The Pigeonhole Principle

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1 Introduction

The **Pigeonhole Principle** is a simple mathematical concept that has far reaching applications in different fields of maths including geometry, combinatorics, mathematical analysis and more. In this handout, we will look at the **Pigeonhole Principle**, the **Generalised Pigeonhole Principle** and the **Infinite Pigeonhole Principle**.

2 The Pigeonhole Principle (PHP)

In its simplest form, the Pigeonhole Principle states that

> "If $n + 1$ pigeons are to be distributed among n pigeonholes, then **at least one** pigeonhole will contain **at least two** pigeons."

Of course, here we assume that the pigeons must remain whole. The power of the PHP comes from the fact that it tells us about the *existence* of something. It does not require any additional assumptions or cases and so can be a powerful tool for *existence* questions. A proof of this principle is left for the problem set.

Remark: This principle gets its name from the historical use of **pigeons** to deliver letters to specific **pigeonholes**.

Exercise 2.1 *Notice that in the statement of the Pigeonhole Principle, we emphasised "**at least one** pigeonhole" and "**at least two** pigeons". Show that the statement is incorrect if we replace either of the "**at least**" with "**exactly**".*

The Pigeonhole Principle sounds very obvious and allows us to make some unsurprising statements such as:

- Among a group of 13 people, at least two of them will have the same birth month.
- Among a selection of 8 days in the year, at least two of them will be on the same day of the week.
- A word with 27 letters must have at least one letter that appears at least twice.

However, the PHP can sometimes be used in problems where its application can be a bit more discrete. We shall investigate a few examples!

Example 2.2 *In an equilateral triangle with side length 2, five points are randomly selected. Prove that there exists two of these points which are at most 1 unit apart.*

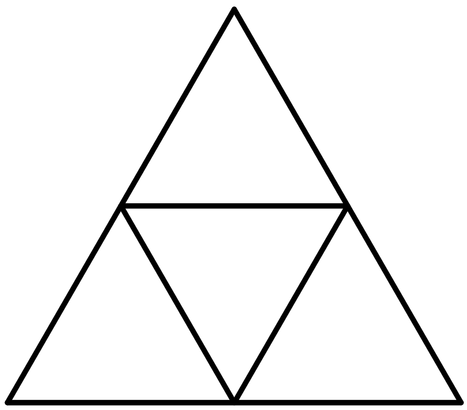
Hint: To familiarise yourself with this problem (and convince yourself that it's true), it would be useful to start with such a triangle and try to fit five points in it such that they are all more than 1 unit away from each other.

You will quickly find that no matter how hard you try, there will always be at least two points that are at most 1 unit away from each other! It might seem like this can be justified by saying that in the *"best"* case scenario, the points would be as spread out as possible which still has a pair of points at most 1 unit apart.

However, this sort of argument has a major problem! What does the *"best"* case scenario mean? How do we know it is the *"best"* case scenario? If we want to make a **rigorous** proof, we would need to address these questions which can be very difficult.

This is why the Pigeonhole Principle is so powerful! It is a general and rigorous argument to show **existence**.

Solution: We first partition the triangle into four smaller equilateral triangles with side length 1 as below.



By the Pigeonhole Principle, among the 5 points inside the big triangle, there will exist two of them that lie inside the same small equilateral triangle (with side length 1). Since the furthest distance between any two points in a triangle is its longest side, these two points will be at most 1 unit away from each other. $\square$

Remark: In this problem, our **pigeons** were the five **points** and the **pigeonholes** were the four smaller **equilateral triangles**. This choice was not immediately obvious and is an example of how the PHP can be used creatively to solve many types of problems!

Example 2.3 *In a face-to-face session, there are n attendees some of whom are friends with each other (assume that friendship is mutual). Prove that there exists at least two attendees that have the same number of friends among the attendees.*

Hint: Notice that we are trying to prove that "**there exists**" **at least two** attendees with the **same number** of friends. This statement sounds suspiciously similar to the PHP! And it gives us some indication as to what our pigeons and pigeonholes might be: the **pigeons** would be the **attendees** and the **pigeonholes** would be the **number of friends** that each attendee has.

However, notice that there are n attendees, so there are **n pigeons** and each attendee might have $0, 1, 2, \ldots, n-1$ friends so there are **n pigeonholes** as well. This means that we cannot apply the PHP just yet!

Solution: As above, notice that among the attendees, each of them has n possible numbers of friends $\{0, 1, \dots, n-1\}$. However, an attendee with 0 friends is not a friend with anyone else and an attendee with $n-1$ friends is a friend with everyone else. Hence, among the attendees there cannot be one with 0 friends AND one with $n-1$ friends.

This means that among the n attendees, each of them only has $n-1$ possible numbers of friends (either $\{0, 1, 2, \dots, n-2\}$ or $\{1, 2, 3, \dots, n-1\}$). By the Pigeonhole Principle, there must exist two attendees with the same number of friends. $\square$

Exercise 2.4 *Among any group of 2023 integers, prove that there exists two of them, say a and b, such that $a-b$ is a multiple of 2022.*

Exercise 2.5 *Prove that if 5 points are chosen in a square of side length 2, then there exist two points whose distance apart is at most $\sqrt{2}$.*

3 Generalised Pigeonhole Principle

The Pigeonhole Principle can be quite easily generalised to the following statement:

> "If $kn+1$ pigeons are to be distributed among n pigeonholes, then **at least one** pigeonhole will contain **at least $\mathbf{k+1}$** pigeons."

