0. 说明

问1: 证明 $(\boldsymbol{C}\boldsymbol{u})^{\wedge} \equiv \boldsymbol{C}\boldsymbol{u}^{\wedge}\boldsymbol{C}^{T}$ 问2: 证明 $(\boldsymbol{C}\boldsymbol{u})^{\wedge} \equiv (2\cos\phi+1)\boldsymbol{u}^{\wedge} - \boldsymbol{u}^{\wedge}\boldsymbol{C} - \boldsymbol{C}^{T}\boldsymbol{u}^{\wedge}$ 问3: 证明 $\exp((\boldsymbol{C}\boldsymbol{u})^{\wedge}) \equiv \boldsymbol{C} \exp(\boldsymbol{u}^{\wedge})\boldsymbol{C}^{T}$ 问4: 证明 $(\boldsymbol{T}\boldsymbol{x})^{\wedge} \equiv \boldsymbol{T}\boldsymbol{x}^{\wedge}\boldsymbol{T}^{-1}$ 问5: 证明 $\exp((\boldsymbol{T}\boldsymbol{x})^{\wedge}) \equiv \boldsymbol{T} \exp(\boldsymbol{x}^{\wedge})\boldsymbol{T}^{-1}$ 问6: 证明 $\boldsymbol{x}^{\wedge}\boldsymbol{p} \equiv \boldsymbol{p}^{\odot}\boldsymbol{x}$ 问7: 证明 $\boldsymbol{p}^{T}\boldsymbol{x}^{\wedge} \equiv \boldsymbol{x}^{T}\boldsymbol{p}^{\odot}$ 问8: 证明 $(\boldsymbol{T}\boldsymbol{p})^{\odot} \equiv \boldsymbol{T}\boldsymbol{p}^{\odot}\boldsymbol{T}^{-1}$

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问1:证明
$$(oldsymbol{C}oldsymbol{u})^\wedge \equiv oldsymbol{C}oldsymbol{u}^\wedge oldsymbol{C}^T$$

证: 与上次作业相同

• 方1: 设
$$oldsymbol{u}=[u_1,u_2,u_3]^T$$
, $oldsymbol{C}=egin{bmatrix} c_{11} & c_{12} & c_{13} \ c_{21} & c_{22} & c_{23} \ c_{31} & c_{32} & c_{33} \end{bmatrix}$

带入公式左右两边计算即可。

• 方2: 因为 $m{C} \in SO(3)$: $\forall m{x} \in \mathbb{R}^3 \qquad (m{C}m{u})^\wedge m{x} = (m{C}m{u}) imes (m{C}m{C}^Tm{x}) = m{C}(m{u} imes C^Tm{x}) = m{C}m{u}^\wedge m{C}^Tm{x}$ $\therefore (m{C}m{u})^\wedge \equiv m{C}m{u}^\wedge m{C}^T$

问2:证明
$$(Coldsymbol{u})^\wedge\equiv (2\cos\phi+1)oldsymbol{u}^\wedge-oldsymbol{u}^\wedge C-oldsymbol{C}^Toldsymbol{u}^\wedge$$

证:

• 方1: 设
$$oldsymbol{u}=[u_1,u_2,u_3]^T$$
, $oldsymbol{C}=egin{bmatrix} c_{11} & c_{12} & c_{13} \ c_{21} & c_{22} & c_{23} \ c_{31} & c_{32} & c_{33} \end{bmatrix}$

带入公式左右两边计算即可。

• 方2: 带入公式 $C = \cos \theta \mathbf{1} + (1 - \cos \theta) \mathbf{a} \mathbf{a}^T + \sin \theta \mathbf{a}^{\wedge}$ 计算

• 方2: 利用李代数的性质
$$(Wu)^{\wedge} \equiv u^{\wedge}(\operatorname{tr}(W)\mathbf{1} - W) - W^{T}u^{\wedge}$$
 可得:

