Finite Group and Representations

February 21, 2022

Contents

1 Finite Groups

1.1 Sylow's Theorem

Sylow's theorem is of fundamental importance in analyzing the structure of finite groups.

Theorem 1. *Let G be a finite group, $|G| = n$, $n = p^r m$, p a prime number and $(p, m) = 1$. Then the following hold*

- *$\forall 1 \leq k \leq r$, $|G|$ has subgroup of order p^k. The one with order p^r is called p-**Sylow subgroup** of G.*
- *All p-Sylow subgroups are conjugate.*

- *Let k_p be the number of p-Sylow subgroups, then $k_p \equiv 1(mod \quad p)$, and $k_p = |[G : N_G(P)]|$, where P is one of the p-Sylow subgroup.*

First we have the following lemma.

Lemma 1. *Let p, r, n, m denote the integers as above, then the combinatorial number $\binom{p^r m}{p^k}$ is of the form $p^{r-k}l$, where $(l, p) = 1$.*

Proof. Fix n and p, let S be the set of all the subsets of G with p^k elements. Then the cardinal of S is computed to be

$$|S| = \binom{n}{p^k} = \binom{p^r m}{p^k}$$

consider the action of right multiplication of G on G itself, which induces the action of G on the set of p^k elements subsets S. By the lemma, $|S|$ is of form $p^{r-k}l, (l, p) = 1$.

We claim that S must have a length $m^{'}$ orbit with $m^{'} = p^s u, s \leq r - k$, u and integer. If not, then all the orbit will have length which is some multiple of p^{r-k+1}, and because S is the sum of lengths of all the orbits, so it must be a multiple of p^{r-k+1}, contradict to the last paragraph that $|S|$ is of form $p^{r-k}l, (l, p) = 1$. So we must have such a length $m^{'}$ orbit. Denote an element in this orbit as s.

By orbit formula we have $n = m^{'}|G_s|$, where G_s is the isotropic subgroup of G with respect to s. Then we have $|G_s| = \frac{n}{m^{'}} \geq p^k \frac{m}{u}$. Because $(p, m^{'}) = 1$, $\frac{m}{u}$ is an integer which is less than m. By induction, G_s has a p-Sylow subgroup, which is obviously also a p-Sylow subgroup of G.

1.2 Solvable and Nilpotent Groups

1.3 Extension of Groups

1.4 Group (Co)homology

1.5 Group Presentations and Free Groups

1.6 Group Actions

Suppose a group G acts on a set X.

Definition 1. *A subset $B \subset X$ is called a **block** (with respect to the G-action), if for any $g \in G$, either*

$$gB = B \text{ or } gB \cap B = \emptyset$$

holds.

It is obvious that the total set X, and the set $\{x\}$ contains only one element x are blocks of X. These blocks are called **trivial blocks**.

Definition 2. *A G-action on X is called* ***primitive****, if any block is trivial.*

It follows from the definition that given a block in X, the distinct subsets

$$\{gB|\, g \in G\}$$

gives a partition of X. Moreover, this induces a G-action on this partition, that is, for any part P and any $g \in G$, gP is also a part in this partition.

The primitivity can be reformulated using partitions. Suppose

$$\mathcal{P} = \coprod_i P_i$$

is a partition of X into disjoint parts P_i. We call G **preserves** $\mathcal{P}$, if any element $g \in G$ sends a part to a part, that is, for any part P_i and $g \in G$, there is a corresponding part P_j with

$$gP_i = P_j.$$

Proposition 1. *A G-action on X is primitive if and only if G preserves a partition $\mathcal{P}$ implies that $\mathcal{P}$ is trivial.*

2 Representations of Finite Groups

2.1 G-modules

Definition 3. *A* ***G-module*** *is a pair (ρ, V) in which V is a k-vector space, and ρ is a homomorphism*

$$\rho : G \to GL(V).$$

In many situations we also say ρ is a representation of G on V, or just say V is a G-module if the pair (ρ, V) is specified without ambiguity, and for any element $g \in G$ we may use g to denote its image $\rho(g)$.

