Topics in Algebra: A Primer of Algebraic D-modules; Exercises

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Remark 0.1. I have used `Gemini 2.5 pro`/`ChatGPT` to discuss exercises. I've mainly tried solving them myself, but it has happened that it has given me ideas for how to proceed in the proofs - it has also helped me with details that were not that important for writing the main proofs, but understanding details in said proofs. I've at times used them to find solutions, but have always written the solutions myself.

Chapter 1

4.2

Find the canonical form of the elements of A_3 listed below:
(a) $\partial_1^2 \cdot x_1^2$.
(b) $\partial_2^3 \cdot x_1 \cdot \partial_3 \cdot x_3 + x_3 \cdot \partial_1 \cdot x_1$.
(c) $\partial_1 \cdot x_1 \cdot \partial_2 \cdot \partial_3 \cdot x_3 \cdot x_2$.
(d) $\partial_1^3 \cdot x_1^2 + \partial_2^2 \cdot x_2^3$.

(a):

$$\begin{aligned}
\partial_1^2 \cdot x_1^2 &= \partial_1 \cdot (\partial_1 \cdot x_1^2) \\
&= \partial_1 \cdot (x_1^2 \cdot \partial_1 + 2x_1) \\
&= \partial_1 \cdot (x_1^2 \cdot \partial_1) + 2x_1\partial_1 + 2 \\
&= x_1^2\partial_1^2 + 2x_1\partial_1 + 2x_1\partial_1 + 2 \\
&= x_1^2\partial_1^2 + 4x_1\partial_1 + 2
\end{aligned}$$

(b):

$$\begin{aligned}
\partial_2^3 \cdot x_1 \cdot \partial_3 \cdot x_3 + x_3 \cdot \partial_1 \cdot x_1 &= \partial_2^2 \cdot (x_1 \cdot \partial_2) \cdot (x_3 \cdot \partial_3 + 1) + x_3 \cdot (x_1 \cdot \partial_1 + 1) \\
&= \partial_2 \cdot (x_1 \cdot \partial_2) \cdot \partial_2 \cdot (x_3 \cdot \partial_3 + 1) + x_3 \cdot x_1 \cdot \partial_1 + x_3 \\
&= x_1 \cdot \partial_2^3 \cdot (x_3 \cdot \partial_3 + 1) + x_3 \cdot x_1 \cdot \partial_1 + x_3 \\
&= x_1 \cdot x_3 \cdot \partial_2^3 \cdot \partial_3 + x_1 \cdot \partial_2^3 + x_3 \cdot x_1 \cdot \partial_1 + x_3.
\end{aligned}$$

(c):

$$\begin{aligned}
\partial_1 \cdot x_1 \cdot \partial_2 \cdot \partial_3 \cdot x_3 \cdot x_2 &= (x_1 \cdot \partial_1 + 1) \cdot \partial_2 \cdot (x_3 \cdot \partial_3 + 1) \cdot x_2 \\
&= (x_1 \cdot \partial_1 \cdot \partial_2 + \partial_2) \cdot (x_3 \cdot \partial_3 + 1) \cdot x_2 \\
&= (x_1 \cdot \partial_1 \cdot \partial_2) \cdot (x_3 \cdot \partial_3 + 1) \cdot x_2 + \partial_2 \cdot (x_3 \cdot \partial_3 + 1) \cdot x_2 \\
&= (x_1 \cdot \partial_1 \cdot \partial_2) \cdot (x_3 \cdot x_2 \cdot \partial_3 + x_2) + \partial_2 \cdot (x_3 \cdot x_2 \cdot \partial_3 + x_2) \\
&= x_1 \cdot \partial_1 \cdot (x_3 \cdot (x_2 \cdot \partial_2 \partial_3) + \partial_3) + x_1 \cdot \partial_1 \cdot x_2 \cdot \partial_2 + x_1 \cdot \partial_1 + x_3 \cdot x_2 \cdot \partial_2 \cdot \partial_3 + x_3 \cdot \partial_3 + x_2 \cdot \partial_2 + 1 \\
&= x_1 \cdot x_3 \cdot x_2 \cdot \partial_1 \cdot \partial_2 \cdot \partial_3 + x_1 \cdot x_3 \cdot \partial_1 \cdot \partial_3 + x_1 \cdot x_2 \cdot \partial_1 \cdot \partial_2 \\
&+ x_1 \cdot \partial_1 + x_3 \cdot x_2 \cdot \partial_2 \cdot \partial_3 + x_3 \cdot \partial_3 + x_2 \cdot \partial_2 + 1.
\end{aligned}$$

(d):

$$\partial_1^3 \cdot x_1^2 + \partial_2^2 \cdot x_2^3 = (x_1^2 \cdot \partial_1^3 + 6 \cdot x_1 \cdot \partial_1^2 + 6 \cdot \partial_1) + (x_2^3 \cdot \partial_2^2 + 6x_2^2 \cdot \partial_2 + 6x_2)$$

4.10

Consider the following two matrices in $M_\infty(K)$:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Show that the K-algebra of $M_\infty(K)$ generated by P and Q are isomorphic to $A_1(K)$.

Remark 0.2. We will assume that the base field K is of characteristic 0.

Using the formulation in [Cou95, 1.§3] we have that $A_1(K) = K\{z_1, z_2\}/\langle z_2 z_1 - z_1 z_2 - 1\rangle$. We define a map $f : K\{z_1, z_2\} \to \langle P, Q\rangle \subset M_\infty(K)$ by $f(z_1) = Q$ and $f(z_2) = P$. We claim that since there are no relations in $K\{z_1, z_2\}$ we can extend f to a homomorphism Θ of K-algebras that acts as f on the generators z_1, z_2 of $K\{z_1, z_2\}$.

Let $\{e_1, e_2, \ldots, \}$ be the infinite standard basis for some K-vector space V with $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$. Then we see that $Q(e_i) = e_{i+1}$ and $P(e_1) = 0$ and $P(e_j) = (j-1)e_{j-1}$ for $j > 1$. Then we have that

$$\begin{aligned}
(PQ - QP)(e_1) &= PQ(e_1) - QP(e_1) \\
&= P(e_2) - Q(0) \\
&= P(e_2) \\
&= e_1
\end{aligned}$$

and for $j > 1$ we have

$$\begin{aligned}
(PQ - QP)(e_j) &= PQ(e_j) - QP(e_j) \\
&= P(e_{j+1}) - (j-1)(Q(e_{j-1})) \\
&= je_j - (j-1)e_j \\
&= e_j.
\end{aligned}$$

Thus $PQ - QP$ acts as the identity on V.

If we instead take the matrix perspective, we see that

$$PQ = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$QP = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

so that

$$\begin{aligned} [P, Q] &= PQ - QP \\ &= I_\infty, \end{aligned}$$

where I_∞ is the infinite identity matrix.

From the definition of Θ, together with what we have just shown, it follows that $\langle z_2 z_1 - z_1 z_2 - 1 \rangle \subset \ker \Theta$. Thus by the universal property of the quotient there is a ring homomorphism $\widetilde{\Theta} : A_1(K) \to \langle P, Q \rangle$ such that $\widetilde{\Theta} \circ \pi = \Theta$ with $\pi : K\{z_1, z_2\} \twoheadrightarrow A_1(K)$. Since Θ and π are K-linear by construction and π is surjective, it follows that $\widetilde{\Theta}$ is K-linear:

$$\begin{aligned} \widetilde{\Theta}(ka) &= \widetilde{\Theta}(k\pi(a')) \\ &= (\widetilde{\Theta} \circ \pi)(ka') \\ &= \Theta(ka') \\ &= k\Theta(a') \\ &= k\widetilde{\Theta}(a), \end{aligned}$$

holds for all $a \in K\{z_1, z_2\}/\langle z_2 z_1 - z_1 z_2 - 1 \rangle$ and $k \in K$, where we used that π is surjective so that for each $a \in K\{z_1, z_2\}/\langle z_2 z_1 - z_1 z_2 - 1 \rangle$ there is some $a' \in K\{z_1, z_2\}$ such that $\pi(a') = a$. Since Θ and π are surjective (that they are surjective follows essentially by definition) so is $\widetilde{\Theta}$. Since $A_1(K)$ is simple ([Cou95, Chap 2, Theorem 2.1]) and $\widetilde{\Theta}$ is not the zero morphism (e.g. $\widetilde{\Theta}(\overline{z}_1) = Q \neq 0$) it follows that $\ker \widetilde{\Theta} = 0$ and so $\widetilde{\Theta}$ is injective, hence is an isomorphism.

Chapter 2

4.2

Show that the left ideal of A_3 generated by $\partial_1, \partial_2, \partial_3$ may also be generated by two elements. Hint: Choose $D_1 = \partial_1$ and $D_2 = \partial_2 + x_1 \partial_3$ to be the generators and calculate $[D_1, D_2]$.

Let $I = \langle D_1, D_2 \rangle \trianglelefteq A_3$. We have that

$$\begin{aligned}
[D_1, D_2] &= [\partial_1, \partial_2 + x_1\partial_3] \\
&= \underbrace{[\partial_1, \partial_2]}_{=0} + [\partial_1, x_1\partial_3] \\
&= \partial_1(x_1\partial_3) - \underbrace{(x_1\partial_3)\partial_1}_{=x_1\partial_1\partial_3} \\
&= x_1\partial_1\partial_3 + \partial_3 - x_1\partial_1\partial_3 \\
&= \partial_3.
\end{aligned}$$

Hence ∂_3 is in I since $[D_1, D_2]$ is in I (I is a left ideal and each term in $[D_1, D_2]$ is on the form of ap_i with p_i in I and a in A_3).

But then also $\partial_2 = D_2 - x_1\partial_3$ is in D, and so $\langle \partial_1, \partial_2, \partial_3 \rangle \subset I$. The other direction is clear, since the generators D_1, D_2 for I are in $\langle \partial_1, \partial_2, \partial_3 \rangle$. Thus $\langle \partial_1, \partial_2, \partial_3 \rangle = \langle D_1, D_2 \rangle$.

4.10

Let $\phi : A_1^2 \to A_1$ be defined by $(a, b) \mapsto a\partial + bx$. Show that ϕ is surjective and that its kernel is isomorphic to the left ideal $A_1\partial^2 + A_1(x\partial - 1)$.Conclude that this is a projective left ideal of $A_1(K)$.

First, we notice that ϕ is linear with respect to $+$ and that both A_1^2 and A_1 are A_1-modules, and that ϕ is A_1-linear, hence is an A_1-module homomorphism.

Using the standard basis $\mathbf{B} = \{x^\alpha \partial^\beta \;:\; \alpha, \beta \in \mathbb{N}\}$ for A_1, we can write any element $D \in A_1$ as $D = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta$. We investigate the terms $c_{\alpha\beta} x^\alpha \partial^\beta$ in D seperately.

$\underline{\alpha = \beta = 0}$: We then see that $(-c_{\alpha\beta}x, c_{\alpha\beta}\partial)$ is sent under ϕ to

$$\begin{aligned}
-c_{\alpha\beta}x\partial + c_{\alpha\beta}\partial x &= c_{\alpha\beta}(\partial x - x\partial) \\
&= c_{\alpha\beta} \underbrace{[\partial, x]}_{=1} \\
&= c_{\alpha\beta}.
\end{aligned}$$

$\underline{\alpha \neq 0 \text{ or } \beta \neq 0}$: Assume first that $\beta \neq 0$. Then $(c_{\alpha\beta} x^\alpha \partial^{\beta-1}, 0)$ is sent to $c_{\alpha\beta} x^\alpha \partial^\beta$.

If $\alpha \neq 0$ then this gets a bit trickier since we need to move the x across ∂^β. One can show by the relations in [Cou95, Chap 1, §1] that $\partial^\beta x = x\partial^\beta + \beta\partial^{\beta-1}$. If $\beta = 0$ then it is enough to take $(0, c_{\alpha\beta}x^{\alpha-1})$ which is sent to $c_{\alpha\beta}x^\alpha$. If $\beta \geqslant 1$ then we have that

$$\begin{aligned}
c_{\alpha\beta} x^{\alpha-1}\partial^\beta x &= c_{\alpha\beta} x^{\alpha-1}(x\partial^\beta + \beta\partial^{\beta-1}) \\
&= c_{\alpha\beta} x^\alpha \partial^\beta + \beta c_{\alpha\beta} x^{\alpha-1}\partial^{\beta-1}.
\end{aligned}$$

Thus we see that if $\beta > 1$ we may take

$$(-\beta c_{\alpha\beta} x^{\alpha-1}\partial^{\beta-2}, c_{\alpha\beta} x^{\alpha-1}\partial^\beta)$$

which is sent (under ϕ) to

$$-\beta c_{\alpha\beta} x^{\alpha-1}\partial^{\beta-1} + c_{\alpha_\beta} x^\alpha \partial^\beta + \beta c_{\alpha\beta} x^{\alpha-1}\partial^{\beta-1} = c_{\alpha_\beta} x^\alpha \partial^\beta.$$

If $\beta = 1$ then we may take $(c_{\alpha\beta}x^\alpha, 0)$ which is sent to $c_{\alpha\beta}x^\alpha\partial = c_{\alpha\beta}x^\alpha\partial^\beta$.

Thus, we have seen that for any summand d in D, there is an element in A_1^2 that is sent under ϕ to d. By linearity it follows that there is an element in A_1^2 that is sent to D, so that ϕ is surjective. Note that by the 1st isomorphism theorem (for modules) we then have $A_1^2/\ker\phi \cong A_1$ as (left) A_1-modules.

Let $J = A_1\partial^2 + A_1(x\partial - 1)$. We find that $\ker\phi = \{(a,b) \in A_1^2 \;:\; a\partial + bx = 0\}$. Notice that if $a\partial + bx = 0$ then on right-multiplying with ∂ we get

$$\begin{aligned} a\partial^2 + bx\partial &= 0 \\ \Leftrightarrow a\partial^2 + b(x\partial - 1) &= -b, \end{aligned}$$

so that for any pair $(a,b) \in \ker\phi$ we have that $a\partial^2 + b(x\partial - 1) = -b \in J \Rightarrow b \in J$. We then have a projection map $\pi : \ker\phi \to J$ defined by $(a,b) \mapsto b$ which is well-defined (since $x\partial$ and 1 are in canonical form) and this map is furthermore A_1-linear.

<u>π injective</u>: If $b = 0$ then since we have the relation $a\partial = -bx$ it follows that (since ∂ is a basis element in A_1) that $a = 0$, and so $(a,b) = (0,0)$. Thus π is injective.

<u>π surjective</u>: Let $b = r\partial^2 + s(x\partial - 1)$. Upon right-multiplying with x we get

$$\begin{aligned} bx &= r\partial^2 x + s(x\partial - 1)x \\ &= rx\partial^2 + 2r\partial + sx^2\partial + sx - sx \\ &= rx\partial^2 + 2r\partial + sx^2\partial \\ &= (rx\partial + 2r + sx^2)\partial. \end{aligned}$$

Thus, if we take $a = -(rx\partial + 2r + sx^2)$ we see that

$$\begin{aligned} a\partial + bx &= -(rx\partial + 2r + sx^2)\partial + (rx\partial + 2r + sx^2)\partial \qquad (0.1) \\ &= 0. \qquad (0.2) \end{aligned}$$

Thus with $a, b \in A_1$ as above we see that $(a,b) \mapsto b$ so that π is surjective. Hence π an A_1-module isomorphism $A_1\partial^2 + A_1(x\partial - 1) \cong \ker\phi$.

Consider the short exact sequence

$$0 \to \ker\phi \hookrightarrow A_1^2 \xrightarrow{\phi} A_1 \to 0$$

Since A_1 with the canonical left A_1-module structure is projective (it is a direct summand of a free module: A_1 itself) and so by an equivalent characterization of A_1 being projective we have that the sequence above splits, thus $A_1^2 \cong A_1 \oplus \ker\phi$. Hence $\ker\phi$ is a direct summand of the free A_1-module A_1^2 and thus is a projective A_1-module.

Chapter 3

3.5

Let J be an ideal of $S = K[x_1, \ldots, x_n]$. Let $\bar{f}$ be the image of $f \in S$ in the quotient ring S/J. Suppose that D is a derivation of S/J and choose $g_i \in S$ such that $\overline{g}_i = D(\overline{x_i})$, for $1 \leqslant i \leqslant n$.

(1) Show that if $B = \sum_i g_i\partial_i$, then $\overline{B} = D$, in the notation of [Cou95, Chap. 3, Exc. 3.2].

(2) Let $\mathrm{Der}_J(S)$ be the set of derivations $D \in \mathrm{Der}(S)$ such that $D(J) \subseteq J$. Conclude, using [Cou95, Chap. 3, Exc. 3.2], that there is an isomorphism of vector spaces between $\mathrm{Der}_J(S)/J\mathrm{Der}_J(S)$ and $\mathrm{Der}(S/J)$.

Remark 0.3. There might be a typo in 3.5.(2) and it should perhaps say $\mathrm{Der}_J(S)/J\mathrm{Der}(S) \cong \mathrm{Der}(S/J)$. We will assume this is the case in the proof.

(1): By [Cou95, Chap. 3, Exc. 3.2.(1)] we have that $\overline{B}(\overline{f}) = \overline{B(f)}$ defines a K-linear endomorphism of $S/J \Leftrightarrow B(J) \subseteq J$. Thus we believe that for $\overline{B}$ to make sense we need $B(J) \subseteq J$. Assume that $D \in \mathrm{Der}(S/J)$ and that $B = \sum_i g_i \partial_i$ with $\overline{g_i} = D(\overline{x_i})$ for $i = 1, \ldots, n$. Consider the projection map $\pi : S \twoheadrightarrow S/J$. We can then write $\pi(g_i) = D(\pi(x_i))$ for $i = 1, \ldots, n$.

Let $f \in S$ be arbitrary. We then have that

$$\begin{aligned}
\pi(B(f)) &= \pi\left(\sum_{i=1}^{n} g_i \partial_i(f)\right) \\
&= \sum_{i=1}^{n} \pi(g_i)\pi(\partial_i f) \\
&= \sum_{i=1}^{n} \pi(\partial_i f) D(\overline{x_i}) \qquad (\text{since } \pi(g_i) = D(\overline{x_i})).
\end{aligned} \tag{0.3}$$

On the other hand, we claim that by Leibniz rule and induction that $D(\overline{x}_i^{\alpha_i}) = \alpha_i \overline{x_i}^{\alpha_i - 1} D(\overline{x_i})$. This provides our base case. Now assume that in variables $\overline{x_1}, \ldots, \overline{x_n}$ we make the inductive assumption that

$$D(\overline{x_1}^{\alpha_1} \cdots \overline{x_n}^{\alpha_n}) = \sum_{i=1}^{n} \alpha_i \overline{x_1}^{\alpha_1} \cdots \overline{x_i}^{\alpha_i - 1} \cdots \overline{x_n}^{\alpha_n} D(\overline{x_i}).$$

We now consider variables $\overline{x_1}, \ldots, \overline{x_{n+1}}$ and a monomial $\overline{x_1}^{\alpha_1} \cdots \overline{x_{n+1}}^{\alpha_{n+1}}$. By Leibniz rule, we then have

$$\begin{aligned}
D(\overline{x_1}^{\alpha_1} \cdots \overline{x_{n+1}}^{\alpha_{n+1}}) &= \overline{x}_{n+1}^{\alpha_{n+1}} D(\overline{x_1}^{\alpha_1} \cdots \overline{x}_n^{\alpha_n}) + \overline{x_1}^{\alpha_1} \cdots \overline{x}_n^{\alpha_n} D(\overline{x_{n+1}}^{\alpha_{n+1}}) \\
&= \overline{x_{n+1}}^{\alpha_{n+1}} \left(\sum_{i=1}^{n} \alpha_i \overline{x_1}^{\alpha_1} \cdots \overline{x_i}^{\alpha_i - 1} \cdots \overline{x_n}^{\alpha_n} D(\overline{x_i})\right) + \overline{x_1}^{\alpha_1} \cdots \overline{x}_n^{\alpha_n} (\alpha_{n+1} \overline{x_{n+1}}^{\alpha_{n+1} - 1} D(\overline{x_{n+1}})) \\
&= \sum_{i=1}^{n+1} \alpha_i \overline{x_1}^{\alpha_1} \cdots \overline{x_i}^{\alpha_i - 1} \cdots \overline{x_{n+1}}^{\alpha_{n+1}} D(\overline{x_i}).
\end{aligned}$$

Thus it follows by induction that this holds for any monomial in any number of variables $\overline{x_1}, \ldots, \overline{x_n}$. We may now apply this together with K-linearity on any $\pi(f) = \pi\left(\sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha\right) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \overline{x}^\alpha$ in S.

