0. 说明

本 PDF 文档为自动生成，如有遗漏的格式错误但不影响阅读请见谅，若影响了阅读请告知！

问1：EKF 推导

- 预测

  $p(\boldsymbol{x_k}|\boldsymbol{\check{x}_0}, \boldsymbol{v_{1:k}}, \boldsymbol{y_{0:k-1}}) = \mathcal{N}(\boldsymbol{\check{x}_k}, \boldsymbol{\check{P}_k})$

  $\because \boldsymbol{f(x_{k-1}, v_k, w_k)} \approx \boldsymbol{\check{x}_k} + \boldsymbol{F_{k-1}(x_{k-1} - \hat{x}_{k-1})} + \boldsymbol{w'_k}$

  $\therefore \boldsymbol{\check{x}_k} = \boldsymbol{f(\hat{x}_{k-1}, v_k, 0)}$

  $\boldsymbol{\check{P}_k} = E[(\boldsymbol{x_k} - E[\boldsymbol{x_k}])(\boldsymbol{x_k} - E[\boldsymbol{x_k}])^T] = E[(\boldsymbol{x_k} - \boldsymbol{\check{x}_k})(\boldsymbol{x_k} - \boldsymbol{\check{x}_k})^T]$

  $\quad = \boldsymbol{F_{k-1}}E[(\boldsymbol{x_{k-1}} - \boldsymbol{\check{x}_{k-1}})(\boldsymbol{x_{k-1}} - \boldsymbol{\check{x}_{k-1}})^T]\boldsymbol{F_{k-1}^T} + E[\boldsymbol{w'_k w'^T_k}] = \boldsymbol{F_{k-1}\check{P}_{k-1}F_{k-1}^T} + \boldsymbol{Q'_k}$

- 卡尔曼增益

  $p(\boldsymbol{x_k}|\boldsymbol{\check{x}_0}, \boldsymbol{v_{1:k}}, \boldsymbol{y_{0:k}}) = \eta p(\boldsymbol{y_k}|\boldsymbol{x_k}) \int p(\boldsymbol{x_k}|\boldsymbol{x_{k-1}}, \boldsymbol{v_k})p(\boldsymbol{x_{k-1}}|\boldsymbol{\check{x}_0}, \boldsymbol{v_{1:k-1}}, \boldsymbol{y_{0:k-1}})d\boldsymbol{x_{k-1}} \quad (1-1)$

  其中，$p(\boldsymbol{y_k}|\boldsymbol{x_k}) \approx \mathcal{N}(\boldsymbol{\check{y}_k} + \boldsymbol{G_k(x_k - \check{x}_k)}, \boldsymbol{R'_k})$

  $\quad p(\boldsymbol{x_k}|\boldsymbol{x_{k-1}}, \boldsymbol{v_k}) \approx \mathcal{N}(\boldsymbol{\check{x}_k} + \boldsymbol{F_{k-1}(x_{k-1} - \hat{x}_{k-1})}, \boldsymbol{Q'_k})$

  $\quad p(\boldsymbol{x_{k-1}}|\boldsymbol{\check{x}_0}, \boldsymbol{v_{1:k-1}}, \boldsymbol{y_{0:k-1}}) \backsim \mathcal{N}(\boldsymbol{\hat{x}_{k-1}}, \boldsymbol{\hat{P}_{k-1}})$

  $\because \int p(\boldsymbol{x_k}|\boldsymbol{x_{k-1}}, \boldsymbol{v_k})p(\boldsymbol{x_{k-1}}|\boldsymbol{\check{x}_0}, \boldsymbol{v_{1:k-1}}, \boldsymbol{y_{0:k-1}})d\boldsymbol{x_{k-1}} = p(\boldsymbol{x_k}|\boldsymbol{\check{x}_0}, \boldsymbol{v_{1:k}}, \boldsymbol{y_{0:k-1}}) = \mathcal{N}(\boldsymbol{\check{x}_k}, \boldsymbol{\check{P}_k})$