We note that there are three categories involved in the group representations: The category of groups, the category of fields, and the category of k-vector spaces. The basic theory of finite group representations need to take the morphisms in all these categories into account.

Proposition 2. *Let V_1, V_2 be two G-modules, then their direct sum*

$$V_1 \oplus V_2$$

is also a G-module with the obvious G-action.

Proposition 3. *Let V be a G-module, and $W \subseteq V$ be a G-submodle, then the quotient*

$$V/W$$

is also a G-module with the obvious G-action.

Definition 4. *A G-moduloe V is called* ***irreducible*** *or* ***simple****, if the only G-submodules are $\{0\}$ and V itself.*

Definition 5. *Given two G-modules $(\rho_1, V_1), (\rho_2, V_2)$, a linear map $f \in Hom(V_1, V_2)$ is called a G-**homomorphism** if it commutes with the G-action, that is*

$$f\rho_1(g) = \rho_2(g)f$$

for any $g \in G$. The set of all G-homomorphisms are denoted by

$$Hom_G(V_1, V_2).$$

We also remark that the vector space $Hom_G(V_1, V_2)$ is also called the **multiplicity space of V_1 in V_2.** and its dimension

$$dim_k(Hom_G(V_1, V_2))$$

which is obviously a non-negative integer, is called the **multiplicity of V_1 in V_2)**. The meaning will be discussed later when considering decomposition of a G-module.

From the definition, it is clear that being a G-homomorphism is much more stronger than just being a linear map, and G-homomorphism are the linear maps we are interested in. One of the basic methods in (either finite or infinite) group representation is that if we can take 'sum' over the group elements, we can obtain G-homomorphisms from arbitrary linear map by taking 'average'.

Theorem 2 (Schur's Lemma)**.** *If k is algebraic closed, and V_1, V_2 are simple G-modules, then for any G-homomorphism $f \in Hom(V_1, V_2)$, either f is the zero map:*

$$f = 0,$$

or f is a k-linear isomorphism. Moreover, If $V = W$ and k is algebraically closed, the invertible lienar operator $f \in End(V)$ in the second case, is always of the form

$$f = \lambda \cdot Id_V$$

for some scalar $\lambda \in k$.

Definition 6. *For any G-module (ρ, V), the dual vector space V^* is also a G-module which is compactible with the canonical pairing between V and V^*. This means that for any $\phi \in V^*$, the action ρ^* is determined by*

$$\langle \rho^*(g)\phi, v\rangle = \langle \phi, \rho(g^{-1})v\rangle$$

for any $\phi \in V^*$. Call (ρ^*, V^*) the ***dual representation of*** *(ρ, V).*

Definition 7. *For any two G-modules $(\rho_1, V_1), (\rho_2, V_2)$, the tensor product vector space*

$$V_1 \otimes_k V_2$$

also has a G-module structure, with the action $(\rho_1 \otimes \rho_2)(\cdot)$ given by

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) := gv_1 \otimes gv_2.$$

We call $V_1 \otimes V_2$ the ***tensor product representations*** *of (ρ_1, V_1) and (ρ_2, V).*

We mention here that for 'modules' of general algebraic structures, the 'dual modules' and 'tensor product modules' with respect to that algebraic structure may not exist. The existence of the 'dual representation' and 'tensor product representation' relies on certain special structures which exists for groups. This pattern will be well illustrated using the language of Hopf algebras.

Recall that for two vector spaces V_1, V_2, we can canonically identify the tensor product space with the space of linear maps

$$V_1^* \otimes V_2 \cong Hom(V_1, V_2).$$

Moreover, if we are given two representations, (ρ_1, V_1) and (ρ_2, V), there is a natural G-module structure on $Hom(V_1, V_2)$, thourgh the tensor product of the dual representation (ρ_1^*, V_1^*) and the representation (ρ_2, V_2). We check that for the pure tensors $\phi_1 \otimes v_2 \in V_1^* \otimes V_2 \cong Hom(V_1, V_2)$ and $\phi_2 \otimes v_1 \in V_2^* \otimes V_1$,

$$(\rho_1^* \otimes \rho_2)(g)(\phi \otimes v) = \rho_2(g) \circ f \circ \rho_1(g)^{-1}.$$

We note that if (ρ, V) is a G-module, and for arbitrary $f \in GL(V)$, the 'adjoint action' of f on ρ

$$\rho_f(g) := f^{-1} g f$$

gives a new G-module structure on the same space V. As there are 'too many' elements in $GL(V)$, it is necessary to view all $\rho_f(\cdot)$ as the 'same' representation.

