Assignment #03

B. Guruvayurappan
18747

Problem #01:

(b) Consider $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \cdot \frac{-b \mp \sqrt{b^2-4ac}}{-b \mp \sqrt{b^2-4ac}}$$

$$= \frac{(-b)^2 - (b^2-4ac)}{2a \cdot \left(-b \mp \sqrt{b^2-4ac}\right)} = \frac{4ac}{2a \cdot \left(-b \mp \sqrt{b^2-4ac}\right)}$$

$$\Rightarrow \boxed{x = \frac{2c}{-b \mp \sqrt{b^2-4ac}}}$$

Problem #04:

(a) At the stationary point, we have:

$$\frac{dx}{dt} = \frac{dy}{dt} = 0$$

$$\Rightarrow \quad -x + ay + x^2 y = 0 \qquad\qquad b - ay - x^2 y = 0$$

$$\Rightarrow \quad x = (a + x^2) y \qquad\qquad b = (a+x^2) y$$

$$\Rightarrow \quad \boxed{x = b} \quad \text{and} \quad \boxed{y = \frac{x}{a+x^2} = \frac{b}{a+b^2}}$$

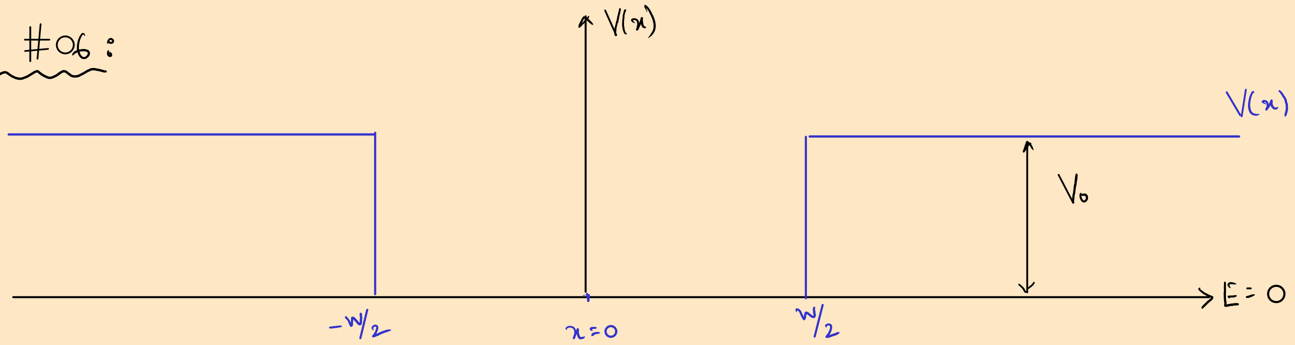
(b) Since $x = b$, we substitute for b in the denominator of the RHS above:

$$\Rightarrow \boxed{y = \frac{b}{a + (b^2)} = \frac{b}{a + x^2}}$$

Similarly substituting $b = x$ in the numerator as well gives:

$$y = \frac{(b)}{a + x^2} = \frac{x}{a + x^2} \quad \Rightarrow \quad \boxed{x = y(a + x^2)}$$

Problem #06:



For Bound States, we must have: $0 < E < V_0$.

In the regions $|x| \geq w/2$, we have:

$$\hat{H}\psi = E\psi \qquad \text{(Time-Independent Schrödinger Eq}^n\text{)}$$

$$\Rightarrow \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$$

$$\Rightarrow \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = -\underbrace{(V_0 - E)}_{>0}\psi$$

$$\Rightarrow \psi \longrightarrow e^{qx/\hbar} \text{ (or) } e^{-qx/\hbar}, \quad \text{where } q = \sqrt{2m(V_0 - E)}$$

To be able to normalise the bound state ψ, we must have:

$$\psi \longrightarrow \begin{cases} e^{qx/\hbar}, & x \leq -w/2 \\ e^{-qx/\hbar}, & x \geq w/2 \end{cases}$$

We assign the co-efficient of $x \geq w/2$ to be 1. Then,

$$\psi = \begin{cases} e^{-qx/\hbar}, & x \geq w/2 \\ a e^{qx/\hbar}, & x \leq -w/2 \end{cases}$$