  $\therefore \mathcal{N}(\boldsymbol{\hat{x}_k}, \boldsymbol{\hat{P}_k}) = \eta\mathcal{N}(\boldsymbol{\check{x}_k}, \boldsymbol{\check{P}_k}) \times \mathcal{N}(\boldsymbol{\check{y}_k} + \boldsymbol{G_k(x_k - \check{x}_k)}, \boldsymbol{R'_k}) \quad (1-2)$

  等式（1-2）只计算指数部分：

  $\boldsymbol{(x_k - \check{x}_k)^T \check{P}_k^{-1}(x_k - \check{x}_k) + [y_k - \check{y}_k - G_k(x_k - \check{x}_k)]^T R'^{-1}_k [y_k - \check{y}_k - G_k(x_k - \check{x}_k)]}$

  $= \boldsymbol{(x_k - \check{x}_k)^T G_k^T R'^{-1}_k G_k (x_k - \check{x}_k) - 2(x_k - \check{x}_k)^T G_k^T R'^{-1}_k (y_k - \check{y}_k) + (x_k - \check{x}_k)^T \check{P}_k^{-1}(x_k - \check{x}_k)}$

  根据归一化积公式可得：

  $\boldsymbol{\hat{P}_k^{-1} = G_k^T R'^{-1}_k G_k + \check{P}_k^{-1}} \Longrightarrow \boldsymbol{\hat{P}_k = (G_k^T R'^{-1}_k G_k + \check{P}_k^{-1})^{-1}}$

  根据SMW公式 $\boldsymbol{(A^{-1} + BD^{-1}C)^{-1} = A - AB(D + CAB)^{-1}CA}$ 可得：

  $\boldsymbol{\hat{P}_k = (G_k^T R'^{-1}_k G_k + \check{P}_k^{-1})^{-1} = \check{P}_k - \check{P}_k G_k^T (R'_k + G_k \check{P}_k G_k^T)^{-1} G_k \check{P}_k} \quad (1-3)$

  卡尔曼增益 $\boldsymbol{K_k = \check{P}_k G_k^T (R'_k + G_k \check{P}_k G_k^T)^{-1}}$

- 更新

  将卡尔曼增益带入式（1-3）可得：

  $\boldsymbol{\hat{P}_k = (1 - K_k G_k)\check{P}_k}$

  $\boldsymbol{\hat{P}_k^{-1}\hat{x}_k = G_k^T R'^{-1}_k G_k \check{x}_k + G_k^T R'^{-1}_k (y_k - \check{y}_k) + \check{P}_k^{-1}\check{x}_k} \Longrightarrow$

  $\boldsymbol{\hat{x}_k = \hat{P}_k(G_k^T R'^{-1}_k G_k \check{x}_k + G_k^T R'^{-1}_k (y_k - \check{y}_k) + \check{P}_k^{-1}\check{x}_k)}$

  $\boldsymbol{\hat{x}_k = \check{x}_k + (G_k^T R'^{-1}_k G_k + \check{P}_k^{-1})^{-1} G_k^T R'^{-1}_k (y_k - \check{y}_k)}$

  根据 SMW公式 $\boldsymbol{(D + CAB)^{-1}CA = D^{-1}C(A^{-1} + BD^{-1}C)^{-1}}$ 可得：

$\hat{\boldsymbol{x}}_{\boldsymbol{k}} = \check{\boldsymbol{x}}_{\boldsymbol{k}} + \boldsymbol{K}_{\boldsymbol{k}}(\boldsymbol{y}_{\boldsymbol{k}} - \boldsymbol{g}(\check{\boldsymbol{x}}_{\boldsymbol{k}}, \boldsymbol{0}))$

问2：高维（L维）高斯分布的 sigma point (共 2L+1 个)：

$\boldsymbol{L}\boldsymbol{L}^{\boldsymbol{T}} = \boldsymbol{\Sigma}_{xx}$，（**Cholesky** 分解，$\boldsymbol{L}$ 为下三角矩阵）

$x_0 = \mu_x$

$x_i = \mu_x + \sqrt{\boldsymbol{L} + k} col_i \boldsymbol{L}$　　　　　其中 $i = 1, 2, \cdots, L$

