Assignment - #02

Problem #05:

We have: $I = I_1 + ch_1^2 = I_2 + ch_2^2$. Thus,

$$I_1 - I_2 = c(h_2^2 - h_1^2)$$

$$\Rightarrow \boxed{ch_2^2 = c(h_2^2 - h_1^2) \cdot \frac{h_2^2}{(h_2^2 - h_1^2)} = \frac{h_2^2 (I_1 - I_2)}{h_2^2 - h_1^2}}$$

Problem 6:

We know that error in Simpson's method is $O(h^4)$. So, following the previous argument,

$$\boxed{e = ch_2^4 = \frac{h_2^4 (I_1 - I_2)}{h_2^4 - h_1^4}}$$

Problem #07:

Romberg Error: We have:

$$I_n^{(k)} = \frac{4^k I_n^{(k-1)} - I_{n/2}^{(k-1)}}{4^k - 1}$$

with error $O(h^{2k+2})$. Thus,

$$I \approx I_n^{(k)} + ch^{2k+2} \approx I_{n/2}^{(k)} + c(2h)^{2k+2}$$

$$\Rightarrow \quad err = |ch^{2k+2}| = \left| \frac{I_{n/2}^{(k)} - I_n^{(k)}}{(2h)^{2k+2} - h^{2k+2}} \right| \cdot h^{2k+2}$$

$$\Rightarrow \boxed{err = \left| \frac{I_{n/2}^{(k)} - I_n^{(k)}}{2^{2k+2} - 1} \right|}$$

Problem #10:

We have: $$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + V(x) = \text{const.}$$

At $x = a$, $\frac{dx}{dt} = 0 \Rightarrow E = V(a)$. Thus, we get:

$$\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 = V(a) - V(x)$$

$$\Rightarrow \frac{dx}{dt} = \sqrt{\frac{2}{m}\left[V(a) - V(x)\right]}$$

Integrating this gives: (with $m = 1$)

$$\boxed{\int_0^a \frac{dx}{\sqrt{2\left[V(a) - V(x)\right]}} = \int_0^{T/4} dt = T/4}$$

Problem 12:

The total power emitted by a black-body about the entire E-M spectrum is:

$$W = \int_0^\infty I(\omega)\, d\omega$$

$$= \int_0^\infty \frac{\hbar}{4\pi^2 c^2} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1}\, d\omega$$

Set $x = \frac{\hbar\omega}{k_B T}$

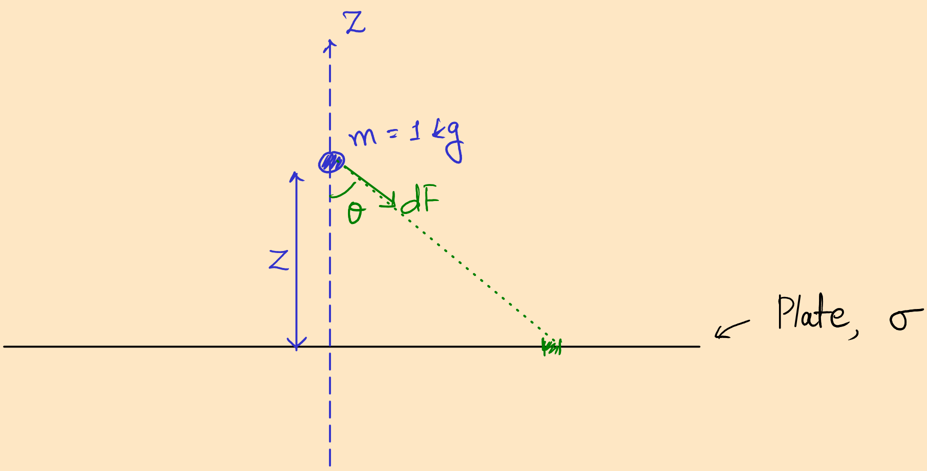
$\Rightarrow \omega = \frac{k_B T x}{\hbar}$

$d\omega = \frac{k_B T}{\hbar} dx$

$$= \frac{\hbar}{4\pi^2 c^2} \int_0^\infty \left(\frac{k_B T}{\hbar} x\right)^3 \cdot \frac{1}{e^x - 1} \cdot \frac{k_B T}{\hbar} dx$$

$$\Rightarrow \boxed{W = \frac{k_B^4 T^4}{4\pi^2 c^2 \hbar^3} \int_0^\infty \frac{x^3}{e^x - 1}\, dx}$$

