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Southern University of Science and Technology

Undergraduate Thesis

Thesis Title: Empirical Study on the Reduction of Model Risk in Risk Measures (VaR / ES) Calculations

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摘 要

VaR(在险值) 与 ES(期望损失) 常共同运用在定量风险管理研究中。风险测度计算结果的不确定性大部分来源于风险组合 (损失 / 回报组合) 的分布特性 (单一风险分布 / 复合风险分布: 齐次边际分布 / 非齐次边际分布)。其中，计算复合风险组合分布的 VaR / ES 颇具挑战性。

本文由两部分组成。首先，我们介绍 VaR 及 ES 的基础知识，引入重排算法 (RA) 及适应性重排算法 (ARA) 的概念，该算法现可通过 R 语言：qrmtools 包中的函数实现，以估算 VaR 的上下界及 ES 的上界，即：$\underline{\mathrm{VaR}}$，$\overline{\mathrm{VaR}}$，$\underline{\mathrm{ES}}$。另，$\overline{\mathrm{ES}}$ 由风险组合中各边际分布的理论 ES 之和 (即：满足 Comonotonicity(同单调性) 条件下所得的 $\overline{\mathrm{ES}}_\alpha(X_n^+)$) 估算而得，由 Matlab 实现。

第二部分，选取 Pareto 分布、指数分布、对数正态分布及其复合的风险分布 (齐次：由独立同分布复合而成；非齐次：除独立同分布复合外的复合分布)，分别运用 RA 与 ARA 算法及同单调性下的 ES 估算各风险组合 VaR 与 ES 的上下界, 并比较两方法所得结果。本文讨论由连续分布作为边际分布组成的复合分布的风险测度值估算，对离散边际分布不作讨论。

所得结论为：RA 与 ARA 算法对于多数复合连续分布 (无严重厚尾现象) 的 $\underline{\mathrm{VaR}}$，$\overline{\mathrm{VaR}}$ 有较精确的计算结果, 而对于 $\underline{\mathrm{ES}}$ 的计算结果不够稳定。RA 与 ARA 算法的结果易受到厚尾边际分布的影响，表现为 $\underline{\mathrm{ES}}$ 以及 $\underline{\mathrm{VaR}}$，$\overline{\mathrm{VaR}}$ 的估计值上下界有较大差异。针对提升 ES 上下界的计算精度问题，RA 与 ARA 算法仍需从调整分布函数尾部权重等方向作进一步改善。

[关键词]: 模型不确定性, 复合连续风险分布, 重排算法 (RA), 适应性重排算法 (ARA)

ABSTRACT

VaR (Value-at-Risk) and ES (Expected Shortfall) work as a pair in quantitative risk management. It is known that model uncertainty of the risk measures, are largely affected by the distribution of risk portfolios (Single / Aggregate: Homogeneous / Inhomogeneous Marginal Distributions). The VaR / ES calculations for aggregate risk portfolios are challenging.

This thesis consists of two parts. Firstly, we introduce the preliminaries of VaR / ES, the notion of Rearrangement Algorithm (RA) and Adaptive Rearrangement Algorithm (ARA) implemented in R Package: qrmtools to approximate the lower / upper-bound for VaR as well as the lower-bound for ES, i.e. $\underline{\mathrm{VaR}}$, $\overline{\mathrm{VaR}}$ and $\underline{\mathrm{ES}}$. $\overline{\mathrm{ES}}$ for aggregate risk is estimated from the sum of theoretical ES(s) of marginal distributions by Matlab, i.e. $\overline{\mathrm{ES}}$ under comonotonicity.

In the second part, we estimate $\underline{\mathrm{VaR}}$, $\overline{\mathrm{VaR}}$, $\underline{\mathrm{ES}}$,and $\overline{\mathrm{ES}}$ for homogeneous and inhomogeneous portfolios consisting of Pareto, Exponential, Lognormal Distribution and their miscellanies. For each portfolio discussed, we compare the results of VaR / ES calculations induced from RA and ARA methods. This thesis focuses on the aggregate risk consisting of all continuous marginal distributions rather than the discrete marginals.

We conclude that RA and ARA are perfectly applicable in most aggregate risk portfolios except for the case consisting of very heavy-tailed marginal distributions. In this case, the value spread of the bounds for $\underline{\mathrm{ES}}$, $\underline{\mathrm{VaR}}$ and $\overline{\mathrm{VaR}}$ appear to be much larger than those with the relatively light-tailed. RA and ARA need improvements to reduce the range for $\underline{\mathrm{ES}}$, such as to readjust weights in divided intervals of tails implemented in RA and ARA.

[Keywords]: Model Uncertainty, Continuous Aggregate Risk / Loss, Rearrangement Algorithm (RA), Adaptive Rearrangement Algorithm (ARA)

Notations

α	Confidence Level α
TVaR	Tail VaR
CTE	Conditional Tail Expectation
CVaR	Conditional VaR
X_i	A Single Risk Position over a period, where $i = 1, \cdots, n$.
$X_i \sim \mathrm{f_i}$	$\mathrm{f_i}$: probability density function (p.d.f), where $i = 1, \cdots, n$.
$X_i \overset{d}{\sim} \mathrm{F_i}$	$\mathrm{F_i}$: cumulative density function (c.d.f), where $i = 1, \cdots, n$.
X_n^+	$X_1 + \cdots + X_n$
X	$(X_1, X_2, \cdots, X_n)'$
$\underline{\mathrm{VaR}}, \overline{\mathrm{VaR}}$	Best VaR, Worst VaR
$\underline{\mathrm{ES}}, \overline{\mathrm{ES}}$	Best ES, Worst ES
$\mathcal{B}_\alpha(X)$	$\frac{\overline{ES}_\alpha(X_n^+)}{\overline{VaR}_\alpha(X_n^+)}$: Ratio between Worst ES and Worst VaR
$\Delta_\alpha(X_n^+)$	$\frac{\mathrm{VaR}_\alpha(X_n^+)}{\sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)} \hat{=} \frac{\mathrm{VaR}_\alpha(X_n^+)}{\mathrm{VaR}_\alpha^+(X_n^+)}$: Ratio between the portfolio VaR and comonotonic VaR

Contents

Introduction

VaR and ES are the benchmark as risk quantitative tools, with their importance for financial institutions. There are currently several approaches for calculating values of risk measures. Embrechts et al.[1, Proposition 4] utilised Dual Bound method to compute $\overline{\text{VaR}}_\alpha(X)$ in homogeneous case. Whereas there is no method for computing $\underline{\text{VaR}}_\alpha(X)$, neither for $\overline{\text{ES}}_\alpha(X)$ nor $\underline{\text{ES}}_\alpha(X)$. Meanwhile, Dual Bound method is complicated in concept and weak in results stability. Wang's approach was proposed by Wang et al.[2], which is simpler than Dual Bound method but still difficult to widely apply. Moreover, it is impossible to apply Dual Bound method or Wang's approach in an inhomogeneous portfolio. Rearrangement Algorithm (RA), firstly proposed by Puccetti et al.[3, SECT.2], and Adaptive Rearrangement Algorithm (ARA) proposed by Hofert et al.[4, SECT.4], are out of the limitation of the type of portfolios.

The rest of the thesis is organized in four chapters. In Chapter 1, we review the preliminaries of VaR and ES, including the definition of $\text{TVaR}_\alpha(X)$, $\text{CTE}_\alpha(X)$, $\underline{\text{VaR}}_\alpha$(X), $\overline{\text{VaR}}_\alpha$(X), $\underline{\text{ES}}_\alpha$(X), $\overline{\text{ES}}_\alpha$(X), etc. Meanwhile, we introduce the concept of Comonotonicity so that we can transfer the estimates $\overline{\text{ES}}_\alpha$(X) and $\text{VaR}^+_\alpha(X^+_n)$ to the sum in comonotonic case, i.e. $\sum_{i=1}^n \text{ES}_\alpha(X_i)$ and $\sum_{i=1}^n \text{VaR}_\alpha(X_i)$. We also introduce the definition made by Embrechts et al.[5]: $\Delta_\alpha(X^+_n)$, $\overline{\Delta}_\alpha(X^+_n)$ and $\mathcal{B}_\alpha(X^+_n)$ to better analyze the relationship and similarity between $\overline{\text{ES}}_\alpha(X^+_n)$, $\overline{\text{VaR}}_\alpha(X^+_n)$ and $\text{VaR}^+_\alpha(X^+_n)$.

In Chapter 2, we list the steps for Rearrangement Algorithm (RA) and Adaptive Rearrangement Algorithm (ARA). The ideas of these methods are simple, to rearrange each column so that every row of the input matrix has the minimal sum, i.e. a minimal average row sum. Appendix A shows an example of the column rearrangements for a 4×4 input matrix.

In Chapter 3 and 4, we put these algorithms into practice. We study the unusual marginal distributions: Pareto, Exponential, Lognormal Distributions. In Chapter 3, we start with homogeneous cases. We choose the number of risk type: $n = 30$ and the number of observations $N = 10^5$ for each type, where all the risks are identically distributed. The performances of RA and ARA are proved to be equally competent. However, these algorithms are weak in dealing with the heavy-tailed marginal distributions. The outcomes shown in Chapter 4 (imhomogeneous portfolios), expose their weakness again. Therefore, more research and improvements of RA and ARA are needed in the future.