$x_{i+L} = \mu_x - \sqrt{\boldsymbol{L} + k} col_i \boldsymbol{L}$　　　　　col_i 表示取第 i 列

请证明：

$$\mu_x = \sum_{i=0}^{2L} \alpha_i x_i$$

$$\boldsymbol{\Sigma}_{xx} = \sum_{i=0}^{2L} \alpha_i (x_i - \mu_x)(x_i - \mu_x)^T \qquad \text{其中 } \alpha_i = \begin{cases} \frac{k}{L+k} & i = 0 \\ \frac{1}{2}\frac{1}{L+k} & \text{其他} \end{cases}$$

证明：

- $\mu_x = \sum_{i=0}^{2L} \alpha_i x_i$

  $$\sum_{i=0}^{2L} \alpha_i x_i = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_{2L} x_{2L} = \frac{k}{L+k}\mu_x + \frac{1}{2(L+k)}(x_1 + x_2 + \cdots + x_{2L})$$
  $$= \frac{k}{L+k}\mu_x + \frac{1}{2(L+k)}(2L\mu_x) = \mu_x$$

- $\boldsymbol{\Sigma}_{xx} = \sum_{i=0}^{2L} \alpha_i (x_i - \mu_x)(x_i - \mu_x)^T$

  $\alpha_0 (x_0 - \mu_x)(x_0 - \mu_x)^T = 0$

  $(x_1 - \mu_x)(x_1 - \mu_x)^T + (x_{L+1} - \mu_x)(x_{L+1} - \mu_x)^T = 2(L+k) col_1 \boldsymbol{L} (col_1 \boldsymbol{L})^T$

  $\cdots$

  $(x_L - \mu_x)(x_L - \mu_x)^T + (x_{L+L} - \mu_x)(x_{L+L} - \mu_x)^T = 2(L+k) col_L \boldsymbol{L} (col_L \boldsymbol{L})^T$

  综上：$\sum_{i=0}^{2L} (x_i - \mu_x)(x_i - \mu_x)^T = 2(L+k)\boldsymbol{L}\boldsymbol{L}^{\boldsymbol{T}}$

  又 $\because \alpha_i = \frac{1}{2(L+k)} \quad , i \neq 0$

  $\therefore \sum_{i=0}^{2L} \alpha_i (x_i - \mu_x)(x_i - \mu_x)^T = \boldsymbol{\Sigma}_{xx}$

问3：考虑如下离散时间系统

$$\begin{bmatrix} x_k \\ y_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} x_{k-1} \\ y_{k-1} \\ \theta_{k-1} \end{bmatrix} + T \begin{bmatrix} \cos\theta_{k-1} & 0 \\ \sin\theta_{k-1} & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} v_k \\ \omega_k \end{bmatrix} + \mathbf{w_k} \right), \quad \mathbf{w_k} \backsim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q})$$

$$\begin{bmatrix} r_k \\ \phi_k \end{bmatrix} = \begin{bmatrix} \sqrt{x_k^2 + y_k^2} \\ \operatorname{atan2}(-y_k, x_k) - \theta_k \end{bmatrix} + \boldsymbol{n_k}, \quad \boldsymbol{n_k} \backsim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R})$$

该系统可以看作是移动机器人在 xy 平面上的移动，测量值为移动机器人距离原点的距离和方位。请建立**EKF**方程来估计移动机器人的姿态，并写出雅可比 $\boldsymbol{F_{k-1}}, \boldsymbol{G_k}$ 和协方差 $\boldsymbol{Q'_k}, \boldsymbol{R'_k}$ 的表达式。