Problem 14 :



We have:

$$F_z = \int dF\cos\theta = \iint_{-1/2}^{1/2} \frac{G\sigma m\,dx\,dy}{r^2}\cos\theta$$

$$= \iint_{-1/2}^{1/2} \frac{G\sigma m}{(x^2+y^2+z^2)} \cdot \frac{z}{(x^2+y^2+z^2)^{1/2}}\,dx\,dy$$

$$\Rightarrow \boxed{F_z = G\sigma m z \iint_{-1/2}^{1/2} \frac{1}{(x^2+y^2+z^2)^{3/2}}\,dx\,dy}$$

Problem 15:

(b) Setting the derivative equal to zero, we obtain:

$$\frac{d}{dx}\,x^{a-1}\cdot e^{-x} = (a-1)\,\cancel{x^{a-2}}\,\cancel{e^{-x}} - \cancelto{x}{x^{a-1}}\,\cancel{e^{-x}} = 0$$

$$\Rightarrow \boxed{x = a-1}$$

The second derivative is:

```python
import sympy as smp

x, a = smp.symbols('x a')
f = x**(a - 1) * smp.exp(-x)

smp.diff(f, x, 2)
```

[1] ✓ 1.5s Python

$$x^{a-1}\left(1 - \frac{2(a-1)}{x} + \frac{(a-2)(a-1)}{x^2}\right)e^{-x}$$

At $x = a-1$, this gives:

```python
import sympy as smp

x, a = smp.symbols('x a')
f = x**(a - 1) * smp.exp(-x)

smp.diff(f, x, 2).subs(x, a - 1)
```

[2] ✓ 0.4s Python

$$(a-1)^{a-1}\left(\frac{a-2}{a-1}-1\right)e^{1-a}$$

For $a > 1$, we have:

$$\left.\frac{d^2f}{dx^2}\right|_{x=a-1} = \underbrace{(a-1)^{a-1}}_{>0}\underbrace{\left(\frac{a-2}{a-1}-1\right)}_{<0}\underbrace{e^{1-a}}_{>0} < 0$$

Thus, $f(x) = x^{a-1}e^{-x}$ has a maxima at $x = a-1$.

(c) Change of vars:

$$Z(x) = \frac{x}{c+x}$$

We wish to get: $Z(a-1) = 1/2$

$$\Rightarrow \quad \frac{a-1}{c+a-1} = \frac{1}{2}$$

$$\Rightarrow \quad \boxed{c = a-1}$$

(d) Overflow issues: We have,

$$f(x) = x^{a-1} \cdot e^{-x}$$

At large values of x, x^{a-1} might cause overflow, and e^{-x} might cause underflows, even if $f(x)$ can be stored as a float without issues. Consider one such value of x:

$$f(x) = x^{a-1} \cdot e^{-x} = e^{(a-1)\ln x} \cdot e^{-x} = e^{(a-1)\ln x - x}$$

For $a > 1$, and $x \gg 1$,

$$\left| (a-1)\ln x - x \right| < |x|$$

So, as long as the value of $f(x)$ can be stored without overflow/underflow, the expression $f(x) = e^{(a-1)\ln x - x}$ can be used to avoid the issues of overflow/underflow.

(e) Expression for the integral:

We have: $z = \dfrac{x}{c+x} = 1 - \dfrac{c}{c+x}$

$$\Rightarrow \frac{c}{c+x} = 1 - z \implies \frac{x+c}{c} = \frac{1}{1-z}$$

$$\Rightarrow x = \frac{c}{1-z} - c \quad \Rightarrow \boxed{x = \frac{cz}{1-z}}$$