Definition 8. *Two G-modules $(\rho_1, V_1), (\rho_2, V_2)$ are called G-equivalent, or just equivalent, if there is a linear map $f \in Hom(V_1, V_2)$ which is a G-isomorphism.*

For any finite group G there are some 'obvious' representations.

Example 1. *Let 1_G be the one dimensional vector space spanned by v such that all elements $g \in G$ acts as 1 on v, that is,*

$$\rho(g) \cdot v = v \text{ for all } g \in G.$$

*It is clear that $(\rho, 1_G)$ is a G-module, which is called the **trival representation of** G.*

Trivial representation is clearly the representation with the 'minimal dimension'. Although it looks quite tautological, we will see that trivial representations can be used to construct more general representations.

Example 2. *Let*

$$k[G]$$

be the $|G|$-dimensional vector space spanned by elements in G which are viewed as basis. The left multiplication on G

$$L_g \cdot h = gh \text{ for any } g, h \in G$$

extends k-linearly as a G-action on $k[G]$. It is clear that $(L_g, k[G])$ is a representation of G, which is called the ***left regular representation of*** *G. Similary, the representation $(R_g, k[G])$ linearly extended by the right multiplication*

$$R_{g^{-1}} \cdot h = hg^{-1}$$

is called the ***right regular representation of*** *G.*

We will see that any irreducible G-module is 'contained' in the regular representation.

We observe that the vector space

$$k[G]$$

is more than just being a G-module, as the product in G

$$g_1 \cdot g_2 = g_1 g_2$$

extends k bilinearly to an associative product in $k[G]$. We call the associative algebra $k[G]$ with this product the **group algebra** of G. It is tautological that any G-module is a $k[G]$-module and vice versa. This useful connection enables us to borrow methods from representation of associative algebras to analyze group representations.

2.2 Complete Reducibility

The 'averaging' can also be taken on an arbitrary linear map to obtain a G-homomorphism.

Proposition 4. *Let $(\rho_1, V_1), (\rho_2, V_2)$ be two G-modules. For any linear map $f \in Hom(V_1, V_2)$, the 'averaged' map*

$$A_G(f) := \frac{1}{|G|} \sum_{g \in G} \rho_2(g) f \rho_1(g^{-1})$$

is a G-homomorphism. In particular, if $f \in Hom_G(V_1, V_2)$ is already a G-homomorphism, then

$$A_G(f) = f.$$

Proof. This can be done by a direct computation, as we note that for any $x \in G$ we have

$$xA_G(f) = \frac{1}{|G|} \sum_{g \in G} xgfg^{-1} = \frac{1}{|G|} \sum_{h \in G} hf(x^{-1}h)^{-1} = \frac{1}{|G|} \sum_{h \in G} hfh^{-1}x = A_G(f)x,$$

Theorem 3. *Let G be a finite group, k be a field such that*

$$char(k) \nmid |G|.$$

Then any G-module is ***completely reducible****, that is, for any G-submodule W of V, there is a complement G-submodule W':*

$$V = W \oplus W'.$$

Proof. Let $\tilde{W}$ be any complement vector subspace of V with respect to W:

$$V = W \oplus \tilde{W}.$$

In general $\tilde{W}$ is not a G-module, but from this decomposition we can take 'average' to obtain a complement G-submodule. Let $\tilde{P}_W \in End(V)$ be the projection operator with respect to the above direct sum decomposition:

$$\tilde{P}_W(w) = w,\ \tilde{P}_W(\tilde{w}) = 0 \text{ for any } w \in W,\ \tilde{w} \in \tilde{W}.$$

We consider the 'averaged' projection operator $P_W \in Hom(V, W)$:

$$P_W := A_G(\tilde{P}_W).$$

It is also easy to verify that

$$P_W|_W = Id_W$$

as $\tilde{P}_W|_W = id_W$, and

$$Id_V - P_W$$

is also a G-projection operator onto a complement subspace of V respect to W, because

$$\begin{aligned}&(Id_V - P_W) + P_W = Id_W,\ (Id_V - P_W)^2 = (Id_V - P_W);\\ &x(Id_V - P_W) = (Id_V - P_W)x \text{ for all } x \in G\end{aligned}$$

by a direct computation. Therefore the subspace

$$W' := (Id_V - P_W)V$$

is a G-submodule and

$$V = W \oplus W',$$

hence W' is the desired complement G-submodule.

Proposition 5. *Any G-module V is isomorphic to a direct sum of irreducible G-modules V_i:*

$$V \cong \bigoplus_i V_i.$$

The following proposition is a kind of converse of Schur's lemma, which is a very useful criteria for irreducibility:

Proposition 6. *A G-module V is irreducible if and only if*

$$dim(Hom_G(V, V)) = 1.$$

which implies that the averaged linear map $A_G(f)$ is a G-homomorphism.

Proposition 7. *If V_1, V_2 are irreducible G-modules, then for any $f \in Hom(V_1, V_2)$, the G homomorphism $A_G(f)$ is either 0, or an G-isomorphism.*

More precisely, we note that for any $f \in Hom(V_1, V_2)$, if $V_1 \ncong V_2$, then

$$\frac{1}{|G|}\sum_{g\in G}\rho_2(g)f\rho_1(g^{-1}) = A_G(f) = 0;$$

and if $V_1 = V_2 = V$,

$$\frac{1}{|G|}\sum_{g\in G}\rho_2(g)f\rho_1(g^{-1}) = T \cdot Id_V$$

where T is a constant to be determined. By taking the trace of both sides, we note that

$$Td(V) = \frac{1}{|G|}\cdot\sum_{g\in G} Tr(\rho_2(g)f\rho_1(g)^{-1}) = Tr(f),$$

Therefore $T = \frac{Tr(f)}{d(V)}$, and

$$\frac{1}{|G|}\sum_{g\in G}\rho_2(g)f\rho_1(g^{-1}) = \frac{Tr(f)}{d(V)}Id_V.$$

We can obtain a very useful relation about two irreducible G-modules, by taking the matrix coefficients of certain averaged map $A_G(f)$, and this relation will be used later. Consider the matrix coefficients of $A_G(\phi_1 \otimes v_2)$ with respect to $\phi_2 \in V_2^*$, $v_1 \in V_1$, we have

$$\begin{aligned}
&\frac{1}{|G|}\sum_{g\in G}\langle\phi_2, \rho_2(g)v_2\rangle\langle\phi_1, \rho_1(g^{-1})v_1\rangle \\
=&\frac{1}{|G|}\sum_{g\in G}\langle\phi_2, \rho_2(g)(\phi_1\otimes v_2)\rho_1(g^{-1})v_1\rangle \\
=&\langle\phi_2, \frac{1}{|G|}\sum_{g\in G}\rho_2(g)(\phi_1\otimes v_2)\rho_1(g^{-1})\cdot v_1\rangle \\
=&\begin{cases} 0 \text{ , if } V_1 \ncong V_2, \\ \\ \frac{\langle\phi_2, v_1\rangle\langle\phi_1, v_2\rangle}{d(V)} \text{ , if } V_1 = V_2 = V. \end{cases}
\end{aligned}$$

2.3 Characters of Representations

For any two k-valued functions f_1, f_2 on G, we define a k-bilinear map

$$(f_1, f_2) := \frac{1}{|G|}\sum_{g\in G} f_1(g)f_2(g^{-1}).$$

By **??**, for two representations ρ_1, ρ_2, let $\{e_i\}$, $\{f_i\}$ be orthonormal basis of V_1, V_2, then

$$(\chi_{\rho_1}, \chi_{\rho_2}) = \frac{1}{|G|}\sum_{g\in G}\chi_{\rho_1}(g)\chi_{\rho_2}(g^{-1})$$

$$=\frac{1}{|G|}\sum_{g\in G,\, i,j=1,\cdots,d}(e_i,\rho_1(g)e_i)(f_j,\rho_2(g^{-1})f_j)$$
$$=\delta_{V_1,V_2}$$

We call the above the **first orthogonal relation of irreducible characers**.

Consider the regular representation $(reg_G, k[G])$, we note that for any $g \in G$,

$$Tr(reg_G(g)) = \begin{cases} 0,\ g \neq 1 \\ |G|,\ g = 1. \end{cases}$$

because $gh = h$ if and only if $g = 1$. For any character χ, we note that

$$(\chi, reg_G) = \frac{1}{|G|}\sum_{g\in G} Tr(reg_G(g))Tr|_V(\rho(g)) = Tr|_V(Id_V) = d(V).$$

If χ is irreducible, it occurs in reg_G with multiplicity equals $d_\chi \neq 0$, which in particular implies that any irreducible representation occurs in $|G|$. By the dimension formula we obtain

$$\sum_i d_\uparrow^2 = |G|.$$

Proposition 8. *There are only finitely many irreducible representations which all occurs in Reg_G.*

We use

$$Rep(G)$$

to denote the category of G-modules, and

$$Irr(G)$$

To denote the set of irreducible G-modules.

For any G-module M and $V \in Irr(G)$, call the tensor product G-module

$$M^V := V \otimes Hom_G(V, M)$$

the **isotypic component M with respect to V**. As vector spaces we have

$$M^V \simeq V^{\oplus m_M(V)},$$

but we should notice that this isomorphism is no canonical, because there may exist different splitting of M^V into $m_M(V)$-pieces of V. However, the direct sum decomposition of M into isotypic components is unique.

Theorem 4. *For any G-module M we have a canonical isomorphism of G-modules*

$$M \simeq \bigoplus_{V\in Irr(G)} M^V.$$

Definition 9. *Let (ρ, V) be a representation of G. We call the trace of $\rho(g)$*

$$\chi_{\rho,V}(g) := Tr|_V(\rho(g))$$

*the **character** of an element $g \in G$ on the representation (ρ, V). It is clear that when fixing a representation (ρ, V), $\chi_{\rho,V}(\cdot)$ can also be viewd as a linear function on $\mathbb{C}[G]$. Therefore we also call $\chi_{\rho,V}(\cdot)$ the **character of a representation** (ρ, V).*

In some situations, we may also write

$$\chi_\rho \text{ or } \chi_V.$$

In many situations just

$$\chi$$

instead of $\chi_{\rho,V}$ for convenience, if there is no ambiguity.

Definition 10. *A function $f : G \to k$ is called a **class function on** G if it is invariant under the conjugate action of G, that is,*

$$f(ghg^{-1}) = f(h) \text{ for any } g, h \in G.$$

It is clear that the class functions of G is a linear subspace of $k[G]^*$, which will be denoted as $R_k(G)$.

Equivalently, $R_k(G)$ can also be viewed as the space of functions on the set conjugacy classes $C(G)$ of G.

Now we consider the particular case when $k = \mathbb{C}$. Let $(\cdot, \cdot)$ be the standard inner product on $\mathbb{C}^d$, and we identify V^* with V with respect to this inner product, and we can take an orthonormal basis $\{e_i\}$ of V:

$$(e_i, e_i) = 1.$$

In particular, for any $f = \chi_\rho$ and $g \in G$, by **??**, we note that

$$(\rho^*(g)e_i, e_i) = (e_i, \rho(g)^{-1}e_i) = \overline{(\rho(g)^{-1}e_i, e_i)},$$

take the summation over the basis we obtain

$$\chi_{\rho^*}(g) = \sum_i (\rho^*(g)e_i, e_i) = \sum_i \overline{(\rho(g)^{-1}e_i, e_i)} = \overline{\chi_\rho(g^{-1})}.$$

A direct computation shows that

Lemma 2. *A function $f : G \to k$ is a class function if and only if*

$$\frac{1}{|G|} f(x^{-1})x \in Z(k[G]).$$

In particular, for any irreducible G-module V,

$$\frac{1}{|G|}\sum_{x\in G}\chi_V(x^{-1})x \in Z(k[G]).$$

Schur's Lemma implies that central elements acts as scalars on an irreducible module, and more explicitly:

Lemma 3. *For a irreducible G-module V, and $z \in Z(k[G])$, z acts as the scalar $\frac{\chi_V(z)}{d(V)}$ on V, that is, for any $v \in V$,*

$$zv = \frac{\chi_V(z)}{d(V)}v.$$

We note that for any irreducible G-module V,

$$\chi_V(\frac{1}{|G|}\sum_{x\in G} f(x^{-1})x) = \frac{1}{|G|}\sum_{x\in G} f(x^{-1})\chi_V(x) = (f, \chi_V).$$

We need the following lemma

Lemma 4. *If a central element $z \in k[G]$ satisfies*

$$\chi_V(z) = 0$$

for any irreducible character χ_V, then

$$z = 0.$$

Proof. We consider the left multiplication of an central element z on $k[G]$. By Lemma **??** and the assumption that $\chi_V(z) = 0$, z acts as 0 on any irreducible submodule isomorphic to V, therefore the only possiblity is that $z = 0$.

Proposition 9. *If a class function $f \in C(G)$ satisfies*

$$(\chi, f) = 0$$

for any (irreducible) character χ, then

$$f = 0.$$

The above proposition implies that irreducible characters $\chi_1, \cdots, \chi_t$ span $C(G)$, and moreover form a(n) (orthonormal) basis of $C(G)$. On the other hand, the characteristic functions on conjugacy classes C_G also form a basis of $C(G)$, hence we conclude that

Theorem 5. *The number of irreducible representations equals the number of conjugacy classes in G.*

Although we have the above equality of two numbers, it is in general not known how to describe the correspondence between irreducible representations and conjugacy classes.

We note that for a given conjugacy class C, The character χ is a constant function as it being a class function, therefore we set

$$\chi(C) := \chi(x) \text{ for any } x \in C.$$

Suppose $\chi_1, \cdots, \chi_t$ are all irreducible characters, and $C_1, \cdots, C_t$ are all conjugacy classes, then we can obtain a $t \times t$ matrix

$$X := (\chi_{i,j})_{t\times t},\ \chi_{i,j} := \chi_i(C_j).$$

Introduce the matrix

$$C = diag(|C_1|, \cdots, |C_t|),$$

we can rewrite the first orthogonal relation as

$$XCX^t = |G|I_t.$$

It follows that

$$CX^tX = |G|I_t$$

which means that irreducible characters with respect to different conjugacy classes are also orthogonal, that is,

$$\sum_i \chi_i(C_j)\chi_i(C_k) = \delta_{j,k}\frac{|G|}{|C_i|}.$$

2.4 Some Number Theoretic Properties and Burnside's p^aq^b Theorem

Lemma 5. *The operators $\rho_V(g)$ are all diagonalizable with eigenvectors being roots of unity.*

It is obvious that

$$\rho_V(g)^{|G|} = \rho_V(g^{|G|}) = \rho_V(e) = Id_V,$$

so that the characteristic polynomial is a divisor of $x^{|G|}-1$. We note that $x^{|G|}-1$ is a polynomial without multiple roots, so is the characteristic polynomial, and therefore $\rho_V(g)$ is diagonoalizable.

Lemma 6. *The (irreducible) character values $\chi(g)$ are all algebraic integers.*

Clearly, $\chi(g)$ is the sum of eigenvalues which are paticular algebraic integers all being roots of unity, therefore $\chi(g)$ is also an algebraic integer.

Lemma 7. *Let (ρ, V) be an irreducible representation of G, and C a conjugacy class of G. Then*

$$\sum_{x \in C} \rho_V(x)$$

is a scalar, and more precisely,

$$\sum_{x \in C} \rho_V(x) = \frac{|C| \chi_V(C)}{d} Id_V.$$

where

$$d = dim(V)$$

is the degree of V.