Chapter 1 Preliminaries

1.1 Risk Measures

VaR (Value-at-Risk) and ES (Expected Shortfall) at confidence level α for a r.v. X with the cumulative distribution function (c.d.f) $\mathrm{F}_X(x)$ is defined as

$$\mathrm{VaR}_\alpha(X) = inf\left\{x \in \mathbb{R} : \mathbb{P}\left[X \le x\right] \ge \alpha\right\}, \text{where } \alpha \in (0,1), X \in L^0, \tag{1.1}$$

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\beta(X)\,\mathrm{d}\beta, \text{where } \alpha \in (0,1), X \in L^0. \tag{1.2}$$

In accordance with the book of Habart et al.[6, Cap.2.4,page 26], the Tail-VaR (TVaR) at confidence level α, the Conditional Tail Expectation (CTE) at level α and Conditional VaR at level α are defined as follows:

$$\mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\xi(X)\,\mathrm{d}\xi, \text{where } \xi = \mathrm{F}_X(x), \tag{1.3}$$

$$\mathrm{CTE}_\alpha(X) = \mathrm{E}\left[X | X > \mathrm{VaR}_\alpha(X)\right], \tag{1.4}$$

$$\mathrm{CVaR}_\alpha(X) = \mathrm{E}\left[X - \mathrm{VaR}_\alpha(X) | X > \mathrm{VaR}_\alpha(X)\right]. \tag{1.5}$$

It is easy to see that

$$\mathrm{CTE}_\alpha(X) = \mathrm{CVaR}_\alpha(X) + \mathrm{VaR}_\alpha(X). \tag{1.6}$$

In Habart et al.'s book[6, Eq.2.34,page 26], ES is defined in Eq.1.7 to guarantee the nonnegativity of it. It is because that we usually discuss the risk distribution in loss but not profit. Since the two notions for ES are equivalent, we adopt the previous and most widely-used definition of ES, with the same definition of TVaR.

$$\mathrm{ES}_\alpha(X) = \mathrm{E}[max(X - VaR_\alpha(X), 0)]. \tag{1.7}$$

Let $x = \mathrm{VaR}_\xi(X)$. The equation can be written as

$$\begin{aligned}\mathrm{TVaR}_\alpha(X) &= \frac{1}{1-\alpha}\int_{\mathrm{VaR}_\alpha(X)}^\infty x\,\mathrm{d}\,\mathrm{F}_X(x) \\ &= \frac{1}{1-\alpha}\left[\mathrm{E}(X) - \int_0^{\mathrm{VaR}_\alpha(X)} x\,\mathrm{d}\,\mathrm{F}_X(x)\right].\end{aligned} \tag{1.8}$$

Since $\mathrm{TVaR}_\alpha(X)$ is the function of α, then $\mathrm{TVaR}_0(X) = \mathrm{TVaR}_\alpha(X)|_{\alpha=0} = \mathrm{E}(X)$ is concluded. As Habart et al.[6] mentioned, it is possible to obtain that TVaR is a decreasing function of α, therefore,

$$\mathrm{TVaR}_\alpha(X) \ge \mathrm{TVaR}_0(X) = \mathrm{E}(X). \tag{1.9}$$

Suppose $X_i, i = 1, \cdots, n$, as the single risk (the losses or profits over a period), with their corresponding loss probability density functions (p.d.f(s)) $X_i \sim \mathrm{f_i}, i = 1, \cdots, n$, and their cumulative distribution functions (c.d.f(s)) $X_i \overset{d}{\sim} \mathrm{F_i}$. Denote the aggregate risk $X_n^+ \hat{=} X_1 + \cdots + X_n$, and $\mathbf{X} \hat{=} (X_1, X_2, \cdots, X_n)'$. Note that the dependence structure between $X_i, i = 1, \cdots, n$, is unknown.

Now define the bounds for $\mathrm{VaR}_\alpha(X_n^+)$ and $\mathrm{ES}_\alpha(X_n^+)$, as well as the concept of Comonotonicity.

Definition 1.1.1. *Denote F as the joint probability density function of $\mathbf{X} = (X_1, X_2, \cdots, X_n)'$.*

Best VaR is defined as:

$$\underline{\mathrm{VaR}}_\alpha(X_n^+) = \inf\left\{\mathrm{VaR}_\alpha(X_1^F + \cdots + X_n^F)\right\}, \tag{1.10}$$

Worst VaR is defined as:

$$\overline{\mathrm{VaR}}_\alpha(X_n^+) = \sup\left\{\mathrm{VaR}_\alpha(X_1^F + \cdots + X_n^F)\right\}, \tag{1.11}$$

Best ES is defined as:

$$\underline{\mathrm{ES}}_\alpha(X_n^+) = \inf\left\{\mathrm{ES}_\alpha(X_1^F + \cdots + X_n^F)\right\}, \tag{1.12}$$

Worst ES is defined as:

$$\overline{\mathrm{ES}}_\alpha(X_n^+) = \sup\left\{\mathrm{ES}_\alpha(X_1^F + \cdots + X_n^F)\right\}. \tag{1.13}$$

Definition 1.1.2 (Comonotonicity)**.** **X** *is called **comonotonic** if there is a group of increasing functions $\psi_\mathrm{i}, i = 1, \cdots, n$, such that for a r.v. Y, we have*

$$X_i = \psi_\mathrm{i}(Y) \text{ almost surely, } i = 1, \cdots, n.$$

Since all $\psi_\mathrm{i}(Y), i = 1, \cdots, n$ are monotonic increasing of Y, we explained the notion 'comonotonic': common-monotonic.

See Embrechts et al.'s paper (2014) [5] for these definitions. Under the comonotonic condition, we have:

1. VaR is comonotonic additive, i.e.

$$\mathrm{VaR}_\alpha(X_n^+) = \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i). \tag{1.14}$$

   Now denote the comonotonic VaR as $\mathrm{VaR}_\alpha^+(X_n^+)$. Therefore,

$$\mathrm{VaR}_\alpha^+(X_n^+) = \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i).$$

2. ES is comonotonic additive, i.e.

$$\mathrm{ES}_\alpha(X_n^+) = \sum_{i=1}^n \mathrm{ES}_\alpha(X_i). \tag{1.15}$$

   For the proof of Eq.1.14 and Eq.1.15, see McNeil et al.'s book[7, Theorem 6.15].

3. Since ES is subadditive, then

$$\mathrm{ES}_\alpha(X_n^+) \leq \sum_{i=1}^{n} \mathrm{ES}_\alpha(X_i). \tag{1.16}$$

Combined with Eq.1.15 and Eq.1.16, we have

$$\overline{\mathrm{ES}}_\alpha(X_n^+) = \sum_{i=1}^{n} \mathrm{ES}_\alpha(X_i) = \mathrm{ES}_\alpha(X_n^+). \tag{1.17}$$

As Embrechts et al.(2014)[5] defined in their paper, the two super-additivity ratios are shown below:

1. The super-additivity ratio for X_n^+ of the model:

$$\Delta_\alpha(X_n^+) = \frac{\mathrm{VaR}_\alpha(X_n^+)}{\sum_{i=1}^{n} \mathrm{VaR}_\alpha(X_i)} = \frac{\mathrm{VaR}_\alpha(X_n^+)}{\mathrm{VaR}_\alpha^+(X_n^+)}. \tag{1.18}$$

$\Delta_\alpha(X_n^+)$ represents the diversification between the portfolio VaR and the comonotonic VaR.

2. The upper-bound of super-additivity ratio (i.e. the worse ratio) for X_n^+:

$$\overline{\Delta}_\alpha(X_n^+) = \frac{\overline{\mathrm{VaR}}_\alpha(X_n^+)}{\mathrm{VaR}_\alpha^+(X_n^+)}. \tag{1.19}$$

As P.Embrechts et al.(2013)[1, page 2755] implied, $\overline{\Delta}_\alpha(X_n^+)$ represents the percent of VaR to be super-additive as the function of n (the number of marginal risks).

3. The ratio between worst ES and worst VaR for X_n^+:

$$\mathcal{B}_\alpha(X_n^+) = \frac{\overline{\mathrm{ES}}_\alpha(X_n^+)}{\overline{\mathrm{VaR}}_\alpha(X_n^+)} = \frac{\sum_{i=1}^{n} \mathrm{ES}_\alpha(X_i)}{\overline{\mathrm{VaR}}_\alpha(X_n^+)}. \tag{1.20}$$

$\mathcal{B}_\alpha(X_n^+)$ represents the diversification between worst portfolio ES and worst portfolio VaR.

1.2 Distributions

1.2.1 Pareto Distribution

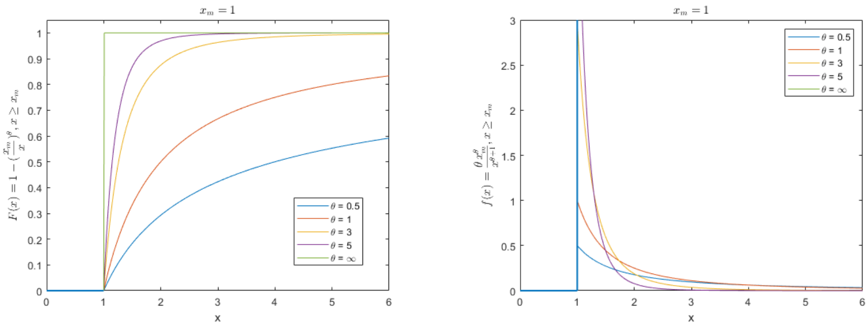
The random variable $X \sim$ Pareto$(x; x_m, \theta)$ if its c.d.f is given by:

$$\mathrm{F}(x) = \begin{cases} 1 - (\frac{x_m}{x})^\theta, & x \geq x_m, \\ 0, & x < x_m, \end{cases} \quad \text{where } x_m = \min\{x : X = x\}, \theta > 0. \tag{1.21}$$

$\theta > 0$ is called the shape parameter / tail index.

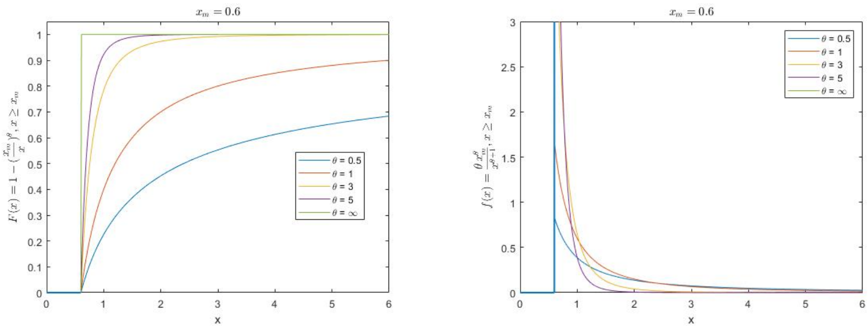
The p.d.f is

$$\mathrm{f}(X) = \begin{cases} \frac{\theta\, x_m^\theta}{x^{\theta+1}} & x \geq x_m, \\ 0, & x < x_m. \end{cases} \quad \text{where } x_m = \min\{x : X = x\}, \theta > 0. \tag{1.22}$$



(a) $x_m = 1$ c.d.f **(b)** $x_m = 1$ p.d.f

Figure 1.1: Pareto Distribution



(a) $x_m = 0.6$ c.d.f **(b)** $x_m = 0.6$ p.d.f

Figure 1.2: Pareto Distribution

The figures for c.d.f(s) and p.d.f(s) of Pareto Distributions are shown in Fig.1.1.

