Problem Set 8

Written by James Bang for the MaPS Correspondence Program

Instructions

- Some of these problems are based off the notes "*Colouring and Invariants*".
- They are in roughly difficulty order and get quite difficult, so you are **not** expected to be able to solve every problem.
- However, please attempt as many questions as you can and submit your solutions to your mentor for marking and feedback.
- You may (and encouraged to) submit incomplete solutions if you can not solve a problem completely.
- You may type your solutions or submit a pdf of a **clear** scan/photo of **legible** written solutions.
- Feel free to discuss these problems with your peers and on the forum but the solutions you submit must be written by yourself.

Problems

1. Suppose that all the MaPS students (of which there are at least 2) are sitting around in a circle with a pile of 1001 chocolates in the middle. Each person takes some number of chocolates from the pile, and by coincidence each person either has 6 less or twice as many chocolates as the person to their right. Prove that not all the chocolates were taken.
2. Show that if $x^2 + y^2$ is divisible by 7, then both x and y are divisible by 7.
3. Suppose $n \equiv 1, 2 \pmod 4$. Is it possible to assign some signs to the expression
$$\pm 1 \pm 2 \pm \cdots \pm n = 0$$
so that equality holds?

   Assigning signs means replacing each $\pm$ with either $+$ or $-$.
4. In this question, we will prove that the $AM - GM$ inequality is true for a set of n positive real numbers, where n can be any positive integer. We will be using an induction-like approach.
   (a) Prove that the $AM - GM$ inequality holds for $n = 2$
   (b) Prove that if the $AM - GM$ inequality holds for $n = k$, for some integer k, then the $AM - GM$ inequality holds for $n = 2k$
   (c) Prove that if the $AM - GM$ inequality holds for $n = k$, for some integer k, then the $AM - GM$ inequality holds for $n = k - 1$
   (d) Explain why the three statements above proves that the $AM - GM$ inequality holds for all positive integers n.
5. The fifteen puzzle is a game in which 15 of the 16 squares of a 4×4 frame are filled with numbered sliding pieces, leaving one space in which to slide one piece at a time. Is it possible to begin with the configuration below and finish with the pieces numbered 14 and 15 swapped?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	