Effects of Collateral on Pricing, Multi-Currency Optionality

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1. Recent developments on the option-adjusted yield curve.
2. Single Currency, Multiple Currency
3. Concept of Convexity due to stochastic spreads correlation to the underlying
   - How does collateral affects Pricing, Options volatility, and Stocks terminal distribution
4. Asymptotic limits of the Option Adjusted Yield curve
   - Counterparties now-days have the optionality to change collateral currency. How will the yield curve be constructed?

Effect of Collateral Posting on Pricing Derivatives

Historically the derivatives pricing theory is based on the fact that the borrowing and lending can be done at the same risk-free rate.
However, this paradigm has been broken down in recent years due to the fact that the rates available on cash transactions depend on the exact nature of the transaction and on what type of collateral can be pledged.

Therefore, work has been done (see Piterbarg[1], Andersen [2], etc.), on adjusting the derivative prices to reflect the reality of the difference in the rates of borrowing and lending money and other assets.

One must start the analysis here by the realization that, in addition to providing protection against default, the collateral carries an interest rate $r_Q(t)$.
The counter-party that posts collateral receives interest for it, $r_Q(t)$, and this is considered risk-free interest. In case counterparty defaults there will be no losses for us as we hold his collateral.

Effect of Collateral Posting on Pricing Derivatives

What other rates are available in the market?

1. From the Repo market, Assets can be posted as collateral against funds borrowed. For instance, a stock can be exchanged against funds in return for an interest rate $r_R(t)$ at the repurchase of the asset.

2. The unsecured funding for the bank. This is done at a very high rate $r_F(t)$ compared with the collateral and the repo rate. Namely,

$$r_Q(t) < r_R(t) < r_F(t) \quad , \tag{1}$$

as $r_F(t)$ considers the default probability of the bank.

Effect of Collateral Posting on Pricing Derivatives

Assume the asset follows:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu_S(t)\mathrm{d}t + \sigma_S(t)\mathrm{d}W_S(t) \quad . \tag{2}$$

and consider a derivative on this of value $V(t, S(t))$.
The derivative can be replicated by holding $\Delta(t)$ amount of the asset and $\gamma(t)$ of cash.

$$V(t) = \Pi(t) = \Delta(t)S(t) + \gamma(t) \tag{3}$$

The cash amount is split among a number of accounts as follows:

- Amount $Q(t)$ is in collateral account.
- Amount $V(t) - Q(t)$ is lent/borrowed at the unsecured rate from the treasury
- Amount $\Delta(t)S(t)$ is borrowed in Repo market to finance the purchase of $\Delta(t)$ of stocks, i.e. overall cash amount is $Q(t) + \big(V(t) - Q(t)\big)$, and there is a need to pay repo rate interest on the amount $-\Delta(t)S(t)$ borrowed in the repo market.

Effect of Collaterals on Pricing Derivatives

The overall change in value of cash is on the interest paid/received on the collateral and the unsecured funding and the rest paid in the repo market. The growth of all cash accounts (collateral, unsecured, stock secured) is given by:

$$\mathrm{d}\gamma(t) = \Big[r_Q(t)Q(t) + r_F(t)\big(V(t) - Q(t)\big) - r_R(t)\Delta(t)S(t)\Big]\mathrm{d}t \qquad (4)$$

The self-financing condition gives:

$$\mathrm{d}\gamma(t) = \mathrm{d}V(t) - \Delta(t)\mathrm{d}S(t) \quad , \qquad (5)$$

The solution to this equation will be shown in next slides to be:

$$V(t) = E_t\Big(e^{-\int_t^T r_F(u)\mathrm{d}u}V(T) + \int_t^T e^{-\int_t^u r_F(v)\mathrm{d}v}\big(r_F(u) - r_Q(u)\big)Q(u)\mathrm{d}u\Big) \quad (6)$$

If the collateral is equal to the value of $V(t)$ then the growth rate is $r_Q(t)$

Effect of Collateral on Pricing Derivatives

From which we conclude that

$$E_t\big(\mathrm{d}V(t)\big) = \Big(r_F(t)V(t) - (r_F(t) - r_Q(t))Q(t)\Big)\mathrm{d}t \tag{7}$$

If the collateral is equal to the value of $V(t)$ then the growth rate is $r_Q(t)$

$$E_t\big(\mathrm{d}V(t)\big) = r_Q(t)V(t)\mathrm{d}t \quad , \quad V(t) = E_t\Big(e^{-\int_t^T r_Q(u)\mathrm{d}u}V(T)\Big) \quad . \tag{8}$$

If the collateral is zero, then the growth rate is the banks unsecured funding rate

$$E_t\big(\mathrm{d}V(t)\big) = r_F(t)V(t)\mathrm{d}t \quad , \quad V(t) = E_t\Big(e^{-\int_t^T r_F(u)\mathrm{d}u}V(T)\Big) \quad . \tag{9}$$

Apparently the measure for the calculations in the above remains the same in the case of full collateral and zero collateral.

But what is this measure, how to find it?

Effect of Collateral on Pricing - RN Measure

$$\mathrm{d}V(t) = \Delta(t)\mathrm{d}S(t) + r_Q(t)Q(t)\mathrm{d}t + r_F(t)(V - Q)\mathrm{d}t - r_R(t)\Delta(t)S(t)\mathrm{d}t \quad (10)$$

This can be re-written as

$$\mathrm{d}V(t) = r_F(t)V(t)\mathrm{d}t + (r_Q - r_F)Q(t)\mathrm{d}t - r_R(t)\Delta(t)S(t)\mathrm{d}t + \Delta(t)S(t) \quad (11)$$

If we assume

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu_S(t)\mathrm{d}t + \sigma_S(t)\mathrm{d}W(t) \quad , \quad (12)$$

then the change in portfolio value will be

$$\begin{aligned} \mathrm{d}V(t) &= r_F(t)V(t)\mathrm{d}t + (r_Q - r_F)Q(t)\mathrm{d}t \quad (13) \\ &- \Delta(t)(\mu_S - r_R)S(t)\mathrm{d}t + \Delta(t)\sigma_S(t)S(t)\mathrm{d}W(t) \end{aligned}$$

To solve this equation we need to go into a measure in which the stock drift will be r_R, so that $S(t)$ is canceled for the drift part of (13).

Effect of Collateral on Pricing - RN Measure

This effect is reached in the "risk-neutral" measure which is indeed defined so it removes the drift of the asset from (13)

$$\Theta(t) = \frac{\mu_S(t) - r_R(t)}{\sigma_S(t)} \tag{14}$$

We can now rewrite the equation for the portfolio value change

$$\begin{aligned} \mathrm{d}V(t) &= r_F(t)V(t)\mathrm{d}t + (r_Q - r_F)Q(t)\mathrm{d}t + \Delta(t)S(t)[\Theta(t)\mathrm{d}t + \mathrm{d}W(t)] \\ &= r_F(t)V(t)\mathrm{d}t + (r_Q - r_F)Q(t)\mathrm{d}t + \Delta(t)S(t)\mathrm{d}\tilde{W}(t) \end{aligned} \tag{15}$$

In this measure the discounted stock price is a martingale

$$\begin{aligned} \mathrm{d}(D_R(t)S(t)) &= (\mu_S - r_R)D_R(t)S(t)\mathrm{d}t + \sigma_S(t)D_R(t)S(t)\mathrm{d}W(t) && (16) \\ &= \sigma_S(t)D_R(t)S(t)[\Theta(t)\mathrm{d}t + \mathrm{d}W(t)] && (17) \\ &= \sigma_S(t)D_R(t)S(t)\mathrm{d}\tilde{W}(t) && (18) \end{aligned}$$

Effect of Collateral on Pricing - RN Measure

The solution for the partly collateralized and partly unsecured derivative is:

$$V(t) = E_t\left[e^{-\int_t^T r_F(u)\mathrm{d}u}V(T) + \int_t^T e^{-\int_t^u r_F(v)\mathrm{d}v}(r_F(u) - r_Q(u))Q(u)\mathrm{d}u\right] \quad (19)$$

If collateral is posted equal in value to the PV of the trade, $Q(t) = V(t)$ then from above or from Eq. (15)

$$V(t) = E_t\left[e^{-\int_t^T r_Q(u)\mathrm{d}u}V(T)\right] \quad (20)$$

If the trade is uncollaterized, $Q(t) = 0$, then from above or from Eq. (15)

$$V(t) = E_t\left[e^{-\int_t^T r_F(u)\mathrm{d}u}V(T)\right] \quad (21)$$

Effect of Collateral, Pricing Implications
Convexity Adjustment

Let us look into the effect of collateral on a simple FRA on a single currency setting. a FRA is defined so its value is zero at inception

$$0 = E_t\left(e^{-\int_t^T r_i(u)\mathrm{d}u}(S(T) - F_i(t,T)\right) \quad , \quad \text{where; } i = Q, F \quad , \tag{22}$$

Q and F stand for collateralized and uncollaterallized trade.

$$F_{\text{noCSA}} = \frac{E_t\left[e^{-\int_t^T r_F(u)\mathrm{d}u} S(T)\right]}{E_t\left[e^{-\int_t^T r_F(u)\mathrm{d}u}\right]} = \tilde{E}^T[S(T)] \quad , \tag{23}$$

$$F_{\text{CSA}} = \frac{E_t\left[e^{-\int_t^T r_Q(u)\mathrm{d}u} S(T)\right]}{E_t\left[e^{-\int_t^T r_Q(u)\mathrm{d}u}\right]} = E^T[S(T)] \quad . \tag{24}$$

What is the difference between these two?

