## 0. 说明

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## 问1：证明 $(\boldsymbol{C}\boldsymbol{u})^\wedge \equiv \boldsymbol{C}\boldsymbol{u}^\wedge\boldsymbol{C}^T$

**证：与上次作业相同**

- 方1：设 $\boldsymbol{u} = [u_1, u_2, u_3]^T$， $\boldsymbol{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

  带入公式左右两边计算即可。

- 方2：因为 $\boldsymbol{C} \in SO(3)$ :

  $\forall \boldsymbol{x} \in \mathbb{R}^3 \qquad (\boldsymbol{C}\boldsymbol{u})^\wedge\boldsymbol{x} = (\boldsymbol{C}\boldsymbol{u}) \times (\boldsymbol{C}\boldsymbol{C}^{\boldsymbol{T}}x) = \boldsymbol{C}(\boldsymbol{u} \times C^T)\boldsymbol{x} = \boldsymbol{C}\boldsymbol{u}^\wedge\boldsymbol{C}^T\boldsymbol{x}$

  $\therefore\ (\boldsymbol{C}\boldsymbol{u})^\wedge \equiv \boldsymbol{C}\boldsymbol{u}^\wedge\boldsymbol{C}^T$

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## 问2：证明 $(\boldsymbol{C}\boldsymbol{u})^\wedge \equiv (2\cos\phi+1)\boldsymbol{u}^\wedge - \boldsymbol{u}^\wedge\boldsymbol{C} - \boldsymbol{C}^T\boldsymbol{u}^\wedge$

**证：**

- 方1：设 $\boldsymbol{u} = [u_1, u_2, u_3]^T$， $\boldsymbol{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

  带入公式左右两边计算即可。

- 方2：带入公式 $\boldsymbol{C} = \cos\theta\boldsymbol{1} + (1-\cos\theta)\boldsymbol{a}\boldsymbol{a}^{\boldsymbol{T}} + \sin\theta\boldsymbol{a}^\wedge$ 计算

- 方2：利用李代数的性质 $(\boldsymbol{W}\boldsymbol{u})^\wedge \equiv \boldsymbol{u}^\wedge(\operatorname{tr}(\boldsymbol{W})\boldsymbol{1} - \boldsymbol{W}) - \boldsymbol{W}^T\boldsymbol{u}^\wedge$ 可得:

  $$\begin{aligned}(\boldsymbol{C}\boldsymbol{u})^\wedge &\equiv \boldsymbol{u}^\wedge(\operatorname{tr}(\boldsymbol{C})\boldsymbol{1} - \boldsymbol{C}) - \boldsymbol{C}^T\boldsymbol{u}^\wedge \\ &= \boldsymbol{u}^\wedge\operatorname{tr}(\boldsymbol{C})\boldsymbol{1} - \boldsymbol{u}^\wedge\boldsymbol{C} - \boldsymbol{C}^T\boldsymbol{u}^\wedge \\ &= (2\cos\phi+1)\boldsymbol{u}^\wedge - \boldsymbol{u}^\wedge\boldsymbol{C} - \boldsymbol{C}^T\boldsymbol{u}^\wedge \qquad (\operatorname{tr}(\boldsymbol{C}) = 2\cos\phi + 1,\text{其中 }\phi\text{是旋转角度}\end{aligned}$$

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## 问3：证明 $\exp\left((\boldsymbol{C}\boldsymbol{u})^{\wedge}\right) \equiv \boldsymbol{C}\exp\left(\boldsymbol{u}^{\wedge}\right)\boldsymbol{C}^{T}$

**证：**

- 根据指数映射$\exp\left(\phi^{\wedge}\right)=\sum\limits_{n=0}^{\infty}\frac{1}{n!}\left(\phi^{\wedge}\right)^{n}$ 可得：

  $\exp\left((\boldsymbol{C}\boldsymbol{u})^{\wedge}\right)=\sum\limits_{n=0}^{\infty}\frac{1}{n!}((\boldsymbol{C}\boldsymbol{u})^{\wedge})^{n}, \quad (3-1)$

  $((\boldsymbol{C}\boldsymbol{u})^{\wedge})^{n}=(\boldsymbol{C}\boldsymbol{u}^{\wedge}\boldsymbol{C}^{T})^{n}=\boldsymbol{C}(\boldsymbol{u}^{\wedge})^{n}\boldsymbol{C}^{T}$ (这一步用到了$C^{T}C=\mathbf{1}$)

  带入公式（3-1）可得：$\sum\limits_{n=0}^{\infty}\frac{1}{n!}((\boldsymbol{C}\boldsymbol{u})^{\wedge})^{n}=\boldsymbol{C}(\sum\limits_{n=0}^{\infty}\frac{1}{n!}((\boldsymbol{u})^{\wedge})^{n})\boldsymbol{C}^{T}=\boldsymbol{C}\exp\left(\boldsymbol{u}^{\wedge}\right)\boldsymbol{C}^{T}\Longrightarrow$
  $\exp\left((\boldsymbol{C}\boldsymbol{u})^{\wedge}\right)\equiv\boldsymbol{C}\exp\left(\boldsymbol{u}^{\wedge}\right)\boldsymbol{C}^{T}$

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## 问4：证明 $(\mathcal{T}\boldsymbol{x})^{\wedge} \equiv \boldsymbol{T}\boldsymbol{x}^{\wedge}\boldsymbol{T}^{-1}$

**证：**

令 $\boldsymbol{T}=\begin{bmatrix}\boldsymbol{C} & \boldsymbol{r}\\ \mathbf{0}^{T} & 1\end{bmatrix}$ $\mathcal{T}=\begin{bmatrix}\boldsymbol{C} & \boldsymbol{r}^{\wedge}\boldsymbol{C}\\ \mathbf{0} & \boldsymbol{C}\end{bmatrix}$ $\boldsymbol{x}=\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{bmatrix}$

恒等式左边 $(\mathcal{T}\boldsymbol{x})^{\wedge}=\left(\begin{bmatrix}\boldsymbol{C} & \boldsymbol{r}^{\wedge}\boldsymbol{C}\\ \mathbf{0} & \boldsymbol{C}\end{bmatrix}\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{bmatrix}\right)^{\wedge}=\begin{bmatrix}\boldsymbol{C}\boldsymbol{u}+\boldsymbol{r}^{\wedge}\boldsymbol{C}\boldsymbol{v}\\ \boldsymbol{C}\boldsymbol{v}\end{bmatrix}^{\wedge}=\begin{bmatrix}(\boldsymbol{C}\boldsymbol{v})^{\wedge} & \boldsymbol{C}\boldsymbol{u}+\boldsymbol{r}^{\wedge}\boldsymbol{C}\boldsymbol{v}\\ \mathbf{0}^{T} & 0\end{bmatrix}$

恒等式右边 $\boldsymbol{T}\boldsymbol{x}^{\wedge}\boldsymbol{T}^{-1}=\begin{bmatrix}\boldsymbol{C}\boldsymbol{v}^{\wedge}\boldsymbol{C}^{T} & \boldsymbol{C}\boldsymbol{u}-\boldsymbol{C}\boldsymbol{v}^{\wedge}(\boldsymbol{C}^{T}\boldsymbol{r})\\ \mathbf{0}^{T} & 0\end{bmatrix}$

又$\because (\boldsymbol{C}\boldsymbol{v})^{\wedge}\equiv\boldsymbol{C}\boldsymbol{v}^{\wedge}\boldsymbol{C}^{T}$

$\boldsymbol{r}^{\wedge}\boldsymbol{C}\boldsymbol{v}=\boldsymbol{C}\boldsymbol{C}^{T}\boldsymbol{r}^{\wedge}\boldsymbol{C}\boldsymbol{v}=\boldsymbol{C}(\boldsymbol{C}^{T}\boldsymbol{r})^{\wedge}\boldsymbol{v}=-\boldsymbol{C}\boldsymbol{v}^{\wedge}(\boldsymbol{C}^{T}\boldsymbol{r})$