Again, we assume that pigeons must remain whole. And again, notice the emphasis on both **"at least"**.

This allows us to assert some unsurprising facts:

- Among a group of 15 people, at least 3 of them will be born on the same day of the week.
- Among 17 cards from a standard deck, at least 5 of them will be of the same suit.
- If 61 standard dice are thrown, at least 11 of them will show the same number.

Let's look at a harder example!

Example 3.1 *Among a group of 6 students, each pair of them are either friends or strangers. Prove that there exists three of them such that they are either all friends with each other or all strangers with each other.*

Hint: Observe that we are trying to prove existence! This means that the PHP could potentially be used here! But it is definitely not obvious how it should be used.

Solution: Suppose that we name the six students S_1, S_2, S_3, S_4, S_5 and S_6. Then let us look at this problem from the perspective of S_1. They will have some type of relationship (friends or strangers) with each of the other 5 students. By the Pigeonhole Principle, S_1 will have the same type of relationship with at least 3 students. Without loss of generality, assume that S_1 is friends with S_2, S_3 and S_4.[1]

[1] When we say "without loss of generality, assume...", this means that our argument can be repeated for all cases in the question. In this particular question, it means that S_1 could have been friends or strangers with any group of 3 of the remaining 5 students. However, if we can solve the problem for the case where S_1 is friends with S_2, S_3 and S_4, then the argument also applies to all the other cases. This is sometimes abbreviated to "WLOG".

Now we consider the type of relationships between S_2, S_3 and S_4.

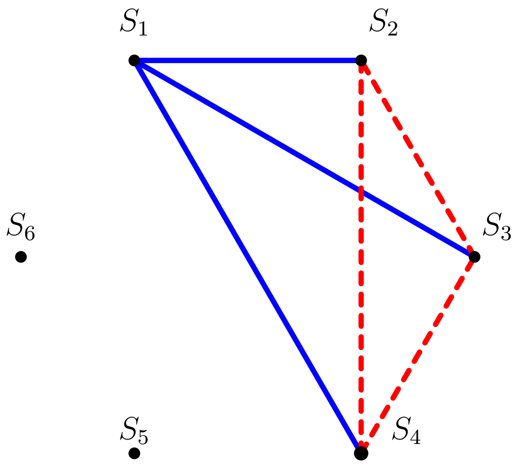


Figure 1: Friends are marked in blue and strangers are marked in red. Unknown or unnecessary relationships are not marked.

If any pair among S_2, S_3 and S_4 are friends, then this will form a group of 3 friends with S_1. Otherwise, S_2 S_3 and S_4 form a group of 3 strangers. In either case, there exists a group of 3 friends or 3 strangers. $\square$

Remark: This result can be extended to show that there must be at least 2 groups of either 3 friends or 3 strangers.

Exercise 3.2 *In James' sock drawer, he has 5 types of socks. He is running late to school and so just grabs a handful of socks. What is the minimum number of socks that he should grab to ensure that he has a matching pair of socks?*

Exercise 3.3 *Prove that among any 17 integers, there are five of which whose sum is divisible by 5.*

4 Infinite Pigeonhole Principle

This is a further (and purely theoretical) extension of the Pigeonhole Principle. It states that:

> "If an infinite number of pigeons are to be distributed among a finite number of pigeonholes, then **at least one** pigeonhole will contain an infinite number of pigeons."

Again, we also assume that the pigeons must remain whole. This result may seem even more obvious but it has some more interesting applications!

Example 4.1 *Prove that every positive integer n has some multiple of the form*

$$1111\ldots11110000\ldots0000,$$

that is a sequence of 1's followed by a sequence of 0's.

Solution: Consider the sequence of numbers

$$1, 11, 111, 1111, 11111, 111111, \ldots.$$

Now consider their remainders when divided by n: there are n possible **remainders** and infinitely many **numbers** in the sequence. Hence by the Pigeonhole Principle, there are at least two (actually infinitely many) **numbers** in this sequence that have the **same remainder** upon division by n. Suppose they are

$$\underbrace{111\ldots111}_{a \text{ digits}} \text{ and } \underbrace{111\ldots111}_{b \text{ digits}}$$

where $a < b$. This means that the difference of these numbers will be a multiple of n, and will be equal to

$$\underbrace{111\ldots111}_{b-a \text{ digits}}\underbrace{000\ldots000}_{a \text{ digits}}$$

which is of the desired form.

Exercise 4.2 *Prove that for any integer n, there exists a Fibonacci number that is a multiple of n.*

5 Practice Problems

These are a few harder problems you can attempt for practice.

1. Prove that among any 27 distinct odd integers less than 100, there exists two of them whose sum is 102.

2. Given any 52 integers, show that there are two of them either whose sum or difference is divisible by 100.

3. Prove that if 4 points are chosen in a square of side length 8, then there exist two points whose distance apart is at most $\sqrt{65}$.

4. Suppose that if we have a 3×7 grid of squares, where each unit square is coloured either black or white. Prove that there exists a rectangle made up of the unit squares whose corner squares are of the same colour.

5. A disk of radius 1 is completely covered by 7 identical smaller disks (which may overlap). Show that the radius of the smaller disks must be at least $\frac{1}{2}$.

6. Each of 14 red balls and 14 green balls is marked with an integer between 1 and 100 inclusive; no integer appears on more than one ball. The value of a pair of balls is the sum of the numbers on the balls. Show there are at least two pairs, consisting of one red and one green ball, with the same value.

7. Among a group of n integer numbers, prove that there is some subset of them whose sum is divisible by n.