Effect of Collateral - $\tilde{E}^T \to E^T$ measure change

Looking into F_{noCSA} we have

$$F_{\mathrm{noCSA}} = \tilde{E}^T[S(T)] = \frac{E_t\left[e^{-\int_t^T r_F(u)\mathrm{d}u} S(T)\right]}{P_F(t,T)} \quad , \tag{25}$$

$$= \frac{E_t\left[e^{-\int_t^T r_Q(u)\mathrm{d}u} e^{-\int_t^T (r_F - r_Q)(u)\mathrm{d}u} S(T)\right]}{P_F(t,T)} \quad , \tag{26}$$

$$= \frac{P_Q(t,T) E_t^T\left[e^{-\int_t^T s(u)\mathrm{d}u} S(T)\right]}{P_F(t,T)} \quad . \tag{27}$$

Therefore

$$\tilde{E}^T[S(T)] = E_t^T\left[\frac{P_Q(t,T) e^{-\int_t^T s(u)\mathrm{d}u}}{P_F(t,T)} \; S(T)\right] \quad . \tag{28}$$

Effect of Collateral, Convexity Adjustment

Which suggests that the Radom-Nikodym derivative is

$$RN = \frac{P_Q(t,T)}{P_F(t,T)} e^{-\int_t^T s(u)\mathrm{d}u} \quad . \tag{29}$$

This can be written as function of martingale ($RN = M(T,T)/M(t,T)$)

$$\begin{aligned}
M(t,T) &= E_t^T\left[e^{-\int_0^T s(u)\mathrm{d}u}\right] = e^{-\int_0^t s(u)\mathrm{d}u} E_t^T\left[e^{-\int_t^T (r_F(u)-r_Q(u))\mathrm{d}u}\right] \quad , \\
&= e^{-\int_0^t s(u)\mathrm{d}u} E_t\left[\frac{e^{-\int_t^T r_Q(u)\mathrm{d}u}}{P_Q(t,T)} e^{-\int_t^T (r_F(u)-r_Q(u))\mathrm{d}u}\right] \quad , \qquad (30) \\
M(t,T) &= \frac{e^{-\int_0^t s(u)\mathrm{d}u}}{P_Q(t,T)} E_t\left[e^{-\int_t^T r_F(u)\mathrm{d}u}\right] = \frac{P_F(t,T)}{P_Q(t,T)} e^{-\int_0^t s(u)\mathrm{d}u} \quad , \\
M(T,T) &= \frac{P_Q(T,T)}{P_F(T,T)} e^{-\int_0^T s(u)\mathrm{d}u} \quad , \quad \frac{M(T,T)}{M(t,T)} = \frac{P_Q(t,T)}{P_F(t,T)} e^{-\int_t^T s(u)\mathrm{d}u} \quad .
\end{aligned}$$

Effect of Collateral, Convexity Adjustment

$$F_{\mathrm{noCSA}} - F_{\mathrm{CSA}} = \frac{E_t\left[e^{-\int_t^T r_F(u)\mathrm{d}u} S(T)\right]}{E_t\left[e^{-\int_t^T r_F(u)\mathrm{d}u}\right]} - \frac{E_t\left[e^{-\int_t^T r_Q(u)\mathrm{d}u} S(T)\right]}{E_t\left[e^{-\int_t^T r_Q(u)\mathrm{d}u}\right]} \tag{31}$$

$$= \frac{E_t\left[e^{-\int_t^T r_Q(u)\mathrm{d}u} e^{-\int_t^T (r_F(u)-r_Q(u))\mathrm{d}u} S(T)\right]}{P_F(t,T)} - E^{Q_T}\left[S(T)\right]$$

$$= \frac{P_Q(t,T) E_t^{Q_T}\left[e^{-\int_t^T s(u)\mathrm{d}u} S(T)\right]}{P_F(t,T)} - E^{Q_T}\left[S(T)\right] \tag{32}$$

where $s(u) = r_Q(u) - r_F(u)$. From previous slide

$$M(t,T) = \frac{P_F(t,T)}{P_Q(t,T)} e^{-\int_0^t s(u)\mathrm{d}u} \quad , \quad E_t^{Q_T}\left[\frac{M(T,T)}{M(t,T)}\right] = 1 \quad . \tag{33}$$

Effect of Collateral, Convexity Adjustment

Then the difference in price between the collaterized and uncollaterized FRA will be:

$$F_{\mathrm{noCSA}} - F_{\mathrm{CSA}} = E_t^{Q_T}\left[\frac{M(T,T)}{M(t,T)}S(T) - S(T)\right] \quad , \tag{34}$$

$$= E_t^{Q_T}\left[\left(\frac{M(T,T)}{M(t,T)} - E_t^{Q_T}\left(\frac{M(T,T)}{M(t,T)}\right)\right)\left(S(T) - E_t^{Q_T}(S(T))\right)\right]$$

$$= \frac{1}{M(t,T)}\mathrm{Cov}_t^T[M(T,T), S(T)] \quad . \tag{35}$$

$$F_{\mathrm{noCSA}} - F_{\mathrm{CSA}} = \mathrm{Cov}_t^T[RN(t,T), S(T)] \quad . \tag{36}$$

Effect of Collateral, Convexity Adjustment

Final point to discuss is the estimation of the convexity adjustment that arises when the spreads are stochastic, by taking for the spreads a simple Vasicek model. We assume here that the risk-free (the collateral discount curve) is deterministic and let the other curve be stochastic, as well as its spread against this curve. As a simple model for the underlying we take lognormal for the underlying and Vasicek for the spreads.
We assume a Vasicek model for the unsecured rate r_F, which translates to Vasicek for unsecured Bond, a Vasicek for the spread, and their sum results in the collateral rate. We also assume a lognormal for the Stock, and to simplify, deterministic for the collateral rate

$$\frac{\mathrm{d}S(t)}{S(t)} = r_R(t)\mathrm{d}t + \sigma_S G(t,T)\mathrm{d}W_S(t) \quad , \tag{37}$$

$$\mathrm{d}s = a(\theta - s(t))\mathrm{d}t + \sigma_F \mathrm{d}W_F(t) \quad , \tag{38}$$

with correlations $\rho_{SF} = \mathrm{d}W_S(t)\mathrm{d}W_F(t)/\mathrm{d}t$ (same vol and Brownian for the spread to cancel the one from r_F to give deterministic for r_c).

Effect of Collateral, Convexity Adjustment

We assume that the collateral rate is deterministic and obtained from the funding and the spread curves which are taken to be stochastic

$$\frac{\mathrm{d}P_F(t,T)}{P_F(t,T)} = r_F(t)\mathrm{d}t - \sigma_F(t)G(t,T)\mathrm{d}W_F(t) \tag{39}$$

$$F_{CSA}(t,T) = E_t^{Q_T}[S(T)] = E_t[S(T)] \quad , \tag{40}$$

meaning that the forward is martingale in the E_t measure

$$\frac{\mathrm{d}F_{CSA}}{F_{CSA}} = \sigma_S \mathrm{d}W_S(t) \quad . \tag{41}$$

The RN factor is martingale in Q_T-forward measure and from Eq. (33)

$$\mathrm{d}M(t,T)/M(t,T) = -\sigma_F G(t,T)\mathrm{d}W_F(t) \quad . \tag{42}$$

Effect of Collateral, Convexity Adjustment

which results in the following dynamics for the product $M(t,T)F_{CSA}(t,T)$

$$\frac{\mathrm{d}\big(M(t,T)F_{CSA}(t,T)\big)}{\big(M(t,T)F_{CSA}(t,T)\big)} = -\sigma_S\sigma_F G(t,T)\rho_{SF}\mathrm{d}t + \sigma\mathrm{d}W(t) \quad , \qquad (43)$$

resulting in (above $\sigma^2 = \sigma_S^2 + \sigma_F^2 + 2\rho\sigma_S\sigma_F$)

$$\begin{aligned} M(T,T)F_{CSA}(T,T) &= M(0,T)F_{CSA}(0,T) \\ &\quad \exp\left\{-\int_0^T \rho\sigma_S\sigma_F G(t,T)\mathrm{d}t + \sigma W(T) - \frac{1}{2}\sigma^2 T\right\} \end{aligned} \qquad (44)$$

Effect of Collateral, Convexity Adjustment

Summing it up

$$F_{noCSA}(0,T) - F_{CSA}(0,T) = \mathrm{Cov}\left(\frac{M(T,T)}{M(0,T)}(F_{CSA}(T,T) - F_{CSA}(0,T))\right) \quad ,$$

$$= F_{CSA}(0,T)\exp\left\{-\int_0^T \rho\sigma_S\sigma_F G(t,T)\mathrm{d}t\right\} \quad .$$