We then have that

$$\begin{aligned}
D(\pi(f)) &= \sum_{\alpha} c_\alpha D(\overline{x}^\alpha) \\
&= \sum_{\alpha} c_\alpha \left(\sum_{i=1}^{n} \alpha_i \overline{x_1}^{\alpha_1} \cdots \overline{x_i}^{\alpha_i - 1} \cdots \overline{x_i}^{\alpha_n} D(\overline{x_i})\right) \\
&= \sum_{i=1}^{n} \left(\sum_{\alpha} \alpha_i c_\alpha \overline{x}^{\alpha - e_i}\right) D(\overline{x_i}) \\
&= \sum_{i=1}^{n} \pi(\partial_i f) D(\overline{x_i}) \\
&= \pi(B(f)) \qquad (\text{by } 0.3).
\end{aligned}$$

In particular, it follows from this that for any $f \in J$, we have that $\pi(B(f)) = D(\pi(f)) = D(0) = 0$, so that $B(f) \subseteq J$.

Furthermore, by the above formula, we then have, for $D \in \mathrm{Der}(S/J)$ and for $B = \sum_i g_i \partial_i$ with g_i choosen such that $\overline{g_i} = D(\overline{x_i})$, that

$$\begin{aligned}
\overline{B}(\overline{x_j}) &= \overline{B(x_j)} \\
&= \overline{\sum_i g_i \partial_i (x_j)} \\
&= \overline{g_j} \\
&= D(\overline{x_j}).
\end{aligned}$$

We claim that any derivation D on the quotient ring S/J is determined by its action on the generators $\overline{x_1}, \ldots, \overline{x_n}$ by (possibly) repeated use of the *Leibniz rule*. By [Cou95, Chap. 3, Exc. 3.2.(3)] $\overline{B}$ is thus a derivation coinciding with D and so $\overline{B} = D$.

<u>(2)</u>: We may note from the previous exercise that there is a surjective map $\phi : \mathrm{Der}_J(S) \to \mathrm{Der}(S/J), \quad B \mapsto \overline{B}$. If we can show that this is a K-linear map with kernel $J \cdot \mathrm{Der}_J(S)$ we are done.

<u>K-linear</u>: Let $B_1, B_2 \in \mathrm{Der}_J(S)$ and $\overline{f} \in S/J$ (with $f \in S$). Then we have that

$$\begin{aligned}
\phi(B_1 + B_2)(\overline{f}) &= \overline{B_1 + B_2}(\overline{f}) \\
&= \overline{(B_1 + B_2)(f)} \\
&= \overline{B_1(f)} + \overline{B_2(f)} \\
&= \overline{B_1}(\overline{f}) + \overline{B_2}(\overline{f}) \\
&= \phi(B_1)(\overline{f}) + \phi(B_2)(\overline{f}).
\end{aligned}$$

We also have that for any $c \in K$, $B \in \mathrm{Der}_J(S)$ and $\overline{f} \in S/J$, it holds that

$$\begin{aligned}
\phi(cB)(\overline{f}) &= \overline{cB}(\overline{f}) \\
&= \overline{cB(f)} \\
&= \pi(cB(f)) \\
&= c\pi(B(f)) \\
&= c\overline{B(f)} \\
&= c\overline{B}(\overline{f}) \\
&= c\phi(B)(f).
\end{aligned}$$

<u>$\ker \phi = J \cdot \mathrm{Der}(S)$</u>: Note that $J \cdot \mathrm{Der}(S) = \{\sum_{i=1}^{n} f_i B_i \ : \ f_i \in J, B_i \in \mathrm{Der}(S)\}$ with $n \in \mathbb{N}_{\geqslant 1}$.

Then we see that $B(f) \in J$ for all $f \in S$ since J is an ideal. Thus $B(S) \subseteq J$ so in particular $B(J) \subseteq J$ and so by [Cou95, Chap. 3, exc. 3.2.(1)] it follows that $\phi(B) = \overline{B} = 0$.

On the other hand, assume that $B \in \ker \phi$. Then by [Cou95, Chap. 3, Exc. 3.2.(2)] we have that $B(S) \subseteq J$. Since $B \in \mathrm{Der}(S)$ we have that $B = \sum_i g_i \partial_i$. Then we see that $B(x_i) = g_i \in J$ for $i = 1, \ldots, n$. Then we see that since $\partial_i \in \mathrm{Der}(S)$ for all $i = 1, \ldots, n$ it follows that $B = \sum_i g_i \partial_i \in J \cdot \mathrm{Der}(S)$. Thus $\ker \phi = J \cdot \mathrm{Der}(S)$.

We conclude that $\mathrm{Der}_J(S)/J \cdot \mathrm{Der}(S) \cong \mathrm{Der}(S/J)$.

3.6

Let $R = K[t^2, t^3]$.

(1) Show that R is isomorphic to $K[x,y]/J$ where J is the ideal of $K[x,y]$ generated by $y^2 - x^3$.

(2) Let $D_1 = 2y\partial_x + 3x^2\partial_y$ and $D_2 = 3y\partial_y - 2x\partial_x$. Use the previous exercise to show that the set of derivatives of $K[x,y]/J$ is generated by D_1 and D_2 as a module over $K[x,y]/J$.

(3) Conclude that $\mathrm{Der}_K(R)$ is an R-module generated by $t\partial_t$ and $t^2\partial_t$, where $\partial_t = \frac{d}{dt}$.

(1): Consider the map $K[x,y] \to R, \quad x \mapsto t^2, y \mapsto t^3$ and extend it by linearity to all of $K[x,y]$. We claim that this defines a homomorphism of K-algebras (ring homomorphism which is K-linear) ϕ, and it is clear that it is surjective by construction (it maps to the generators and are polynomial rings over the same base-fields). We see that $y^2 - x^3 \xmapsto{\phi} (t^3)^2 - (t^2)^3 = t^6 - t^6 = 0$ so that $\langle y^2 - x^3 \rangle \subset \ker\phi$. If we can show that $\ker\phi \subset \langle y^2 - x^3\rangle$ then we are done (using the 1st isomorphism theorem for rings).

Let $a = \sum_{(\alpha,\beta)\in\mathbb{N}^2} c_{\alpha\beta}x^\alpha y^\beta \in \ker\phi$, so that

$$\begin{aligned} \phi(a) &= \sum_{(\alpha,\beta)\in\mathbb{N}^2} c_{\alpha\beta}(t^2)^\alpha (t^3)^\beta \\ &= 0 \\ \Leftrightarrow \sum_{(\alpha,\beta)\in\mathbb{N}^2} c_{\alpha\beta}t^{2\alpha+3\beta} &= 0. \end{aligned}$$

We here note that if $\beta \geqslant 2$ for some term $c_{\alpha\beta}x^\alpha y^\beta$ we may rewrite this as $c_{\alpha\beta}x^\alpha y^{\beta-2}y^2$ and this is sent under ϕ to the same element in $K[t^2,t^3]$ as $x^\alpha y^{\beta-2}x^3 = x^{\alpha+3}y^{\beta-2}$. After relabeling $\beta^{(0)} := \beta - 2$ one may repeat this process until at some step m we have that $\beta^{(m)} < 2$. Hence we end up with $c_{\alpha\beta}x^{\alpha+3m}y^{\beta^{(m)}}$ with $\beta^{(m)} \in \{0,1\}$.

Explicitly, what reduction m does is subtracting $c_{\alpha\beta}x^{\alpha+3(m-1)}y^{\beta^{(m-1)}-2}(y^2 - x^3)$ from the term $c_{\alpha\beta}x^{\alpha+3(m-1)}y^{\beta^{(m-1)}-2}y^2$.

If $\beta^{(m)} = 0$ then we get that

$$\begin{aligned} \phi(c_{\alpha\beta}x^{\alpha+3m}y^{\beta^{(m)}}) &= c_{\alpha\beta}(t^2)^{\alpha+3m} \\ &= c_{\alpha\beta}t^{2\alpha+6m} \\ &= c_{\alpha\beta}t^{2(\alpha+3m)} \end{aligned}$$

so that the exponent is of the form $2\alpha' + \beta^{(m)}$ where $\alpha' := \alpha + 3m$. If $\beta^{(m)} = 1$ we get

$$\begin{aligned} \phi(c_{\alpha\beta}x^{\alpha+3m}y^{\beta^{(m)}}) &= c_{\alpha\beta}(t^2)^{\alpha+3m}t^3 \\ &= c_{\alpha\beta}t^{2\alpha+6m+3} \\ &= c_{\alpha\beta}t^{2(\alpha+3m)+3} \end{aligned}$$

so that the exponent is of the form $2\alpha' + 3\beta^{(m)}$ with $\alpha' := 2(\alpha + 3m)$.

Assume that we have $(\alpha_1,\beta_1) \neq (\alpha_2,\beta_2)$ such that $2\alpha_1 + 3\beta_1 = 2\alpha_2 + 3\beta_2$ in the reduced form as above, i.e. $\beta_1, \beta_2 < 2$. Then

$$2(\alpha_1 - \alpha_2) = 3(\beta_2 - \beta_1).$$

Modulo 2 we get that $\beta_2 - \beta_1 = 0$, so that $\beta_1 - \beta_2 = 2k$ for some $k \in \mathbb{Z}$. But this is only possible if $k = 0$ so that $\beta_1 = \beta_2$. Thus it follows that $\alpha_1 = \alpha_2$ so that $(\alpha_1,\beta_1) = (\alpha_2,\beta_2)$.