$$egin{aligned} oldsymbol{(Cu)}^\wedge &\equiv oldsymbol{u}^\wedge(\operatorname{tr}(oldsymbol{C})\mathbf{1} - oldsymbol{C}) - oldsymbol{C}^Toldsymbol{u}^\wedge \ &= oldsymbol{u}^\wedge\operatorname{tr}(oldsymbol{C})\mathbf{1} - oldsymbol{u}^\wedgeoldsymbol{C} - oldsymbol{C}^Toldsymbol{u}^\wedge \ &= (2\cos\phi + 1)oldsymbol{u}^\wedge - oldsymbol{u}^\wedgeoldsymbol{C} - oldsymbol{C}^Toldsymbol{u}^\wedge \ &= (2\cos\phi + 1)oldsymbol{u}^\wedge - oldsymbol{u}^\wedgeoldsymbol{C} - oldsymbol{C}^Toldsymbol{u}^\wedge \ &= (2\cos\phi + 1)oldsymbol{u}^\wedge - oldsymbol{u}^\wedgeoldsymbol{C} - oldsymbol{C}^Toldsymbol{u}^\wedge \ &= (2\cos\phi + 1)oldsymbol{u}^\wedge - oldsymbol{u}^\wedgeoldsymbol{C} - oldsymbol{C}^Toldsymbol{u}^\wedge \ &= (2\cos\phi + 1)oldsymbol{u}^\wedge - oldsymbol{u}^\wedgeoldsymbol{C} - oldsymbol{C}^Toldsymbol{u}^\wedge \ &= (2\cos\phi + 1)oldsymbol{u}^\wedge - oldsymbol{u}^\wedgeoldsymbol{C} - oldsymbol{C}^Toldsymbol{u}^\wedge \ &= (2\cos\phi + 1)oldsymbol{u}^\wedge - oldsymbol{u}^\wedgeoldsymbol{C} - oldsymbol{C}^Toldsymbol{u}^\wedge \ &= (2\cos\phi + 1)oldsymbol{u}^\wedge - oldsymbol{u}^\wedgeoldsymbol{C} - oldsymbol{C}^Toldsymbol{u}^\wedge - oldsymbol{U}^\wedgeoldsymbol{U}^\wedge - oldsymbol{U}^\wedgeoldsymbol{U}^\wedgeoldsymbol{U}^\wedge - oldsymbol{U}^\wedgeoldsymbol{U}^\wedge - oldsymbol{U}^$$

问3:证明 $\exp\left(\left(oldsymbol{C}oldsymbol{u} ight)^\wedge ight)\equivoldsymbol{C}\exp\left(oldsymbol{u}^\wedge ight)oldsymbol{C}^T$

证:

• 根据指数映射 $\exp\left(\phi^{\wedge}\right)=\sum\limits_{n=0}^{\infty}\frac{1}{n!}(\phi^{\wedge})^{n}$ 可得:

$$\exp\left((\boldsymbol{C}\boldsymbol{u})^{\wedge}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} ((\boldsymbol{C}\boldsymbol{u})^{\wedge})^{n}, \quad (3-1)$$
 $((\boldsymbol{C}\boldsymbol{u})^{\wedge})^{n} = (\boldsymbol{C}\boldsymbol{u}^{\wedge}\boldsymbol{C}^{T})^{n} = \boldsymbol{C}(\boldsymbol{u}^{\wedge})^{n}\boldsymbol{C}^{T} \qquad (这一步用到了 $\boldsymbol{C}^{T}\boldsymbol{C} = \boldsymbol{1})$
带入公式 (3-1) 可得: $\sum_{n=0}^{\infty} \frac{1}{n!} ((\boldsymbol{C}\boldsymbol{u})^{\wedge})^{n} = \boldsymbol{C}(\sum_{n=0}^{\infty} \frac{1}{n!} ((\boldsymbol{u})^{\wedge})^{n}) \boldsymbol{C}^{T} = \boldsymbol{C} \exp\left(\boldsymbol{u}^{\wedge}\right) \boldsymbol{C}^{T} \Longrightarrow \exp\left((\boldsymbol{C}\boldsymbol{u})^{\wedge}\right) \equiv \boldsymbol{C} \exp\left(\boldsymbol{u}^{\wedge}\right) \boldsymbol{C}^{T}$$

问4:证明 $(\mathcal{T}oldsymbol{x})^\wedge\equivoldsymbol{T}oldsymbol{x}^\wedgeoldsymbol{T}^{-1}$

证:

令
$$m{T} = egin{bmatrix} m{C} & m{r} \\ m{0}^T & 1 \end{bmatrix} & \mathcal{T} = egin{bmatrix} m{C} & m{r}^\wedge m{C} \\ m{0} & m{C} \end{bmatrix} & m{x} = egin{bmatrix} m{u} \\ m{v} \end{bmatrix} \end{pmatrix}$$
恒等式左边 $(\mathcal{T}m{x})^\wedge = \left(egin{bmatrix} m{C} & m{r}^\wedge m{C} \\ m{0} & m{C} \end{bmatrix} egin{bmatrix} m{u} \\ m{v} \end{bmatrix} \right)^\wedge = m{C}m{C}m{u} + m{r}^\wedge m{C}m{v} \\ m{C}m{v} \end{bmatrix}^\wedge = m{C}m{U} + m{r}^\wedge m{C}m{v} \\ m{0}^T & 0 \end{bmatrix}$
恒等式右边 $m{T}m{x}^\wedge m{T}^{-1} = m{C}m{V}^\wedge m{C}^T & m{C}m{u} - m{C}m{v}^\wedge (m{C}^Tm{r}) \\ m{0}^T & 0 \end{bmatrix}$

