left[x^2+2x-(x+1)(x-1)\right]}{x^2(x+2)^2}$$

$$= \frac{2(x-1)(2x+1)}{x^2(x+2)^2}$$

let $y' = 0$

$$\frac{2(x-1)(2x+1)}{x^2(x+2)^2} = 0$$

$$2(x-1)(2x+1) = 0$$

$$x = 1, -\frac{1}{2}$$

(f)

$$y' = \frac{2\cdot 3(3x-1)(x^2-x)-(2x-1)(3x-1)^2}{(x^2-x)^2}$$

$$= \frac{(3x-1)\left[6(x^2-x)-(2x-1)(3x-1)\right]}{(x^2-x)^2}$$

$$= \frac{(3x-1)\left(6x^2-6x-6x^2+5x-1\right)}{x^2(x-1)^2}$$

$$= \frac{(3x-1)(-x-1)}{x^2(x-1)^2}$$

$$= \frac{(1-3x)(x+1)}{x^2(x-1)^2}$$

let $y' = 0$

$$\frac{(1-3x)(x+1)}{x^2(x-1)^2} = 0$$

$$(1-3x)(x+1) = 0$$

$$x = -1, \frac{1}{3}$$

(g)

$$y' = \frac{3\sqrt{2-x} - \frac{1}{2} \cdot -1(3x-4)(2-x)^{-\frac{1}{2}}}{2-x}$$

$$= \frac{3\sqrt{2-x} + \frac{3x-4}{2\sqrt{2-x}}}{2-x}$$

$$= \frac{6(2-x)+3x-4}{2(2-x)\sqrt{2-x}}$$

$$= \frac{8-3x}{2(2-x)\sqrt{2-x}}$$

let $y' = 0$

$$\frac{8-3x}{2(2-x)\sqrt{2-x}} = 0$$

$$8-3x = 0$$

$$x = \frac{8}{3}$$

(h)

$$y' = \frac{1\sqrt{x^2+1} - \frac{1}{2} \cdot 2x(x+1)(x^2+1)^{-\frac{1}{2}}}{x^2+1}$$

$$= \frac{\sqrt{x^2+1} - \frac{x^2+x}{\sqrt{x^2+1}}}{x^2+1}$$

$$= \frac{x^2+1-x^2-x}{(x^2+1)\sqrt{x^2+1}}$$

$$= \frac{1-x}{(x^2+1)\sqrt{x^2+1}}$$

let $y' = 0$

$$x = 1$$

(i)

$$y=\left(\frac{3-x}{x-2}\right)^{\frac{1}{2}}$$

$$\left(\frac{3-x}{x-2}\right)'=\frac{-1(x-2)-1(3-x)}{(x-2)^2}$$

$$=\frac{-1}{(x-2)^2}$$

$$y'=\frac{1}{2}\cdot\frac{-1}{(x-2)^2}\cdot\left(\frac{3-x}{x-2}\right)^{-\frac{1}{2}}$$

$$=\frac{-1}{2(x-2)^2}\cdot\sqrt{\frac{x-2}{3-x}}$$

$$=\frac{-(x-2)}{2(x-2)^2\sqrt{x-2}\sqrt{3-x}}$$

$$=\frac{-1}{2(x-2)\sqrt{5x-x^2-6}}$$

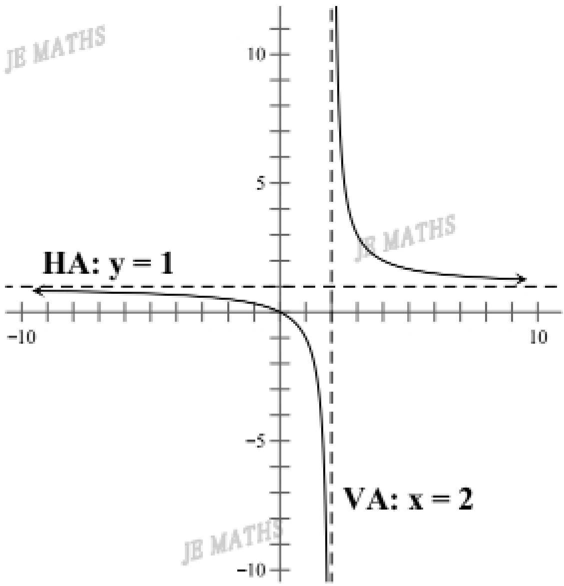
let $y'=0$

no values of x

2. (i)