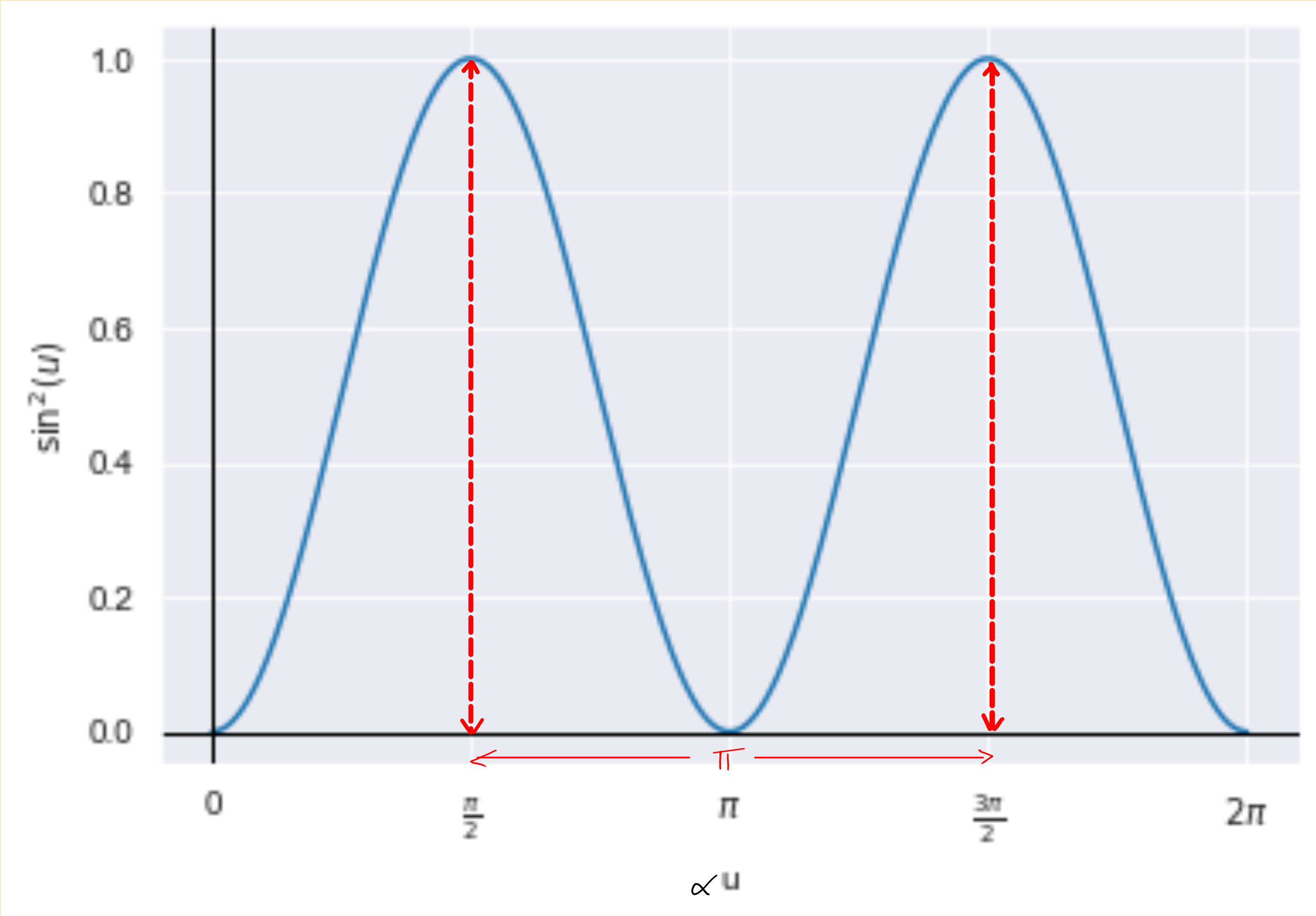
$$\Rightarrow dx = \left[\frac{c}{1-z} + \frac{cz}{(1-z)^2}\right] dz$$

So,
$$\int_0^\infty x^{a-1} e^{-x}\, dx$$

$$= \int_0^1 e^{(a-1)\ln\left[\frac{(a-1)z}{1-z}\right] - \frac{(a-1)z}{1-z}} \times \left[\frac{a-1}{1-z} + \frac{(a-1)z}{(1-z)^2}\right] dz$$

Problem # 16 :

(a) Slit Separation:



The separation between the slits d satisfies:

$$\alpha d = \pi \quad \Rightarrow \quad \boxed{d = \frac{\pi}{\alpha}}$$

(e) (ii) Pattern of Slits: