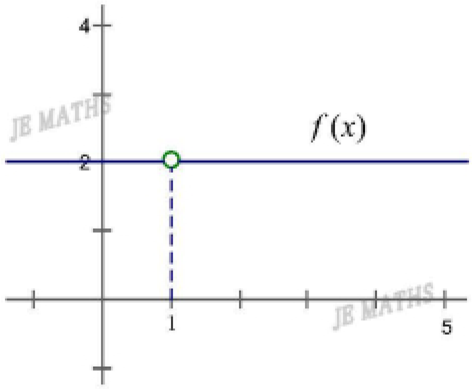
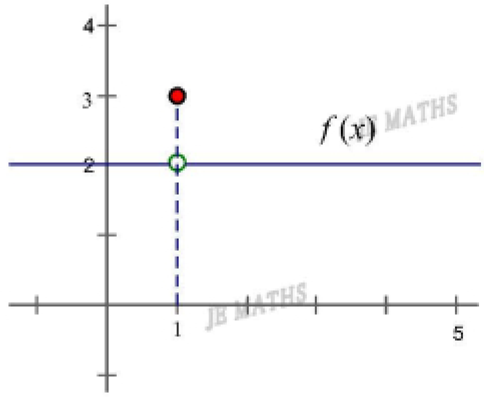
Stage 2

1. Given the graph of $f(x)$, find the limit of $f(x)$.

(a)

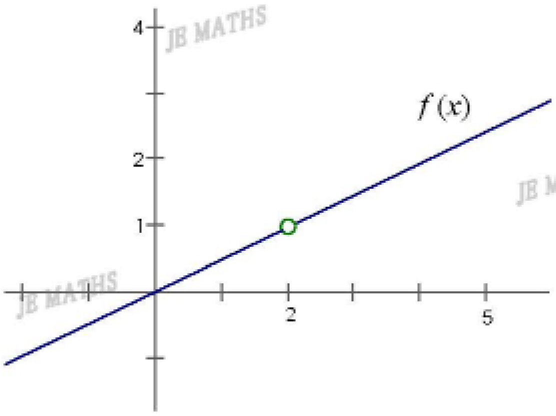


$\lim_{x \to 1} f(x) =$ _____

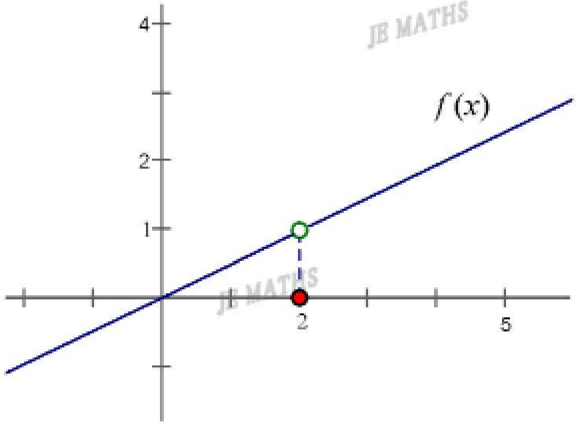


$\lim_{x \to 1} f(x) =$ _____

(b)

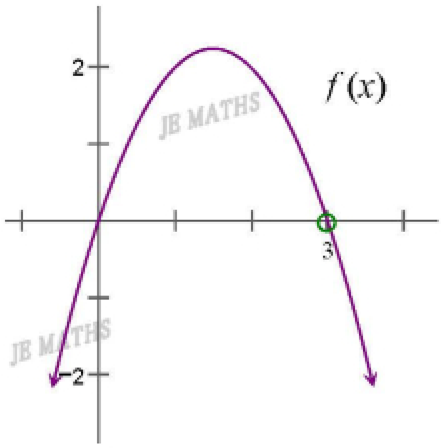


$\lim_{x \to 2} f(x) =$ _____

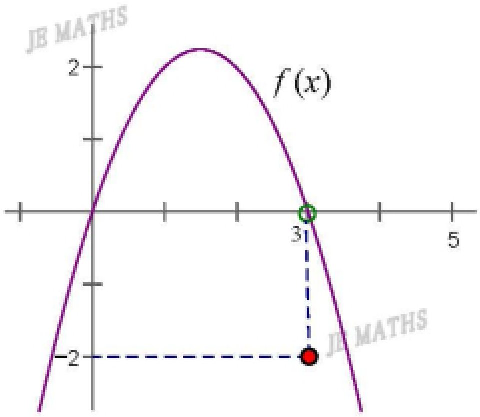


$\lim_{x \to 2} f(x) =$ _____

(c)

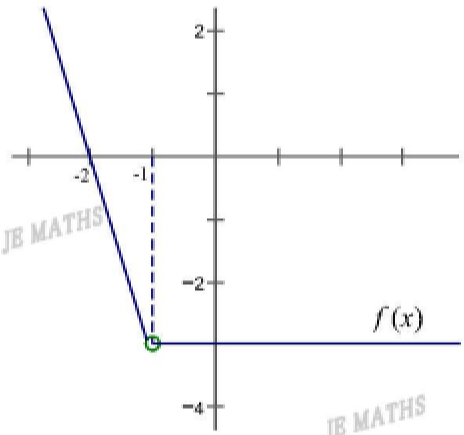


$\lim_{x \to 3} f(x) =$ _____

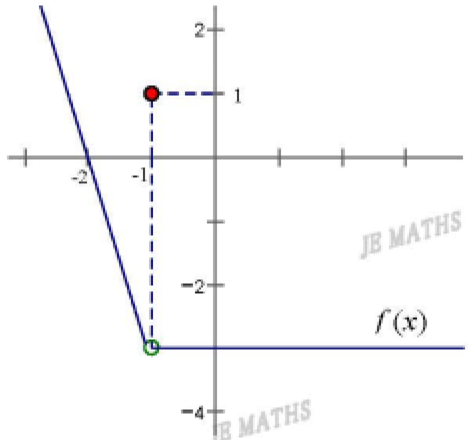


$\lim_{x \to 3} f(x) =$ _____

(d)

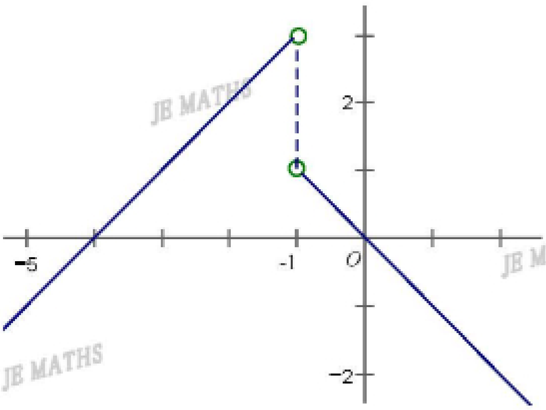


$\lim_{x \to -1} f(x) =$ _____

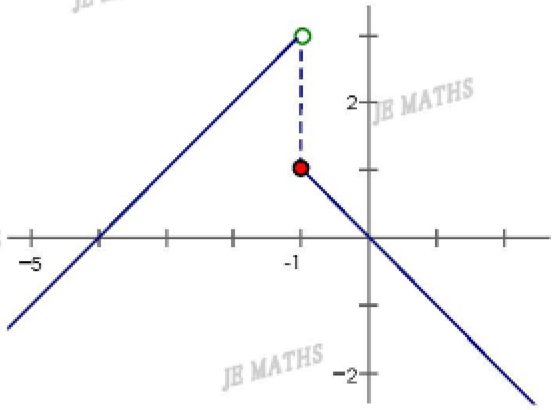


$\lim_{x \to -1} f(x) =$ _____

(e)



$\lim_{x \to -1} f(x) =$ _____



$\lim_{x \to -1} f(x) =$ _____

2. Sketch the **graph** and find the limit of each piecewise function.

(a) Let $f(x) = \begin{cases} x & \text{for } x > 0 \\ 2x & \text{for } x < 0 \end{cases}$.

Find $\lim_{x \to 0} f(x)$.

(b) Let $f(x)=\begin{cases} 3-x & for\ x\neq 1 \\ 4 & for\ x=1 \end{cases}$.

Find $\lim\limits_{x\to 1} f(x)$.

(c) Let $f(x)=\begin{cases} x-1 & for\ x<0 \\ x+2 & for\ x\geq 0 \end{cases}$.

Find $\lim\limits_{x\to 0} f(x)$.

(d) Let $f(x)=\begin{cases} x-2 & for\ x>2 \\ 2-x & for\ x<2 \\ 3 & for\ x=2 \end{cases}$.

Find $\lim\limits_{x\to 2} f(x)$.

3. Consider the function $f(x) = \dfrac{x^2 - 4}{x + 2}$.

(i) State the domain of $f(x)$.

(ii) Simplify $f(x)$ and hence sketch the graph of $f(x)$.

(iii) Use the graph of $f(x)$ to find $\lim\limits_{x \to -2} \dfrac{x^2 - 4}{x + 2}$.

4. Simplify each function first and then find the limit.

(a) $\lim\limits_{x \to 0} \dfrac{x^2 + 3x}{x}$

(b) $\lim\limits_{x \to 4} \dfrac{(x-1)(x-4)}{x-4}$

(c) $\lim_{x\to 1}\frac{x^2-1}{x-1}$

(d) $\lim_{x\to 3}\frac{x^2-x-6}{x-3}$

(e) $\lim_{x\to 0}\frac{3x}{x^2-2x}$

(f) $\lim_{x\to -1}\frac{x+1}{x^2-x-2}$

(g) $\lim_{x\to -1}\frac{x+1}{x^3+1}$

(h) $\lim_{x\to 2}\frac{x^3-8}{x-2}$

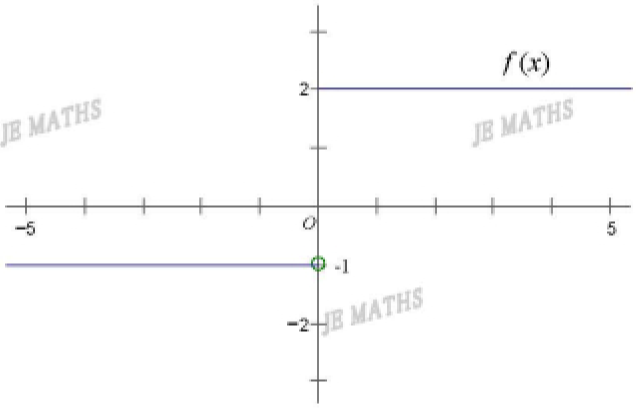
Stage 3

1. Given the graph of $f(x)$, find the limit of $f(x)$.

(a) $\lim\limits_{x\to 0^+} f(x) =$ _____

$\lim\limits_{x\to 0^-} f(x) =$ _____

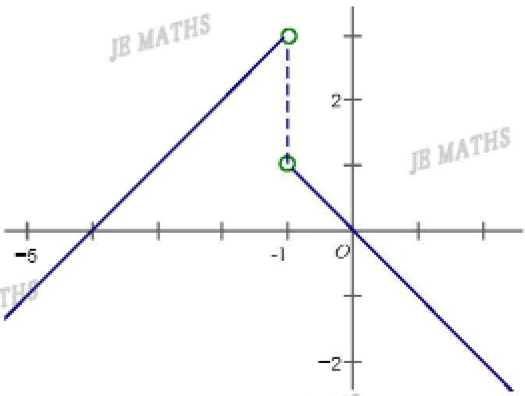
$\lim\limits_{x\to 0} f(x) =$ _____



(b) $\lim\limits_{x\to -1^+} f(x) =$ _____

$\lim\limits_{x\to -1^-} f(x) =$ _____

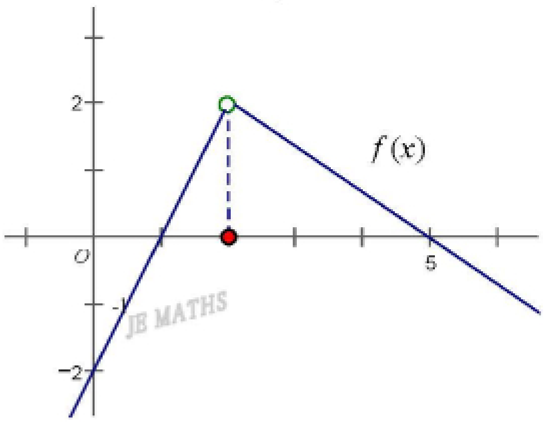
$\lim\limits_{x\to -1} f(x) =$ _____



(c) $\lim\limits_{x\to 2^+} f(x) =$ _____

$\lim\limits_{x\to 2^-} f(x) =$ _____

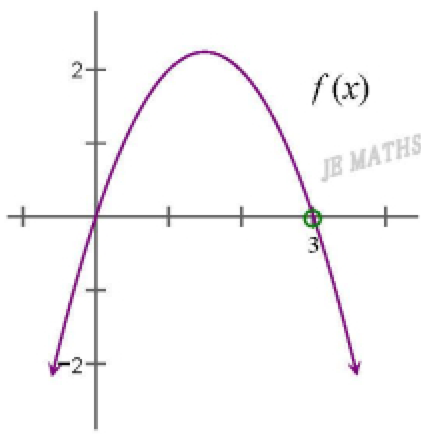
$\lim\limits_{x\to 2} f(x) =$ _____



(d) $\lim\limits_{x\to 3^+} f(x) =$ _____

$\lim\limits_{x\to 3^-} f(x) =$ _____

$\lim\limits_{x\to 3} f(x) =$ _____



2. Let $f(x)=\begin{cases} x & for\ x \ge 1 \\ -x & for\ x<1 \end{cases}$.

(i) Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ by substitution.

(ii) Hence, explain why $\lim_{x\to 1} f(x)$ does NOT exist.

(iii) Sketch the graph of $f(x)$ and confirm your results in part (i) (ii) by the graph.

3. Let $f(x)=\begin{cases} x^2+1 & for\ x>0 \\ 2 & for\ x<0 \end{cases}$.

(i) Find $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ by substitution.

(ii) Hence, explain why $\lim_{x\to 0} f(x)$ does NOT exist.

(iii) Sketch the graph of $f(x)$ and confirm your results in part (i) (ii) by the graph.

4. Let $f(x)=\begin{cases} 2x & for\ x>1 \\ 4-2x & for\ x<1 \end{cases}$.

(i) Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ by substitution.

(ii) Hence, explain why $\lim_{x\to 1} f(x)$ does exist and find the value of it.

(iii) Sketch the graph of $f(x)$ and confirm your results in part (i) (ii) by the graph.

5. Let $f(x)=|x+2|$.

(i) Express $f(x)$ as a piecewise function.

(ii) Find $\lim_{x\to -2^+} f(x)$ and $\lim_{x\to -2^-} f(x)$ by substitution.

(iii) Does $\lim_{x\to -2} f(x)$ exist? If it does, find its value. If not, explain your reasons.

(iv) Sketch the graph of $f(x)$ and confirm your results in part (i) (ii) by the graph.

6. Find $\lim_{x\to 0^+} f(x)$ if $f(x) = x+3$:

   (a) for domain all real x

   (b) for domain such that $x > 0$

7. Find each one-sided limit:

   (a) $\lim_{x\to 0^+} \frac{x}{|x|}$

   (b) $\lim_{x\to 0^-} \frac{x}{|x|}$

   (c) $\lim_{x\to 2^+} \frac{|x-2|}{x-2}$

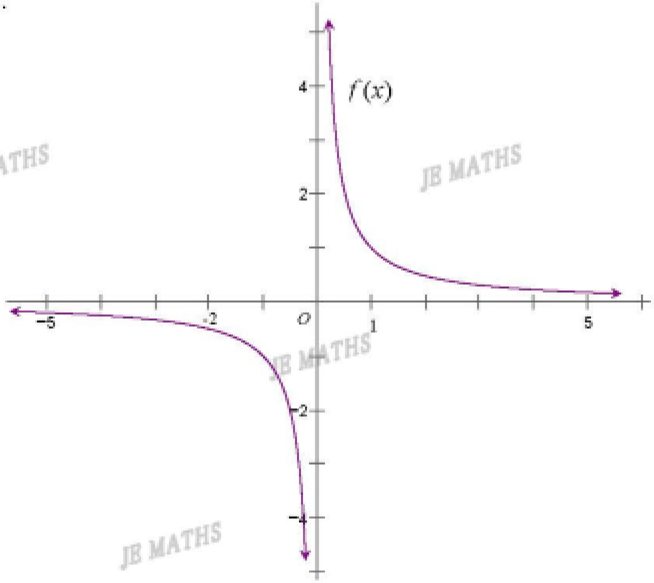
   (d) $\lim_{x\to 2^-} \frac{|x-2|}{x-2}$

   (e) $\lim_{x\to 2^+} \sqrt{x-2}$

   (f) $\lim_{x\to 1^-} \sqrt{1-x^2}$

Stage 4

1. Given the graph of the function $f(x) = \dfrac{1}{x}$, find:



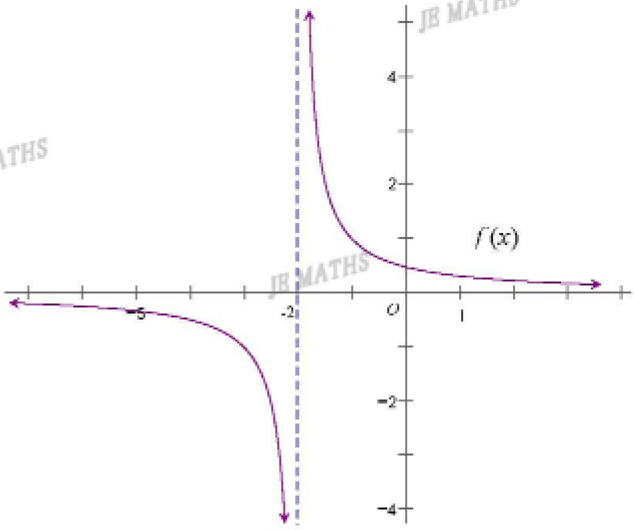
$\lim\limits_{x \to 0^+} \dfrac{1}{x} =$ ______

$\lim\limits_{x \to 0^-} \dfrac{1}{x} =$ ______

$\lim\limits_{x \to +\infty} \dfrac{1}{x} =$ ______

$\lim\limits_{x \to -\infty} \dfrac{1}{x} =$ ______

2. Given the graph of the function $f(x) = \dfrac{1}{x+2}$, find:



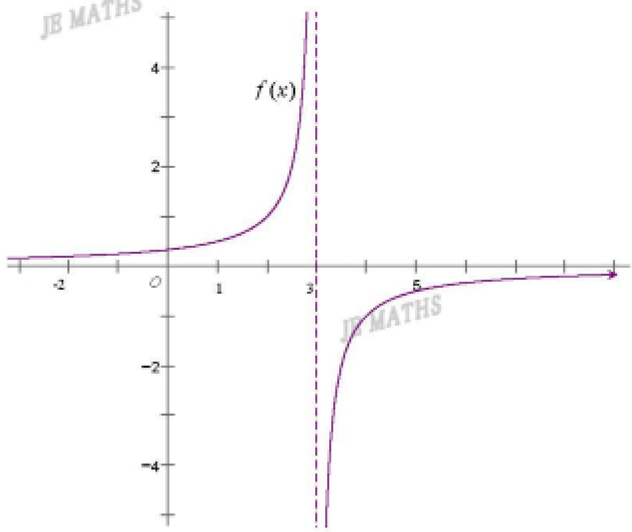
$\lim\limits_{x \to -2^+} \dfrac{1}{x+2} =$ ______