D: $x\neq 2$

R: $y\neq 1$



(ii)

$$y' = \frac{1(x-2)-1\cdot x}{(x-2)^2}$$

$$= \frac{-2}{(x-2)^2}$$

$$y - \frac{p}{p-2} = -\frac{2}{(p-2)^2}(x-p)$$

$$-2x + 2p = (p-2)^2 y - p(p-2)$$

$$0 = 2x + (p-2)^2 y - p^2$$

(iii)

$$0 = 2x + (p-2)^2 \cdot 0 - p^2$$

$$2x = p^2$$

$$x = \frac{p^2}{2} \to A(\frac{p^2}{2}, 0) \to d_{OA} = \frac{p^2}{2}$$

$$x = p \to B(p, 0) \to d_{BO} = p$$

$$\frac{p^2}{2} = 2|p|$$

$$p^2 = 4|p|$$

$p^2 = 4p$	$p^2 = -4p$
$p(p-4) = 0$	$p(p+4) = 0$
$p = 0, 4$	$p = 0, -4$

$\therefore p = \pm 4$ ($A = B$ at $p = 0$, omit 0)

(iv)

(a)

$$0 = 2\cdot 0 + (p-2)^2 c - p^2$$

$$0 = p^2 c - 4pc + 4c - p^2$$

$$0 = p^2 c - p^2 - 4pc + 4c$$

$$0 = p^2(c-1) + 4pc - p^2$$

$$p = \frac{4c \pm \sqrt{16c^2 - 4\cdot 4c(c-1)}}{2(c-1)}$$

$$= \frac{4c \pm \sqrt{16c}}{2(c-1)}$$

$$= \frac{2c \pm \sqrt{4c}}{c-1}$$

$$= \frac{2c \pm 2\sqrt{c}}{c-1}$$

$p = \dfrac{2c + 2\sqrt{c}}{c-1}$ or $p = \dfrac{2c - 2\sqrt{c}}{c-1}$

$= \dfrac{2\sqrt{c}}{\sqrt{c}-1}$ $\qquad$ $= \dfrac{2\sqrt{c}}{\sqrt{c}-1}$

(b)

$$\sqrt{c} \ge 0 \to c \ge 0$$

$p = \dfrac{2c \pm \sqrt{4c}}{c-1}$, since $4c$ must be real number

tangent at $\dfrac{2}{2} = x = 1$ exists for $c = 1$

(c)

$c = 0$ or $c = 1$

$$\frac{2\sqrt{c}}{\sqrt{c}-1} = \frac{2\sqrt{c}}{\sqrt{c}+1}$$

$$2c + 2\sqrt{c} = 2c - 2\sqrt{c}$$

$$4\sqrt{c} = 0$$

$c = 0$

also, when $c = 1$

only $p = \dfrac{2\sqrt{c}}{\sqrt{c}+1}$ exists

(d)

(1) $0 < 2$

$2\sqrt{c} \le 2 + 2\sqrt{c}$

$\dfrac{2\sqrt{c}}{\sqrt{c}+1} < \dfrac{2+2\sqrt{c}}{\sqrt{c}+1}$

$p < 2$

since p is on the left side of the asymptote $x = 2$

$\therefore$ the tangent is on the left branch of the hyperbola regardless of c

(2) let $p > 2$

$\dfrac{2\sqrt{c}}{\sqrt{c}-1} > 2$

$2\sqrt{c} > 2\sqrt{c} - 2$

$0 > -2$

$\sqrt{c} - 1 > 0$

$\sqrt{c} > 1$

$c > 1$

(3)

since $c \ge 0$ for a tangent to exist and $c = 0$ for only one,

$c > 0$ for there to be two tangents ($c \ne 1$)

$p = \dfrac{2\sqrt{c}}{\sqrt{c}-1}$ must be on the left side as $p = \dfrac{2\sqrt{c}}{\sqrt{c}+1}$ is always on the left branch

$\therefore c < 1$ and $c > 0$

$0 < c < 1$

3. $\dfrac{dy}{dx} = \dfrac{\dfrac{du}{dx} \cdot x - 1 \cdot u}{x^2}$

$\times x^2$: $\dfrac{dy}{dx} x^2 = \dfrac{du}{dx} \cdot x - u$

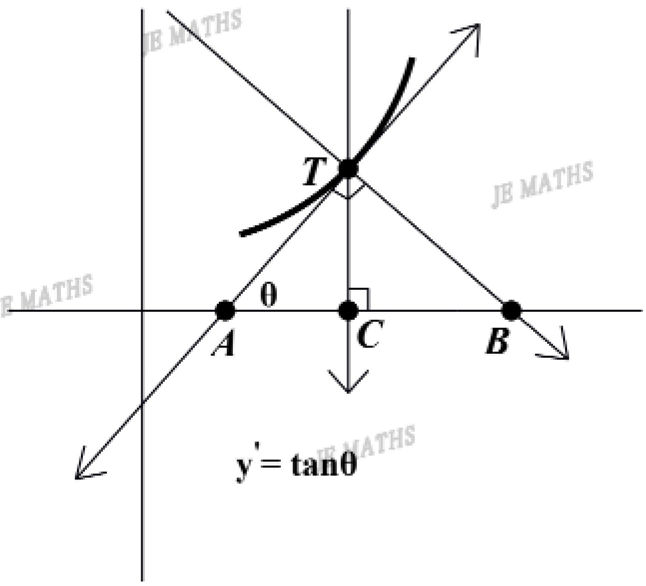
$\div x$: $x\dfrac{dy}{dx} = \dfrac{du}{dx} - \dfrac{u}{x}$ (since $y = \dfrac{u}{x}$)

$x\dfrac{dy}{dx} = \dfrac{du}{dx} - y$

$\dfrac{du}{dx} = x\dfrac{dy}{dx} + y$

$\dfrac{du}{dx} = y + x\dfrac{dy}{dx}$

4. (i)



(ii)

(a) $\tan\theta = \dfrac{TC}{AC}$

$y' = \dfrac{y}{AC}$

$AC = \dfrac{y}{y'}$

(b)

$\angle TBC = 180° - 90° - \theta = 90° - \theta$

$\tan(90° - \theta) = \cot\theta$

$\tan\angle TBC = \dfrac{TC}{BC}$

$\cot\theta = \dfrac{TC}{BC}$

$\dfrac{1}{\tan\theta} = \dfrac{y}{BC}$

$y' = \dfrac{BC}{y}$

$BC = yy'$

(c)

$AT = \sqrt{\dfrac{y^2}{(y')^2} + y^2}$

$= \sqrt{\dfrac{y^2\left[1+(y')^2\right]}{(y')^2}}$

$= \dfrac{y\sqrt{1+(y')^2}}{y'}$

$\sec\theta = \dfrac{AT}{AC}$ Notice: $\sec^2\theta = 1 + \tan^2\theta$ (since $\tan\theta = y'$)

$= \dfrac{y\sqrt{1+(y')^2}}{y'} \cdot \dfrac{y'}{y}$ $\sec = \sqrt{1+(y')^2}$

$= \sqrt{1+(y')^2}$

(d)

$\cos ec\theta = \dfrac{AT}{TC}$ Notice: $\csc^2\theta = 1 + \cot^2\theta$

$= \dfrac{y\sqrt{1+(y')^2}}{y'} \cdot \dfrac{1}{y}$ $\csc^2\theta = 1 + \dfrac{1}{(y')^2}$

$= \dfrac{\sqrt{1+(y')^2}}{y'}$ $\csc^2\theta = \dfrac{1+(y')^2}{(y')^2}$

$\csc\theta = \dfrac{\sqrt{1+(y')^2}}{y'}$

(e)

$$AT^2 = AC^2 + TC^2$$

$$AT = \sqrt{\frac{y^2}{(y')^2} + y^2}$$

$$= \sqrt{\frac{y^2\left[1+(y')^2\right]}{(y')^2}}$$

$$= \frac{y\sqrt{1+(y')^2}}{y'}$$

(f)

$$BT^2 = TC^2 + BC^2$$

$$BT = \sqrt{y^2 + (yy')^2}$$

$$= \sqrt{y^2 + y^2(y')^2}$$

$$= y\sqrt{1+(y')^2}$$

(iii)

$$f'(x)=\frac{2(x+1)-1(2x-3)}{(x+1)^2}$$

$$=\frac{2x+2-2x+3}{(x+1)^2}$$

$$=\frac{5}{(x+1)^2}$$

$$f'(4)=\frac{5}{x^2}=\frac{1}{5}$$

$$f(4)=\frac{2\cdot 4-3}{4+1}=1$$

$$AC=\frac{y}{y'}=1\cdot 5=5$$

$$BC=yy'=1\cdot\frac{1}{5}=\frac{1}{5}$$

$$AT=y\sqrt{1+(y')^2}\cdot\frac{1}{y'}$$

$$=1\sqrt{1+0.2^2}+5$$

$$=5\sqrt{\frac{26}{25}}$$

$$=\sqrt{\frac{26}{25}}$$

$$=\frac{\sqrt{26}}{5}$$