Thus we see that after reduction of all terms in a we have

$$a = \underbrace{\sum_{(\alpha',\beta')\in\mathbb{N}^2} c_{\alpha'\beta'} x^{\alpha'} y^{\beta'}}_{\text{reduced part}} + \underbrace{\sum_{(\gamma,\ell)\in\mathbb{N}^2} d_{\gamma\ell} x^{\gamma} y^{\ell}(y^2 - x^3)}_{\text{from reduction}}$$

being sent under ϕ to the same element as a. The sum from the reduction above is clearly in $\langle y^2 - x^3\rangle$.

Under ϕ we see that the "from reduction" part is sent to 0 by construction, and so we are left with

$$\sum_{(\alpha',\beta')\in\mathbb{N}^2} c_{\alpha'\beta'} t^{2\alpha'+3\beta'} = 0$$

where we saw that each $2\alpha' + 3\beta'$ is unique. Since the $t^{2\alpha'+3\beta'}$ are linearly independent elements in $K[t^2, t^3]$ this foces $c_{\alpha'\beta'} = 0$ for all $(\alpha', \beta') \in \mathbb{N}^2$. Thus $a \in \langle y^2 - x^3\rangle$.

We conclude that $\ker\phi = \langle y^2 - x^3\rangle$ so that $K[x, y]/J \cong K[t^2, t^3]$ are isomorphic K-algebras.

<u>(2)</u>: By [Cou95, Chap. 3, Prop. 1.3] any derivation D on $K[x, y]$ is on the form $D(x)\partial_x + D(y)\partial_y$, that is, a derivation D on $[x, y]$ is determined by its action on the generators x, y of the K-algebra $K[x, y]$.

By [Cou95, Chap 3., exc. 3.5.(1)] each derivation D on $K[x, y]/(y^2 - x^3)$ is on the form $\overline{B}$ with $B = g_x\partial_x + g_y\partial_y$ with $\overline{g}_x = D(\overline{x})$ and $\overline{g}_y = D(\overline{y})$. By [Cou95, Chap 3., exc. 3.5.(2)] with $J = \langle y^2 - x^3\rangle$ and $S = K[x, y]$ we have that $\mathrm{Der}_J(S)/J\mathrm{Der}(S) \cong \mathrm{Der}(S/J)$, where $\mathrm{Der}_J(S)$ are all derivations B on S such that $B(J) \subseteq J$. We hence claim that if we write B as $B = g(x, y)\partial_x + h(x, y)\partial_y$ we need

$$\begin{aligned}(g(x,y)\partial_x + h(x,y)\partial_y)(y^2 - x^3) &= g(x,y)\partial_x(y^2 - x^3) + h(x,y)\partial_y(y^2 - x^3)\\ &= -3x^2 g(x,y) + 2yh(x,y) \in J.\end{aligned}$$

Therefore, we see that in S/J we need $\overline{g}3x^2 = \overline{h}2y$. One obvious candidate choice is to choose $\overline{g} = 2yf(x, y)$ and $\overline{h} = 3x^2 f(x, y)$ to get $0 \in J$.

We may also note that

$$-3x^2 \cdot (-2x) + 2y \cdot (3y) = 6y^2 - 6x^3 = 6(y^2 - x^3) \in J$$

and so is zero in S/J.

We note that $\mathrm{Der}_J(S) \cong \{(g, h) \in S^2 \ : \ -3x^2 g + 2yh \in J\}$. If we define $\varphi : S^2 \to S, (g, h) \mapsto -3x^2 g + 2yh$.

Consider the kernel $\ker\varphi$ - since $-3x^2 g + 2yh = 0 \Leftrightarrow 3x^2 g = 2yh$ and S is a UFD, it follows that x^2 must divide h, so that $h = h'x^2$. In the same way, we need y to divide g so that $g = g'y$. We then get

$$\begin{aligned} -3x^2 g + 2yh &= 0\\ \Leftrightarrow -3x^2(g'y) + 2y(h'x^2) &= 0\\ \Leftrightarrow yx^2(2h' - 3g') &= 0\\ \Rightarrow 2h' &= 3g'\\ \Leftrightarrow h' &= \frac{3}{2}g'\end{aligned}$$

If we set $f = \frac{1}{2}g'$ then $2f = g'$ and $3f = h'$. Thus $g = 2fy$ and $h = 3fx^2$ so that $(g, h) = f(2y, 3x^2)$. This shows that $\ker\varphi \subseteq S(2y, 3x^2)$. On the other hand, we have already seen that $S(2y, 3x^2) \subseteq \ker\varphi$ and so $\ker\varphi = S(2y, 3x^2)$.

We notice that $\varphi^{-1}(J) = \{(g,h) \in S^2 \ : \ \varphi(g,h) \in J\} = \mathrm{Der}_J(S)$. We note that $(y^2 - x^3)$ is principal, and one can then show that $\varphi^{-1}(J) = \varphi^{-1}((y^2 - x^3)) = \ker\varphi + S(2x, 3y)$ since $\varphi(2x, 3y) = -6x^3 + 6y^2 = 6(y^2 - x^3)$. Thus we have that $\varphi^{-1}(J) = \mathrm{Der}_J(S) = S(2y, 3x^2) + S(2x, 3y)$. Thus we want to claim that $D_1 = 2y\partial_x + 3x^2\partial_y$ and $D_2 = 2x\partial_x + 3y\partial_y$ generate $\mathrm{Der}_J(S)$ so that their images $\overline{D_1}, \overline{D_2}$ in S/J generate $\mathrm{Der}(S/J)$.

(3): Under the parametrization $x = t^2$ and $y = t^3$ we have that

$$\partial_t(f(t^2, t^3)) = 2t\partial_x f + 3t^2\partial_y f$$

by the multivariable chain rule.

Thus, we may identify (we claim) $\partial_t = 2t\partial_x + 3t^2\partial_y$. Hence we have

$$\begin{aligned} t\partial_t &= 2t^2\partial_x + 3t^3\partial_y \\ &= 2x\partial_x + 3y\partial_y \\ &= D_2 \end{aligned}$$

and

$$\begin{aligned} t^2\partial_t &= 2t^3\partial_x + 3t^4\partial_y \\ &= 2y\partial_x + 3x^2\partial_y \\ &= D_1. \end{aligned}$$

Chapter 4

5.5

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. Suppose that the coordinate functions of F are at most of degree 2. Show that if $\Delta F \neq 0$ everywhere in $\mathbb{C}^n$, then

$$F(p) - F(q) = JF\left(\frac{p+q}{2}\right)(p-q).$$

Remark 0.4. This is enough to prove the Jacobian conjecture for quadratic maps because by [Bass, Connell & Wright, 82, Theorem 2.1] an injective polynomial map of $\mathbb{C}^n$ to itself must be bijective.

Remark 0.5. We think that *bijective* here should be replaced by *bijective and has a polynomial inverse* - although we are not sure those statements are not equivalent.

By hypothesis, we have that $F_i \in \mathbb{C}[x_1, \ldots, x_n]$ are all of degree 2, i.e. $F_i = \sum_\alpha c_\alpha x^\alpha$ is such that $|\alpha| \leqslant 2$ for all α, and that $\Delta F = \det J(F) \neq 0$ everywhere on $\mathbb{C}^n$. Since the F_i are polynomials, it follows that ΔF itself is a polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. But $\mathbb{C}$ is algebraically closed, so any non-constant polynomial must have a zero (e.g. by fixing appropriate variables to get a non-constant single-variable polynomial over $\mathbb{C}$ and using the fundamental theorem of algebra). Thus $\Delta F \neq 0$ implies that $\Delta F = d$ for some $d \in \mathbb{C}^\times$. This in turn implies that $JF(p)$ is invertible for all $p \in \mathbb{C}^n$. In our understanding, we have the Taylor-expansion of F around a point a as

$$F(z) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D^\alpha F(a)(z-a)^\alpha$$

with $\alpha! = \alpha_1! \cdots \alpha_n!$ (as in [Cou95, Chap. 1, §2]), $(z-a)^\alpha = (z_1 - a_1)^{\alpha_1} \cdots (z_n - a_n)^{\alpha_n}$, and $D^\alpha F(a) = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n} F(a)$.

We let $a = \frac{p+q}{2}$. Then we see that $p - a = p - \frac{p+q}{2} = \frac{p-q}{2}$ and $q - a = q - \frac{p+q}{2} = \frac{q-p}{2} = -\frac{(p-q)}{2}$. We then have that

$$F(p) = F(a) + \sum_{j=1}^{n} \frac{\partial F}{\partial z_j}(a)(p_j - a_j) + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 F}{\partial z_j \partial z_k}(a)(p_j - a_j)(p_k - a_k)$$

and

$$F(q) = F(a) + \sum_{j=1}^{n} \frac{\partial F}{\partial z_j}(a)(q_j - a_j) + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 F}{\partial z_j \partial z_k}(a)(q_j - a_j)(q_k - a_k).$$

Thus, we have that

$$\begin{aligned} F(p) - F(q) &= \sum_{j=1}^{n} \frac{\partial F}{\partial z_j}(a)(p_j - q_j) + \underbrace{\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 F}{\partial z_j \partial z_k}(a)(p_j - a_j)(p_k - a_k) - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 F}{\partial z_j \partial z_k}(a)(q_j - a_j)(q_k - a_k)}_{=0} \\ &= JF(a)(p-q) \\ &= JF\left(\frac{p+q}{2}\right)(p-q) \end{aligned}$$

where we interpret $(p-q)$ as a column-vector. The reason for $= 0$ in RHS of the first equality is that

$$\begin{aligned} q_j - a_j &= q_j - \frac{(p_j + q_j)}{2} \\ &= \frac{q_j - p_j}{2} \end{aligned}$$

and

$$\begin{aligned} p_j - a_j &= p_j - \frac{(p_j + q_j)}{2} \\ &= \frac{p_j - q_j}{2} \end{aligned}$$

Thus we have that

$$\begin{aligned} (q_j - a_j)(q_k - a_k) &= \left(\frac{q_j - p_j}{2}\right)\left(\frac{q_k - p_k}{2}\right) \\ &= (-1)^2\left(\frac{p_j - a_j}{2}\right)\left(\frac{p_k - q_k}{2}\right) \\ &= \left(\frac{p_j - a_j}{2}\right)\left(\frac{p_k - q_k}{2}\right) \\ &= (p_j - a_j)(p_k - a_k) \end{aligned}$$

so that any term $\frac{\partial^2 F}{\partial z_j \partial z_k}(a)(p_j - a_j)(p_k - a_k)$ gets removed by the subtraction.

If $F(p) = F(q) \Leftrightarrow F(p) - F(q) = 0 = JF\left(\frac{p+q}{2}\right)(p-q)$ then since $\frac{p+q}{2} \in \mathbb{C}^n$ by assumption we have that $JF\left(\frac{p+q}{2}\right)$ is invertible since it has a non-zero determinant ($\Delta F \neq 0$) - thus $(JF\left(\frac{p+q}{2}\right))^{-1} JF\left(\frac{p+q}{2}\right)(p-q) = (JF\left(\frac{p+q}{2}\right))^{-1} 0 \Leftrightarrow p - q = 0 \Leftrightarrow p = q$.

5.6

Which of the following derivations are locally nilpotent?

(1) $x_1\partial_1 + x_2\partial_2$

(2) $x_1\partial_1 + \partial_2$.

Remark 0.6. We will use the following definition for **locally nilpotent**.

Definition 0.7. We say that a derivation D on a (commutative) K-algebra S (for K field of characteristic 0) is **locally nilpotent** if for each $a \in S$, there exists a $k \in \mathbb{N}_{\geqslant 1}$ such that $D^k(a) = 0$.

We claim that $D_1 = x_1\partial_1 + x_2\partial_2$ is not locally nilpotent, since e.g. $x_1 \in K[x_1, x_2]$ is such that

$$\begin{aligned} D_1(x_1) &= (x_1\partial_1 + x_2\partial_2)(x_1) \\ &= (x_1\partial_1)(x_1) + (x_2\partial_2)(x_1) \\ &= (x_1\partial_1)(x_1) \\ &= x_1. \end{aligned}$$

Thus

$$D_1^k(x_1) = x_1$$

for all $k \in \mathbb{N}_{\geqslant 1}$.

In the same fashion, we see that $D_2 = x_1\partial_1 + \partial_2$ is not locally nilpotent, since the same problem arises with $x_1 \in K[x_1, x_2]$, i.e. we have that $D_2^k(x_1) = x_1$ for $k \in \mathbb{N}_{\geqslant 1}$.

Chapter 5

4.1

Let $g_1, \ldots, g_n$ be polynomials in $K[X]$.

(1) Show, by induction on m, that ∂_i^m may be written in the form $D(\partial_i - g_i) + f$ where $D \in A_n$ and $f \in K[X]$.

(2) Conclude that every element of A_n can be put in the form $Q+f$, where $Q = \sum_{i=1}^n D(\partial_i - g_i)$ and $f \in K[X]$.

(1):

Fix polynomials $g_1, \ldots, g_n \in K[X]$. For the base case, let $m = 1$ and consider ∂_i. Then if we take $D = 1$ and let $f = g_i$ we have that $1 \cdot (\partial_i - g_i) + f = \partial_i - g_i + g_i = \partial_i$.

Assume by induction that it holds for m; we fix such D and f such that $D(\partial_i - g_i) + f = \partial_i^m$ and we want to show that it holds for $m+1$: We then have that

$$\begin{aligned} \partial_i(D(\partial_i - g_i) + f) &= \partial_i(\partial_i^m) \\ &= \partial_i^{m+1}. \end{aligned}$$

It remains to show that the LHS in the first equality, i..e $\partial_i(D(\partial_i - g_i) + f)$, is on the required form. We find that

$$\begin{aligned} \partial_i(D(\partial_i - g_i) + f) &= \partial_i D(\partial_i - g_i) + \partial_i f \\ &= \partial_i D(\partial_i - g_i) + f\partial_i + \frac{\partial f}{\partial x_i} \\ &= \partial_i D(\partial_i - g_i) + f(\partial_i - g_i) + fg_i + \frac{\partial f}{\partial x_i} \\ &= \underbrace{(\partial_i D + f)(\partial_i - g_i)}_{D'} + \underbrace{fg_i + \frac{\partial f}{\partial x_i}}_{=f'} \end{aligned}$$

is on the required form with D', f' as indicated above.

(2): Recall that in canonical form any element $a \in A_n$ can be written on the form

$$a = \sum_{\alpha,\beta\in\mathbb{N}} c_{\alpha\beta} x^\alpha \partial^\beta.$$

Assuming the result of (1), we have that any ∂^β occuring in the terms of the canonical forms may be written as $D_\beta(\partial_i - g_i) + f_\beta$. Thus we may rewrite a as

$$\begin{aligned} a &= \sum_{\alpha,\beta\in\mathbb{N}} c_{\alpha\beta} x^\alpha (D_\beta(\partial_i - g_i) + f_\beta) \\ &= \underbrace{\sum_{\alpha,\beta\in\mathbb{N}} c_{\alpha\beta} x^\alpha D_\beta(\partial_i - g_i)}_{=Q} + \underbrace{\sum_{\alpha,\beta\in\mathbb{N}} c_{\alpha\beta} x^\alpha f_\beta}_{=f} \end{aligned}$$

4.11

Show that $\cos z$ and $\sin z$ are torsion elements of $\mathcal{H}(\mathbb{C})$.

To say that $\cos z$ and $\sin z$ are torsion elements of $\mathcal{H}(\mathbb{C})$ it is enough to show that $\cos z$ and $\sin z$ are holomorphic functions (they are!) which satisfy an ordinary differential equation with polynomial coefficients, i.e. that there is an element $P = \sum_{\alpha,\beta\in\mathbb{N}} c_{\alpha\beta} z^\alpha \partial^\beta \in A_1(\mathbb{C})$ such that $P \cdot \cos z = 0$ and similarly for $\sin z$.

In fact, we see that the same $P = \left(1 + \frac{d^2}{dz^2}\right)$ works for both $\cos z$ and $\sin z$, since we have that

$$\frac{d^2}{dz^2}(\cos z) = -\cos z$$

and

$$\frac{d^2}{dz^2}(\sin z) = -\sin z.$$

Thus, we have that

$$\begin{aligned} P \cdot \cos z &= \left(1 + \frac{d^2}{dz^2}\right)(\cos z) \\ &= \cos z - \cos z \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} P \cdot \sin z &= \left(1 + \frac{d^2}{dz^2}\right)(\sin z) \\ &= \sin z - \sin z \\ &= 0. \end{aligned}$$

Chapter 6

Explanation for $\tau_{\epsilon\epsilon'}$

Let $D(\epsilon)$ be the open unit disk with centre 0 and radius $\epsilon \in \mathbb{R}_{>0}$. Let $D'(\epsilon) = D(\epsilon) \setminus \{0\}$ and let $\tilde{D}(\epsilon) = \{z \in \mathbb{C} \ : \ \text{Re } z < \ln \epsilon\}$. We note that $D'(\epsilon) \subseteq D(\epsilon)$ so that $\mathcal{H}(D(\epsilon)) \subseteq \mathcal{H}(D'(\epsilon))$ as $A_1(\mathbb{C})$-modules, where $\mathcal{H}(U)$ for U any open subset of $\mathbb{C}$, is the set of holomorphic functions $f : U \to \mathbb{C}$.

Let $\mathcal{M}_\epsilon$ denote the quotient module $\mathcal{H}(\tilde{D}(\epsilon))/\pi^*(\mathcal{H}(D(\epsilon)))$. If $\epsilon' \leqslant \epsilon$, then $\tilde{D}(\epsilon') \subseteq \tilde{D}(\epsilon)$ so that $\mathcal{H}(\tilde{D}(\epsilon)) \subseteq \mathcal{H}(\tilde{D}(\epsilon'))$.

We then have the following diagram:

$$\begin{tikzcd}
\mathcal{H}(D(\epsilon)) \arrow[r, "\mathsf{res}"] \arrow[d, "\pi_\epsilon^*"'] & \mathcal{H}(D(\epsilon')) \arrow[d, "\pi_{\epsilon'}^*"] \\
\mathcal{H}(\tilde{D}(\epsilon)) \arrow[r, "\widetilde{\mathsf{res}}"] \arrow[d, "\widetilde{\pi_\epsilon}"'] & \mathcal{H}(\tilde{D}(\epsilon')) \arrow[d, "\widetilde{\pi_{\epsilon'}}"] \\
\mathcal{M}_\epsilon \arrow[r, dashed, "\tau_{\epsilon\epsilon'}"'] & \mathcal{M}_{\epsilon'}
\end{tikzcd} \tag{0.4}$$

1. For $f \in \mathcal{H}(D(\epsilon))$ we have that

$$\begin{aligned}
\widetilde{\mathsf{res}}(\pi_\epsilon^*(f)) &= (f \circ \pi)|_{\tilde{D}(\epsilon')} \\
&= \left(f|_{D(\epsilon')}\right) \circ \pi \qquad \text{(since if } z \in \tilde{D}(\epsilon') \text{ then } \pi(z) \in D(\epsilon')) \\
&= \pi_{\epsilon'}^*(f|_{D(\epsilon')}) \\
&= (\pi_{\epsilon'}^* \circ \mathsf{res})(f) \in \pi_{\epsilon'}^*(\mathcal{H}(D(\epsilon'))),
\end{aligned}$$

so that the uppermost square in diagram 0.4 commutes. In particular, $\widetilde{\mathsf{res}}(\pi_\epsilon^*(\mathcal{H}(D(\epsilon))) \subseteq \pi_{\epsilon'}^*(\mathcal{H}(D(\epsilon')))$.

2. From 1. we see that $\widetilde{\mathsf{res}}(\pi_\epsilon^*(\mathcal{H}(D(\epsilon))) \subseteq \ker \widetilde{\pi_{\epsilon'}}$ so that $\pi_\epsilon^*(\mathcal{H}(D(\epsilon))) \subseteq \ker(\widetilde{\pi}_{\epsilon'} \circ \widetilde{\mathsf{res}})$. Thus by the universal property of the quotient we claim that we get a unique $A_1(\mathbb{C})$-module homomorphism $\tau_{\epsilon\epsilon'} : \mathcal{M}_\epsilon \to \mathcal{M}_{\epsilon'}$ such that

$$\tau_{\epsilon\epsilon'} \circ \widetilde{\pi}_\epsilon = \widetilde{\pi_{\epsilon'}} \circ \widetilde{\mathsf{res}},$$

i.e. so that the lowermost square in diagram 0.4 commutes.

4.7

Let $\Omega = (0,1)$. The Heaviside hyperfunction is $Y = \left[\frac{\log(-z)}{2\pi i}\right]$ and the Dirac delta hyperfunction is $\delta = \left[\frac{1}{2\pi i}\right]$. Show that $\partial \cdot Y = \delta$.

We will assume that the action of $A_1(\mathbb{R})$ on $\mathcal{B}(\Omega)$ (cf. [Cou95, Chap. 6, Exc. 4.6]) is defined so that $\partial \cdot [h] = [h']$.

Thus we have that

$$\begin{aligned}
\partial \cdot Y &= \partial \cdot \left[\frac{\log(-z)}{2\pi i}\right] \\
&= \left[\frac{1}{-z} \cdot (-1) \cdot \frac{1}{2\pi i}\right] \\
&= \left[\frac{1}{2\pi i z}\right] \\
&= \delta.
\end{aligned}$$

4.9

Let δ' be the first derivative of the Dirac microfunction. Let $A_1(\mathbb{C})\delta'$ be the submodule of $\mathcal{M}$ generated by δ. Show that
(1) $A_1(\mathbb{C})\delta = A_1(\mathbb{C})\delta'$.
(2) $A_1(\mathbb{C})\delta' \cong A_1(\mathbb{C})/J$, where J is the left ideal of $A_1(\mathbb{C})$ generated by x^2 and $x\partial + 2$.
Hint: If $Q \in A_1(\mathbb{C})$ satisfies $Q\delta' = 0$ then we have $Q \cdot \partial \in A_1(\mathbb{C})x$, the annihilator of δ. Write $Q = Q_2x^2 + Q_1x + Q_0$, where $Q_2 \in A_1(\mathbb{C}), Q_0, Q_1 \in \mathbb{C}[\partial]$ and calculate $Q \cdot \partial$.

(1):

We have that $\delta' = \partial \bullet \delta \in A_1(\mathbb{C})\delta$ so that it is clear that $A_1(\mathbb{C})\delta' \subseteq A_1(\mathbb{C})\delta$.

For the other inclusion, we note that

$$\begin{aligned}\delta' &= \partial \bullet \delta \\ &= \partial \bullet \mathsf{can}\left(\frac{e^{-z}}{2\pi i}\right) \\ &= \mathsf{can}\left(\partial \bullet \frac{e^{-z}}{2\pi i}\right) \\ &= \mathsf{can}\left(\frac{-e^{-z}}{2\pi i} \cdot e^{-z}\right) \\ &= \mathsf{can}\left(-\frac{e^{-2z}}{2\pi i}\right).\end{aligned}$$

We then have that

$$\begin{aligned}(-x) \bullet \delta' &= (-x) \bullet \mathsf{can}\left(\frac{e^{-2z}}{2\pi i}\right) \\ &= \mathsf{can}\left((-x) \bullet \frac{-e^{-2z}}{2\pi i}\right) \\ &= \mathsf{can}\left(e^{z} \cdot \frac{e^{-2z}}{2\pi i}\right) \\ &= \mathsf{can}\left(\frac{e^{-z}}{2\pi i}\right) \\ &= \delta,\end{aligned}$$

so that $\delta \in A_1(\mathbb{C})\delta' \Rightarrow A_1(\mathbb{C})\delta \subseteq A_1(\mathbb{C})\delta'$, so that $A_1(\mathbb{C})\delta = A_1(\mathbb{C})\delta'$.

Alternative proof: One may note that $x \bullet \delta = 0$. Since $[\partial, x] = 1$ in $A_1(\mathbb{C})$ and by the action of $A_1(\mathbb{C})$ on $\mathcal{M}$ (as we understand it: As an extension of the action on $\mathcal{H}(\tilde{D}(\epsilon))$) we have that

$$\begin{aligned}\delta &= [\partial, x] \bullet \delta \\ &= (\partial x) \bullet \delta - x\partial \bullet \delta \\ &= -x\partial \bullet \delta \\ &= -x \bullet \delta'\end{aligned}$$

Thus it follows that $\delta \in A_1(\mathbb{C})\delta' \Rightarrow A_1(\mathbb{C})\delta \subseteq A_1(\mathbb{C})\delta'$. Since the other direction is clear from the fact that $\delta' = \partial \bullet \delta \in A_1(\mathbb{C})\delta$ so that $A_1(\mathbb{C})\delta' \subseteq A_1(\mathbb{C})\delta$, we are done.

(2): First, it is easy to see that $A_1(\mathbb{C})x \subseteq \mathrm{Ann}_{A_1(\mathbb{C})}(\delta)$ (with the left $A_1(\mathbb{C})$-module structure on $\mathcal{M}$). On the other hand, let $P \in A_1(\mathbb{C})$ such that $P \bullet \delta = 0$. Then note that using the *canonical basis* for

$A_1(\mathbb{C})$, we may write $P = Qx + R$ where $Q \in A_1(\mathbb{C})$ and $R \in \mathbb{C}[\partial]$. Since $\bullet$ is an action, we then have that

$$\begin{aligned}
P \bullet \delta &= (Qx + R) \cdot \delta \\
&= (Qx \bullet \delta) + (R \bullet \delta) \\
&= (Q \bullet \underbrace{(x \bullet \delta)}_{=0}) + (R \bullet \delta) \\
&= R \bullet \delta \\
&= 0.
\end{aligned}$$

We have that $R = \sum_{\beta \in \mathbb{N}} c_\beta \partial^\beta$ where only *finitely many* c_β are *non-zero.*

Lemma 0.8. $\partial^n \bullet \delta = \mathsf{can}\left((-1)^n n! \frac{e^{-(n+1)z}}{2\pi i}\right)$.

Proof. We proceed by induction: We saw earlier that this holds for the base case $n = 1$. Assume by induction that it holds for some n, we then want to show that it holds for $n + 1$. We then have that

$$\begin{aligned}
\partial^{n+1} \bullet \delta &= \partial \bullet (\partial^n \bullet \delta) \\
&= \partial \bullet \mathsf{can}\left((-1)^n n! \frac{e^{-(n+1)z}}{2\pi i}\right) \\
&= \mathsf{can}\left((-1)^{n+1} (n+1)! \frac{e^{-(n+1)z}}{2\pi i} \cdot e^{-z}\right) \\
&= \mathsf{can}\left((-1)^{n+1} (n+1)! \frac{e^{-(n+2)z}}{2\pi i}\right).
\end{aligned}$$

□

From lemma 0.8 and the $A_1(\mathbb{C})$ action we then get that

$$\begin{aligned}
R \bullet \delta &= \left(\sum_{\beta \in \mathbb{N}} c_\beta \partial^\beta\right) \bullet \delta \\
&= \sum_{\beta \in \mathbb{N}} c_\beta \mathsf{can}\left((-1)^\beta \beta! \frac{e^{-(\beta+1)z}}{2\pi i}\right) \\
&= \sum_{\beta \in \mathbb{N}} C_\beta \mathsf{can}\left(e^{-(\beta+1)z}\right) \qquad \left(\text{with } C_\beta = \frac{(-1)^\beta \beta! c_\beta}{2\pi i}\right) \\
&= \mathsf{can}\left(\sum_{\beta \in \mathbb{N}} C_\beta e^{-(\beta+1)z}\right) \\
&= 0.
\end{aligned}$$

We see that

$$\sum_{\beta \in \mathbb{N}} \frac{C_\beta}{z^{\beta+1}}$$

is such that its image under the pullback π^* is

$$\sum_{\beta \in \mathbb{N}} C_\beta e^{-(\beta+1)}.$$

We will assume without proof that a (complex-valued) function defined on $D(\epsilon)$ must be such that for it to be *holomorphic* on $D(\epsilon)$, then its *principal part* must be zero. We note that for $\mathsf{can}(a) = 0$ to hold, we must have that $a \in \pi^*(\mathcal{H}(D(\epsilon))$. The only way this can happen then is that $C_\beta = 0$ for all β. This in turn implies that $c_\beta = 0$ so that $R = 0$. Thus it follows that $\mathrm{Ann}_{A_1(\mathbb{C})}(\delta) \subseteq A_1(\mathbb{C})x \Rightarrow \mathrm{Ann}_{A_1(\mathbb{C})}(\delta) = A_1(\mathbb{C})x$.

Remark 0.9. A reason for this last part of the argument, not spelled out, is that we believe that $\mathsf{can} : \mathcal{H}(\tilde{D}(\epsilon)) \to \mathcal{M}$ is explicitly defined as $i_\epsilon \circ \widetilde{\pi}_\epsilon$ where

$$\widetilde{\pi}_\epsilon : \mathcal{H}(\tilde{D}(\epsilon)) \twoheadrightarrow \mathcal{M}_\epsilon$$

and

$$i_\epsilon : \mathcal{M}_\epsilon \hookrightarrow \mathcal{M}$$

is an injection, thus forcing $\ker \mathsf{can} = \ker \widetilde{\pi}_\epsilon$.

Continuing with the proof: By the hint, we may assume that $Q \in A_1(\mathbb{C})$ satisfies $Q \bullet \delta' = 0$. We may rewrite this as

$$\begin{aligned} Q \bullet \delta' &= Q \bullet (\partial \bullet \delta) \\ &= (Q \cdot \partial) \bullet \delta \qquad \text{(since it is an action)} \\ &= 0. \end{aligned}$$

Thus we have that $Q \cdot \partial \in \mathrm{Ann}_{A_1(\mathbb{C})}(\delta) = A_1(\mathbb{C})x$.