Proof. We note that

$$yC = Cy$$

for any $y \in G$ as C is a conjugacy class of G, therefore

$$\rho_V(y) \sum_{x \in C} \rho_V(x) = \sum_{x \in C} \rho_V(x) \rho_V(y),$$

and we deduce that

$$\sum_{x \in C} \rho_V(x) \in Hom_G(V, V).$$

By Schur's Lemma,

$$\sum_{x \in C} \rho_V(x) = \lambda \cdot Id_V,$$

for some constant λ, and we compute that

$$\lambda = \frac{|C| \chi_V(C)}{d}$$

by taking the trace of both sides.

Let $R, S, T \subseteq G$ be three subsets of G, and define the number

$$c_{R,S}(t) := |\{rs = t | \, r \in R, \, s \in S, \, t \in T\},$$

that is, the number of elements $r \in R$, $s \in S$ such that

$$rs = t,$$

and we also define

$$c_{R,S}(t) := |\{rs \in T | \, r \in R, \, s \in S\} = \sum_{t \in T} C_{R,S}(t).$$

In particular, if R, S, T are conjugacy classes, we note that $C_{R,S}(t)$ are the same for any $t \in T$, and therefore

$$C_{R,S}^T := C_{R,S}(t)|T|.$$

for any element $t \in T$.

Proposition 10. *Let $C_1, \cdots, C_t$ be all the conjugacy classes of G, and for an irreducible G-module V, let*

$$\lambda_{ij} := \frac{|C_j|\chi_{ij}}{\chi_{i1}}.$$

Then

$$\lambda_{ij}\lambda_{ik} = \sum_l c_{jk}^l \lambda_{il}.$$

where

$$c_{jk}^l := c_{C_j,C_k}^{C_l}.$$

Proof. It is clear that for any two conjugacy classes $C_1, C_2, \subseteq G$ we have

$$\sum_{x\in C_1, y\in C_2} xy = \sum_{z\in G} c_{C_1,C_2}(z)z.$$

We note that both sides are elements in $k[G]$ which acts on V. For the left hand side we note that

$$\begin{aligned}\rho_V(\sum_{x\in C_1, y\in C_2} xy) =& \sum_{x\in C_1, y\in C_2} \rho_V(x)\rho_V(y)\\ =&(\sum_{x\in C_1}\rho_V(x))(\sum_{y\in C_2}\rho_V(y))\\ =&\lambda_i\lambda_j\end{aligned}$$

by **??**. For the right hand side, we note that

$$\begin{aligned}\rho_V(\sum_{z\in G} c_{C_1,C_2}(z)z) =& \sum_{z\in G} c_{C_1,C_2}(z)\rho_V(z)\\ =& \sum_{C\in C_G}\sum_{z\in C} c_{C_1,C_2}(z)\rho_V(z)\\ =& \sum_{C\in C_G} c_{C_1,C_2}(C)\sum_{z\in C}\rho_V(z)\\ =&\sum_l c_{jk}^l\lambda_l,\end{aligned}$$

which concludes the proof.

We note that for fixed i and j, the entries of the $t\times t$ matrix

$$(c_{jl}^l)_{t\times t}$$

are all (non-negative) integers, and $\lambda_{i,k}$ is the eigenvalue corresponding to the eigenvector

$$(\lambda_{i,1}, \cdots, \lambda_{i,t})^T.$$

Therefore

Proposition 11. *The numbers $\lambda_{i,j}$ are all algebraic integers.*

On the other hand, we note that the integers c_{ij}^k can be read from the multiplication table of G, therefore we can obtain the the above proposition provides an algorithm which can (in principle) compute the character table numerically, from the multiplication table of G.

Theorem 6. *If G is a finite group of order $p^a q^b$, p, q are primes, $a + b > 1$, then G is solvable.*

2.5

2.6 Induced Modules; The Theory of Frobenius, Clifford, and Mackey

Definition 11. *Let H be a subgroup of G and V an G-module. The restriction H-module is still the space V with the action the same as that of G.*

Definition 12. *Let H be a subgroup of G and W an H-module. The induced G-module is defined by*

$$\uparrow_H^G(W) := k[G] \otimes_{k[H]} W.$$

Theorem 7. *The induction functor is the left adjoint of restriction functor, that is, there is a canonical isomorphism*

$$Hom_G(\uparrow_H^G(W), V) \simeq Hom_H(W, \downarrow_H^G(V)).$$

It is necessary to discuss the effects of induction and restriction when changing the subgroup of G.