$\therefore (\mathcal{T}\boldsymbol{x})^{\wedge}\equiv\boldsymbol{T}\boldsymbol{x}^{\wedge}\boldsymbol{T}^{-1}$

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## 问5：证明 $\exp\left((\mathcal{T}\boldsymbol{x})^{\wedge}\right) \equiv \boldsymbol{T}\exp\left(\boldsymbol{x}^{\wedge}\right)\boldsymbol{T}^{-1}$

**证：**

与**问3**类似，先将 $\exp\left((\mathcal{T}\boldsymbol{x})^{\wedge}\right)$ 展开，然后利用性质 $(\mathcal{T}\boldsymbol{x})^{\wedge}\equiv\boldsymbol{T}\boldsymbol{x}^{\wedge}\boldsymbol{T}^{-1}$ 和 $\boldsymbol{T}^{-1}\boldsymbol{T}=\mathbf{1}$ 化简即可得到恒等式右边；

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## 问6：证明 $\boldsymbol{x}^{\wedge}\boldsymbol{p} \equiv \boldsymbol{p}^{\odot}\boldsymbol{x}$

**证：** 令 $\boldsymbol{x}=\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{bmatrix}$ $\boldsymbol{p}=\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}$

$\boldsymbol{x}^{\wedge}\boldsymbol{p}=\begin{bmatrix}\boldsymbol{v}^{\wedge} & \boldsymbol{u}\\ \mathbf{0}^{T} & 0\end{bmatrix}\begin{bmatrix}\varepsilon\\ \eta\end{bmatrix}=\begin{bmatrix}\boldsymbol{v}^{\wedge}\varepsilon+\eta\boldsymbol{u}\\ 0\end{bmatrix}$

$\boldsymbol{p}^{\odot}\boldsymbol{x}=\begin{bmatrix}\eta\mathbf{1} & -\varepsilon^{\wedge}\\ \mathbf{0}^{T} & \mathbf{0}^{T}\end{bmatrix}\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{bmatrix}=\begin{bmatrix}\eta\mathbf{1}\boldsymbol{u}-\varepsilon^{\wedge}\boldsymbol{v}\\ 0\end{bmatrix}$

$\because \boldsymbol{v}^{\wedge}\varepsilon=-\varepsilon^{\wedge}\boldsymbol{v}$

$\therefore \boldsymbol{x}^{\wedge}\boldsymbol{p}\equiv\boldsymbol{p}^{\odot}\boldsymbol{x}$

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## 问7：证明 $\boldsymbol{p}^T\boldsymbol{x}^\wedge \equiv \boldsymbol{x}^T\boldsymbol{p}^{\circledcirc}$

**证：** 令 $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$ $\boldsymbol{p} = \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix}$

$\boldsymbol{p}^T\boldsymbol{x}^\wedge = [\boldsymbol{\varepsilon^T} \quad \eta] \begin{bmatrix} \boldsymbol{v}^\wedge & \boldsymbol{u} \\ \boldsymbol{0}^T & 0 \end{bmatrix} = [\boldsymbol{\varepsilon^T}\boldsymbol{v}^\wedge \quad \boldsymbol{\varepsilon^T}\boldsymbol{u}]$

$\boldsymbol{x}^T\boldsymbol{p}^{\circledcirc} = [\boldsymbol{u^T} \quad \boldsymbol{v^T}] \begin{bmatrix} \boldsymbol{0} & \varepsilon \\ -\varepsilon^\wedge & \boldsymbol{0} \end{bmatrix} = [-\boldsymbol{v^T}\boldsymbol{\varepsilon}^\wedge \quad \boldsymbol{u^T}\boldsymbol{\varepsilon}]$

$\because \boldsymbol{\varepsilon}^T\boldsymbol{u} = \boldsymbol{u}^T\boldsymbol{\varepsilon} = \text{标量}$

$\boldsymbol{\varepsilon^T}\boldsymbol{v}^\wedge = -(\boldsymbol{v}^\wedge\boldsymbol{\varepsilon})^T = -(-\boldsymbol{\varepsilon}^\wedge\boldsymbol{v})^T = -\Big(-\boldsymbol{v}^T(\boldsymbol{\varepsilon}^{\boldsymbol{\wedge}})^T\Big) = -\boldsymbol{v^T}\boldsymbol{\varepsilon}^\wedge$