In the region $|x| \leq w/2$, the Schrödinger Eq^n gives:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\Rightarrow \psi = b e^{ipx/\hbar} + c e^{-ipx/\hbar}, \quad \text{where } p = \sqrt{2mE}$$

Combining these:

$$\psi = \underset{\text{For Normalisation}}{N} \begin{cases} a e^{qx/\hbar}, & x \leq -w/2 \\ b e^{ipx/\hbar} + c e^{-ipx/\hbar}, & -w/2 \leq x \leq w/2 \\ e^{-qx/\hbar}, & x \geq w/2 \end{cases}$$

Consider the following transformation:

$$U\hat{x}U^\dagger = -\hat{x}$$
$$U\hat{p}U^\dagger = -\hat{p}$$

Now,

$$U\hat{H}U^\dagger = \hat{H} \qquad \left(\text{since } V(-\hat{x}) = V(x)\right)$$

Suppose ψ satisfies $\hat{H}\psi = E\psi$. Then,

$$UH\psi = EU\psi$$

$$\Rightarrow UH\underbrace{(U^\dagger U)}_{=I}\psi = EU\psi$$

$$\Rightarrow (UHU^\dagger)U\psi = E(U\psi)$$

$$\Rightarrow H(U\psi) = E(U\psi)$$

Suppose the energy levels of ψ are degenerate.

$$\Rightarrow U\psi = \lambda\psi$$

$$\Rightarrow \psi(-x) = \lambda \cdot \psi(x)$$

$$\Rightarrow \psi(x) = \psi(-(-x)) = \lambda\psi(-x) = \lambda^2\psi(x)$$

$$\Rightarrow \underline{\underline{\lambda = \pm 1}}$$

$\Rightarrow \psi$ is either odd or even.

First, we consider ψ to be even. Then,

$$\psi(x) = N\begin{cases} e^{qx/\hbar}, & x \leq -w/2 \\ A\cos(px/\hbar), & -w/2 \leq x \leq w/2 \\ e^{-qx/\hbar}, & x \geq w/2 \end{cases}$$

As ψ is differentiable everywhere,

$$e^{-qw/2\hbar} = A\cos\left(\frac{pw}{2\hbar}\right) \quad \text{- - - - - Continuity at } x = w/2$$

$$-\frac{q}{\cancel{2\hbar}}e^{-qw/2\hbar} = -\frac{pA}{\cancel{2\hbar}}\sin\left(\frac{pw}{2\hbar}\right) \quad \text{- - — - differentiability at } x = w/2$$

Dividing these gives: (and using the defⁿ of p and q)

$$q = p \tan\left(\frac{p\omega}{2\hbar}\right)$$

$$\Rightarrow \boxed{\sqrt{\frac{V-E}{E}} = \tan\left(\frac{\omega\sqrt{2mE}}{2\hbar}\right)} \longrightarrow \text{for even } \psi$$

Similarly, for odd ψ,

$$\boxed{\sqrt{\frac{V-E}{E}} = -\cot\left(\frac{\omega\sqrt{2mE}}{2\hbar}\right)} \longrightarrow \text{for odd } \psi$$

The sol^{ns} of these eq^{ns} satisfy:

<u>Even $\psi(x)$:</u> $\quad n\pi \leq \frac{\omega}{2\hbar}\sqrt{2mE_n} \leq \left(n+\frac{1}{2}\right)\pi$

$$\Rightarrow \boxed{\frac{1}{2m}\left(\frac{2n\pi\hbar}{\omega}\right)^2 \leq E_n \leq \frac{1}{2n}\left[\frac{(2n+1)\pi\hbar}{\omega}\right]^2}$$