Let K, H be subgroups of G. Recall that the double cosets in G with respect to subgroups $K, H \leq G$:

$$K\backslash G/H$$

are subsets of the form

$$KsH,\ s \in G.$$

It is easy to verify that any two different double cosets are disjoint, therefore they form a partition of G into disjoint double cosets.

Lemma 8. *We have the following two correspondences of double cosets:*

$$HsK$$

Lemma 9. *Suppose $\{s_i\}$ are representatives of double cosets $H\backslash G/K$:*

$$G = \cup_i K s_i K,$$

and for each s_i, $\{\gamma_{ij}\}$ are representatives of right cosets $H \cap s_i K s_i^{-1}\backslash H$:

$$H = \cup_j (H \cap s_i K s_i^{-1})\gamma_{ij},$$

then $\gamma_{ij} s_i$ are right coset representatives of $H\backslash G$:

$$G = \cup_{ij} H \gamma_{ij} s_i.$$

Suppose (ρ, V) is a representation of H, then for any $s \in G$, we can obtain a new representation (ρ^s, V_s) of the subgroup sHs^{-1} such that $V_s = V$ are the same vector spaces, but the action is given by:

$$\rho^s(h)v := \rho(s^{-1}hs)v.$$

Theorem 8. *Suppose V is a K-module, then as H-modules we have an isomorphism of H-modules:*

$$\uparrow_K^G(V) \cong \bigoplus_{s \in H\backslash G/K} \uparrow_{H \cap sKs^{-1}}^H (V_s).$$

2.7 The Theorem of Artin and Brauer

Theorem 9. *Any character $\chi \in R(G)$ is a linear combination of characters induced from (1-dimensional) characters of cyclic subgroups, with rational coefficients. That is, let*

$$\mathcal{CS}(G)$$

be the set of cyclic subgroups of G, then for any $\chi \in R(G)$ there are rational numbers a_C and 1-dimensional characters χ_C associated with each $C \in \mathcal{CS}(G)$ such that

$$\chi = \sum_{C \in \mathcal{CS}(G)} a_C \uparrow_C^G (\chi_C).$$

Define the following function over G:

$$\theta_C(x) = \begin{cases} |C|, & \text{if } C = \langle x \rangle \\ 0, & \text{else.} \end{cases}$$

Lemma 10.

$$\sum_{C \in \mathcal{CS}(G)} \uparrow_C^G \theta_C = |G|.$$

Theorem 10. *Any character $\chi \in R(G)$ is a linear combination of characters induced from (1-dimensional) characters of elementary subgroups, with integer coefficients. That is, let*

$$\mathcal{ES}(G)$$

be the set of elementary subgroups of G, then for any $\chi \in R(G)$ there are integers a_E and 1-dimensional characters χ_E associated with each $E \in \mathcal{ES}(G)$ such that

$$\chi = \sum_{E \in \mathcal{ES}(G)} a_E \uparrow_E^G (\chi_E).$$

2.8 Representations of Finite Abelian Groups and Pontryajin Duality

2.9 Representations of Symmetric groups S_n

Lemma 11. *The conjugacy classes of S_n are labelled by Young diagrams with n-boxes.*

Let T be a Young tableaux, Let P_T be the subgroup of S_n which preserves the rows of T, and similarly Q_T be the subgroup of S_n which preserves the Columns of T. Define two elements in $\mathbb{C}[S_n]$:

$$a_T := \sum_{g \in P_T} g,\ b_T := \sum_{g \in Q_T} (-1)^{l(g)} g.$$

The **Young symmetrizer** c_λ is defined by

$$c_\lambda := \sum_{T \in T_\lambda} a_T b_T.$$

Theorem 11. *The subspace*

$$\mathbb{C}[S_n]c_\lambda$$

*is an irreducible S_n-module called the **Specht module**.*

2.10 Representations of $GL_2(\mathbb{F}_q)$