解：

- 写出表达式

令 $\mathbf{w_k}=\begin{bmatrix}\mathrm{w_{kxy}}\\ \mathrm{w_{k\theta}}\end{bmatrix}$，$\boldsymbol{\Omega_k}=\begin{bmatrix}x_k\\ y_k\\ \theta_k\end{bmatrix}$，$\boldsymbol{o_k}=\begin{bmatrix}v_k\\ \omega_k\end{bmatrix}$，$\boldsymbol{n_k}=\begin{bmatrix}\mathrm{n_{kr}}\\ \mathrm{n_{k\phi}}\end{bmatrix}$，根据题目可知：

$$\begin{bmatrix}x_k\\ y_k\\ \theta_k\end{bmatrix}=\begin{bmatrix}x_{k-1}+T\cos\theta_{k-1}(v_k+\mathrm{w_{kxy}})\\ y_{k-1}+T\sin\theta_{k-1}(v_k+\mathrm{w_{kxy}})\\ \theta_{k-1}+T(\omega_k+\mathrm{w_{k\theta}})\end{bmatrix}=\boldsymbol{f(\Omega_k,o_k,\mathbf{w_k})}$$

$$\begin{bmatrix}r_k\\ \phi_k\end{bmatrix}=\begin{bmatrix}\sqrt{x_k^2+y_k^2}+n_{kr}\\ \operatorname{atan2}(-y_k,x_k)-\theta_k+n_{k\phi}\end{bmatrix}=\boldsymbol{g(\Omega_k,n_k)}$$

$$\boldsymbol{F_{k-1}}=\frac{\partial\boldsymbol{f(\Omega_k,o_k,\mathbf{w_k})}}{\partial\boldsymbol{\Omega_{k-1}}}=\begin{bmatrix}1&0&-T\sin\theta_{k-1}(v_k+\mathrm{w_{kxy}})\\ 0&1&T\cos\theta_{k-1}(v_k+\mathrm{w_{kxy}})\\ 0&0&1\end{bmatrix}$$

$$\mathbf{w_k'}=\frac{\partial\boldsymbol{f(\Omega_k,o_k,\mathbf{w_k})}}{\partial\mathbf{w_k}}\mathbf{w_k}=\begin{bmatrix}T\cos\theta_{k-1}\mathrm{w_{kxy}}\\ T\sin\theta_{k-1}\mathrm{w_{kxy}}\\ T\mathrm{w_{k\theta}}\end{bmatrix}$$

$$\boldsymbol{Q_k'}=E[\mathbf{w_k'}\mathbf{w_k'}^T]=\begin{bmatrix}T\cos\theta_{k-1}&0\\ T\sin\theta_{k-1}&0\\ 0&T\end{bmatrix}E[\mathbf{w_k}\mathbf{w_k}^T]\begin{bmatrix}T\cos\theta_{k-1}&0\\ T\sin\theta_{k-1}&0\\ 0&T\end{bmatrix}=\begin{bmatrix}T\cos\theta_{k-1}&0\\ T\sin\theta_{k-1}&0\\ 0&T\end{bmatrix}\boldsymbol{Q}\begin{bmatrix}T\cos\theta_{k-1}&0\\ T\sin\theta_{k-1}&0\\ 0&T\end{bmatrix}$$

$$\boldsymbol{G_k}=\frac{\partial\boldsymbol{g(\Omega_k,n_k)}}{\partial\boldsymbol{\Omega_k}}=\begin{bmatrix}\frac{x_k}{\sqrt{x_k^2+y_k^2}}&\frac{y_k}{\sqrt{x_k^2+y_k^2}}&0\\ \frac{-y_k}{x_k^2+y_k^2}&\frac{x_k}{x_k^2+y_k^2}&-1\end{bmatrix}$$

$$\boldsymbol{n_k'}=\frac{\partial\boldsymbol{g(\Omega_k,n_k)}}{\partial\boldsymbol{n_k}}\boldsymbol{n_k}=\boldsymbol{n_k}$$

$$\boldsymbol{R_k'}=E[\boldsymbol{n_k'}\boldsymbol{n_k'}^T]=E[\boldsymbol{n_k}\boldsymbol{n_k}^T]=\boldsymbol{R_k}$$

- 建立 EKF 方程来估计移动机器人的姿态

  注意这里是求移动机器人的**姿态**，即θ_k

  将上面求得的表达式带入EKF 方程中，然后将表达式中的姿态 θ 项拿出来即可。