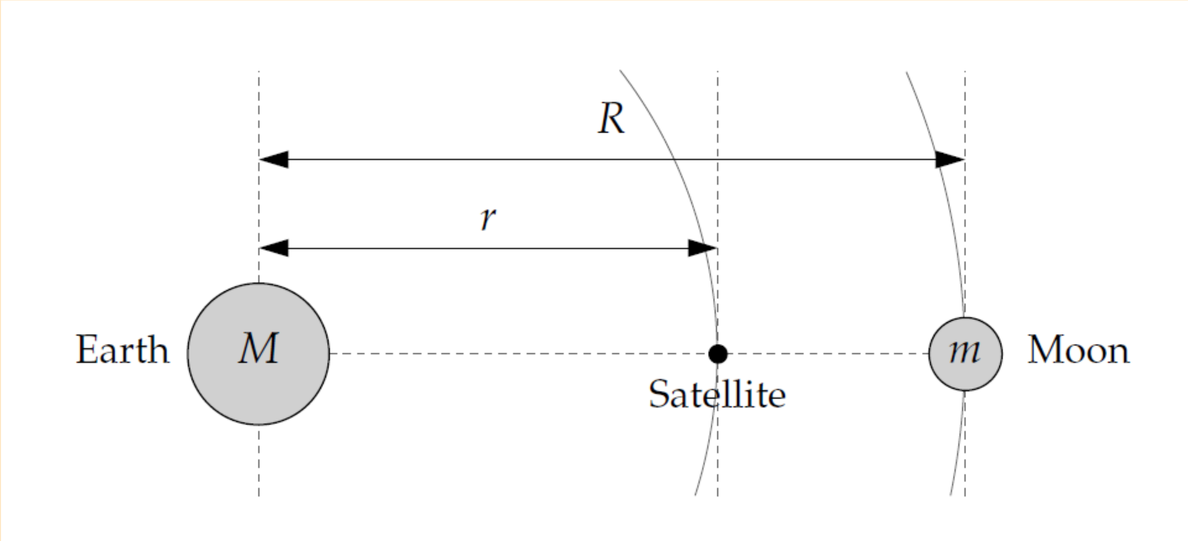
<u>Odd $\psi(x)$:</u> $\quad \left(n+\frac{1}{2}\right)\pi \leq \frac{\omega}{2\hbar}\sqrt{2mE_n} \leq (n+1)\pi$

$$\Rightarrow \boxed{\frac{1}{2n}\left[\frac{(2n+1)\pi\hbar}{\omega}\right]^2 \leq E_n \leq \frac{1}{2n}\left[\frac{(2n+2)\pi\hbar}{\omega}\right]^2}$$

We use the above bounds for Regula-Falsi Method.

Problem #08:

(a)



The condition for Lagrange Point can be written as:

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r$$

$\omega \longrightarrow$ Angular velocity of satellite/moon

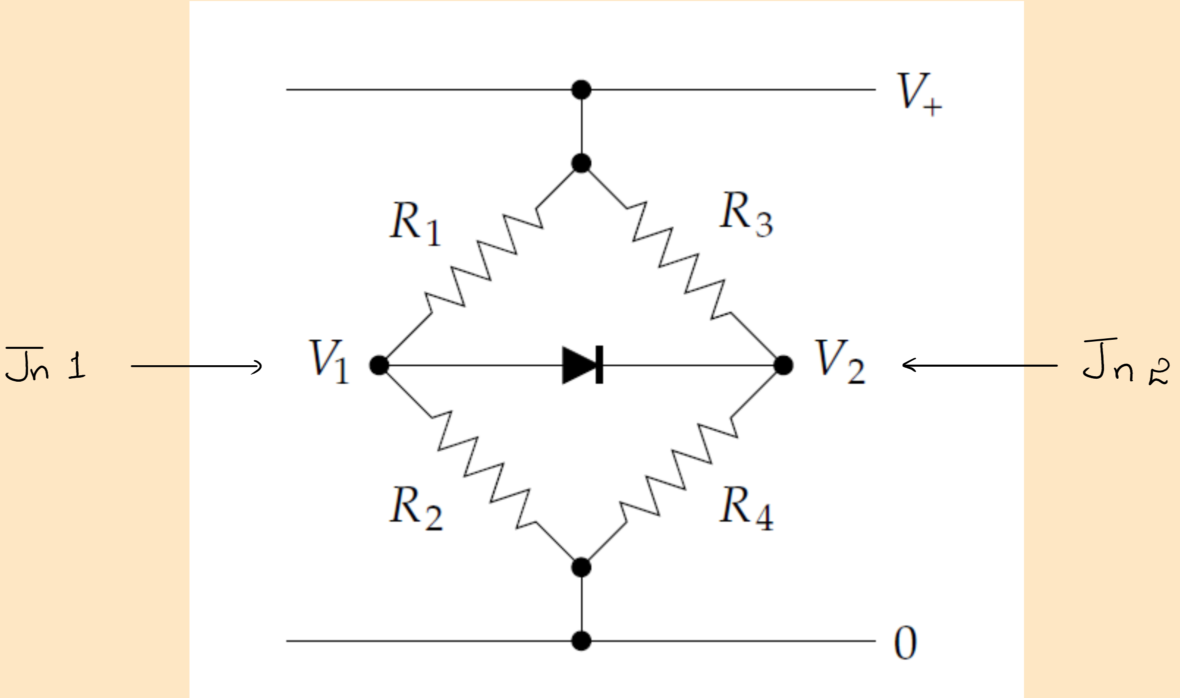
We wish to solve for r using Secant Method. Since the satellite has to be closer than the moon, we have:

$$r_{min} = 6400 \text{ km} \longrightarrow \text{Radius of Earth}$$

$$r_{max} = R - \Delta R \longrightarrow$$ We choose a slightly smaller value to avoid divide by zero error.

Thus, our two initial guesses are r_{min}, r_{max}.

Problem #09:



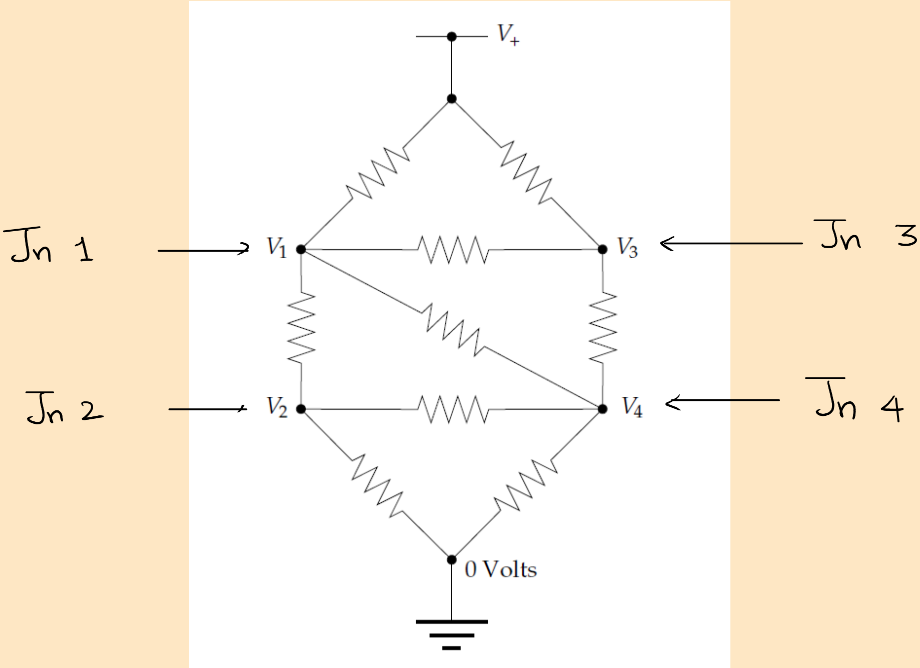
Given that the diode is forward biased, we assume that $V_1 > V_2$.