For the convexity correction of the simple FRA due to collateralization

$$F_{\mathrm{noCSA}}(0,T) - F_{\mathrm{CSA}}(0,T) = F_{\mathrm{CSA}}(0,T)\left(e^{-\sigma_S\sigma_F\rho\left(\frac{T-G(0,T)}{\kappa}\right)} - 1\right) \quad (45)$$

Taking $\kappa \to 0$

$$F_{\mathrm{noCSA}}(0,T) - F_{\mathrm{CSA}}(0,T) = F_{\mathrm{CSA}}(0,T)\left(e^{-\sigma_S\sigma_F\rho\frac{T^2}{2}} - 1\right) \quad (46)$$

Effect of Collateral, Convexity Adjustment

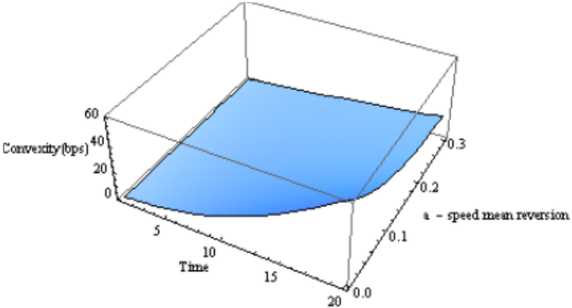

Figure: Convexity adjustment in forward contracts due to correlation of the stochastic spread between the unsecured and OIS funding and the underlying, is shown. For the conservative parameters that we have taken here, the convexity correction can reach up to tens of basis points.

Effect of Collateral, Libor Convexity Adjustment

For the case of equity forward in the risky and non-risky trades gives a difference of

$$F_{\mathrm{noCSA}} - F_{\mathrm{CSA}} = \mathrm{Cov}_t^T[RN(t,T), S(T)] \quad . \tag{47}$$

This can be translated to interest rates. In the IR case, the Libor short rate closely resembles the funding rate r_F and the collateral rate is similar to the OIS rate which is obtained by the funding rate minus a spread $r_C(t) = r_F(t) - s(t)$.

If we observe closely the expressions for the ratio of two consecutive bonds which enter into ratio for the Libor rate

$$\begin{aligned}\frac{B(t,T_{i-1})}{B(t,T_i)} &= \frac{B(0,T_{i-1})}{B(0,T_i)} \exp\left\{-\sigma \int_0^t \Big(G(u,T_{i-1}) - G(u,T_i)\Big)\mathrm{d}W^{T_i}(u)\right. \\ &\quad \left. - \frac{1}{2}\sigma^2 \int_0^t \Big(G(u,T_{i-1}) - G(u,T_i)\Big)^2 \mathrm{d}u\right\} \quad ,\end{aligned} \tag{48}$$

which is a martingale process under the **risky** T_i-forward measure.

Effect of Collateral, Libor Convexity Adjustment

The Libor

$$L(t;T_{i-1},T_i) \;=\; \frac{1}{\tau_i}\left(\frac{B(t,T_{i-1})}{B(t,T_i)}-1\right)=\frac{1}{\tau_i}\left(\frac{B(0,T_{i-1})}{B(0,T_i)}\exp\{\cdots\}-1\right),$$

$$=\;\left[\frac{1}{\tau_i}\left(\frac{B(0,T_{i-1})}{B(0,T_i)}-1\right)+\frac{1}{\tau_i}\right]\exp\big\{Eq.(48)\big\}-\frac{1}{\tau_i}\quad ,$$

$$L(t;T_{i-1},T_i) \;=\; \left[L(0;T_{i-1},T_i)+\frac{1}{\tau_i}\right]\exp\left\{-\alpha_i x-\frac{1}{2}\alpha_i^2\right\}-\frac{1}{\tau_i}\quad . \qquad (49)$$

It has resulted in a martingale **shifted lognormal distribution for the Libor**.

The role of the $S(T)$ in the interest rate is played by the $1+\tau L(T_{i-1};T_{i-1},T_i)$.

Effect of Collateral, Libor Convexity Adjustment

Therefore let us see the situation in the calculation of the expectation of the forward rate under the OIS discounting and/or Libor discounting. Here $1 + \tau L(T_{i-1}; T_{i-1}, T_i)$ has the same distribution in the T_i-forward measure of the index curve.

$$E_0\left[e^{-\int_0^{T_i} r_{OIS}(u)\mathrm{d}u}(1+\tau L(T_{i-1};T_{i-1},T_i)\right] = P_{OIS}(0,T_i)E_0^{T_i^{OIS}}\left[(1+\tau L(T_{i-1};T_{i-1},T_i)\right]$$

$$= P_{OIS}(0,T_i)\tilde{E}_0^{T_i^L}\left[\frac{P_L(0,T_i)}{P_{OIS}(0,T_i)}e^{\int_0^{T_i} s_{OIS}(u)\mathrm{d}u}(1+\tau L(T_{i-1};T_{i-1},T_i)\right]$$

$$= P_L(0,T_i)\tilde{E}_0^{T_i^L}\left[e^{\int_0^{T_i} s_{OIS}(u)\mathrm{d}u}(1+\tau L(T_{i-1};T_{i-1},T_i)\right]$$

where the measure change is opposite way to the one of Eq. (28).

Effect of Collateral, Libor Convexity Adjustment

We can calculate (substitute here $\tilde{L}_i(0) = (1 + \tau L(0; T_{i-1}, T_i))$

$$P_{OIS}(0, T_i) E_0^{T_i^{OIS}} \left[\tilde{L}_i(0) \exp \left\{ -\sigma \int_0^{T_{i-1}} \Big(G(u, T_{i-1}) - G(u, T_i) \Big) \mathrm{d}W_L^{T_i}(u) \right. \right.$$

$$\left. \left. - \frac{1}{2}\sigma^2 \int_0^{T_{i-1}} \Big(G(u, T_{i-1}) - G(u, T_i) \Big)^2 \mathrm{d}u \right\} \right] \qquad (50)$$

Here it is needed a simple measure change from the index T_i-forward measure to the OIS T_i-forward measure. This will give

$$P_{OIS}(0, T_i) E_0^{T_i^{OIS}} \left[\tilde{L}_i(0) \exp \left\{ -\sigma \int_0^{T_{i-1}} \Big(G(u, T_{i-1}) - G(u, T_i) \Big) \mathrm{d}\tilde{W}_L^{T_i}(u) \right. \right.$$

$$+ \rho_{Ls} \sigma_L \sigma_s \int_0^{T_{i-1}} \Big(G(u, T_{i-1}) - G(u, T_i) \Big) G_s(u, T_i) \mathrm{d}u$$

$$\left. \left. - \frac{1}{2}\sigma^2 \int_0^{T_{i-1}} \Big(G(u, T_{i-1}) - G(u, T_i) \Big)^2 \mathrm{d}u \right\} \right] \qquad (51)$$

Effect of Collateral, Libor Convexity Adjustment

This results in

$$E_0\left[e^{-\int_0^{T_i} r_{OIS}(u)\mathrm{d}u}(1+\tau L(T_{i-1};T_{i-1},T_i)\right] =$$

$$P_{OIS}(0,T_i)(1+\tau L(0;T_{i-1},T_i))$$

$$\times \exp\left\{\rho_{Ls}\sigma_L\sigma_s\int_0^{T_{i-1}}\Big(G(u,T_{i-1})-G(u,T_i)\Big)G_s(u,T_i)\mathrm{d}u\right\} \quad (52)$$

as compared to the risky-discounted expectation

$$E_0\left[e^{-\int_0^{T_i} r(u)\mathrm{d}u}(1+\tau L(T_{i-1};T_{i-1},T_i)\right] = \quad P_L(0,T_i)(1+\tau L(0;T_{i-1},T_i)) \quad (53)$$

The same result can be calculated by taking the covariance of the Radon-Nykodim derivative and the Libor rate. (do this for comparison reasons!)

Effect of Collateral, Libor Convexity Adjustment

Similarly This results in

$$E_0^{T_i^{OIS}}\left[(1+\tau L(T_{i-1};T_{i-1},T_i)\right]=$$

$$(1+\tau L(0;T_{i-1},T_i))$$

$$\times \exp\left\{\rho_{Ls}\sigma_L\sigma_s\int_0^{T_{i-1}}\Big(G(u,T_{i-1})-G(u,T_i)\Big)G_s(u,T_i)\mathrm{d}u\right\} \quad , \qquad (54)$$

as compared to the risky-discounted expectation

$$E_0^{T_i^{L}}\left[(1+\tau L(T_{i-1};T_{i-1},T_i)\right]= \quad (1+\tau L(0;T_{i-1},T_i)) \quad . \qquad (55)$$

With

$$E_0^{T_i^{OIS}}\left[L(T_{i-1};T_{i-1},T_i)\right]-E_0^{T_i^{L}}\left[L(T_{i-1};T_{i-1},T_i)\right]=$$

$$\exp\left\{\rho_{Ls}\sigma_L\sigma_s\int_0^{T_{i-1}}\Big(G(u,T_{i-1})-G(u,T_i)\Big)G_s(u,T_i)\mathrm{d}u\right\}-1 \quad . \; (56)$$

Effect of Collateral, Libor Convexity Adjustment

This calculation can be extended analogously also to calculation of Option prices with stochastic spreads, like Caps, Floors, or Libor pay-delay convexity corrections.

$$P_{OIS}(0,T)E^{T^{OIS}}\left[\left((L_0+\frac{1}{\tau})\exp\{\}\times CVX-\frac{1}{\tau}\right)-K\right]^{+} \tag{57}$$

Effect of Collateral - Multiple Currencies

For two currencies, two points to pay attention. First, extract the effective growth rate of the derivative when collateral is posted in foreign ccy. Second, get some information for the dynamics of this rate. The basic equation in the foreign ccy collateral, are ones that relate the spot $FX(t)$ (dom/for), the Fx-forward $FX(t, t + \mathrm{d}t)$.

For a fully collateralized trade, the amount posted in foreign ccy will be $V(t)/FX(t)$. The collateral will accrue at the foreign rate $r_Q^f(t)$ over $t, t + \mathrm{d}t$, to the amount $V(t)/FX(t)(1 + r_Q^f(t)\mathrm{d}t)$ in foreign currency. At time $t + \mathrm{d}t$, $V(t)/FX(t)(1 + r_Q^f(t)\mathrm{d}t)FX(t + \mathrm{d}t)$.

We can reach at the same amount through the domestic ccy only if we add a cross-currency basis spread $s(t)$ to the domestic rate $r_Q(t)$. Therefore

$$
\begin{aligned}
Q(t + \mathrm{d}t) &= \left[\frac{V(t)}{FX(t)}(1 + r_Q^f(t)\mathrm{d}t)\right] FX(t + \mathrm{d}t) \quad , & (58) \\
&= V(t)\Big(1 + r_Q(t)\mathrm{d}t + s(t)\mathrm{d}t\Big) \quad . & (59)
\end{aligned}
$$

Effect of Collateral - Multiple Currencies

It is obvious from the above that the growth rate of the collateral account is done neither with the domestic nor with foreign overnight rate, but with an effective rate that equals the domestic overnight rate $r_Q(t)$ adjusted by the domestic-foreign currency basis spread $s(t)$

$$r_{\tilde{Q}}(t) = r_Q(t) + s(t) \quad . \tag{60}$$

In expected terms we have

$$F(t, t + \mathrm{d}t) = E_t^{t+\mathrm{d}t}[FX(t + \mathrm{d}t] = FX(t)\Big(1 + r_Q(t)\mathrm{d}t - r_Q^f(t)\mathrm{d}t + s(t)\mathrm{d}t\Big) \quad ,$$

inferring the following dynamics for the FX rate

$$\frac{\mathrm{d}FX(t)}{FX(t)} = \big(r_Q(t) - r_Q^f(t) + s(t)\big)\mathrm{d}t + \sigma_{FX}(t)\mathrm{d}W_{FX}(t) \tag{61}$$

and if we note $FX^*(t) = FX(t)e^{-\int_t^T s(u)\mathrm{d}u}$

$$E_t^{Q_T}[FX^*(T)] = FX^*(t)\frac{P_Q^f(t, T)}{P_Q(t, T)} \quad , \tag{62}$$

with fully collateralized domestic and foreign bonds.

Effect of Collateral - Multiple Currencies

Now we have to rethink the way of our cash account growing. We can have partly collateral in domestic currency, partly in foreign and the rest uncollaterlized and growing at the unsecured rate $r_F(t)$

$$\begin{aligned}
\mathrm{d}\gamma(t) &= r_Q(t)Q_d(t)\mathrm{d}t + \left[\frac{Q_f(t)}{FX(t)}\left(1 + r_Q^f(t)\mathrm{d}t\right) FX(t+\mathrm{d}t)\right] \quad (63)\\
&+ r_F(t)\left(V(t) - Q_d(t) - Q_f(t)\right)\mathrm{d}t - r_R(t)\Delta(t)S(t)\mathrm{d}t\\
&= r_Q(t)Q_d(t)\mathrm{d}t + \left(r_Q(t) + s(t)\right) Q_f(t)\mathrm{d}t \quad (64)\\
&\quad +r_F(t)\left(V(t) - Q_d(t) - Q_f(t)\right)\mathrm{d}t - r_R(t)\Delta(t)S(t)\mathrm{d}t
\end{aligned}$$

The solution to this equation is

$$\begin{aligned}
V(t) &= E_t\Bigg(e^{-\int_t^T r_F(u)\mathrm{d}u}V(T) + \int_t^T e^{-\int_t^u r_F(v)\mathrm{d}v}(r_F(u) - r_Q(u))\mathrm{d}u\\
&\quad + \int_t^T e^{-\int_t^u r_F(v)\mathrm{d}v}(r_F(u) - r_Q(u) - s(u))Q_f(u)\mathrm{d}u\Bigg) \quad (65)
\end{aligned}$$

Effect of Collateral - Multiple Currencies

If the collateral is placed in the foreign currency is equal to the value of the derivative then

$$V(t) = E_t\left[e^{-\int_t^T (r_Q(u)+s(u))\mathrm{d}u} V(T)\right] = P_Q(t,T)E_t\left[e^{-\int_t^T s(u)\mathrm{d}u} V(T)\right] \quad . \tag{66}$$

Therefore if collateral is posted in domestic or foreign currency the price of the derivative is:

$$V^d(t) = P_Q(t,T)E_t^{Q_T}\left[V(T)\right] \quad , \tag{67}$$

$$V^f(t) = P_Q(t,T)E_t^{Q_T}\left[e^{-\int_t^T s(u)\mathrm{d}u} V(T)\right] \quad . \tag{68}$$

Effect of Collateral - Multiple Currencies

A second modification to CSA trading has occurred when banks offered the counterparties the option to change currency of collateral. The counterparties receive interest from their collateral and they are interested to post in the currency that offers the highest returns.
Let us assume that the different currencies, when rates are converted in domestic ccy, offer "effective" short-rate returns (overnight rates) $c_1(t)$, $c_2(t)$, ...
In the risk-neutral measure, the spot rate at time T is given by a Gaussian process (details in following slides)

$$c_i(T) = \mu_i(T) + \Sigma_i(T) W_i(T) \quad . \tag{69}$$

In the new reality we are looking for maximum discounting:

$$C(0,T) = E_0\left[e^{-\int_0^T \max(c_1(u),c_2(u))\mathrm{d}u}\right] \quad , \tag{70}$$

$$= E_0\left[e^{-\sum_{i=0}^{N} \Delta T_i \max(c_1(T_i),c_2(T_i))}\right] \quad . \tag{71}$$

We discretize the time interval, and need to calculate the European options in a number of points, lets say daily points T_i, $T_i + \delta$, where

Effect of Collateral - Multiple Currencies

A simple way of doing the calculations is the T-forward measure of the first bond $P_1(0,T)$

$$C(0,T) = E_0\left[e^{-\int_0^T \max(c_1(u),c_2(u))\mathrm{d}u}\right] \quad , \qquad (72)$$

$$= E_0\left[e^{-\int_0^T \left[c_1(u)+\max(c_2(u)-c_1(u),0)\right]\mathrm{d}u}\right] \quad , \qquad (73)$$

$$= P_1(0,T)E_0^{T_{P_1}}\left[e^{-\int_0^T \max(c_2(u)-c_1(u),0)\mathrm{d}u}\right] \quad . \qquad (74)$$

Therefore we write the dynamics of $c_1(t)$ and $c_2(t)$ in the T-forward measure where $P_1(t,T)$ is the numeraire.
We discretize the time interval, and need to calculate the European options in a number of points, lets say daily points T_i, $T_i+\delta$, where $\delta = 1/365$

$$C(0,T) = P_1(0,T)E_0^{T_{P_1}}\left[e^{-\sum_{i=0}^{N}\Delta T_i \max(c_2(T_i)-c_1(T_i),0)}\right] \quad . \qquad (75)$$

Effect of Collateral - Multiple Currencies

Due to Girsanov's theorem, the Brownian motion in the $T + \delta$-forward measure will be given by

$$W^{T+\delta}(t) = W(t) - \int_0^t \left(\frac{\sigma_q}{q(s)} - \frac{\sigma_p}{p(s)} \right) \mathrm{d}s \quad , \tag{76}$$

where q-measure is the $T + \delta$ forward measure and the p-measure is the originally known risk-neutral (money market) measure. Since $\sigma_p \equiv 0$, and

$$\frac{\sigma_q}{q} = -\sigma_r G(t, T + \delta) \quad , \tag{77}$$

we have

$$\mathrm{d}W_1^{T+\delta}(t) = \mathrm{d}W_1(t) + \sigma_1 G_1(t, T + \delta)\mathrm{d}t \quad , \tag{78}$$

$$\mathrm{d}W_2^{T+\delta}(t) = \mathrm{d}W_2(t) + \sigma_1 G_1(t, T + \delta)\mathrm{d}t \quad , \tag{79}$$

and

$$\mathrm{d}x_1(t) = \Big(y(t) - \sigma_1^2 G(t, T + \delta) - \kappa(t)x(t) \Big)\mathrm{d}t + \sigma_1(t)\mathrm{d}W_1^{T+\delta}(t) \quad . \tag{80}$$