Take $x_m = 1$ as an example. As θ goes greater, F(x) converges to 1 more rapidly. On the contrary, as θ goes smaller, the fatter the tail of its p.d.f will be. Then fix θ. The smaller x_m is, the faster its p.d.f will tend to 0.

It can be proved that its E(X) and Var(X) are shown below:

$$\mathrm{E}(X) = \begin{cases} \infty, & \theta \leq 1, \\ \frac{\theta}{\theta-1} x_m, & \theta > 1. \end{cases} \tag{1.23}$$

$$\mathrm{Var}(X) = \begin{cases} \infty, & \theta \leq 2, \\ \frac{\theta\, x_m^2}{(\theta-1)^2(\theta-2)} x_m, & \theta > 2. \end{cases} \tag{1.24}$$

Similarly, its theoretical VaR$_\alpha(X)$ and theoretical ES$_\alpha(X)$ can be easily verified:

$$\mathrm{VaR}_\alpha(X) = x_m\,(1-\alpha)^{-\frac{1}{\theta}}. \tag{1.25}$$

$$\begin{aligned} \mathrm{ES}_\alpha(X) &= \int_\alpha^1 \frac{1}{1-\alpha} x_m (1-\beta)^{-\frac{1}{\theta}}\, \mathrm{d}\beta \\ &= \frac{\theta x_m}{1-\theta}(1-\alpha)^{-\frac{1}{\theta}} \\ &= \frac{\theta}{\theta-1}\,\mathrm{VaR}_\alpha(X),\ \theta > 1. \end{aligned} \tag{1.26}$$

α	$\theta = 1.1$	$\theta = 1.5$	$\theta = 3$	$\theta = 5$
0.99	11.00000	3.000000	1.500000	1.250000
0.995	11.00000	3.000000	1.500000	1.250000
0.999	11.00000	3.000000	1.500000	1.250000
$\alpha \to 1$	11.00000	3.000000	1.500000	1.250000

Table 1.1: ES/VaR Ratio for Pareto(θ;$x_m = 1$) Distributions

Therefore,

$$\frac{\mathrm{ES}_\alpha(X)}{\mathrm{VaR}_\alpha(X)} = \frac{\theta}{\theta - 1}. \tag{1.27}$$

Tab.1.1, Tab.1.2 and Tab.1.3 show the ES/VaR ratio for the three distributions in condition of different parameters. Since the ratio is irrelevant with α from Eq.1.27, it remains the same when different α is taken. The ES/VaR Ratio for Exponential Distribution can be easily confirmed (see Tab.1.2), which is proved to be the same as Embrechts et al.'s work [see 5, Tab.3, page 39]. As θ grows larger, theoretical ES and theoretical VaR will become very close.

1.2.1.1 Alternative Forms of Pareto Distribution

Another form of Pareto Distribution is given by:

Supoose $X \sim$ Pareto($x; \beta, \theta$),

$$\begin{aligned} \mathrm{F}(x) &= 1 - (1 + \frac{x}{\beta})^{-\theta}, \\ &= 1 - (\frac{\beta}{x + \beta})^{\theta}. \end{aligned} \quad \text{where } x \geq 0, \theta > 0, \beta > 0. \tag{1.28}$$

Similarly, it can be proved that:

$$\mathrm{E}(X) = \frac{\beta}{\theta}. \tag{1.29}$$

$$\mathrm{Var}(X) = \frac{\beta^2\,(\theta + 2)}{\theta^2\,(\theta - 2)},\ \ \theta > 2. \tag{1.30}$$

$$\mathrm{VaR}_\alpha(X) = \beta\,(1 - \alpha)^{\theta} - \beta. \tag{1.31}$$

$$\mathrm{ES}_\alpha(X) = \frac{\beta}{\theta + 1}\,(1 - \alpha)^{\theta} - \beta. \tag{1.32}$$

Therefore,

$$\frac{\mathrm{ES}_\alpha(X)}{\mathrm{VaR}_\alpha(X)} = \frac{1}{\theta + 1}\,(1 - \frac{\theta\,\beta}{\mathrm{VaR}_\alpha(X)}). \tag{1.33}$$

This form is utilised in the R package qrmtools. See details in the description of R Package 'qrmtools'.

Eq.1.21 and Eq.1.28 are equivalent in their variances but not in their means. See Habart-Corlosquet et al. [6, Cap.3.4, page 57] for details. Take $\beta = 1$, then we get the form used in Embrechts et al.'s paper [see 5, SECT.4, page 37].

1.2.1.2 General Pareto Distribution(GPD)

The general Pareto Distribution is given by:

$$\mathrm{F}(x) = \begin{cases} 1-(1+\frac{\xi\, x}{\beta})^{-\frac{1}{\xi}}, & \begin{cases} x \geq 0, & \text{if}\, \xi \geq 0, \\ x \in [0, -\frac{\beta}{\xi}], & \text{if}\, \xi < 0, \end{cases} \\ 1-e^{-\frac{x}{\beta}}, & \xi = 0, \end{cases} \quad \text{where } \beta > 0. \tag{1.34}$$

1.2.2 Exponential Distribution

Supoose $X \sim \mathrm{Exp}(x;\theta)$,

$$\mathrm{f}(x) = \begin{cases} \theta\, e^{-\theta\, x}, & x > 0, \\ 0, & x \leq 0, \end{cases} \quad \text{where } \theta > 0. \tag{1.35}$$

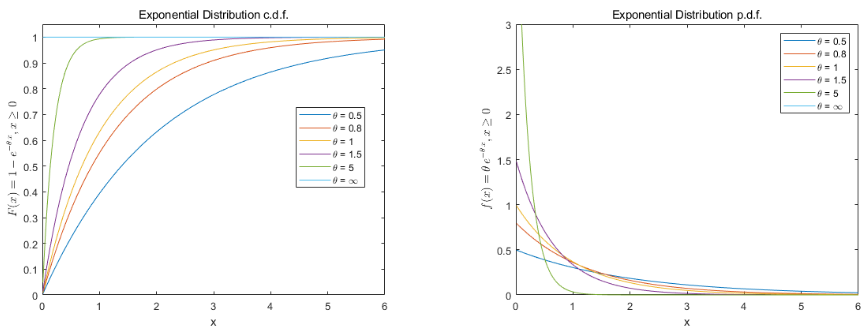
Then its theoretical VaR and theoretical ES are:

$$\mathrm{VaR}_{\alpha}(X) = -\frac{ln(1-\alpha)}{\theta}. \tag{1.36}$$

$$\mathrm{ES}_{\alpha}(X) = \frac{1}{\theta} + \mathrm{VaR}_{\alpha}(X). \tag{1.37}$$

Therefore, its ES/VaR Ratio is concluded. See Tab.1.2. ES/VaR Ratio is not afftectd by any change of θ.

$$\frac{\mathrm{ES}_{\alpha}(X)}{\mathrm{VaR}_{\alpha}(X)} = 1 + \frac{1}{\theta\ \mathrm{VaR}_{\alpha}(X)}. \tag{1.38}$$



(a) c.d.f **(b)** p.d.f

Figure 1.3: Exponential Distribution

The c.d.f(s) and p.d.f(s) of Exponential Distribution are shown in Fig.1.3. The smaller the θ is, the heavier the tail of its p.d.f will be.

1.2.3 Lognormal Distribution

Supoose $lnX \sim \mathrm{N}(\mu, \sigma^2)$,

$$\mathrm{f}(x) = \begin{cases} \frac{1}{\sigma\, x\sqrt{2\,\pi}}\, e^{-\frac{(lnx-\mu)^2}{2\,\sigma^2}}, & x > 0, \\ 0, & x \leq 0. \end{cases} \tag{1.39}$$

α	$\theta = 0.5$	$\theta = 1$	$\theta = 1.5$	$\theta = 2$
0.99	1.217147	1.217147	1.217147	1.217147
0.995	1.188739	1.188739	1.188739	1.188739
0.999	1.144765	1.144765	1.144765	1.144765
$\alpha \to 1$	1.000000	1.000000	1.000000	1.000000

Table 1.2: ES/VaR Ratio for Exponential(θ) Distributions

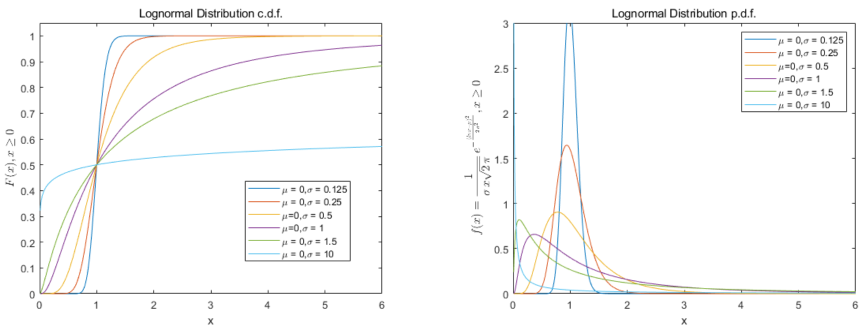
Then we have,

$$\mathrm{E}(X) = e^{\mu + \frac{\sigma^2}{2}}. \tag{1.40}$$

$$\mathrm{Var}(X) = (e^{\sigma^2} - 1)\, e^{2\,\mu + \sigma^2}. \tag{1.41}$$

The c.d.f(s) and p.d.f(s) of Lognormal Distribution are shown in Fig.1.4 and Fig.1.5. Fix μ, the smaller the σ is, the more rapidly the c.d.f will tend to 1; the heavier the tail of p.d.f will be. Fix σ, the larger the μ is, the slower the c.d.f will grow to 1; the heavier the tail of p.d.f will be.

Since the explicit form of c.d.f(s) is impossible to be concluded from p.d.f(s), VaR and ES cannot be induced by c.d.f(s). However, we can adapt the linear transformation inspired by Normal Distribution to solve out the theoretical VaR and theoretical ES of Lognormal Distribution [8, see SECT.2,page 3]. ES/VaR Ratios are shown in Tab.1.1. The shape of p.d.f that is close to Normal Distribution reveals a smaller diversification between ES and VaR. Meanwhile, as α close to 1, the ratio is more likely to be 1.



(a) $\mu = 0$ c.d.f. **(b)** $\mu = 0$ p.d.f.

Figure 1.4: Lognormal Distribution

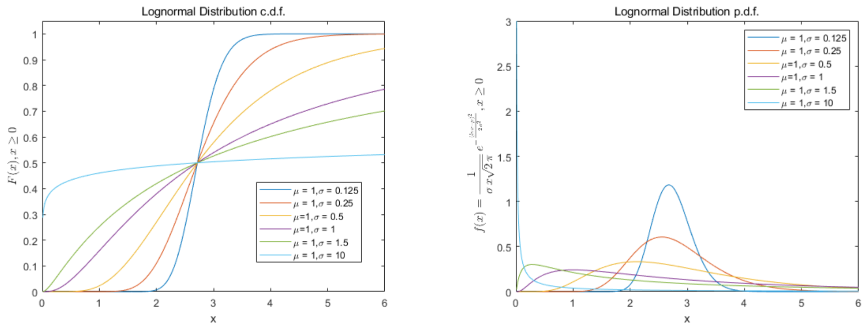
Since

$$\alpha = \int_{-\infty}^{\mathrm{VaR}_\alpha(X)} f(t)\, \mathrm{d}t = \int_{-\infty}^{\mathrm{VaR}_\alpha(Z)} \varphi(h)\, \mathrm{d}h, \tag{1.42}$$

where $Z \sim \mathrm{N}(0,1)$, $\varphi(z)$ is the p.d.f of Z. Denote $\sigma_\alpha \hat{=} \mathrm{VaR}_\alpha(Z)$, i.e. the quantile of Standard Normal Distribution.

We have

$$P(X < x) \hat{=} P(X < \mathrm{VaR}_\alpha(X)) = P(\frac{X - \mathrm{E}(X)}{\sqrt{\mathrm{Var}(X)}} < \frac{\mathrm{VaR}_\alpha(X) - \mathrm{E}(X)}{\sqrt{\mathrm{Var}(X)}}) = \alpha. \tag{1.43}$$



(a) $\mu = 1$ c.d.f **(b)** $\mu = 1$ p.d.f

Figure 1.5: Lognormal Distribution

Then

$$\begin{aligned} \frac{\mathrm{VaR}_\alpha(X) - \mathrm{E}(X)}{\sqrt{\mathrm{Var}(X)}} &= \sigma_\alpha, \\ \mathrm{VaR}_\alpha(X) &= \sigma_\alpha \sqrt{\mathrm{Var}(X)} + \mathrm{E}(X). \end{aligned} \tag{1.44}$$

Theoretical ES is concluded in Eq.1.45:

$$\begin{aligned} \mathrm{ES}_\alpha(X) &= \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_\xi(X)\, \mathrm{d}\xi \\ &= \frac{1}{1-\alpha} \int_\alpha^1 \sigma_\xi \sqrt{\mathrm{Var}(X)} + \mathrm{E}(X)\, \mathrm{d}\xi \\ &= e^{\mu + \frac{\sigma^2}{2}} + \frac{\sqrt{(e^{\sigma^2} - 1)e^{2\mu+\sigma^2}}}{1-\alpha} \int_\alpha^1 \sigma_\xi\, \mathrm{d}\xi. \end{aligned} \tag{1.45}$$

α	$\mu = 0, \sigma = 1$	$\mu = 0, \sigma = 0.25$	$\mu = 0, \sigma = 0.5$	$\mu = 0, \sigma = 3$
0.99	1.4870367	1.0918496	1.2003638	6.2845204
0.995	1.4435193	1.0852625	1.1849592	5.2686648
0.999	1.3724330	1.0740254	1.1590193	3.9325902
$\alpha \to 1$	1.0000000	1.0000000	1.0000000	1.0000000

Table 1.3: ES/VaR Ratio for Lognormal(μ,σ) Distributions

Chapter 2 Rearrangement Algorithm (RA) and Adaptive Rearrangement Algorithm (ARA)

In this chapter, we introduce RA and ARA to compute the lower bound and upper bound of best / worst VaR and best / worst ES. Rearrangement Algorithm (RA) was proposed by Puccetti and Rüschendorf [9]. P.Embrechts et al.(2013)[1, SECT.2.2] greatly improved RA in their papers. Based on the previous researches, Hofert et al.[4, SECT.4] concluded a revised version of RA: Adaptive Rearrangement Algorithm (ARA). These methods can be implemented by the R package qrmtools.

To better understand the workforce of RA() and ARA(), see Appendix A for the example of rearranging a $N \times n = 4 \times 4$ matrix.

2.1 Rearrangement Algorithm (RA)

The Rearrangement Algorithm (RA) is applied to minimize the distribution of X_n^+, i.e. to approximate the $\underline{\mathrm{VaR}}_\alpha(X_n^+), \overline{\mathrm{VaR}}_\alpha(X_n^+)$ and $\underline{\mathrm{ES}}_\alpha(X_n^+)$* for arbitrary marginal distribution functions, in both homogeneous and inhomogeneous cases.

Before we list the details of RA, we summarize the definition made by Hofert et al.[4, ALGO 3.1]:

Definition 2.1.1 (Oppositely Ordered)**.** *Two columns* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *are* ***oppositely ordered*** *if* $\forall i, j \in 1, \cdots, n$, $(x_i - x_j)(y_i - y_j) \leq 0$.

The specific description of RA is shown below (see [4, ALGO 3.1]):

1. Construct $X_n^+ = X_1 + \cdots + X_n$. The quantile function of $X_j, j \in 1, \cdots, n$ is denoted as F_j^{-1}. Fix an absolute convergence tolerance $\epsilon \geq 0$.
2. For $i \in 1, \cdots, N$ (N: the number of discretization points for each risk type), define
$$\underline{x}_{ij}^\alpha \hat{=} F_j^{-1}(\alpha + \frac{(1-\alpha)(i-1)}{N})$$
to form the matrix $\underline{X}^\alpha$ so that it can approximate the values for risk measures mentioned (see 2.1) from below.
Define
$$\overline{x}_{ij}^\alpha \hat{=} F_j^{-1}(\alpha + \frac{(1-\alpha)\,i}{N})$$
to form the matrix $\overline{X}^\alpha$. Since $F_j^{-1}(1) = \infty$, the endpoint is adjusted to
$$F_N^{-1}(\alpha + \frac{(1-\alpha)(N-\frac{1}{2})}{N}).$$

*RA() and ARA() both provide three methods:best.VaR, worst.VaR and best.ES. There is no direct method to compute $\overline{\mathrm{ES}}_\alpha(X_n^+)$.

Form the matrix

$$\overline{X}^{\alpha} = (\overline{x}_{ij}^{\alpha})$$

so that it can approximate the values for risk measures mentioned (see 2.1) from above.

3. For each $j \in 1, \cdots, n$, rearrange each element of the j-th column so that every element of this column is oppositely ordered compared with the row sum except for itself. We call the resulting matrix $\underline{Y}^{\alpha}$ and $\overline{Y}^{\alpha}$.*

4. Denote the minimal row sum of matrix X as $s(X) \hat{=} \min_{1 \le i \le N} \sum_{j=1}^{n} x_{ij}$. If

$$s(\overline{Y}^{\alpha}) - s(\overline{X}^{\alpha}) \le \epsilon$$

and

$$s(\underline{Y}^{\alpha}) - s(\underline{X}^{\alpha}) \le \epsilon,$$

then the estimated bound of each risk measure ((see 2.1)) is set as

$$[\underline{s}_N \hat{=} s(\underline{Y}^{\alpha}), \overline{s}_N \hat{=} s(\overline{Y}^{\alpha})].$$

2.2 Adaptive Rearrangement Algorithm (ARA)

ARA was proposed by Hofert et al.[4, ALGO 4.1] and works similarly to RA() except for several parts.

1. Construct $X_n^+ = X_1 + \cdots + X_n$. The quantile function of $X_j, j \in 1, \cdots, n$ is denoted as F_j^{-1}. Fix a vector of convergence tolerance (relative) $\epsilon = (\epsilon_1, \epsilon_2)$, where ϵ_1 is defined as the individual convergence and ϵ_2 as the joint convergence. Take $N = 2^k$, where k is the maximum of the times for rearrangements, $k \in \mathbb{N}$.

2. For $i \in 1, \cdots, N$, define

$$\underline{x}_{ij}^{\alpha} \hat{=} F_j^{-1}(\alpha + \frac{(1-\alpha)(i-1)}{N})$$

to form the matrix $\underline{X}^{\alpha}$ so that it can approximate the values for risk measures mentioned (see 2.1) from below.

Define

$$\overline{x}_{ij}^{\alpha} \hat{=} F_j^{-1}(\alpha + \frac{(1-\alpha)\, i}{N})$$

*This step aims to minimize the variance of the group consisting of each row sum of the matrix.

to form the matrix $\overline{X}^{\alpha}$. Similar to RA, set the endpoint as

$$F_N^{-1}(\alpha + \frac{(1-\alpha)(N-\frac{1}{2})}{N}).$$

Form the matrix

$$\overline{X}^{\alpha} = (\overline{x}_{ij}^{\alpha})$$

so that it can approximate the values for risk measures mentioned (see 2.1) from above.

3. For each $j \in 1, \cdots, n$, rearrange each element of the j-th column so that it is oppositely ordered compared with the sum of the elements except for the j-th one in each row. We call the resulting matrix $\underline{Y}^{\alpha}$ and $\overline{Y}^{\alpha}$. When the number of rearrangements reaches or when the following two conditions hold after each j-th column has been rearranged:

$$|\frac{s(\underline{Y}^{\alpha}) - s(\underline{X}^{\alpha})}{s(\underline{X}^{\alpha})}| \leq \epsilon_1, \tag{2.1}$$

$$|\frac{s(\overline{Y}^{\alpha}) - s(\overline{X}^{\alpha})}{s(\overline{X}^{\alpha})}| \leq \epsilon_1, \tag{2.2}$$

then the rearrangement of j-th column is finished.

4. If

$$|\frac{s(\overline{Y}^{\alpha}) - s(\underline{Y}^{\alpha})}{s(\overline{Y}^{\alpha})}| \leq \epsilon_2$$

holds, then ARA is finished.

Compared with RA, ARA adopts the relative tolerances in two ways. The single tolerance is used after rearrangements of the j-th column finish, and the joint one is used for checking the relative spread between $s(\overline{Y}^{\alpha}$ and $s(\underline{Y}^{\alpha})$.

Since $\overline{\mathrm{ES}}_{\alpha}(X_n^+)$ cannot be calculated through RA() or ARA() because of the lack of direct method*, we use Eq.1.17, i.e. the sum of $\mathrm{X_i}, i = 1, \cdots, n.$ to estimate $\overline{\mathrm{ES}}_{\alpha}(X_n^+)$. Moreover, as BCBS[10] mentioned, $\mathrm{VaR}_{\frac{1+\alpha}{2}}(X_n^+)$ can be compared with $\mathrm{ES}_{\alpha}(X_n^+)$. Therefore, in another way, we can compare the results of $\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$ with the comonotonic ES: $\sum_{i=1}^{n} \mathrm{ES}_{\alpha}(X_i)$, where $\mathrm{ES}(X_i)$ is concluded from the theoretical ES_{α}, see Eq.1.32 and Eq.1.37.

*$\overline{\mathrm{ES}}_{\alpha}(X_n^+)$ is usually regarded as infinity in academy. Therefore, we can explain why there is no direct method in qrmtools package.

Chapter 3 VaR / ES Bounds for Homogeneous Portfolios

In the next two sections, we assume that the r.v.s in their loss vector $\mathbf{X} = (X_1, X_2, \cdots, X_n)'$ with their marginal distribution function $(\mathrm{F_1, F_2, \cdots, F_n})'$ of unknown dependence.

For comparability, fix the number of risk types: $n = 30$ and the number of observation values for each risk type: $N = 10^5$*. In RA(), we choose the absolute tolerance $\epsilon = 10^{-4}$. Concerning the relative tolerance for ARA(), we adopt the conservative choice $\epsilon = (0.001, 0.01)$ recommended by Hofert et al.[4, SECT.4.1]. All the values were computed with R and MATLAB on the laptop: 2.50GHz Intel CORE i5, 4GB RAM.

3.1 Homogeneous Risk Portfolios

Firstly, we begin with different homogeneous portfolios.

1. Pareto Distribution:
   - 1.1 Hom-Par-HT1: nPareto(1.1), with all heavy-tailed marginal distributions.
   - 1.2 Hom-Par-HT2: nPareto(1.5), with all relatively heavy-tailed marginal distributions.
   - 1.3 Hom-Par-LT1: nPareto(2), with all relatively light-tailed marginal distributions.
   - 1.4 Hom-Par-LT2: nPareto(5), with all light-tailed marginal distributions.
2. Exponential Distribution:
   - 2.1 Hom-Exp-HT1: nExp(0.5), with all heavy-tailed marginal distributions.
   - 2.2 Hom-Exp-HT2: nExp(0.8), with all relatively heavy-tailed marginal distributions.
   - 2.3 Hom-Exp-LT1: nExp(1), with all relatively light-tailed marginal distributions.
   - 2.4 Hom-Exp-LT2: nExp(1.5), with all light-tailed marginal distributions.
3. Lognormal Distribution:
   - 3.1 Hom-Logn-LT1: nLogn(0,0.25), with light-tailed marginal distributions (close to Normal Distribution).
   - 3.2 Hom-Logn-LT2: nLogn(0,3), with relatively light-tailed marginal distributions.
   - 3.3 Hom-Logn-HT1: nLogn(0,0.5), with relatively heavy-tailed marginal distributions.
   - 3.4 Hom-Logn-HT2: nLogn(0,1), with heavy-tailed marginal distributions.

*This is the decision made by RAM. On the laptop with 4GB RAM, $N = 10^6$ is tested to be out of storage in R procession. Meanwhile, $N = 10^5$ is also large enough as the number of observations for each risk type.

3.1.1 Homogeneous Pareto Cases

From Tab.3.1 to Tab.3.4, it is easy to conclude that

1. For portfolios consisting of heavy-tailed marginals (see Tab.3.1 and Tab.3.2), results for $\overline{\mathrm{VaR}}_\alpha(X_n^+)$ and $\underline{\mathrm{ES}}_\alpha(X_n^+)$ in any value θ by RA() or ARA() are not performing well because of the large spread between their upper and lower bound.

2. When $\alpha = 0.999$, there is no answer (i.e. "N.A."*) to $\underline{\mathrm{ES}}_\alpha(X_n^+)$ for Pareto Distribution by ARA() except for Pareto(5). The possible reasons are as follows:

   1) max.ra† is set to be equal to N by default. However, after many tests, the range for $\underline{\mathrm{ES}}_\alpha(X_n^+)$ remains unchanged as max.ra increases. Meanwhile, increasing max.ra is time-wasting.

   2) As Aas et al.[11, Tab.3,page 704] stated in their paper, the range for $\overline{\mathrm{VaR}}_\alpha(X_n^+)$ shrinks as N increases. The limited RAM of the laptop makes it impossible to enlarge N. Therefore, the procession paused before the convergence of its upper and lower-bound was obtained.

3. In general, RA() and ARA() perform well in the light-tailed marginals case. Results for $\overline{\Delta}_\alpha(X_n^+)$ and $\mathcal{B}_\alpha(X_n^+)$ by the two methods are fairly close to each other in any case. For light-tailed portfolios, $\overline{\Delta}_\alpha(X_n^+)$ and $\mathcal{B}_\alpha(X_n^+)$ are closer to 1 compared with the heavy-tailed.

Overall, the results for $\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$ and $\overline{\mathrm{ES}}_\alpha(X_n^+)$ are not close due to the heavy tails.

Pareto$(\theta = 1.1)$, n = 30			RA						
α	Compile Time	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\mathrm{VaR}^+_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	49.52s	66.4325	4052.4936	13849.4411 (2537.359 - 25161.523)	9439.1332	858.1030	7636.3585	4.72261908	2.329216066
0.99	53.94s	84.0223	9360.5095	70021.7304 (4971.901 - 135071.560)	21711.7964	1973.7997	17604.0792	4.742380648	2.319510108
0.995	49.71s	122.4430	17604.0794	252344.1277 (8034.99 - 496653.27)	40771.75353	3706.5230	33084.3646	4.749486082	2.316040084
0.999	27.03s	530.2733	76139.3170	5297381.7606 (21017.61 - 10573745.91)	176111.0746	16010.0977	143005.4519	4.755705957	2.313010959
			ARA						
α		$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$			$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975		66.4339	4052.2862 (4038.721 - 4065.851)	12881.8297 (3148.818 - 22614.841)	9439.1332			4.722377384	2.329335277
0.99		84.0339	9360.0738 (9328.803 - 9391.345)	62469.7643 (6491.537 - 118447.992)	21711.7964			4.742159906	2.319618078
0.995		122.2244	17603.1839 (17544.59 - 17661.78)	221282.2187 (11044.28 - 431520.16)	40771.75353			4.749244481	2.316157904
0.999		530.8376	76134.2787 (75879.25 - 76389.30)	N.A.	176111.0746			4.755391262	2.313164026

Table 3.1: RA and ARA Results for $X_n = n\,\mathrm{Pareto}(1.1)$,$n = 30$

Pareto$(\theta = 1.5)$, n = 30			RA						
α	Compile Time	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\mathrm{VaR}^+_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	58.38s	35.3168	925.2117	514.7298 (275.4073 - 754.0524)	1052.6464	350.8821	1486.3041	2.636816469	1.13773572
0.99	52.61s	41.3233	1729.5150	1570.1098 (468.0207 - 2672.1989)	1938.9912	646.3304	2763.0560	2.6758992	1.121118464
0.995	50.57s	44.9852	2763.0561	4181.9020 (682.9908 - 7680.8131)	3077.9567	1025.9856	4003.7001	2.693074932	1.113968225
0.999	30.17s	98.6726	8136.9454	52601.6794 (1448.725 - 103754.634)	9000.0000	3000.0000	12934.2178	2.712315133	1.106066166
			ARA						
α		$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$			$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975		35.3184	925.1895	439.7413 (301.7172 - 577.7654)	1052.6464			2.6367532	1.13776302
0.99		41.3247	1729.4240 (1720.841 - 1738.007)	1168.4097 (532.7065 - 1804.1129)	1938.9912			2.675758405	1.121177456
0.995		45.0183	2762.9143 (2749.293 - 2776.536)	2827.6978 (809.4664 - 4845.9311)	3077.9567			2.692936723	1.114025397
0.999		98.7499	8136.5023 (8096.661 - 8176.343)	N.A.	9000.0000			2.712167433	1.1061264

Table 3.2: RA and ARA Results for $X_n = n\,\mathrm{Pareto}(1.5)$,$n = 30$

*"N.A." appears when the R procession stops at if (iter == max.ra || tol.reached) break.

†max.ra means the maximal number for column rearrangements in R procession.

Pareto($\theta = 2, n = 30$)			RA						
α	Compile Time	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\mathrm{VaR}_\alpha^+(X_n^+)$	$\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	54.88s	21.8084	343.0951	79.8878	379.4733	189.7367	497.6362	1.808269565	1.106029494
0.99	51.55s	24.5455	559.9152	161.8381	600	300	804.266	1.866384	1.071590841
0.995	49.36s	26.0376	804.2661	315.9556	848.5281	424.2641	1149.8304	1.895673237	1.055034024
0.999	22.30s	30.5465	1835.4756	2326.6984 (231.5411 - 4421.8557)	1897.3666	948.6833	2608.1809	1.934761158	1.033719326
			ARA						
α		$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$			$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975		21.8093	343.0718	73.4514 (66.0793 - 80.8234)	379.4733			1.808146763	1.106104611
0.99		24.5461	559.8749	131.8742 (102.7291 - 161.0193)	600			1.866249667	1.071667974
0.995		26.0388	804.2091	224.7414 (142.3067 - 307.176)	848.5281			1.895538887	1.055108802
0.999		30.5072	1835.3642	N.A.	1897.3666			1.934643732	1.033782069

Table 3.3: RA and ARA Results for $X_n = n\,\mathrm{Pareto}(2)$,$n = 30$

Pareto($\theta = 5$), n = 30			RA						
α	Compile Time	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\mathrm{VaR}_\alpha^+(X_n^+)$	$\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	55.69s	6.4507	48.4199	7.6123	78.4230	62.7384	60.0809	0.771774873	1.619643296
0.99	51.79s	6.9273	64.1921	7.8553	94.1957	75.3566	78.1983	0.85184451	1.467403951
0.995	63.59s	7.1147	78.1983	8.3407	108.2025	86.5620	94.2872	0.90337914	1.383693673
0.999	17.78s	7.3881	119.2844	13.5973 (8.011618 - 19.183070)	149.2902	119.4322	141.4828	0.998762886	1.251548308
			ARA						
α		$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$			$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975		6.4513	48.3538	7.5710	78.4230			0.770721292	1.621857361
0.99		6.9277	64.1127	7.6990	94.1957			0.850790853	1.469221249
0.995		7.1453	78.1071	7.9336	108.2025			0.90232556	1.385309312
0.999		7.3887	119.0609	N.A.	149.2902			0.996891531	1.253897702

Table 3.4: RA and ARA Results for $X_n = n\,\mathrm{Pareto}(5)$,$n = 30$

3.1.2 Homogeneous Exponential Cases

Compared with Pareto cases, pure Exponential portfolios have a more regular performance with more specific results, no matter in the light-tailed or heavy-tailed case. The spread between the results for $\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$ and $\overline{\mathrm{ES}}_\alpha(X_n^+)$ are reasonable. See Tab.3.5 to Tab.3.8.

Exp($\theta = 0.5$), n = 30		RA							
α	Compile Time	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\mathrm{VaR}_\alpha^+(X_n^+)$	$\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	14.59s	54.3248	281.3324	60.0008	281.3328	221.3328	322.9212	1.271083183	1.000001422
0.99	13.92s	57.2090	336.3098	60.0011	336.3102	276.3102	377.8987	1.217145802	1.000001189
0.995	13.04s	58.4025	377.8987	60.0013	377.8990	317.8990	419.4875	1.188738247	1.000000794
0.999	11.19s	59.5851	474.4649	60.0017	474.4653	414.4653	516.0538	1.144763868	1.000000843
		ARA							
α		$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$			$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975		54.3495	281.1903	60.0382	281.3328			1.270441164	1.000506774
0.99		57.2153	336.1678	60.0517	336.3102			1.216631887	1.000423598
0.995		58.4057	377.7557	60.0619	377.8990			1.188288419	1.000379346
0.999		59.5922	474.3213	N.A.	474.4653			1.144417398	1.000303592

Table 3.5: RA and ARA Results for $X_n = n\,\mathrm{Exp}(0.5)$,$n = 30$

Exp($\theta = 0.8$), n = 30		RA							
α	Compile Time	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\mathrm{VaR}_\alpha^+(X_n^+)$	$\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	15.08s	33.9530	175.8327	37.5005	175.8330	138.3330	201.8258	1.271082822	1.000001706
0.99	13.4s	35.7556	210.1937	37.5007	210.1928	172.6939	236.1867	1.21714606	0.999995718
0.995	12.1s	36.5016	236.1867	37.5008	236.1869	198.6869	262.1797	1.18873816	1.000000847
0.999	11.25s	37.2407	296.5406	37.5011	296.5408	259.0408	322.5336	1.144764068	1.000000674
		ARA							
α		$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$			$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975		33.9559	175.7439	37.5238	175.8330			1.270440893	1.000506988
0.99		35.7596	210.1049	37.5323	210.1928			1.216631856	1.000418362
0.995		36.5036	236.0973	37.5387	236.1869			1.188288206	1.000379505
0.999		37.2451	296.4508	N.A.	296.5408			1.144417405	1.000303592

Table 3.6: RA and ARA Results for $X_n = n\,\mathrm{Exp}(0.8)$,$n = 30$

Exp($\theta = 1$), n = 30		RA							
α	Compile Time	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\mathrm{VaR}^+_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	15.95s	27.1624	140.6662	30.0004	140.6664	110.6664	161.4606	1.271083183	1.000001422
0.99	13.40s	28.6045	168.1549	30.0005	168.1551	138.1551	188.9493	1.217145802	1.000001189
0.995	12.92s	29.2013	188.9493	30.0006	188.9495	158.9495	209.7438	1.188737932	1.000001058
0.999	11.65s	29.7926	237.2325	30.0009	237.2327	207.234	258.0269	1.144756652	1.000000843

	ARA					
α	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	27.1647	140.5952	30.0191	140.6664	1.270441616	1.000506418
0.99	28.6077	168.0839	30.0258	168.1551	1.216631887	1.000423598
0.995	29.2029	188.8779	30.031	188.9495	1.188288733	1.000379081
0.999	29.7961	237.1607	N.A.	237.2327	1.144410184	1.000303592

Table 3.7: RA and ARA Results for $X_n = n\,\mathrm{Exp}(1)$,$n = 30$

Exp($\theta = 1.5$), n = 30		RA							
α	Compile Time	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\mathrm{VaR}^+_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	16.84s	18.1083	93.7775	20.0003	93.7776	73.7776	107.6404	1.271083823	1.00000095
0.99	13.56s	19.0697	112.1033	20.0004	112.1034	92.1034	125.9662	1.217146115	1.000000925
0.995	12.07s	19.4675	125.9662	20.0004	125.9663	105.9663	139.8292	1.188737775	1.00000117
0.999	10.55s	19.8617	158.1550	20.0006	158.1551	138.1551	172.0179	1.144764063	1.000000668

	ARA					
α	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	18.1098	93.7301	20.0127	93.7776	1.270441352	1.000506658
0.99	19.0718	112.0559	20.0172	112.1034	1.216631476	1.000423929
0.995	19.4686	125.9186	20.0206	125.9663	1.188288576	1.000379192
0.999	19.8641	158.1071	N.A.	158.1551	1.144417351	1.000303627

Table 3.8: RA and ARA Results for $X_n = n\,\mathrm{Exp}(1.5)$,$n = 30$

3.1.3 Homogeneous Lognormal Cases

The results for Lognormal portfolios are more case-dependent since the p.d.f images with different parameters differ. Tab.3.10 represents the portfolios consisting of the quasi-t or quasi-normal marginal distributions. The heavier the tail of marginal is, the more uncertain the estimation will be. The results for the spread between $\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$ and $\overline{\mathrm{ES}}_\alpha(X_n^+)$ are similar to homogeneous Pareto case. See Tab.3.9 and Tab.3.12.

Logn($\mu = 0, \sigma = 1$), n = 30			RA						
α	Compile Time	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\mathrm{VaR}^+_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	61.84s	42.1801	333.4213	50.7503	333.4440	212.9721	424.2747	1.565562977	1.000068084
0.99	60.29s	45.3467	456.8181	53.1974 (50.82295 - 55.57182)	456.8388	307.2142	569.1108	1.486969305	1.000045353
0.995	58.38s	46.8503	569.1107	57.7749 (52.21419 - 63.33559)	569.1311	394.2663	699.9319	1.443467611	1.000035805
0.999	17.81s	48.6052	905.0540	100.4798 (62.35641 - 138.60309)	905.0722	659.4655	1087.6841	1.372405339	1.000020148

		ARA				
α	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	42.1830	332.9006 (331.4082 - 334.3931)	50.4478	333.4440	1.563118057	1.001632322
0.99	45.3489	456.2005 (454.4037 - 457.9972)	52.0772	456.8388	1.484958982	1.001399205
0.995	46.8537	568.4141 (566.3582 - 570.4700)	54.9132 (53.40815 - 56.41815)	569.1311	1.441700785	1.001261364
0.999	48.6096	904.1360 (901.3693 - 906.9028)	N.A.	905.0722	1.371013303	1.001035502

Table 3.9: RA and ARA Results for $X_n = n\,\mathrm{Logn}(0, 1)$,$n = 30$

Logn($\mu = 0, \sigma = 0.25$), n = 30		RA							
α	Compile Time	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\mathrm{VaR}^+_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	15.36s	30.3606	54.0257	30.9523	54.0257	48.9690	57.4928	1.10326226	1.000000322
0.99	17.19s	30.6730	58.5954	30.9523	58.5955	53.6663	61.9902	1.091848187	1.000001315
0.995	13.15s	30.7963	61.9902	30.9524	61.9902	57.1200	65.3520	1.085262182	1.00000025
0.999	12.56s	30.9134	69.7675	30.9524	69.7676	64.9589	73.0984	1.074024521	1.000000864

	ARA					
α	$\underline{\mathrm{VaR}}_\alpha(X_n^+)$	$\overline{\mathrm{VaR}}_\alpha(X_n^+)$	$\underline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\mathrm{ES}}_\alpha(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	30.3367	54.0141	30.9516	54.0257	1.103025375	1.000215081
0.99	30.6494	58.5838	30.9570	58.5955	1.091632037	1.000199322
0.995	30.7733	61.9785	30.9604	61.9902	1.08505735	1.000189025
0.999	30.8942	69.7556	30.9604	69.7676	1.073841328	1.00017146

Table 3.10: RA and ARA Results for $X_n = n\,\mathrm{Logn}(0, 0.25)$,$n = 30$

Logn($\mu = 0, \sigma = 0.5$), n = 30		RA							
α	Compile Time	$\underline{\text{VaR}}_\alpha(X_n^+)$	$\overline{\text{VaR}}_\alpha(X_n^+)$	$\underline{\text{ES}}_\alpha(X_n^+)$	$\overline{\text{ES}}_\alpha(X_n^+)$	$\text{VaR}_\alpha^+(X_n^+)$	$\overline{\text{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	15.4s	32.3505	98.1077	33.9949	98.1080	79.9322	110.9753	1.227385725	1.000002819
0.99	14.79s	33.1738	115.2373	33.9951	115.2376	96.0022	128.8718	1.20036078	1.000002548
0.995	14.05s	33.5177	128.8718	33.9952	128.8721	108.7566	143.1364	1.184956516	1.000002296
0.999	14.24s	33.8653	163.0221	33.9955	163.0224	140.6555	178.8865	1.159017009	1.000001948