Applying KCL at Jn 1 and Jn 2 gives:

$$\boxed{\begin{aligned} \frac{V_+ - V_1}{R_1} &= \frac{V_1 - 0}{R_2} + I_0\left[\exp\left(\frac{V_1 - V_2}{V_T}\right) - 1\right] &&\longrightarrow \text{Jn 1} \\ \frac{V_+ - V_2}{R_3} &= \frac{V_2 - 0}{R_4} - I_0\left[\exp\left(\frac{V_1 - V_2}{V_T}\right) - 1\right] &&\longrightarrow \text{Jn 2} \end{aligned}}$$

We wish to solve this system for V_1, V_2

Problem #10:



Applying KCL at the above Jns:

$$\frac{V_1 - V_+}{R} + \frac{V_1 - V_2}{R} + \frac{V_1 - V_3}{R} + \frac{V_1 - V_4}{R} = 0 \longrightarrow \text{Jn 1}$$

$$\frac{V_2 - 0}{R} + \frac{V_2 - V_1}{R} + 0 + \frac{V_2 - V_4}{R} = 0 \longrightarrow \text{Jn 2}$$

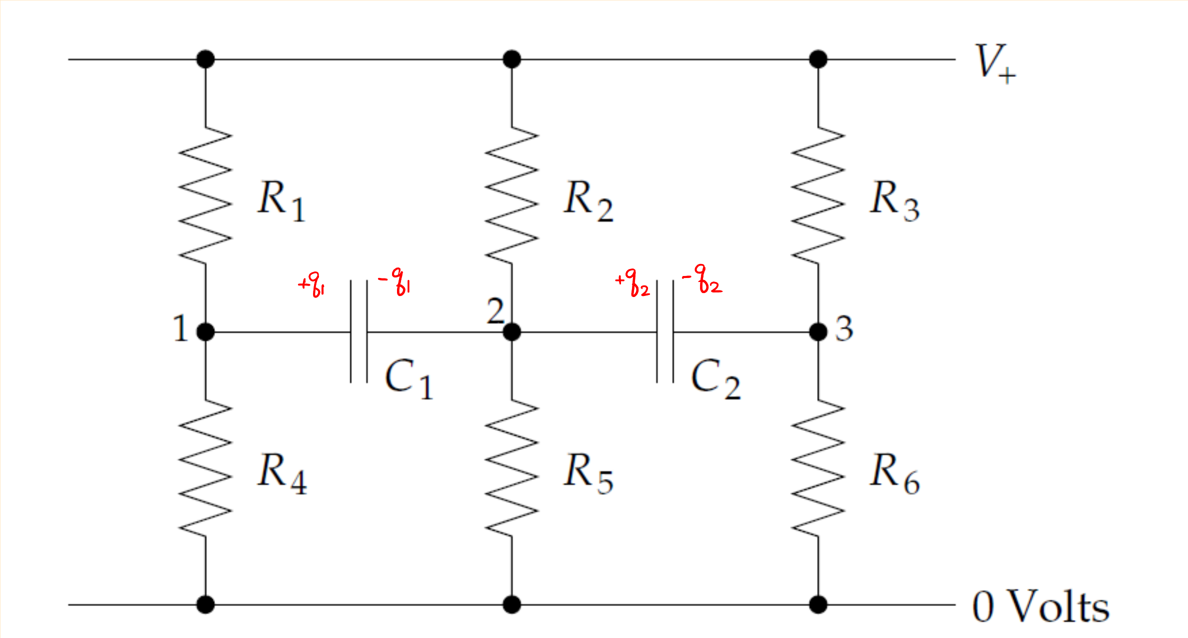
$$\frac{V_3 - V_+}{R} + \frac{V_3 - V_1}{R} + 0 + \frac{V_3 - V_4}{R} = 0 \longrightarrow \text{Jn 3}$$

$$\frac{V_4 - 0}{R} + \frac{V_4 - V_1}{R} + \frac{V_4 - V_2}{R} + \frac{V_4 - V_3}{R} = 0 \longrightarrow \text{Jn 4}$$

Simplifying these gives:

$$\boxed{\begin{aligned} 4V_1 - V_2 - V_3 - V_4 &= V_+ \\ -V_1 + 3V_2 - V_4 &= 0 \\ -V_1 + 3V_3 - V_4 &= V_+ \\ -V_1 - V_2 - V_3 + 4V_3 &= 0 \end{aligned}}$$

Problem #13:



We Apply KCL at the Jns. 1, 2, and 3.