$\lim\limits_{x \to -2^-} \dfrac{1}{x+2} =$ ______

$\lim\limits_{x \to +\infty} \dfrac{1}{x+2} =$ ______

$\lim\limits_{x \to -\infty} \dfrac{1}{x+2} =$ ______

3. Given the graph of the function $f(x) = \dfrac{1}{3-x}$, find:



$\lim\limits_{x \to 3^+} \dfrac{1}{3-x} =$ ______

$\lim\limits_{x \to 3^-} \dfrac{1}{3-x} =$ ______

$\lim\limits_{x \to +\infty} \dfrac{1}{3-x} =$ ______

$\lim\limits_{x \to -\infty} \dfrac{1}{3-x} =$ ______

4. Find the limit of each function:

(a) $\lim_{x\to\infty}\frac{1}{x-1}$

(b) $\lim_{x\to\infty}\frac{1}{x+4}$

(c) $\lim_{x\to\infty}\frac{1}{2x+3}$

(d) $\lim_{x\to\infty}\frac{1}{5-x}$

(e) $\lim_{x\to\infty}\frac{x-1}{x+1}$

(f) $\lim_{x\to\infty}\frac{x}{2x+3}$

(g) $\lim_{x\to\infty}\frac{x-1}{5-x}$

(h) $\lim_{x\to\infty}\frac{3x-1}{x+4}$

(i) $\lim_{x\to\infty}\frac{1}{x^2+1}$

(j) $\lim_{x\to\infty}\frac{2x}{x^2+1}$

(k) $\lim\limits_{x \to \infty} \dfrac{3x^2}{x^2+1}$

(l) $\lim\limits_{x \to \infty} \dfrac{x^2+x}{x^2+1}$

5. Find the limit of each function:

(a) $\lim\limits_{x \to 1^+} \dfrac{1}{x-1}$

(b) $\lim\limits_{x \to 1^-} \dfrac{1}{x-1}$

(c) $\lim\limits_{x \to -4^+} \dfrac{1}{x+4}$

(d) $\lim\limits_{x \to -4^-} \dfrac{1}{x+4}$

(e) $\lim\limits_{x \to 2^+} \dfrac{1}{2x-4}$

(f) $\lim\limits_{x \to 2^-} \dfrac{1}{2x-4}$

(g) $\lim\limits_{x \to \frac{1}{3}^+} \dfrac{1}{1-3x}$

(h) $\lim\limits_{x \to \frac{1}{3}^-} \dfrac{1}{1-3x}$

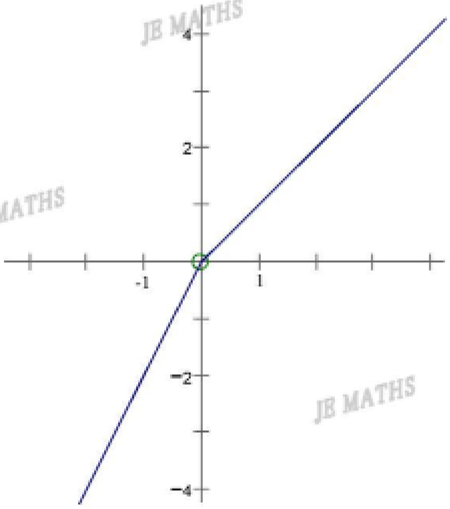
Stage 2

1.

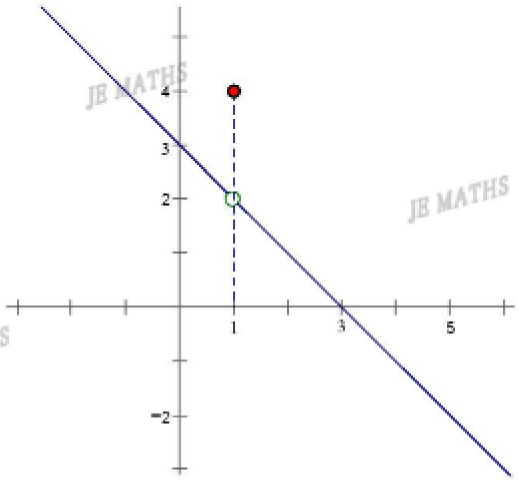
(a) (2), (2)
(b) (1), (1)
(c) (0), (0)
(d) (-3), (-3)
(e) (not exist), (not exist)

2.