		ARA							
α		$\underline{\text{VaR}}_\alpha(X_n^+)$	$\overline{\text{VaR}}_\alpha(X_n^+)$	$\underline{\text{ES}}_\alpha(X_n^+)$	$\overline{\text{ES}}_\alpha(X_n^+)$			$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975		32.3432	98.0451	34.0245	98.1080			1.226602562	1.000641302
0.99		33.1708	115.1708	34.0374	115.2376			1.199668088	1.000579953
0.995		33.5154	128.8022	33.5154	128.8721			1.184316554	1.000542661
0.999		33.8681	162.9453	34.0477	163.0224			1.158470994	1.000473273

Table 3.11: RA and ARA Results for $X_n = n\,\text{Logn}(0, 0.5)$,$n = 30$

Logn($\mu = 0, \sigma = 3$), n = 30		RA							
α	Compile Time	$\underline{\text{VaR}}_\alpha(X_n^+)$	$\overline{\text{VaR}}_\alpha(X_n^+)$	$\underline{\text{ES}}_\alpha(X_n^+)$	$\overline{\text{ES}}_\alpha(X_n^+)$	$\text{VaR}_\alpha^+(X_n^+)$	$\overline{\text{VaR}}_{\frac{1+\alpha}{2}}(X_n^+)$	$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975	56.39s	413.1993	73830.7247 (73820.07 - 73841.38)	70575.4650 (39170.34 - 101980.59)	91908.04828	10733.11751	141148.6815 (141129.4 - 141167.9)	6.878777261	1.244848248
0.99	55.3s	1073.2936	172596.7977 (172573.7 - 172619.9)	201605.0883 (71337.52 - 331872.65)	202466.666	32216.72491	315756.4354 (315716.2 - 315796.7)	5.357366343	1.173061544
0.995	54.07s	2267.5550 (2265.216 - 2269.893)	315756.438 (315716.2 - 315796.7)	486415.3567 (108044.5 - 864786.2)	358779.1411	68096.78577	561487.6357 (561419.3 - 561556.0)	4.636877269	1.136252814
0.999	33.46s	10575.3027 (10528.45 - 10622.15)	1156978.762 (1156846 - 1157112)	4607202.0720 (240006.8 - 8974397.3)	1253177.133	318664.5625	1949326.3162 (1949110 - 1949542)	3.630710466	1.083146185

		ARA							
α		$\underline{\text{VaR}}_\alpha(X_n^+)$	$\overline{\text{VaR}}_\alpha(X_n^+)$	$\underline{\text{ES}}_\alpha(X_n^+)$	$\overline{\text{ES}}_\alpha(X_n^+)$			$\overline{\Delta}_\alpha(X_n^+)$	$\mathcal{B}_\alpha(X_n^+)$
0.975		414.5493 (413.1615 - 415.9371)	73827.8077 (73567.82 - 74087.80)	58336.1001 (43491.35 - 73180.85)	91908.04828			6.878505486	1.244897433
0.99		1070.2632 (1066.633 - 1073.893)	172590.2033 (172025.7 - 173154.7)	142021.3273 (81973.08 - 202069.58)	202466.666			5.357161654	1.173106365
0.995		2262.7860 (2255.677 - 2269.895)	315743.2497 (314762.5 - 316724.0)	300189.6368 (128839.1 - 471540.2)	358779.1411			4.636683599	1.136300274
0.999		10586.3329 (10550.51 - 10622.15)	1156938.415(1153689 - 1160188)	300189.6368 (128839.1 - 471540.2)	1253177.133			3.630583852	1.083183959

Table 3.12: RA and ARA Results for $X_n = n\,\text{Logn}(0, 3)$,$n = 30$

Chapter 4 VaR / ES Bounds for Inhomogeneous Risk Portfolios

Inhomogeneous portfolios can be divided into several groups of homogeneous portfolios. This chapter aims to combine Pareto, Exponential and Lognormal Distributions to a mixed portfolio so that we can further testify the performance of RA() and ARA(). We reset α to $\alpha = 0.995$ for two reasons: 1) We discovered the loss of data calculated under $\alpha = 0.999$ in some homogeneous cases in Chapter 3; 2) In this section, we can conclude the change of $\overline{\Delta}_\alpha(X_n^+)$ and $\mathcal{B}_\alpha(X_n^+)$ with respect to n, the change of the number of single risk types.

We select the following inhomogeneous cases $X_n^+ = X_1 + \cdots + X_n, n \in \mathbb{N}$:

(A.) Inhom1 (Heavy-tailed)*:

$$X_i \sim \begin{cases} \text{Pareto}(2), & i = 1, \cdots, k, k \in \mathbb{N} \\ \text{Exp}(1), & i = k+1, \cdots, 2k; \\ \text{Logn}(0,1) & i = 2k+1, \cdots, 3k. \end{cases}$$

(B.) Inhom2 (Light-tailed)

$$X_i \sim \begin{cases} \text{Pareto}(5), & i = 1, \cdots, k, k \in \mathbb{N} \\ \text{Exp}(5), & i = k+1, \cdots, 2k; \\ \text{Logn}(0,0.25) & i = 2k+1, \cdots, 3k. \end{cases}$$

(C.) Inhom3 (Miscellanies for heavy-tailed and light-tailed)

$$X_i \sim \begin{cases} \text{Pareto}(0.8 + 0.1i), & i = 0, \cdots, k-1, k \in \mathbb{N} \\ \text{Exp}(0.5 + 0.1(i-k)), & i = k, \cdots, 2k-1; \\ \text{Logn}(0, 1 + 0.1(i-2k)) & i = 2k, \cdots, 3k-1. \end{cases}$$

Portfolio A, $\alpha = 0.995$	$k = 6$			$k = 10$			$k = 20$		
RA	Best	Worst	Spread	Best	Worst	Spread	Best	Worst	Spread
$\text{VaR}_\alpha(X_n^+)$	20.4178	305.9342	285.5164	34.0297	516.0898	482.0601	68.0594	1041.5732	973.5138
$\text{ES}_\alpha(X_n^+)$	104.2506 (65.35686 - 143.14426)	321.32174	217.0711	146.1906 (81.55845 - 210.82280)	535.5362	389.3456	239.0590 (110.7701 - 367.3479)	1071.0725	832.0135
$\text{VaR}_{\frac{1+\alpha}{2}}(X_n^+)$	19.4039	401.7081	382.3042	34.8412	678.9209	644.0797	69.6825	1372.0953	1302.4128
$\text{VaR}_\alpha^+(X_n^+)$	195.4960			325.8266			651.6533		
$\overline{\Delta}_\alpha(X_n^+)$	1.564912941			1.583939827			1.598354855		
$\mathcal{B}_\alpha(X_n^+)$	1.050296917			1.037680352			1.02832186		

	$k = 6$			$k = 10$			$k = 20$		
ARA	Best	Worst	Spread	Best	Worst	Spread	Best	Worst	Spread
$\text{VaR}_\alpha(X_n^+)$	20.4204	305.7649	285.3445	34.0327	515.9248	481.8921	68.0638	1041.3844 (1036.416 - 1046.353)	973.3206
$\text{ES}_\alpha(X_n^+)$	85.4352 (68.59033 - 102.28015)	321.32174	235.8865	114.8313 (86.79263 - 142.86999)	535.5362	420.7049	176.3789 (120.4875 - 232.2703)	1071.0725	894.6936
$\text{VaR}_{\frac{1+\alpha}{2}}(X_n^+)$	20.4207	305.7715	285.3508	34.0326	515.9184	481.8858	68.0639	1041.3883 (1036.420 - 1046.357)	973.3244
$\text{VaR}_\alpha^+(X_n^+)$	195.4960			325.8266			651.6533		
$\overline{\Delta}_\alpha(X_n^+)$	1.564046938			1.583433423			1.59806513		
$\mathcal{B}_\alpha(X_n^+)$	1.050878459			1.038012216			1.028508292		

Table 4.1: RA and ARA Results for Mixed Portfolio A (Heavy-tailed)

We have the following observations from Tab.4.1 to Tab.4.3.

*This portfolio is proposed by Embrechts [5, page 40,Tab.5].

Portfolio B, $\alpha = 0.995$	$k=6$			$k=10$			$k=20$		
RA	Best	Worst	Spread	Best	Worst	Spread	Best	Worst	Spread
$\mathrm{VaR}_\alpha(X_n^+)$	8.7563	35.5565	26.8002	14.5938	59.3133	44.7195	29.1875	118.6523	89.4648
$\mathrm{ES}_\alpha(X_n^+)$	9.0931	41.5965	32.5034	15.0283	69.3275	54.2992	29.8423	138.6551	108.8128
$\mathrm{VaR}_{\frac{1+\alpha}{2}}(X_n^+)$	8.8117	40.2663	31.4546	14.6861	67.1787	52.4926	29.3723	134.3915	105.0192
$\mathrm{VaR}_\alpha^+(X_n^+)$	35.0944			58.4906			116.9813		
$\overline{\Delta}_\alpha(X_n^+)$	1.0131678			1.01406481			1.014284503		
$\mathcal{B}_\alpha(X_n^+)$	1.185275751			1.185275751			1.185275751		

	$k=6$			$k=10$			$k=20$		
ARA	Best	Worst	Spread	Best	Worst	Spread	Best	Worst	Spread
$\mathrm{VaR}_\alpha(X_n^+)$	8.7549	35.5482	26.7933	14.5907	59.2636	44.6729	29.1808	118.5289	89.3481
$\mathrm{ES}_\alpha(X_n^+)$	9.0135	41.5965	32.5830	14.8879	69.3275	54.4396	29.6818	138.6551	108.9733
$\mathrm{VaR}_{\frac{1+\alpha}{2}}(X_n^+)$	8.7548	35.5477	26.7929	14.5907	59.2638	44.6731	29.1808	118.5291	89.3483
$\mathrm{VaR}_\alpha^+(X_n^+)$	35.0944			58.4906			116.9813		
$\overline{\Delta}_\alpha(X_n^+)$	1.012931295			1.013215101			1.013229633		
$\mathcal{B}_\alpha(X_n^+)$	1.185275751			1.185275751			1.185275751		