$$egin{aligned} egin{aligned} oldsymbol{\mathcal{R}} & \cdot \cdot \cdot (oldsymbol{C}oldsymbol{v})^\wedge \equiv oldsymbol{C}oldsymbol{v}^\top oldsymbol{C}oldsymbol{v} - oldsymbol{C}oldsymbol{v}^\top oldsymbol{C}^Toldsymbol{r}) \end{aligned} & = oldsymbol{C}oldsymbol{v}^\top oldsymbol{C}^Toldsymbol{r} - oldsymbol{C}oldsymbol{v}^\top oldsymbol{C}^Toldsymbol{r}) \end{aligned} & = oldsymbol{C}oldsymbol{v}^\top oldsymbol{C}oldsymbol{v}^\top oldsymbol{v} - oldsymbol{C}oldsymbol{v}^\top oldsymbol{C}^Toldsymbol{r} - oldsymbol{C}oldsymbol{v}^\top oldsymbol{C}^Toldsymbol{r} - oldsymbol{C}oldsymbol{v}^\top oldsymbol{v}^\top oldsymbol{v} - oldsymbol{C}oldsymbol{v}^\top oldsymbol{v}^\top oldsymbol{v} - oldsymbol{C}oldsymbol{v}^\top oldsymbol{v}^\top oldsymbol{v} - oldsymbol{C}oldsymbol{v}^\top oldsymbol{v}^\top oldsymbol{v} - oldsymbol{C}oldsymbol{v}^\top oldsymbol{v} - oldsymbol{C}oldsymbol{v}^\top oldsymbol{v} - oldsymbol{V} oldsymbol{v}^\top oldsymbol{v} - oldsymbol{V} oldsymbol{v}^\top oldsymbol{v} - oldsymbol{V} - oldsymbol{V} oldsymbol{v}^\top oldsymbol{v} - oldsymbol{V} - oldsymbol{V} oldsymbol{v}^\top oldsymbol{v} - old$$

问5:证明 $\exp\left((\mathcal{T}oldsymbol{x})^\wedge ight)\equivoldsymbol{T}\exp\left(oldsymbol{x}^\wedge ight)oldsymbol{T}^{-1}$

证:

与**问3**类似,先将 $\exp\left((\mathcal{T}\boldsymbol{x})^{\wedge}\right)$ 展开,然后利用性质 $(\mathcal{T}\boldsymbol{x})^{\wedge}\equiv \boldsymbol{T}\boldsymbol{x}^{\wedge}\boldsymbol{T}^{-1}$ 和 $\boldsymbol{T}^{-1}\boldsymbol{T}=\boldsymbol{1}$ 化简即可得到恒等式右边;

问6:证明
$$oldsymbol{x}^\wedge oldsymbol{p} \equiv oldsymbol{p}^\odot oldsymbol{x}$$

证:
$$\Rightarrow \boldsymbol{x} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} \boldsymbol{p} = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix}$$

$$\boldsymbol{x}^{\wedge} \boldsymbol{p} = \begin{bmatrix} \boldsymbol{v}^{\wedge} & \boldsymbol{u} \\ \boldsymbol{0}^{T} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}^{\wedge} \boldsymbol{\varepsilon} + \eta \boldsymbol{u} \\ 0 \end{bmatrix}$$

$$\boldsymbol{p}^{\odot} \boldsymbol{x} = \begin{bmatrix} \eta \mathbf{1} & -\boldsymbol{\varepsilon}^{\wedge} \\ \boldsymbol{0}^{T} & \boldsymbol{0}^{T} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \eta \mathbf{1} \boldsymbol{u} - \boldsymbol{\varepsilon}^{\wedge} \boldsymbol{v} \\ 0 \end{bmatrix}$$

$$:: \boldsymbol{v}^{\wedge} \boldsymbol{\varepsilon} = -\boldsymbol{\varepsilon}^{\wedge} \boldsymbol{v}$$

$$\therefore oldsymbol{x}^{\wedge} oldsymbol{p} \equiv oldsymbol{p}^{\odot} oldsymbol{x}$$

问7:证明 $oldsymbol{p}^Toldsymbol{x}^\wedge\equivoldsymbol{x}^Toldsymbol{p}^\odot$

证:
$$\diamondsuit$$
 $m{x} = egin{bmatrix} m{u} \\ m{v} \end{bmatrix}$ $m{p} = egin{bmatrix} egin{bmatrix} m{\varepsilon} \\ \eta \end{bmatrix}$

$$\boldsymbol{p}^T\boldsymbol{x}^\wedge = [\boldsymbol{\varepsilon^T} \quad \eta] \begin{bmatrix} \boldsymbol{v}^\wedge & \boldsymbol{u} \\ \boldsymbol{0}^T & 0 \end{bmatrix} = [\boldsymbol{\varepsilon^T}\boldsymbol{v}^\wedge \quad \boldsymbol{\varepsilon^T}\boldsymbol{u}]$$

$$m{x}^Tm{p}^{\odot} = [m{u^T} \quad m{v^T}] egin{bmatrix} m{0} & m{arepsilon} \ -m{arepsilon}^{\wedge} & m{0} \end{bmatrix} = [-m{v^T}m{arepsilon}^{\wedge} & m{u^T}m{arepsilon}]$$

$$:: \boldsymbol{\varepsilon}^T \boldsymbol{u} = \boldsymbol{u}^T \boldsymbol{\varepsilon} =$$
标量

$$oldsymbol{arepsilon^T} oldsymbol{v}^\wedge = -(oldsymbol{v}^\wedge oldsymbol{arepsilon})^T = -igg(-oldsymbol{v}^\wedge oldsymbol{v})^T = -igg(-oldsymbol{v}^ op oldsymbol{v}^ op)^T = -igg(-oldsymbol{v}^ op oldsymbol{v}^ op oldsymbol{v}^ op)^T = -igg(-oldsymbol{v}^ op oldsymbol{v}^ o$$

$$\therefore \boldsymbol{p}^T \boldsymbol{x}^\wedge \equiv \boldsymbol{x}^T \boldsymbol{p}^\odot$$

问8:证明 $({m T}{m p})^{\odot} \equiv {m T}{m p}^{\odot}{m \mathcal{T}}^{-1}$

证:
$$\diamondsuit m{p} = egin{bmatrix} m{arepsilon} \\ m{\eta} \end{bmatrix} m{T} = egin{bmatrix} m{C} & m{r} \\ m{0}^T & 1 \end{bmatrix} \ \ m{\mathcal{T}}^{-1} = egin{bmatrix} m{C}^T & -m{C}^T m{r}^\wedge \\ m{0} & m{C}^T \end{bmatrix}$$

$$(oldsymbol{Tp})^{\odot} = egin{bmatrix} \eta \mathbf{1} & -(oldsymbol{C}oldsymbol{arepsilon} + \eta oldsymbol{r})^{\wedge} \ \mathbf{0}^{T} \end{bmatrix}$$

$$m{T}m{p}^{\odot}m{\mathcal{T}}^{-1} = egin{bmatrix} \eta \mathbf{1} & -\eta m{r}^{\wedge} - m{C}m{arepsilon}^{\wedge}m{C}^{T} \ m{0}^{T} & m{0}^{T} \end{bmatrix}$$

$$\because -\eta \boldsymbol{r}^{\wedge} - \boldsymbol{C}\boldsymbol{\varepsilon}^{\wedge}\boldsymbol{C}^{T} = --\eta \boldsymbol{r}^{\wedge} - (\boldsymbol{C}\boldsymbol{\varepsilon})^{\wedge} = -(\boldsymbol{C}\boldsymbol{\varepsilon} + \eta \boldsymbol{r})^{\wedge}$$

$$\therefore (\boldsymbol{T}\boldsymbol{p})^{\odot} \equiv \boldsymbol{T}\boldsymbol{p}^{\odot}\mathcal{T}^{-1}$$

问9:证明 $(Tp)^{\odot^T}(Tp)^{\odot}\equiv \mathcal{T}^{-T}p^{\odot^T}p^{\odot}\mathcal{T}^{-1}$

证:

根据 **问8** 可以得到: $(Tp)^{\odot^T}(Tp)^{\odot} = \mathcal{T}^{-T}p^{\odot^T}T^TTp^{\odot}\mathcal{T}^{-1}$

 $\mathbf{T} \cdot \mathbf{p}^{\odot^T} \mathbf{T}^T \mathbf{T} \mathbf{p}^{\odot} = \mathbf{p}^{\odot^T} \mathbf{p}^{\odot}$ (带入矩阵计算即可得到该等式)

換个角度理解: $p^{\odot^T}T^TTp^{\odot}=(Tp^{\odot})^T(Tp^{\odot})$ 因为 T 是旋转平移变换,所以 Tp^{\odot} 没有改变 p^{\odot} 本身的性质,因此变换后的 $(Tp^{\odot})^T(Tp^{\odot})$ 等于变换前的 $p^{\odot^T}p^{\odot}$ 。