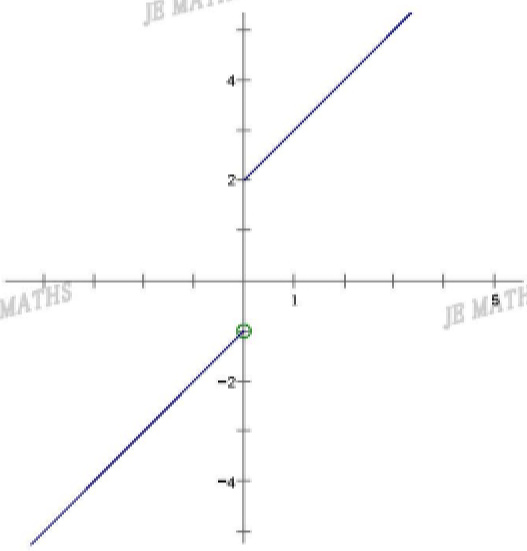
(a) $\lim_{x \to 0} f(x) = 0$



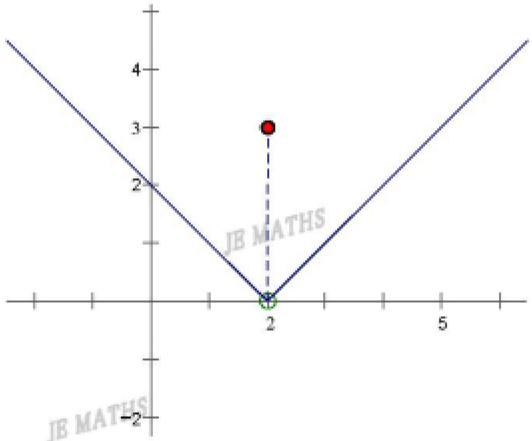
(b) $\lim_{x \to 1} f(x) = 2$



(c) $\lim_{x \to 0} f(x)$ does NOT exist

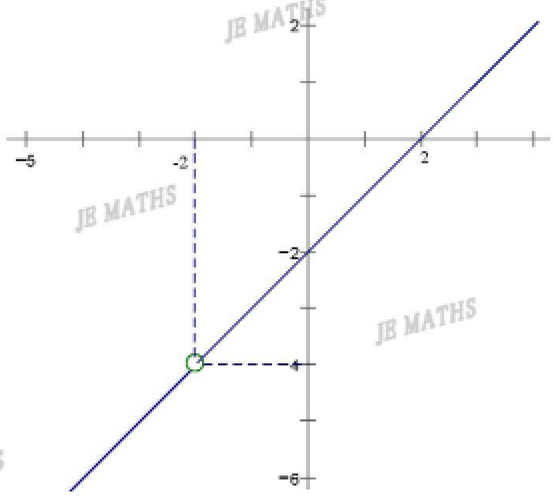


(d) $\lim_{x \to 2} f(x) = 0$



3. (x ≠ -2)
(f(x) = x – 2 for x ≠ -2)

$$\lim_{x \to -2} \frac{x^2 - 4}{x + 2} = \lim_{x \to -2}(x - 2) = -2 - 2 = -4$$



4.

(a) $\lim_{x \to 0} \frac{x^2 + 3x}{x} = \lim_{x \to 0}(x + 3) = 0 + 3 = 3$

(b) $\lim_{x \to 4} \frac{(x-1)(x-4)}{x-4} = \lim_{x \to 4}(x - 1) = 4 - 1 = 3$

(c) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1}(x + 1) = 1 + 1 = 2$

(d) $\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+2)}{x-3} = \lim_{x \to 3}(x + 2) = 3 + 2 = 5$

(e) $\lim_{x \to 0} \frac{3x}{x^2 - 2x} = \lim_{x \to 0} \frac{3x}{x(x-2)} = \lim_{x \to 0} \frac{3}{x-2} = \frac{3}{0-2} = -\frac{3}{2}$

(f) $\lim_{x \to -1} \frac{x+1}{x^2 - x - 2} = \lim_{x \to -1} \frac{x+1}{(x-2)(x+1)} = \lim_{x \to -1} \frac{1}{x-2} = \frac{1}{-1-2} = -\frac{1}{3}$

(g) $\lim_{x \to -1} \frac{x+1}{x^3 + 1} = \lim_{x \to -1} \frac{x+1}{(x+1)(x^2 - x + 1)} = \lim_{x \to -1} \frac{1}{x^2 - x + 1} = \frac{1}{1+1+1} = \frac{1}{3}$

(h) $\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)}{x - 2} = \lim_{x \to 2}(x^2 + 2x + 4) = 4 + 4 + 4 = 12$

Stage 3

1.

(a) (2)

(-1)

(not exist)

(b) (1)

(3)

(not exist)

(c) (2)

(2)

(2)

(d) (0)

(0)

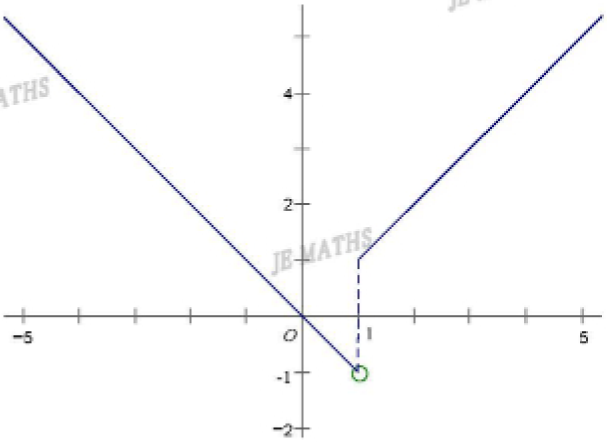
(0)

2. (i) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(-x) = -1$

(ii) As $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$, hence $\lim_{x\to 1} f(x)$ does NOT exist.

(iii)



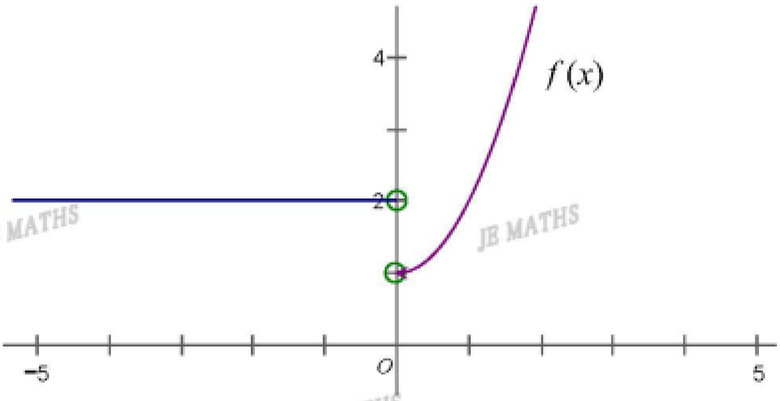
From the graph, $\lim_{x\to 1} f(x)$ does NOT exist.