Table 4.2: Mixed Portfolio B (Light-tailed)

Portfolio C, $\alpha = 0.995$	$k=6$			$k=10$			$k=20$		
RA	Best	Worst	Spread	Best	Worst	Spread	Best	Worst	Spread
$\mathrm{VaR}_\alpha(X_n^+)$	750.2040	5804.4081	5054.2041	750.2053	7497.1237	6746.9184	1752.7111 (1750.945 - 1754.477)	33796.2804 (33792.71 - 33799.85)	32043.5693
$\mathrm{ES}_\alpha(X_n^+)$	3529154.9486 (17339.83 - 7040970.07)	8692.2614	N.A.	3529692.7073 (17353.06 - 7042032.35)	9851.9474	N.A.	3557178.9265 (27570.74 - 7086787.11)	39165.9738	N.A.
$\mathrm{VaR}_{\frac{1+\alpha}{2}}(X_n^+)$	1783.4301 (1778.972 - 1787.888)	12046.4210	10262.9909	1783.4314 (1778.972 - 1787.891)	14468.6769	12685.2455	3423.9315 (3417.543 - 3430.319)	60518.2566 (60512.18 - 60524.34)	57094.3251
$\mathrm{VaR}_\alpha^+(X_n^+)$	1787.471252502554			2310.843060609775			9591.937352522287		
$\overline{\Delta}_\alpha(X_n^+)$	3.247273539			3.244324043			3.523405039		
$\mathcal{B}_\alpha(X_n^+)$	4.862881788			4.263356334			4.083218266		

	$k=6$			$k=10$			$k=20$		
ARA	Best	Worst	Spread	Best	Worst	Spread	Best	Worst	Spread
$\mathrm{VaR}_\alpha(X_n^+)$	748.3077 (745.4498 - 751.1655)	5802.5328 (5783.165 - 5821.901)	5054.2251	748.3097 (745.4498 - 751.1696)	7394.6166 (7371.795 - 7417.438)	6646.3069	1749.1535 (1743.817 - 1754.490)	33775.8791 (33650.20 - 33901.56)	32026.7256
$\mathrm{ES}_\alpha(X_n^+)$	5227445.4050 (27604.79 - 10427286.02)	8692.2614	N.A.	5227774.3967 (27634.73 - 10427914.06)	9851.9474	N.A.	5246493.3839 (39011.33 - 10453975.43)	39165.9738	N.A.
$\mathrm{VaR}_{\frac{1+\alpha}{2}}(X_n^+)$	748.3077 (745.4498 - 751.1655)	5802.6316 (5783.268 - 5821.995)	5054.3239	748.3097 (745.4498 - 751.1696)	7394.6128 (7371.786 - 7417.440)	6646.3031	1749.1535 (1743.817 - 1754.490)	33775.9182 (33650.20 - 33901.64)	32026.7647
$\mathrm{VaR}_\alpha^+(X_n^+)$	1787.471252502554			2310.843060609775			9591.937352522287		
$\overline{\Delta}_\alpha(X_n^+)$	3.246224403			3.199964864			3.521278117		
$\mathcal{B}_\alpha(X_n^+)$	4.862881788			4.263356334			4.083218266		

Table 4.3: Mixed Portfolio C (Heavy-tailed and light-tailed)

1. Portfolio A reveals the typical features for a heavy-tailed distribution: the large scale of $\underline{\mathrm{ES}}_\alpha(X_n^+)$.

   (a) The VaR values at α and $\frac{1+\alpha}{2}$ are almost equivalent.

   (b) When n is large, the bounds for VaR at any level start to enlarge and fluctuate.

   (c) $\overline{\Delta}_\alpha(X_n^+)$, describing the similarity between worst VaR and comonotonic VaR, remains in the interval [1.56,1.60]. The heavy-tailed portfolios tend to more easily deviate from the comonotonic case.

   $\mathcal{B}_\alpha(X_n^+)$ goes to 1 steadily as k, i.e. n grows.

2. In general, Portfolio B performs regularly by RA() and ARA() methods. All the figures are reasonable and convergent in their corresponding bounds.

   (a) The results of the bounds for $\mathrm{VaR}_{\frac{1+\alpha}{2}}(X_n^+)$ and $\mathrm{ES}_\alpha(X_n^+)$ are close.

   (b) The best VaR at level α and $\frac{1+\alpha}{2}$ are very close, with the spread within 1.

   (c) $\overline{\Delta}_\alpha(X_n^+)$ remains in the interval [1.013, 1.014] and slightly increases as n grows larger.

   $\mathcal{B}_\alpha(X_n^+)$ is invariant no matter obtained by RA() or ARA(), or whether n increases.

3. Portfolio C with both light and heavy-tailed marginals, illustrates the most fluctuating performance in the best and worst bound for the values of risk measures.

    (a) Though some of the average VaR values obtained from its bound at α and $\frac{1+\alpha}{2}$ still remains close to each other, their bounds for the best or the worst VaR diverge.

    (b) $\overline{\Delta}_\alpha(X_n^+)$ reveals that Portfolio C is largely far from the comonotonic case. Though $\mathcal{B}_\alpha(X_n^+)$ lies in the interval [4.00,4.90], it still shows the tendency to a smaller number (possibly 1) as n grows larger.

Chapter 5 Conclusions

This thesis focuses on the calculation of the best and the worst values for risk measures in continuous aggregate risk portfolios. Rearrangement Algorithm (RA) and Adaptive Rearrangement Algorithm (ARA) are applicable for the aggregate risk of which the dependence of marginal distributions is unknown. From this point of view, RA and ARA have removed the limit caused by the risk model while calculating the values for risk measures.

After introducing the preliminaries for VaR and ES, we implemented RA and ARA into different portfolios: homogeneous and inhomogeneous, to analyze and compare the results. Since $\overline{\mathrm{ES}}_\alpha(X_n^+)$ cannot be directly obtained by the two methods, we use comonotonic ES to approximate it. The results turn out that RA and ARA both work competitively for each portfolio listed in Chapter 3 and Chapter 4. RA and ARA give specific and convergent values for Best and Worst VaR and a reasonable range for $\underline{\mathrm{ES}}_\alpha(X_n^+)$ in homogeneous light-tailed Pareto, homogeneous Exponential, homogeneous quasi-Normal and inhomogeneous light-tailed portfolios. In light-tailed portfolios, $\overline{\Delta}_\alpha(X_n^+)$ and $\mathcal{B}_\alpha(X_n^+)$ tend more likely to 1. However, RA and ARA fail to conclude convergent values for VaR / ES when the aggregate risk consists of very heavy-tailed marginal distributions, especially for $\underline{\mathrm{ES}}_\alpha(X_n^+)$. $\overline{\mathrm{VaR}}_\alpha(X_n^+)$ with heavy-tailed marginals appears to deviate from comonotonic VaR: $\mathrm{VaR}_\alpha^+(X_n^+)$. The estimation of the values for risk measures (i.e. to reduce the spread in the value of upper and lower-bounds for risk measures) by RA and ARA still needs a thorough study. In the future work, the reallocation of weights in divided intervals of tails implemented in RA / ARA might be an appropriate direction to investigate.

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Appendix A Rearrangements for Matrices

This example is originated from Aas et al.'s paper[11, page 70]. Based on their example, we illustrate the extended version.

Assume an input $N \times n = 4 \times 4$ matrix for rearrangements

$$X = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 4 & 2 & 6 & 2 \\ 7 & 8 & 4 & 8 \\ 10 & 1 & 9 & 13 \end{bmatrix}.$$

The rearrangements are shown as follows:

$$\begin{array}{cc} sum & \begin{array}{cccc} \downarrow & & & \end{array} \\ \begin{array}{c} 8 \\ 10 \\ 20 \\ 23 \end{array} & \begin{pmatrix} 1 & 0 & 3 & 5 \\ 4 & 2 & 6 & 2 \\ 7 & 8 & 4 & 8 \\ 10 & 1 & 9 & 13 \end{pmatrix} \end{array} \quad \begin{array}{cc} sum & \begin{array}{cccc} & \downarrow & & \end{array} \\ \begin{array}{c} 18 \\ 15 \\ 16 \\ 23 \end{array} & \begin{pmatrix} 10 & 0 & 3 & 5 \\ 7 & 2 & 6 & 2 \\ 4 & 8 & 4 & 8 \\ 1 & 1 & 9 & 13 \end{pmatrix} \end{array} \quad \begin{array}{cc} sum & \begin{array}{cccc} & & \downarrow & \end{array} \\ \begin{array}{c} 16 \\ 17 \\ 14 \\ 14 \end{array} & \begin{pmatrix} 10 & 1 & 3 & 5 \\ 7 & 8 & 6 & 2 \\ 4 & 2 & 4 & 8 \\ 1 & 0 & 9 & 13 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} sum & \begin{array}{cccc} & & & \downarrow \end{array} \\ \begin{array}{c} 15 \\ 18 \\ 12 \\ 10 \end{array} & \begin{pmatrix} 10 & 1 & 4 & 5 \\ 7 & 8 & 3 & 2 \\ 4 & 2 & 6 & 8 \\ 1 & 0 & 9 & 13 \end{pmatrix} \end{array} \quad \begin{array}{cc} sum & \\ \begin{array}{c} 15 \\ 18 \\ 12 \\ 10 \end{array} & \begin{pmatrix} 10 & 1 & 4 & 5 \\ 7 & 8 & 3 & 2 \\ 4 & 2 & 6 & 8 \\ 1 & 0 & 9 & 13 \end{pmatrix} \end{array}$$

The workforce of RA() and ARA() aims to rearrange each element within a column so that the opposite order meets, i.e. to minimize every row sum. In the first matrix, we rearrange the first column so that each element is oppositely ordered to the sum of the next three columns in each row, i.e. to match the smallest element of the first column (1), with the largest sum (23); to match the second smallest element of the first column (4), with the second largest sum (20), etc. Then we conclude the second matrix. Rearrange every column in Matrix X as described, then we obtain the last resulting matrix, with the smallest average row sum: 13.75. In this example, we used 6 steps of rearrangements to obtain the resulting matrix.

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