At Jn. 1:

$$\frac{V_1 - V_+}{R_1} + \frac{V_1 - 0}{R_4} + \frac{dq_1}{dt} = 0 \quad \Rightarrow \quad V_1\left(\frac{1}{R_1} + \frac{1}{R_4}\right) = \frac{V_+}{R_1} - C_1 \frac{d}{dt}(V_1 - V_2)$$

At Jn. 2:

$$\frac{V_2 - V_+}{R_2} + \frac{V_2 - 0}{R_5} - \frac{dq_1}{dt} + \frac{dq_2}{dt} = 0 \quad \Rightarrow \quad V_2\left(\frac{1}{R_2} + \frac{1}{R_5}\right) = \frac{V_+}{R_2} + C_1 \frac{d}{dt}(V_1 - V_2) - C_2 \frac{d}{dt}(V_2 - V_3)$$

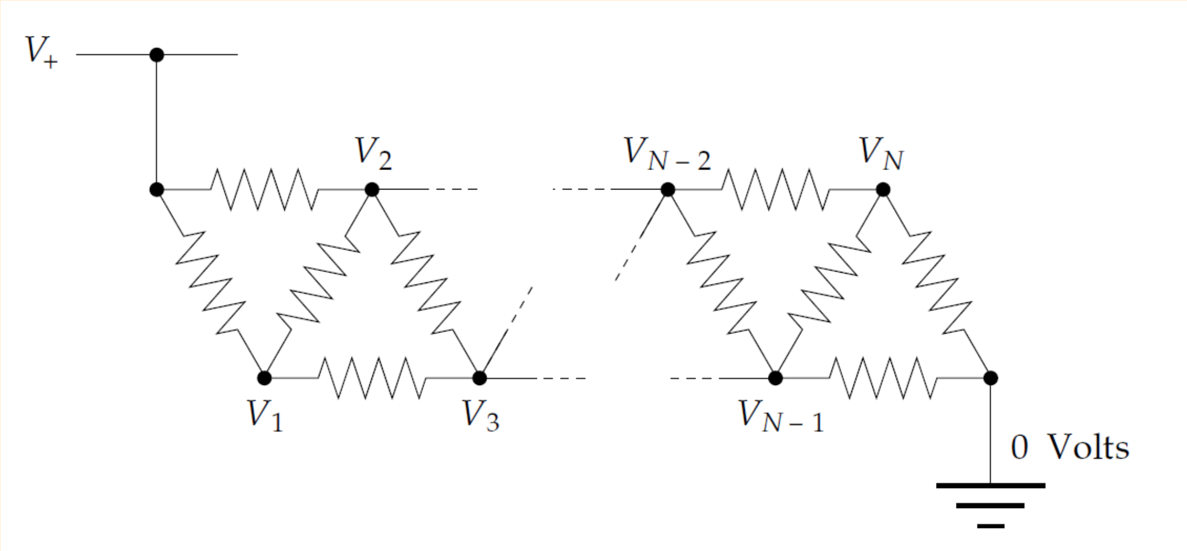
At Jn. 3:

$$\frac{V_3 - V_+}{R_3} + \frac{V_3 - 0}{R_6} - \frac{dq_2}{dt} = 0 \quad \Rightarrow \quad V_3\left(\frac{1}{R_3} + \frac{1}{R_6}\right) = \frac{V_+}{R_3} - C_2 \frac{d}{dt}(V_2 - V_3)$$

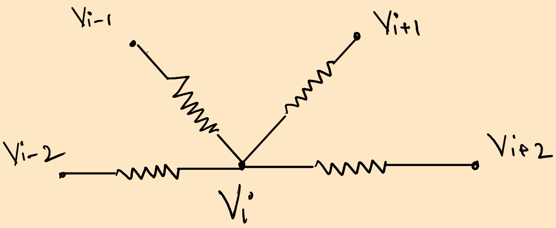
Now, writing $V_+ = x_+ e^{i\omega t}$, and $V_j = x_j e^{i\omega t}$, $j = 1, 2, 3$, We get:

$$\boxed{\begin{aligned}
&\left(\frac{1}{R_1} + \frac{1}{R_4} + i\omega C_1\right) x_1 - i\omega C_1 x_2 = \frac{x_+}{R_1} \\
&- i\omega C_1 x_1 + \left[\frac{1}{R_2} + \frac{1}{R_5} + i\omega(C_1 + C_2)\right] x_2 - i\omega C_2 x_3 = \frac{x_+}{R_2} \\
&- i\omega C_2 x^2 + \left(\frac{1}{R_3} + \frac{1}{R_6} + i\omega C_2\right) x_3 = \frac{x_+}{R_3}
\end{aligned}}$$

Problem # 14:



An arbitrary Jn i is shown below:



for $i = 3, 4 \cdots N-2$

Applying kirchoff Current Law gives:

$$\frac{V_i - V_{i-2}}{R} + \frac{V_i - V_{i-1}}{R} + \frac{V_i - V_{i+1}}{R} + \frac{V_i - V_{i+2}}{R} = 0$$

$$\therefore \boxed{-V_{i-2} - V_{i-1} + 4V_i - V_{i+1} - V_{i+2} = 0} \quad \text{for } i = 3, 4 \cdots N-2$$

Considering the remaining values of i gives:

$$\boxed{\begin{aligned} 3V_1 - V_2 - V_3 &= V_+ \\ -V_1 + 4V_2 - V_3 - V_4 &= V_+ \\ -V_{N-3} - V_{N-2} + 4V_{N-1} - V_N &= 0 \\ -V_{N-2} - V_{N-1} + 3V_N &= 0 \end{aligned}}$$

In matrix form,

$$\underbrace{\begin{pmatrix} 3 & -1 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & -1 & \cdots & 0 \\ 0 & -1 & 4 & -1 & -1 \cdots & 0 \\ \vdots & & \vdots & & & \vdots \\ 0 & \cdots & 0 & -1 & -1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix}}_{V} = \underbrace{\begin{pmatrix} V_+ \\ V_+ \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{W}$$

Problem # 16 :

(a) We have:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \hat{H}\, \psi(x)\, dx = \int_0^L \sin\left(\frac{m\pi x}{L}\right) E\, \psi\, dx$$

Since $\hat{H}$ is a linear operator, using $\psi(x) = \sum_n \psi_n \sin\left(\frac{n\pi x}{L}\right)$ gives:

$$\sum_n \psi_n \int_0^L \sin\left(\frac{m\pi x}{L}\right) \hat{H} \sin\left(\frac{n\pi x}{L}\right) dx = \sum_n \int_0^L \psi_n E \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \psi_m E \frac{L}{2}$$

$$\Rightarrow \boxed{\sum_n \psi_n \int_0^L \sin\left(\frac{m\pi x}{L}\right) \hat{H} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2} \psi_m E L} \quad \text{- - - -} \quad (*)$$

Defining the matrix H as:

$$H_{mn} = \frac{2}{L} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \hat{H} \sin\left(\frac{n\pi x}{L}\right) dx$$

and the vector ψ as: $\psi = (\psi_1, \psi_2, \dots)$, we see that:

$$(H\psi)_m = \sum_n H_{mn} \psi_n = \frac{2}{L}\left(\frac{1}{2} \psi_m E L\right) \qquad \text{using } (*)$$

$$= E \psi_m$$

$$\Rightarrow \boxed{H\psi = E\psi}$$

(b) For $V(x) = \frac{ax}{L}$, we have:

$$H_{mn} = \frac{2}{L} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{ax}{L} \right] \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_0^L \frac{+\hbar^2}{2m} \cdot \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) + \frac{a}{L} x \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \begin{cases} \dfrac{(n\hbar\pi)^2}{2mL^2} + \dfrac{a}{2}, & \text{if } m = n \\ 0, & \text{if } m \neq n, \text{ and both even or both odd} \\ \dfrac{-8a}{\pi^2} \cdot \dfrac{mn}{(m^2-n^2)^2}, & \text{if } m \neq n, \text{ and one even, one odd} \end{cases}$$

Clearly, from the above expression,

- H_{mn} is Real for all m, n
- $H_{mn} = H_{nm}$ for all m, n

$\Rightarrow$ H is real and Symmetric.