3. (i) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x^2+1) = 0+1 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2 = 2$

(ii) As $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, hence $\lim_{x\to 0} f(x)$ does NOT exist.

(iii)



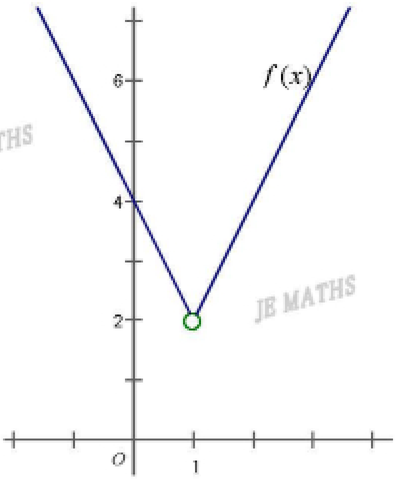
From the graph, $\lim_{x \to 0} f(x)$ does NOT exist.

4. (i) $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x) = 2 \times 1 = 2$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (4 - 2x) = 4 - 2 \times 1 = 2$

(ii) As $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 2$, hence $\lim_{x \to 0} f(x)$ does exist and $\lim_{x \to 1} f(x) = 2$.

(iii)



From the graph, $\lim_{x \to 1} f(x) = 2$.

5. (i) $f(x) = \begin{cases} x + 2 & \text{for } x \geq -2 \\ -x - 2 & \text{for } x < -2 \end{cases}$

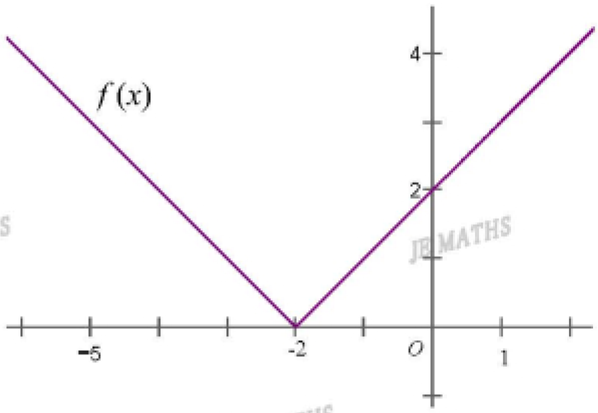
(ii) $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x + 2) = -2 + 2 = 0$

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (-x - 2) = -(-2) - 2 = 0$

(iii) yes, $\lim_{x \to -2} f(x)$ does exist as $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^-} f(x) = 0$.

$\lim_{x \to -2} f(x) = 0$

(iii)



From the graph, $\lim_{x \to -2} f(x) = 0$.

6. (a) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+3) = 0+3 = 3$ (for domain all real x)

(b) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+3) = 0+3 = 3$ (for domain such that $x > 0$)

7.

(a) $\lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$ (as when $x \to 0^+$, $x > 0$ and $|x| = x$)

(b) $\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} \frac{x}{-x} = \lim_{x \to 0^-} (-1) = -1$ (as when $x \to 0^-$, $x < 0$ and $|x| = -x$)

(c) $\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{x \to 2^+} \frac{x-2}{x-2} = \lim_{x \to 2^+} 1 = 1$ (as when $x \to 2^+$, $x > 2$ and $|x-2| = x-2$)

(d) $\lim_{x \to 2^-} \frac{|x-2|}{x-2} = \lim_{x \to 2^-} \frac{-(x-2)}{x-2} = \lim_{x \to 2^-} (-1) = -1$ (as when $x \to 2^-$, $x < 2$ and $|x-2| = -(x-2)$)

(e) $\lim_{x \to 2^+} \sqrt{x-2} = \sqrt{2-2} = 0$

(f) $\lim_{x \to 1^-} \sqrt{1-x^2} = \sqrt{1-1} = 0$

Stage 4

1. $\lim_{x\to 0^+}\frac{1}{x}=\frac{1}{0^+}=+\infty$

$\lim_{x\to 0^-}\frac{1}{x}=\frac{1}{0^-}=-\infty$

$\lim_{x\to +\infty}\frac{1}{x}=\frac{1}{+\infty}=0$

$\lim_{x\to -\infty}\frac{1}{x}=\frac{1}{-\infty}=0$

2. $\lim_{x\to -2^+}\frac{1}{x+2}=\frac{1}{(-2)^+ +2}=\frac{1}{0^+}=+\infty$

$\lim_{x\to -2^-}\frac{1}{x+2}=\frac{1}{(-2)^- +2}=\frac{1}{0^-}=-\infty$

$\lim_{x\to +\infty}\frac{1}{x+2}=\frac{1}{+\infty+2}=\frac{1}{+\infty}=0$

$\lim_{x\to -\infty}\frac{1}{x+2}=\frac{1}{-\infty+2}=\frac{1}{-\infty}=0$

3. $\lim_{x\to 3^+}\frac{1}{3-x}=\frac{1}{3-3^+}=\frac{1}{0^-}=-\infty$

$\lim_{x\to 3^-}\frac{1}{3-x}=\frac{1}{3-3^-}=\frac{1}{0^+}=+\infty$

$\lim_{x\to +\infty}\frac{1}{3-(+\infty)}=\frac{1}{-\infty}=0$

$\lim_{x\to -\infty}\frac{1}{3-x}=\frac{1}{3-(-\infty)}=\frac{1}{+\infty}=0$

4.

