Adjunction space as pushout of two morphisms

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Remark 0.1. `GPT4o`.

> **Problem**
>
> Show that $(X \cup_f Y, \iota_X : X \to X \cup_f Y, \iota_Y : X \cup_f Y)$ is the **pushout** of the morphisms (ι_A, f), where $f : A \to X$ and $\iota : A \to Y$ for $A \subset Y$ a closed subset.

We want to show that $(X \cup_f Y, \iota_X : X \to X \cup_f Y, \iota_Y : X \cup_f Y)$ satisfies the **universal property** of the pushout. Assume that we have an object $B \in \mathbf{Top}$ and morphisms $h_Y : Y \to B$ and $h_X : X \to B$ such that

$$h_Y \circ \iota_A = h_X \circ f.$$

Define $g : X \coprod Y \to B$ such that $g(x) = h_X(x)$ and $g(y) = h_Y(y)$. Then we see that $g|_Y = h_Y$ and $g|_X = h_X$, so that by the **characteristic property of the disjoint union topology**, g is *continuous*. We know that $h_X(f(a)) = h_Y(a)$ for all $a \in A$. Therefore, we have the following diagram.

This means that

Remark 0.2. Pay attention to the fact that $\iota_Y : Y \to X \cup_f Y$ is really a composition of $i_Y : Y \to X \coprod Y$ and the projection $q : X \coprod Y \to (X \coprod Y)/\sim_f = X \cup_f Y$. Same remark holds for ι_X.
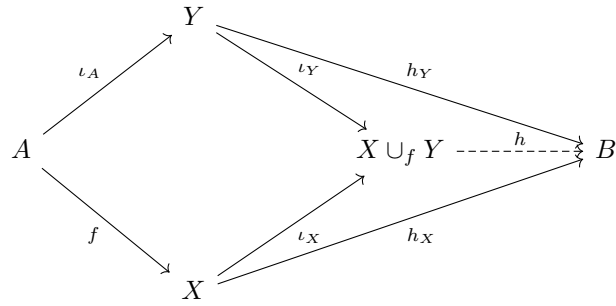
Note that if $q(z) = q(z')$ for $z \neq z'$, then we must have that $z = a$ and $z' = f(a)$ or $z = f(a)$ and $z' = f(a)$. Either way, we find that (do we?)

$$
\begin{aligned}
g(a) &= h_Y(a) \\
&= h_Y(\iota_A(a)) \\
&= h_X(f(a)) \\
&= g(f(a)),
\end{aligned}
$$

so that g makes the same identifications as q, and therefore g **passes to the quotient** to define $h : X \cup_f Y \to B$ such that $h \circ q = g$.

We claim then that $h \circ \iota_X = h \circ \iota_Y$.

We have the following diagram:

We have

$$
\begin{aligned}
h \circ \iota_Y(y) &= (h \circ (q \circ i_Y))(y) \\
&= ((h \circ q) \circ i_Y)(y) \\
&= (g \circ i_Y)(y) \\
&= h_Y(i_Y(y))
\end{aligned}
$$

and

$$
\begin{aligned}
(h \circ \iota_X)(x) &= (h \circ (q \circ i_X))(x) \\
&= ((h \circ q) \circ i_X)(x) \\
&= (g \circ i_X)(x) \\
&= h_X(i_X(x)).
\end{aligned}
$$