$\therefore \boldsymbol{p}^T\boldsymbol{x}^\wedge \equiv \boldsymbol{x}^T\boldsymbol{p}^{\circledcirc}$

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## 问8：证明 $(\boldsymbol{T}\boldsymbol{p})^\odot \equiv \boldsymbol{T}\boldsymbol{p}^\odot\mathcal{T}^{-1}$

**证：** 令$\boldsymbol{p} = \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix}$ $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{r} \\ \boldsymbol{0}^T & 1 \end{bmatrix}$ $\mathcal{T}^{-1} = \begin{bmatrix} \boldsymbol{C^T} & -\boldsymbol{C}^T\boldsymbol{r}^\wedge \\ \boldsymbol{0} & \boldsymbol{C}^T \end{bmatrix}$

$(\boldsymbol{T}\boldsymbol{p})^\odot = \begin{bmatrix} \eta\boldsymbol{1} & -(\boldsymbol{C}\varepsilon + \eta\boldsymbol{r})^\wedge \\ \boldsymbol{0}^T & \boldsymbol{0}^T \end{bmatrix}$

$\boldsymbol{T}\boldsymbol{p}^\odot\mathcal{T}^{-1} = \begin{bmatrix} \eta\boldsymbol{1} & -\eta\boldsymbol{r}^\wedge - \boldsymbol{C}\varepsilon^\wedge\boldsymbol{C}^T \\ \boldsymbol{0}^T & \boldsymbol{0}^T \end{bmatrix}$

$\because -\eta\boldsymbol{r}^\wedge - \boldsymbol{C}\varepsilon^\wedge\boldsymbol{C}^T = --\eta\boldsymbol{r}^\wedge - (\boldsymbol{C}\varepsilon)^\wedge = -(\boldsymbol{C}\varepsilon + \eta\boldsymbol{r})^\wedge$

$\therefore (\boldsymbol{T}\boldsymbol{p})^\odot \equiv \boldsymbol{T}\boldsymbol{p}^\odot\mathcal{T}^{-1}$

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## 问9：证明 $(\boldsymbol{T}\boldsymbol{p})^{\odot^T}(\boldsymbol{T}\boldsymbol{p})^\odot \equiv \mathcal{T}^{-T}\boldsymbol{p}^{\odot^T}\boldsymbol{p}^\odot\mathcal{T}^{-1}$

**证：**

根据 **问8** 可以得到：$(\boldsymbol{T}\boldsymbol{p})^{\odot^T}(\boldsymbol{T}\boldsymbol{p})^\odot = \mathcal{T}^{-T}\boldsymbol{p}^{\odot^T}\boldsymbol{T^T}\boldsymbol{T}\boldsymbol{p}^\odot\mathcal{T}^{-1}$

$\because \boldsymbol{p}^{\odot^T}\boldsymbol{T^T}\boldsymbol{T}\boldsymbol{p}^\odot = \boldsymbol{p}^{\odot^T}\boldsymbol{p}^\odot$ (带入矩阵计算即可得到该等式)

**换个角度理解：** $\boldsymbol{p}^{\odot^T}\boldsymbol{T^T}\boldsymbol{T}\boldsymbol{p}^\odot = (\boldsymbol{T}\boldsymbol{p}^\odot)^T(\boldsymbol{T}\boldsymbol{p}^\odot)$ 因为 $\boldsymbol{T}$ 是旋转平移变换，所以 $\boldsymbol{T}\boldsymbol{p}^\odot$ 没有改变 $\boldsymbol{p}^\odot$ 本身的性质，因此变换后的 $(\boldsymbol{T}\boldsymbol{p}^\odot)^T(\boldsymbol{T}\boldsymbol{p}^\odot)$ 等于变换前的 $\boldsymbol{p}^{\odot^T}\boldsymbol{p}^\odot$ 。