(a) $\lim_{x\to\infty}\frac{1}{x-1}=\frac{1}{\infty-1}=\frac{1}{\infty}=0$

(b) $\lim_{x\to\infty}\frac{1}{x+4}=\frac{1}{\infty+4}=\frac{1}{\infty}=0$

(c) $\lim_{x\to\infty}\frac{1}{2x+3}=\frac{1}{2\times\infty+3}=\frac{1}{\infty}=0$

(d) $\lim_{x\to\infty}\frac{1}{5-x}=\frac{1}{5-\infty}=\frac{1}{-\infty}=0$

(e) $\lim_{x\to\infty}\frac{x-1}{x+1}=\lim_{x\to\infty}\frac{1-\frac{1}{x}}{1+\frac{1}{x}}=\frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}}=\frac{1-0}{1+0}=1$

(f) $\lim_{x\to\infty}\frac{x}{2x+3}=\lim_{x\to\infty}\frac{1}{2+\frac{3}{x}}=\lim_{x\to\infty}\frac{1}{2+\frac{3}{\infty}}=\frac{1}{2+0}=\frac{1}{2}$

(g) $\lim\limits_{x\to\infty}\dfrac{x-1}{5-x}=\lim\limits_{x\to\infty}\dfrac{1-\dfrac{1}{x}}{\dfrac{5}{x}-1}=\dfrac{1-\dfrac{1}{\infty}}{\dfrac{5}{\infty}-1}=\dfrac{1-0}{0-1}=-1$

(h) $\lim\limits_{x\to\infty}\dfrac{3x-1}{x+4}=\lim\limits_{x\to\infty}\dfrac{3-\dfrac{1}{x}}{1+\dfrac{4}{x}}=\dfrac{3-\dfrac{1}{\infty}}{1+\dfrac{4}{\infty}}=\dfrac{3-0}{1+0}=3$

(i) $\lim\limits_{x\to\infty}\dfrac{1}{x^2+1}=\dfrac{1}{\infty^2+1}=\dfrac{1}{\infty}=0$

(j) $\lim\limits_{x\to\infty}\dfrac{2x}{x^2+1}=\lim\limits_{x\to\infty}\dfrac{2}{x+\dfrac{1}{x}}=\dfrac{2}{\infty+\dfrac{1}{\infty}}=\dfrac{2}{\infty+0}=\dfrac{2}{\infty}=0$

(k) $\lim\limits_{x\to\infty}\dfrac{3x^2}{x^2+1}=\lim\limits_{x\to\infty}\dfrac{3}{1+\dfrac{1}{x^2}}=\dfrac{3}{1+\dfrac{1}{\infty^2}}=\dfrac{3}{1+0}=3$

(l) $\lim\limits_{x\to\infty}\dfrac{x^2+x}{x^2+1}=\lim\limits_{x\to\infty}\dfrac{1+\dfrac{1}{x}}{1+\dfrac{1}{x^2}}=\dfrac{1+\dfrac{1}{\infty}}{1+\dfrac{1}{\infty^2}}=\dfrac{1+0}{1+0}=1$

5.

(a) $\lim\limits_{x\to 1^+}\dfrac{1}{x-1}=\dfrac{1}{1^+-1}=\dfrac{1}{0^+}=+\infty$

(b) $\lim\limits_{x\to 1^-}\dfrac{1}{x-1}=\dfrac{1}{1^--1}=\dfrac{1}{0^-}=-\infty$

(c) $\lim\limits_{x\to -4^+}\dfrac{1}{x+4}=\dfrac{1}{(-4)^++4}=\dfrac{1}{0^+}=+\infty$

(d) $\lim\limits_{x\to -4^-}\dfrac{1}{x+4}=\dfrac{1}{(-4)^-+4}=\dfrac{1}{0^-}=-\infty$

(e) $\lim\limits_{x\to 2^+}\dfrac{1}{2x-4}=\dfrac{1}{2\times 2^+-4}=\dfrac{1}{4^+-4}=\dfrac{1}{0^+}=+\infty$

(f) $\lim\limits_{x\to 2^-}\dfrac{1}{2x-4}=\dfrac{1}{2\times 2^--4}=\dfrac{1}{4^--4}=\dfrac{1}{0^-}=-\infty$

(g) $\lim\limits_{x\to \frac{1}{3}^+}\dfrac{1}{1-3x}=\dfrac{1}{1-3\times\left(\dfrac{1}{3}\right)^+}=\dfrac{1}{1-1^+}=\dfrac{1}{0^-}=-\infty$

(h) $\lim\limits_{x\to \frac{1}{3}^-}\dfrac{1}{1-3x}=\dfrac{1}{1-3\times\left(\dfrac{1}{3}\right)^-}=\dfrac{1}{1-1^-}=\dfrac{1}{0^+}=+\infty$