Recall that Q in $A_1(\mathbb{C})$ may be written as $Q = Q'x + Q_0$ with $Q' \in A_1(\mathbb{C})$ and $Q_0 \in \mathbb{C}[\partial]$. Applying this again to Q', we may write $Q' = Q_2 x + Q_1$ with $Q_2 \in A_1(\mathbb{C})$ and $Q_1 \in \mathbb{C}[\partial]$. Thus we get that

$$\begin{aligned} Q &= Q'x + Q_0 \\ &= (Q_2 x + Q_1)x + Q_0 \\ &= Q_2 x^2 + Q_1 x + Q_0. \end{aligned}$$

We then have that

$$\begin{aligned} Q \cdot \partial &= (Q_2 x^2 + Q_1 x + Q_0) \cdot \partial \\ &= Q_2 x^2 \partial + Q_1 x \partial + Q_0 \partial \qquad \text{(with } Q_2 \in A_1(\mathbb{C}) \text{ and } Q_1, Q_0 \in \mathbb{C}[\partial]). \end{aligned}$$

Consider the following diagram:

$$\begin{tikzcd} A_1(\mathbb{C}) \arrow[rr, "f"] \arrow[rr, bend left, mapsto, "g \mapsto g\delta'"] \arrow[dr, "\pi"'] & & A_1(\mathbb{C})\delta' \\ & A_1(\mathbb{C})/J \arrow[ur, dotted, "\exists! \tilde{f}"'] & \end{tikzcd}$$

We claim that with f as defined above, i.e. $g \mapsto g\delta'$, $J \subseteq \ker f$ so that f factorizes as $f = \tilde{f} \circ \pi$. It is clear that f is a surjective $A_1(\mathbb{C})$-module homomorphism, and so it is enough to show that $\ker f = J$ to conclude by the first isomorphism theorem for modules that $A_1(\mathbb{C})/J \cong A_1(\mathbb{C})\delta'$. Notice that $\ker f = \mathrm{Ann}_{A_1(\mathbb{C})}(\delta')$.

$\underline{J \subseteq \ker f}$: We note that

$$\begin{aligned}
[\partial, x] &= (\partial x - x\partial) \\
&= 1 \\
\Leftrightarrow x\partial &= \partial x - 1 \\
\Leftrightarrow x\partial + 2 &= (\partial x - 1) + 2 \\
&= \partial x + 1.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
(x\partial + 2) \bullet \delta' &= (\partial x + 1) \bullet \delta' \\
&= \partial x \bullet \delta' + \delta' \\
&= \partial \bullet (x \bullet \delta') + \delta' \\
&= \partial \bullet \left(\mathsf{can} \left(e^z \cdot \frac{-e^{-2z}}{2\pi i} \right) \right) + \delta' \\
&= \partial \bullet \left(\mathsf{can} \left(-\frac{e^{-z}}{2\pi i} \right) \right) + \delta' \\
&= \partial \bullet (-\delta) + \delta' \\
&= -\delta' + \delta' \\
&= 0.
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
x^2 \bullet \delta' &= x \bullet \left(x \bullet \mathsf{can} \left(\frac{-e^{-2z}}{2\pi i} \right) \right) \\
&= x \bullet \mathsf{can} \left(e^z \cdot \frac{-e^{-2z}}{2\pi i} \right) \\
&= x \bullet \mathsf{can} \left(\frac{-e^{-z}}{2\pi i} \right) \\
&= (-x) \bullet \delta \\
&= (-1) \bullet \underbrace{(x \bullet \delta)}_{=0} \\
&= 0.
\end{aligned}$$

Since the *generators* of J annihilates δ', and J is a *left ideal* of $A_1(\mathbb{C})$, we have that $J \subseteq \mathrm{Ann}_{A_1(\mathbb{C})}(\delta') = \ker f$.

$\underline{\ker f \subseteq J}$:

Notice that $Q_2 \bullet \underbrace{(x^2\partial \bullet \delta)}_{=0} = 0$, and that $Q_1 \bullet (x\partial \bullet \delta) = -Q_1\delta$.

Thus we get that

$$\begin{aligned}(Q \cdot \partial) \bullet \delta &= -Q_1\delta + (Q_0\partial \bullet \delta) \\ &= 0 \\ \Leftrightarrow (Q_0\partial - Q_1) \bullet \delta &= 0 \\ \Rightarrow Q_0\partial - Q_1 &\in A_1(\mathbb{C})x.\end{aligned}$$

We notice that since $Q_0, Q_1 \in \mathbb{C}[\partial]$ the only way this can happen is if $Q_0, Q_1 = 0$. But then we see that $Q = Q_2x^2$ so that $Q \in J = \langle x^2, x\partial + 2\rangle$.

We conclude that $J = \ker f$, i.e. $J = \mathrm{Ann}_{A_1(\mathbb{C})}(\delta')$ so that $A_1(\mathbb{C})\delta' \cong A_1(\mathbb{C})/J$ as left $A_1(\mathbb{C})$-modules.

Chapter 7

6.6

Let R be a filtered algebra with a filtration $\mathcal{F}$. Show that if $\mathsf{gr}^{\mathcal{F}}R$ is a domain, then R is a domain.

We show the contrapositive:

Assume that R is not a domain. This means that there non-zero elements a, b in R such that $ab = 0$. Since $R = \bigcup_i \mathcal{F}_i$ we have that there are $\mathcal{F}_i, \mathcal{F}_j$ such that $a \in \mathcal{F}_i$ and $b \in \mathcal{F}_j$ (i and j choosed to be minimal for a and b). By minimality of i and j, it follows that with respect to the associated *symbol map of order i or j*, we have that

$$\sigma_i(a) \neq 0$$

and

$$\sigma_j(b) \neq 0,$$

since otherwise i or j would not be minimal. But we then see that

$$\begin{aligned}\sigma_i(a) \cdot \sigma_j(b) &= (a + \mathcal{F}_{i-1}) \cdot (b + \mathcal{F}_{j-1}) \\ &= (ab + \mathcal{F}_{i+j-1}) \\ &= \sigma_{i+j}(ab) \\ &= 0.\end{aligned}$$

But $\sigma_i(a), \sigma_j(b) \in \mathsf{gr}^{\mathcal{F}}R = \bigoplus_{k\geqslant 0} \mathcal{F}_k/\mathcal{F}_{k-1}$. Thus $\mathsf{gr}^{\mathcal{F}}R$ is not a domain.

6.9

Let V be the K-vector space of A_n with basis $\{x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\}$. Let $\mathsf{Sp}(V)$ be the symplectic group on V.

(1) Show that an element $\sigma \in \mathsf{Sp}(V)$ can be extended to an automorphism of A_n that preserves the Bernstein filtration.

(2) Show that this automorphism induces an automorphism of $S_n = \mathsf{gr}^{\mathcal{B}}A_n$.

(1): Let $I_\omega := \langle u \otimes v - v \otimes u - \omega(u, v) \ : \ u, v \in V\rangle$ where ω is a fixed *symplectic* form associated to $\mathsf{Sp}(V, \omega)$. We will make the identification $A_n := \mathcal{T}(V)/I_\omega$ where $\mathcal{T}(V)$ is the tensor-algebra on V (cf. [DF04, Chap. 11.5]).

Let $g \in \mathsf{Sp}(V,\omega)$ so that $g : V \to V$ is an automorphism of V. We have that

$$\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} \mathcal{T}^k(V)$$

with the convention that $V^{\otimes 0} = K$ and $\mathcal{T}^k(V) := V^{\otimes k}$. Multiplication in $\mathcal{T}(V)$ is defined via pure tensors by *concatenation*, in the sense that if $v = v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$ and $w = w_1 \otimes \cdots \otimes w_\ell \in V^{\otimes \ell}$ then

$$v \cdot w := v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell \in V^{\otimes(k+\ell)}.$$

Let $\tau : \mathcal{T}(V) \to \mathcal{T}(V)$ be defined on pure tensors $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$ as

$$\tau(v) := g(v_1) \otimes \cdots \otimes g(v_k)$$

and extend it by linearity to all of $\mathcal{T}(V)$.

Since g is K-linear and by definition of τ, we see that τ respects the multiplication of pure tensorers. This extends to multiplication of (finite) sums of pure tensors, so that τ is an endomorphism of the K-algebra $\mathcal{T}(V)$.

We claim that τ is in fact an automorphism of $\mathcal{T}(V)$: This follows directly if we define τ' defined on pure tensors in the same way by $\tau'(v_1 \otimes \cdots \otimes v_k) = g^{-1}(v_1 \otimes \cdots \otimes v_k)$.

$\underline{\tau \text{ respects } \omega(-,-)}$: By assumption, we have that $\omega(g(u), g(v)) = \omega(u,v)$, for all $u, v \in V$.

We then see that for

$$\begin{aligned} \tau(u \otimes v - v \otimes u - \omega(u,v)) &= g(u) \otimes g(v) - g(v) \otimes g(u) - \omega(u,v) && (\text{since } \omega(u,v) \in K \text{ and } g \text{ is } K\text{-linear}) \\ &= g(u) \otimes g(v) - g(v) \otimes g(u) - \omega(g(u), g(v)) && (\text{since } \omega(u,v) = \omega(g(u), g(v))). \end{aligned}$$

Thus we see that τ takes generators of I_ω to *generators* of I_ω. If we think of elements in I_ω, with $\zeta(u,v) = u \otimes v - v \otimes u - \omega(u,v)$ as

$$\left\{ \sum_{i=1}^{k} a_i \zeta(u_i, v_i) b_i \ : \ a_i, b_i \in \mathcal{T} \right\}$$

then since τ respects multiplication we get that any such elements is sent to

$$\sum_{i=1}^{k} \tau(a_i) \zeta'(u_i, v_i) \tau(b_i)$$

where $\zeta'(u_i, v_i)$ is a generator for I_ω. Hence we have that $\tau(I_\omega) \subseteq I_\omega$.