Effect of Collateral - Multiple Currencies

The solution for $x(T)$ in the risk-neutral measure is given by

$$x(T) \quad = \quad h(T)\int_0^T g^2(u)\left(\int_u^T h(s)\mathrm{d}s\right)\mathrm{d}u + h(T)\int_0^T g(u)\mathrm{d}W^{T+\delta}(u) \quad ,$$

whereas in the $(T+\delta)$-forward measure it is given by

$$x_1(T) \;=\; h_1(T)\int_0^T g_1^2(u)\left(\int_u^T h_1(s)\mathrm{d}s\right)\mathrm{d}u - \sigma_1 h_1(T)\int_0^T g_1(u)G_1(u, T+\delta)\mathrm{d}u$$

$$+\; h_1(T)\int_0^T g_1(u)\mathrm{d}W_1^{T+\delta}(u) \quad .$$

The T-forward equation for $x_2(T)$ will be given by

$$x_2(T) \;=\; h_2(T)\int_0^T g_2^2(u)\left(\int_u^T h_2(s)\mathrm{d}s\right)\mathrm{d}u - \sigma_1 h_2(T)\int_0^T g_2(u)G_1(u, T+\delta)\mathrm{d}u$$

$$+ \quad h_2(T)\int_0^T g_2(u)\mathrm{d}W_2^{T+\delta}(u) \quad ,$$

Effect of Collateral - Multiple Currencies

The mean value of $x(T)$ in the $(T+\delta)$-forward measure is different from the one in the risk-neutral measure. In the new measure the mean of $x(T)$ is given by the first two terms

$$\bar{x}_1(T) = h_1(T)\int_0^T g_1^2(u)\left(\int_u^T h_1(s)\mathrm{d}s\right)\mathrm{d}u - \sigma_1 h_1(T)\int_0^T g_1(u)G_1(u,T+\delta)\mathrm{d}u$$

$$\bar{x}_2(T) = h_2(T)\int_0^T g_2^2(u)\left(\int_u^T h_2(s)\mathrm{d}s\right)\mathrm{d}u - \sigma_1 h_2(T)\int_0^T g_2(u)G_1(u,T+\delta)\mathrm{d}u$$

For the case when the mean reversion is constant we get

$$\begin{aligned}\bar{x}_1(T) &= e^{-kT}\int_0^T \sigma_r^2 e^{2ku}\left(\int_u^T e^{-ks}\mathrm{d}s\right)\mathrm{d}u - \sigma_r^2 e^{-kT}\int_0^T e^{ku}\frac{1}{k}\left(1-e^{-k(T+\delta-u)}\right)\mathrm{d}u \\ &= \frac{\sigma_r^2}{2k^2}\left(e^{-k\delta}-1\right)\left(1-e^{-2kT}\right) \quad , \\ &= -y_1(T)G_1(T,T+\delta) \quad .\end{aligned}$$

Effect of Collateral - Multiple Currencies

For $x_2(T)$ we have:

$$\bar{x}_2(T) = e^{-k_2 T} \int_0^T \sigma_2^2 e^{2k_2 u} \left(\int_u^T e^{-k_2 s} \mathrm{d}s \right) \mathrm{d}u$$

$$- \sigma_1 \sigma_2 e^{-k_2 T} \int_0^T e^{k_2 u} \frac{1}{k_1} \left(1 - e^{-k_1 (T+\delta-u)} \right) \mathrm{d}u \quad ,$$

$$= \frac{\sigma_2^2}{2k_2^2} \left(1 - e^{-k_2 T}\right)^2 - \frac{\sigma_1 \sigma_2}{k_1 k_2} \left[\left(1 - e^{-k_2 T}\right) - \frac{k_1}{k_1 + k_2} e^{-k_1 \delta} \left(1 - e^{-(k_1+k_2)T} \right) \right]$$

which when $k_1 = k_2 = k$ and the volatilities $\sigma_1 = \sigma_2 = \sigma$ gives:

$$\bar{x}_2(T) = \frac{\sigma^2}{2k^2} \left(e^{-k\delta} - 1\right) \left(1 - e^{-2kT}\right) = -y_2(T) G_2(T, T+\delta) \quad ,$$

which is the same as the one obtained for $\bar{x}_1(T)$. In the following we will split the Gaussian part in $x_i(T) = \bar{x}_i(T) + x_{0i}(T)$.

Effect of Collateral - Multiple Currencies

The bond reconstruction formula is one of the most important in the short-rate modeling. In the T-forward measure it becomes

$$P_1(t,T) = \frac{P_1(0,T)}{P_1(0,t)} \exp\left(-\frac{1}{2} y_1(t) G_1^2(t,T) - (x_{01}(t) - y_1(t)G_1(t,T))G_1(t,T)\right)$$

$$= \frac{P_1(0,T)}{P_1(0,t)} \exp\left(\frac{1}{2} y_1(t) G_1^2(t,T) - x_{01}(t) G_1(t,T)\right) \quad . \qquad (81)$$

The Gaussian variable $x_{01}(t)$ in the T-forward measure

$$x_{01}(t) = h_1(t) \int_0^t g_1(u) \mathrm{d}W_1^T(u) \quad , \quad y_1(t) = \frac{\sigma_1^2}{2k_1}\left(1 - e^{-2k_1 T}\right) \quad , \ (82)$$

mean value of x_{01} being zero and variance $y_1(t)$. Here we use the notation

$$K_i(t) = \int_0^t \kappa_i(u) \mathrm{d}u \quad , \quad h_i(t) = e^{-K_i(t)} \quad , \quad g_i(t) = \sigma_i e^{K_i(t)} \quad . \quad (83)$$

Effect of Collateral - Multiple Currencies

$$\mathrm{Var}[\mathrm{x}_0(\mathrm{t})] = h^2(t)\int_0^t g^2(u)\mathrm{d}u = e^{-2\int_0^t \kappa(s)\mathrm{d}s}\int_0^t e^{2\int_0^u \kappa(s)\mathrm{d}s}\mathrm{d}u \quad (84)$$

$$= \int_0^t e^{-2\int_u^t \kappa(s)\mathrm{d}s}\mathrm{d}u \quad , \quad (85)$$

$$\mathrm{Var}[\mathrm{x}_{01}(\mathrm{t})] = \int_0^t e^{-2\int_u^t \kappa(s)\mathrm{d}s}\mathrm{d}u = y_1(t) \quad . \quad (86)$$

Notice that the inverse of the bond is now martingale in the T-forward measure

$$E^T\left[\left.\frac{1}{P_1(t,T)}\right|\mathcal{F}_0\right] = E^T\left[\left.\frac{P_1(t,t)}{P_1(t,T)}\right|\mathcal{F}_0\right] = \frac{P_1(0,t)}{P_1(0,T)} \quad . \quad (87)$$

Effect of Collateral - Multiple Currencies

On the other hand we can calculate the same expected value by using the bond formula

$$E^T\left[\frac{1}{P_1(t,T)}\bigg|\mathcal{F}_0\right]=E^T\left[\frac{P_1(0,t)}{P_1(0,T)}\ e^{-\frac{1}{2}y_1(t)G_1^2(t,T)+x_{01}(t)G_1(t,T)}\bigg|\mathcal{F}_0\right]=\frac{P_1(0,t)}{P_1(0,T)}\quad,$$

and both results agree since the exponential function is a martingale in the T-forward measure. The second bond after measure change

$$P_2(t,T)=\frac{P_2(0,T)}{P_2(0,t)}\exp\left(-\frac{1}{2}y_2(t)G_2^2(t,T)-(x_{02}(t)+\bar{x}_2(T))G_2(t,T)\right)\quad,\tag{89}$$

$$=\frac{P_2(0,T)}{P_2(0,t)}\exp\left(-\frac{1}{2}y_2(t)G_2^2(t,T)-\bar{x}_2(T)G_2(t,T)-x_{02}(t)G_2(t,T)\right)\tag{90}$$

where $\mathrm{Var}[x_{02}(t)]=\int_0^t e^{-2\int_u^t \kappa_2(s)\mathrm{d}s}\mathrm{d}u=y_2(t)$, and

$$\bar{x}_2(T)=\frac{\sigma_2^2}{2k_2^2}\left(1-e^{-k_2T}\right)^2-\frac{\sigma_1\sigma_2}{k_1k_2}\left[\left(1-e^{-k_2T}\right)-\frac{k_1}{k_1+k_2}e^{-k_1\delta}\left(1-e^{-(k_1+k_2)T}\right)\right]$$

Effect of Collateral - Multiple Currencies

To extract the dynamics of $c_1(t)$ and $c_2(t)$ in the T_{n+1}-forward measure (denoted E^T below) we use the following equality:

$$E_0^T\left[e^{\int_{T_n}^{T_{n+1}} c_1(u)\mathrm{d}u}\right] = E_0^T\left[\frac{1}{P_1(T_n, T_{n+1})}\right] = E_0^T\left[\frac{P_1(T_n, T_n)}{P_1(T_n, T_{n+1})}\right] = \frac{P_1(0, T_n)}{P_1(0, T_{n+1})}$$

$$= E_0^T\left[\frac{P_1(0, T_n)}{P_1(0, T_{n+1})}\, e^{-\frac{1}{2}y_1(T_n)G_1^2(T_n, T_{n+1})+x_{01}(T_n)G_1(T_n, T_{n+1})}\middle|\, \mathcal{F}_0\right]$$

then, in the T_{n+1}-forward measure we have:

$$c_1(T_n) = \frac{1}{\Delta T_n}\left[-\frac{1}{2}y_1(T_n)G_1^2(T_n, T_{n+1}) + x_{01}(T_n)G_1(T_n, T_{n+1}) + \ln\frac{P_1(0, T_n)}{P_1(0, T_{n+1})}\right]$$

and

$$c_2(T_n) \;\; = \;\; \frac{1}{\Delta T_n}\left[\frac{1}{2}y_2(T_n)G_2^2(T_n, T_{n+1}) + \bar{x}_2(T)G_2(T_n, T_{n+1}) + x_{02}(T_n)G_2(T_n, T_{n+1}\right.$$

$$\left. + \ln\frac{P_2(0, T_n)}{P_2(0, T_{n+1})}\right] \quad .$$

Effect of Collateral - Multiple Currencies

We can summarize the drifts as follows:

$$\mathrm{Drift}_1\Big(c_1(T_n)(T_{n+1}-T_n)\Big) = -\frac{1}{2}y_1(T_n)G_1^2(T_n,T_{n+1}) + \ln\frac{P_1(0,T_n)}{P_1(0,T_{n+1})} \quad ,$$

$$\begin{aligned}\mathrm{Drift}_2\Big(c_2(T_n)(T_{n+1}-T_n)\Big) &= \frac{1}{2}y_2(T_n)G_2^2(T_n,T_{n+1}) + \bar{x}_2(T)G_2(T_n,T_{n+1}) \\ &+ \ln\frac{P_2(0,T_n)}{P_2(0,T_{n+1})} \quad .\end{aligned}$$

The variances of the processes are

$$\mathrm{Variance}_1\Big(c_1(T_n)(T_{n+1}-T_n)\Big) = y_1(T_n)G_1^2(T_n,T_{n+1})N(0,1) \quad , \qquad (93)$$

and

$$\mathrm{Variance}_2\Big(c_2(T_n)(T_{n+1}-T_n)\Big) = y_2(T_n)G_2^2(T_n,T_{n+1})N(0,1) \quad . \qquad (94)$$

Effect of Collateral - Multiple Currencies

The variance of the whole process can be found from the fact that

$$c_2(T_n)\Delta T_n - c_1(T_n)\Delta T_n = m(T_n) \;+\; G_2(T_n, T_{n+1})h_2(T_n)\int_0^{T_n} g_2(u)\mathrm{d}W_2(u)$$

$$-\; G_1(T_n, T_{n+1})h_1(T_n)\int_0^{T_n} g_1(u)\mathrm{d}W_1(u) \quad ,$$

where $\langle \mathrm{d}W_1(t)\mathrm{d}W_2(t)\rangle = \rho\mathrm{d}t$. The variance of the difference will be

$$\mathrm{Variance}\Big(c_2(T_n)\Delta T_n - c_1(T_n)\Delta T_n\Big) \;=\; \mathrm{Variance}_1 + \mathrm{Variance}_2$$

$$-2G_1G_2h_1(T_n)h_2(T_n)\int_0^{T_n} g_1(u)g_2(u)\rho(u)\mathrm{d}u \quad .$$

Effect of Collateral - Multiple Currencies

If we take the correlation to be constant, then the correlation factors out of the integral and the total variance will be given by

$$\mathrm{Variance}\Big(c_2(T_n)\Delta T_n - c_1(T_n)\Delta T_n\Big) = \frac{\sigma_1^2}{2\kappa_1}\left(1-e^{-2\kappa_1 T_n}\right) G_1^2(T_n, T_{n+1})$$

$$+\frac{\sigma_2^2}{2\kappa_2}\left(1-e^{-2\kappa_2 T_n}\right) G_2^2(T_n, T_{n+1})$$

$$-2\rho\frac{\sigma_1\sigma_2}{\kappa_1+\kappa_2}\left(1-e^{-(\kappa_1+\kappa_2)T_n}\right) G_1(T_n, T_{n+1})G_2(T_n, T_{n+1}) \quad .$$

Now we can use the algorithm written at the beginning of the section:

$$C(0,T) = P_1(0,T)E_0^T\left[e^{-\int_0^T \max(c_2(u)-c_1(u),0)\mathrm{d}u}\right] \quad , \qquad (95)$$

with the dynamics of the short rates $c_1(t)$ and $c_2(t)$ in the T-forward measure of $P_1(0,T)$ are given in the above equations.

Effect of Collateral - Multiple Currencies

In the following we can make a first order approximation for the calculation of the expected values, as follows

$$C(0,T) = P_1(0,T)E_0^T\left[e^{-\sum_{n=0}^{N}\Delta T_n \max(c_2(T_n)-c_1(T_n),0)}\right] \quad (96)$$

$$\approx P_1(0,T)e^{-\sum_{n=0}^{N}E_0^T[\max((c_2(T_n)-c_1(T_n))\Delta T_n,0)]} \quad . \quad (97)$$

$$c_1(T_n)\Delta T_n = \mu_1(T_n) + \Sigma_1(T_n)W_1(T_n) \quad , \quad (98)$$

where

$$\Sigma_i(T_n) = \left[\frac{y_i(T_n)}{T_n}\right]^{1/2} G_i(T_n, T_n+\delta) \quad , \quad (99)$$

$$y_i(T_n) = \frac{\sigma_i^2}{2k_i}\left(1 - e^{-2k_i T_n}\right) \quad , \quad (100)$$

and

$$G_i(T_n, T_n + \delta) = \frac{1}{k_i}\left(1 - e^{-k_i\delta}\right) \quad . \tag{101}$$

The spread between the two rates can be written as

$$\Big(c_2(T_n) - c_1(T_n)\Big)\Delta T_n = m(T_n) + \Sigma(T_n)W(T_n) \quad , \tag{102}$$

where

$$\Sigma(T_n) = \sqrt{\Sigma_1^2(T_n) + \Sigma_2^2(T_n) - 2\rho\Sigma_{12}(T_n)} \quad , \tag{103}$$

and

$$W(T_n) = \frac{\Sigma_2(T_n)}{\Sigma(T_n)}W_2(T_n) - \frac{\Sigma_1(T_n)}{\Sigma(T_n)}W_1(T_n) \quad . \tag{104}$$

Effect of Collateral - Multiple Currencies

The total variance of the standard normal process $N(0,1)$ will be

$$\text{Total Variance} = y_1(T_n)G_1^2(T_n,T_{n+1}) + y_2(T_n)G_2^2(T_n,T_{n+1}) - 2\rho\Sigma_{12}(T_n) \quad (105)$$

$$= \Sigma^2(T_n)T_n \quad . \quad (106)$$

Here we have

$$\Sigma_{12} = \frac{\sigma_1\sigma_2}{\kappa_1+\kappa_2}\left(1-e^{-(\kappa_1+\kappa_2)T_n}\right)G_1(T_n,T_{n+1})G_2(T_n,T_{n+1}) \quad . \quad (107)$$

The price $p(T_n)$ of the call spread will be given by

$$p(T_n) = m(T_n)\Phi\left(\frac{m(T_n)}{\Sigma(T_n)\sqrt{T_n}}\right) + \Sigma(T_n)\sqrt{T_n}\,\varphi\left(\frac{m(T_n)}{\Sigma(T_n)\sqrt{T_n}}\right) , \quad (108)$$

Φ and φ being the standard CDF and PDF for a normal Gaussian distribution.

Effect of Collateral - Multiple Currencies

As a reminder the above formula is obtained as follows:

$$p_n = \frac{m(T_n)}{\sqrt{2\pi}} \int_{-\frac{m(T_n)}{\beta_N}}^{\infty} e^{-\frac{1}{2}z^2} \mathrm{d}z + \frac{\beta_N}{\sqrt{2\pi}} \int_{\frac{m^2(T_n)}{\beta_N^2}}^{\infty} e^{-z} \mathrm{d}z \tag{109}$$

$$= m(T_n) N\left(\frac{m(T_n)}{\beta_N}\right) + \frac{\beta_N}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{m^2(T_n)}{\beta_N^2}} \quad , \tag{110}$$

where the above is a standard random variable with mean zero and standard deviation one, and $\beta_N = \sigma_N \sqrt{T_n}$. In the case when $m(T_n) = 0$, i.e. when the option is ATM, the call option pv is $p_n = \beta_N / \sqrt{2\pi}$. For the case of uncorrelated curves, we have $\beta_N = \sqrt{(\Sigma_1^2 + \Sigma_2^2) T_n}$, which simplifies when $\sigma_1 = \sigma_2 = \sigma$ and $\kappa_1 = \kappa_2 = \kappa$ to $\beta_N = \sqrt{2\Sigma_1^2(T_n) T_n}$ and when we use the same curves like GBP.GBPOIS and GBP.GBPOIS instead of GBP.USDOIS, which also give
$m(T_n) = \ln(P_2(0, T_n)/P_2(0, T_{n+1}) - \ln(P_1(0, T_n)/P_1(0, T_{n+1}) \equiv 0$.

Effect of Collateral - Multiple Currencies

In the simple case when the speed of mean reversion and the volatilities are equal, we have

$$m(T_n) = \ln \frac{P_2(0, T_n)}{P_2(0, T_{n+1})} - \ln \frac{P_1(0, T_n)}{P_1(0, T_{n+1})} \quad . \tag{111}$$

To this order of approximation the collateral bond will be given by

$$C(0, T) = P_1(0, T) e^{-\sum_{n=0}^{N} p_n(T_n)} < P_1(0, T) \quad , \tag{112}$$

where $T_{N+1} = T$.
To avoid unnecessary complications coming from the drift term $m(T_n)$ we assume also that the curve components are both taken to be flat 3%. In this case $m(T_n) = 0$ for all T_n. Then the Option-Adjusted rate will be given by

$$e^{-\sum_{n=1}^{N} p_n(T_n)} = e^{-\sum_{n=1}^{N} \frac{\beta_n(T_n)}{\sqrt{2\pi}}} = e^{-r_n T_n} \quad . \tag{113}$$

Effect of Collateral - Multiple Currencies

This results in

$$r_N = \frac{1}{T_N} \sum_{n=1}^{N} \frac{\beta_n(T_n)}{\sqrt{2\pi}} \quad . \tag{114}$$

Let us look at the behavior of r_N at short maturities. β_n is given by

$$\beta_n = \Sigma(T_n)\sqrt{T_n} \tag{115}$$

For the case when $\kappa_1 = \kappa_2 = \kappa$ and $\rho = 0.5$, we have

$$\begin{aligned} \beta_n = \Sigma(T_n)\sqrt{T_n} &= y^{\frac{1}{2}}(T_n)G(T_n, T_n+\delta) \quad , & (116) \\ &= \sqrt{\frac{\sigma^2}{2\kappa}(1-e^{-2\kappa T_n})}G(T_n, T_n+\delta) \quad , & (117) \\ &\approx \sigma\sqrt{T_n}\delta \quad , & (118) \end{aligned}$$

where $T_n = n\delta$, and $\delta = 1/365$.

Effect of Collateral - Multiple Currencies

From (114), the short maturity behavior is as follows:

$$
\begin{aligned}
r_N = \frac{1}{T_N}\sum_{n=1}^{N}\frac{\beta_n(T_n)}{\sqrt{2\pi}} &= \frac{1}{\sqrt{2\pi}}\frac{\sigma}{T_N}\sum_{n=1}^{N}\sqrt{T_n = n\delta}\Big((n+1)\delta - n\delta\Big) \quad , \\
&= \frac{1}{\sqrt{2\pi}}\frac{\sigma}{T_N}\int_0^{T_N}\sqrt{x}\mathrm{d}x \quad , \\
&= \frac{1}{\sqrt{2\pi}}\frac{\sigma}{T_N}\frac{2}{3}T_N^{\frac{3}{2}} \quad , \\
&= \frac{2}{3}\frac{\sigma}{\sqrt{2\pi}}\sqrt{T_N} \quad , \quad (\text{short maturity behavior} \quad T_n \rightarrow
\end{aligned}
$$

In general, at short maturities the approximation $(1 - \exp\{-2k_i T_n\})/(2\kappa_i) \approx T_n$ can be used.

Effect of Collateral - Multiple Currencies

And for any σ_i and κ_i we get the following short maturity behavior

$$r_N = \frac{2}{3}\left[\frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{2\pi}\right]^{\frac{1}{2}} \sqrt{T_N} \quad . \tag{123}$$

Here and in the following when $\sigma_1 \neq \sigma_2$ and $\kappa_1 \neq \kappa_2$, we assume that the drift terms $m(T_n) \approx 0$.
Now let us look at the long maturity asymptotic $T_N \to \infty$. We have

$$r_N = \frac{1}{T_N}\sum_{n=1}^{N} \frac{\beta_n(T_n)}{\sqrt{2\pi}} \quad , \tag{124}$$

$$= \frac{1}{T_N}\sum_{n=1}^{N} \frac{\sqrt{y(T_n)}}{\sqrt{2\pi}} G(T_n, T_n + \delta) \quad , \tag{125}$$

$$= \frac{1}{T_N}\sum_{n=1}^{N} \sqrt{\frac{\sigma^2}{2\pi\kappa}\left(1 - e^{-2\kappa T_n}\right)} G(T_n, T_n + \delta) \quad , \tag{126}$$

Effect of Collateral - Multiple Currencies

For the long maturity behavior $T_N \to \infty$, the exponential terms vanish, and the long maturity asymptotic behavior for any σ_i, κ_i, and ρ will be

$$r_N \approx \frac{1}{\sqrt{2\pi}} \left[\frac{\sigma_1^2}{2\kappa_1} + \frac{\sigma_2^2}{2\kappa_2} - 2\rho \frac{\sigma_1 \sigma_2}{\kappa_1 + \kappa_2} \right]^{\frac{1}{2}} \quad , \quad (T_n \to \infty) \quad . \qquad (128)$$

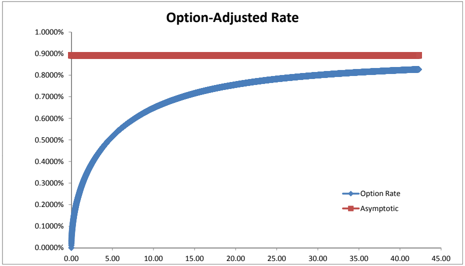

Figure: The Option-Adjustment of the yield curve. The red line is the analytic asymptotic value, and the blue line is the numerical. Here the volatilities of the curves are $\sigma_1 = \sigma_2 = 1\%$, speed of mean reversion is $\kappa_1 = \kappa_2 = 10\%$ and $\rho = 0.5$.

Effect of Collateral - Multiple Currencies; Second Order Approximation

A better approximation would be the second order approximation, as follows

$$E\left[e^{-I(t)}\right] = e^{-E[I(T)]+\frac{1}{2}\mathrm{Var}[I(T)]} \quad , \tag{129}$$

where

$$\mathrm{Var}[I(T)] = E(I^2(T)) - (E(I(T)))^2 \quad , \tag{130}$$

and

$$E(I^2(T)) = \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, E(X^+(t)X^+(s)) \quad . \tag{131}$$

Effect of Collateral - Multiple Currencies; Second Order Approximation

We write the integrals in terms of sums

$$\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, E(X^+(t)X^+(s)) \;=\; \sum_{n=1}^{N}\sum_{k=1}^{N} E\left[X^+(T_n)X^+(T_k)\right] \quad ,$$

$$= 2\sum_{n=1}^{N}\sum_{k=1}^{n-1} E\left[X^+(T_n)X^+(T_k)\right] + \sum_{n=1}^{N} E\left[X^+(T_n)X^+(T_n)\right] \quad .$$

The terms $k = n$ would be double counted in the first term (so the diagonal elements taken in the other term of the sum). Also the terms ΔT_n and ΔT_k have been accounted for in the rates terms themselves, $c_i(T_n)\Delta T_n$, no need to put them explicitly in the sums.

Effect of Collateral - Multiple Currencies; Second Order Approximation

The variables $X(T_k)$ and $X(T_n)$ are normally distributed. We split the expected value into a conditional value. In the summation $T_k < T_n$,

$$E\left[X^+(T_n)X^+(T_k)\right] = \int_0^\infty X(T_k)P\Big(X(T_k) \in (x, x+\mathrm{d}x)\Big)E[X^+(T_n)|X(T_k) = x > 0]$$

The conditional variable is Gaussian and Bachelier can be used

$$\int_0^\infty xP\Big(X(T_k) \in (x, x+\mathrm{d}x)\Big)E[X^+(T_n)|X(T_k) = x]$$

$$= \sum_{l=1}^{L} x_l \varphi\left(\frac{x_l - m(T_k)}{v(T_k)}\right) E[X^+(T_n)|X(T_k) = x_l]\Delta x_l \quad .$$

To complete the above calculation we will need

$$\mathrm{corr}(X(T_n), X(T_k)) = \frac{\mathrm{cov}(X(T_n), X(T_k))}{\sqrt{\mathrm{var}(X(T_n), X(T_n))}\sqrt{\mathrm{var}(X(T_k), X(T_k))}}$$

$$\mathrm{cov}\Big(x_{0i}(T_n), x_{0j}(T_k)\Big) = (\rho, 1)G_i G_j h_i(T_n)h_j(T_k)\int_0^{T_k} g_i(u)g_j(u)\mathrm{d}u \quad .$$

Effect of Collateral - Multiple Currencies

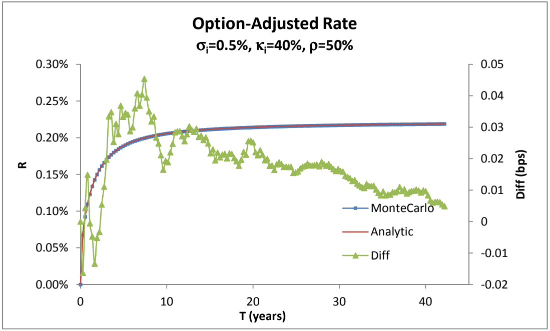

Figure: The Option-Adjustment of the yield curve under normal market scenarios. The red line is the analytic asymptotic value, and the blue line is the numerical. Here the volatilities of the curves are $\sigma_5 = \sigma_2 = 0.5\%$, speed of mean reversion is $\kappa_1 = \kappa_2 = 40\%$ and correlation $\rho = 50\%$. On the secondary plot the difference in bps between the Monte Carlo and the first order analytics is shown. The underlying curves are taken to be flat and equal in value at 3%.

Effect of Collateral - Multiple Currencies

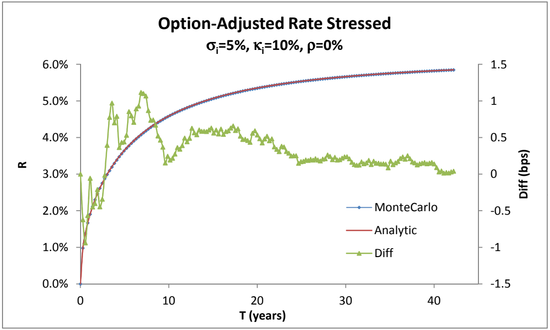

Figure: The Option-Adjustment of the yield curve under very stressed scenarios. The red line is the analytic asymptotic value, and the blue line is the numerical. Here the volatilities of the curves are $\sigma_5 = \sigma_2 = 5\%$, speed of mean reversion is $\kappa_1 = \kappa_2 = 10\%$ and $\rho = 0$. On the secondary plot the difference in bps between the Monte Carlo and the first order analytics is shown. The difference goes to zero at long maturities. The underlying curves are taken to be flat and equal in value at 3%.

Effect of Collateral - Multiple Currencies

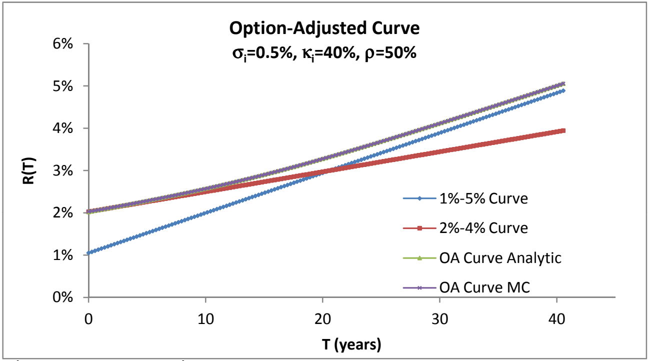

Figure: (normal market). Red line is analytic asymptotic value, and blue line is numerical. Here $\sigma_5 = \sigma_2 = 0.5\%$, $\kappa_1 = \kappa_2 = 40\%$, $\rho = 50\%$. On secondary plot the difference in bps between MC and first order Analytics is shown. The underlying curves are taken to be of linear rates varying from 1% to 5% for the first and from 2% to 4% for the second. The option value is highest at the crossing point of the curves.

Effect of Collateral - Multiple Currencies

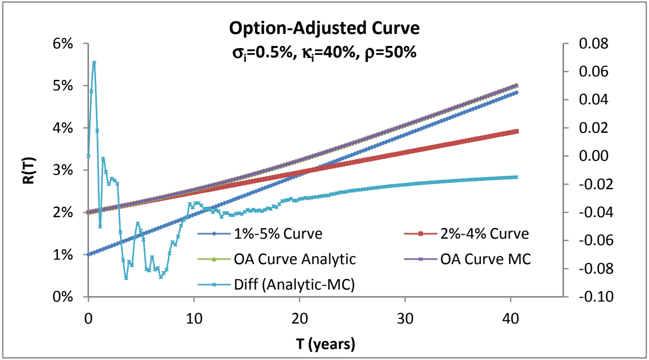

Figure: (normal market scenarios). Red line is the analytic asymptotic value, and blue line is numerical. Here $\sigma_5 = \sigma_2 = 0.5\%$, $\kappa_1 = \kappa_2 = 40\%$, $\rho = 50\%$. The difference in bps between MC and first order Analytics is shown. The underlying curves are taken to be of linear rates varying from 1% to 5% for the first and from 2% to 4% for the second. OA curve stays above both of them.

Effect of Collateral - Multiple Currencies

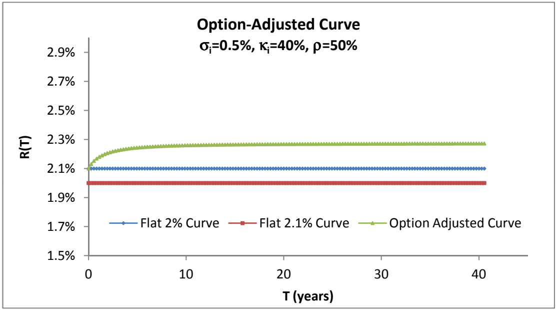

Figure: The Option-Adjusted yield curve under normal market scenarios. The two underlying yield curves are flat and parallel at 2% and 2.1%, respectively. The stochastic effect in this case is expected to be higher than when the underlying curves are further apart.

Effect of Collateral - Multiple Currencies

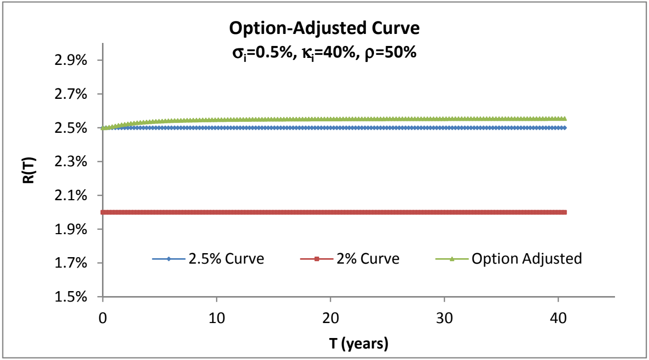

Figure: The Option-Adjusted yield curve under normal market scenarios. The two underlying yield curves are flat and parallel at 2% and 2.3%, respectively. Compare this with the previous figure.

Effect of Collateral - Multiple Currencies

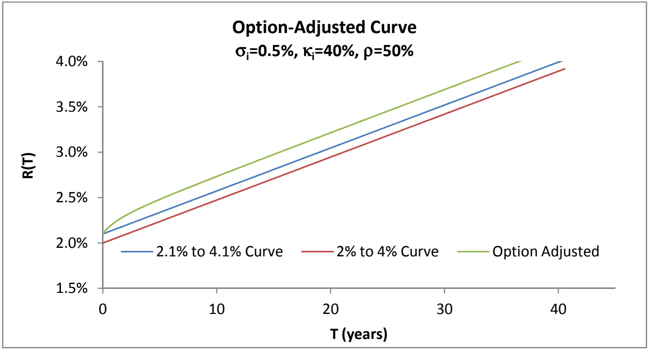

Figure: The Option-Adjusted yield curve under normal market scenarios. The two underlying yield curves are linear but parallel 2% to 4% and 2.1% to 4.1%, respectively. The stochastic effect in this case is expected to be higher than when the underlying curves are further apart.

Effect of Collateral - Multiple Currencies

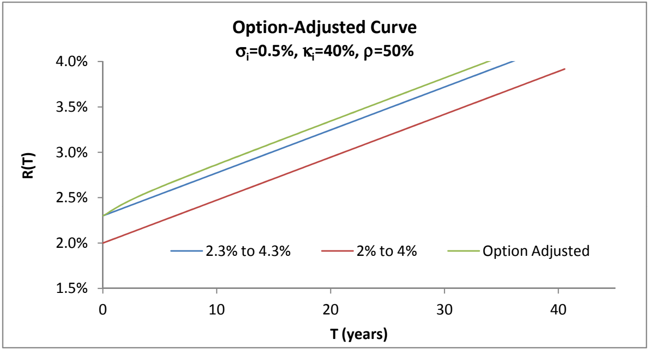

Figure: The Option-Adjusted yield curve under normal market scenarios. The two underlying yield curves are linear but parallel 2% to 4% and 2.3% to 4.3%, respectively. Compare this with the previous figure.

Effect of Collateral - Concluding Remarks

Collateral Posting has unexpected effects on trading

- PV is the first impact as discounting has changed.
- PV differences between CSA trades and non-CSA depend on volatilities and can be significant, particularly for the linear and very liquid products like FRAs, Swaps.
- Another point (mentioned only superficially) the CSA posting causes changes in terminal distribution of Stocks and Libor rates, affecting Swaptions and Caps/Floors.
- The stochastic Spreads introduce volatility-dependent Convexity in Swaps resulting in Vega for linear products (to be discussed at a future time).
- The Option-Adjusted Yield curve is an arean for interesting mathematics in banking.