Consider the following diagram:

$$\begin{tikzcd} \mathcal{T}(V) \arrow[rr, "\tau"] \arrow[d, "\pi"'] \arrow[drr, "\phi = \pi \circ \tau"] & & \mathcal{T}(V) \arrow[d, "\pi"] \\ \mathcal{T}(V)/I_\omega \arrow[rr, dotted, "\widetilde{\tau}"'] & & \mathcal{T}(V)/I_\omega \end{tikzcd}$$

Notice that since τ is an automorphism, we have that $\ker \phi = \ker \pi = I_\omega$. Thus by the universal property of the quotient, there is a unique ring-homomorphism $\widetilde{\tau}$ such that

$$\widetilde{\tau} \circ \pi = \phi.$$

If also consider the diagram

$$\begin{tikzcd}
\mathcal{T}(V) \arrow[rr, "\tau'"] \arrow[d, "\pi"'] \arrow[drr, "\phi' = \pi \circ \tau'"] & & \mathcal{T}(V) \arrow[d, "\pi"] \\
\mathcal{T}(V)/I_\omega \arrow[rr, dotted, "\widetilde{\tau}'"'] & & \mathcal{T}(V)/I_\omega
\end{tikzcd}$$

with τ' the inverse of τ, we have that

$$\begin{aligned}
(\widetilde{\tau}' \circ \widetilde{\tau}) \circ \pi &= \widetilde{\tau}' \circ (\widetilde{\tau} \circ \pi) \\
&= \widetilde{\tau}' \circ (\pi \circ \tau) \\
&= (\widetilde{\tau}' \circ \pi) \circ \tau \\
&= (\pi \circ \tau') \circ \tau \\
&= \pi \circ (\tau' \circ \tau) \\
&= \pi \circ \mathrm{id}_{\mathcal{T}(V)} \\
&= \pi
\end{aligned}$$

and

$$\begin{aligned}
(\widetilde{\tau} \circ \widetilde{\tau}') \circ \pi &= \widetilde{\tau} \circ (\widetilde{\tau}' \circ \pi) \\
&= \widetilde{\tau} \circ (\pi \circ \tau') \\
&= (\widetilde{\tau} \circ \pi) \circ \tau' \\
&= (\pi \circ \tau) \circ \tau' \\
&= \pi \circ (\tau \circ \tau') \\
&= \pi \circ \mathrm{id}_{\mathcal{T}(V)} \\
&= \pi.
\end{aligned}$$

Since π is surjective, this means that both $\widetilde{\tau}' \circ \widetilde{\tau}$ and $\widetilde{\tau} \circ \widetilde{\tau}'$ acts as the identity on $\mathcal{T}(V)/I_\omega = A_n$.

Remark 0.10. Just a note on why we can draw the conclusion from the last paragraph that $\widetilde{\tau} \circ \widetilde{\tau}' = \widetilde{\tau}' \circ \widetilde{\tau} = \mathrm{id}_{A_n}$: Notice that since π is surjective, for each $y \in Y$, we have some $x \in \mathcal{T}(V)$ such that $\pi(x) = y$. Thus we see that

$$\begin{aligned}
(\widetilde{\tau}' \circ \widetilde{\tau})(y) &= (\widetilde{\tau}' \circ \widetilde{\tau})(\pi(x)) \\
&= \pi(x) \\
&= y
\end{aligned}$$

and similarly for $(\widetilde{\tau} \circ \widetilde{\tau}') \circ \pi$. Assuming that an endomorphism $F : A_n \to A_n$ is such that $F = \mathrm{id}_{A_n}$ if and only if $F(y) = y$ for all $y \in A_n$, we must conclude that $\widetilde{\tau}$ is an automorphism.

Furthermore, since τ' and π are K-linear, it follows that $\widetilde{\tau}$ is K-linear, hence is a ring-isomorphism that respects the K-vector space structure on A_n.

Lastly, we want to check that $\widetilde{\tau}$ respects the *filtration* on $\mathcal{T}(V)$. Notice that we have a "natural" filtration $\mathcal{T}^{\leqslant k}(V) = \bigoplus_{j=0}^{k} V^{\otimes j}$ on $\mathcal{T}(V)$.

In our understanding, the induced filtration on A_n we want is then

$$\begin{aligned}
\mathcal{B}_k &= \pi\left(\bigoplus_{j=0}^{k} V^{\otimes j}\right) \\
&= \left\{ s + I_\omega \ : \ s \in \bigoplus_{j=0}^{k} V^{\otimes j} \right\} \\
&= \left(\bigoplus_{j=0}^{k} V^{\otimes j} + I_\omega\right) \Big/ I_\omega
\end{aligned}$$

Remark 0.11. We will not check explicitly that this is the appropriate filtration, but it atleast seem straightforward to check that it fulfills the three criteria given in [Cou95, p. 56].

Since $\tau(\mathcal{T}^k(V)) \subseteq \mathcal{T}^k(V)$ and linearity, it follows that $\tau\left(\bigoplus_{j=0}^{k} V^{\otimes j}\right) \subseteq \bigoplus_{j=0}^{k} V^{\otimes j}$. Furthermore, we saw that $\tau(I_\omega) \subseteq I_\omega$. Thus again, by linearity, we have that

$$\tau\left(\bigoplus_{j=0}^{k} V^{\otimes j} + I_\omega\right) \subseteq \bigoplus_{j=0}^{k} V^{\otimes j} + I_\omega .$$

Since $\widetilde{\tau} \circ \pi = \pi \circ \tau$ we see that we have that it follows that

$$\widetilde{\tau}(s + I_\omega) = \tau(s) + I_\omega \in \left\{ s + I_\omega \ : \ s \in \bigoplus_{j=0}^{k} V^{\otimes j} \right\}.$$

Thus indeed we have that

$$\begin{aligned}
\widetilde{\tau}(\mathcal{B}_k) &= \widetilde{\tau}\left(\left(\bigoplus_{j=0}^{k} V^{\otimes j} + I_\omega\right) \Big/ I_\omega\right) \\
&\subseteq \left(\bigoplus_{j=0}^{k} V^{\otimes j} + I_\omega\right) \Big/ I_\omega \\
&= \mathcal{B}_k .
\end{aligned}$$

We conclude that $\widetilde{\tau}$ preserves the filtration on A_n.

(2): We want to show that the K-linear ring-automorphism $\widetilde{\tau} : A_n \to A_n$ induces an automorphism of S_n.

Notice that $S_n = \bigoplus\limits_{k=0}^{\infty} \mathcal{B}_k / \mathcal{B}_{k-1}$.

Recall that $\widetilde{\tau}(\mathcal{B}_k) \subseteq \mathcal{B}_k$. We consider the following diagram:

$$\begin{tikzcd}
\mathcal{B}_k \arrow[rr, "\widetilde{\tau}|_{\mathcal{B}_k}"] \arrow[d, "\sigma_k"'] \arrow[drr, "\psi_k"] & & \mathcal{B}_k \arrow[d, "\sigma_k"] \\
\mathcal{B}_k/\mathcal{B}_{k-1} \arrow[rr, dotted, "\theta_k"'] & & \mathcal{B}_k/\mathcal{B}_{k-1}
\end{tikzcd}$$

Since $\widetilde{\tau}|_{\mathcal{B}_k}(\mathcal{B}_{k-1}) = \widetilde{\tau}(\mathcal{B}_{k-1}) \subseteq \mathcal{B}_{k-1}$ (it is in fact not only an inclusion but an equality, since $\mathcal{B}_k$ is a finite-dimensional K-vector space) we find that $\ker \psi_k = \ker \sigma_k = \mathcal{B}_{k-1}$ so that by the universal property of the quotient for K-modules we get a unique K-module homomorphism θ_k such that

$$\begin{aligned} \theta_k \circ \sigma_k &= \psi_k \\ &= \sigma_k \circ \widetilde{\tau}|_{\mathcal{B}_k}. \end{aligned}$$

By looking at $\widetilde{\tau}'|_{\mathcal{B}_k}$ where $\widetilde{\tau}'$ is induced from τ' corresponding to $g^{-1} \in \mathsf{Sp}(V, \omega)$ we find that θ_k has a two-sided induced inverse θ_k' (similarly to how we showed that $\widetilde{\tau}$ had a two-sided inverse in (1)). Thus it follows that θ_k are isomorphisms of K-vector spaces.

The multiplication in S_n is defined with respect to the *symbol maps of order* k, $\sigma_k : \mathcal{B}_k \to \mathcal{B}_k/\mathcal{B}_{k-1}$ as follows: Let $\overline{a} = \sigma_k(a) \in \mathcal{B}_k/\mathcal{B}_{k-1}$ and let $\overline{b} = \sigma_\ell(b) \in \mathcal{B}_\ell/\mathcal{B}_{\ell-1}$. Then

$$\begin{aligned} \overline{a} \cdot \overline{b} &= \sigma_k(a) \cdot \sigma_\ell(b) \\ &= \sigma_{k+\ell}(ab). \end{aligned}$$

We may now define a map $\Theta := \bigoplus_{k=0}^{\infty} \theta_k : S_n \to S_n$. We claim that this map respects the multiplication in S_n, as defined above, so that Θ is ring homomorphism. To see this, notice that for $\overline{a} = \sigma_k(a)$ and $\overline{b} = \sigma_\ell(b)$ ($a \in \mathcal{B}_k$ and $b \in \mathcal{B}_\ell$) we have that

$$\begin{aligned} \Theta(\overline{a} \cdot \overline{b}) &= \Theta \circ \sigma_{k+\ell}(ab) \\ &= \theta_{k+\ell} \circ \sigma_{k+\ell}(ab) \\ &= \sigma_{k+\ell} \circ \widetilde{\tau}|_{\mathcal{B}_{k+\ell}}(ab) \\ &= \sigma_{k+\ell} \circ \widetilde{\tau}(ab) \\ &= \sigma_{k+\ell}(\widetilde{\tau}(a)\widetilde{\tau}(b)) \\ &= \sigma_k(\widetilde{\tau}(a)) \cdot \sigma_\ell(\widetilde{\tau}(b)) \\ &= \sigma_k(\widetilde{\tau}|_{\mathcal{B}_k}(a)) \cdot \sigma_\ell(\widetilde{\tau}|_{\mathcal{B}_\ell}(b)) \\ &= (\theta_k \circ \sigma_k)(a) \cdot (\theta_\ell \circ \sigma_\ell)(b) \\ &= \theta_k(\overline{a}) \cdot \theta_\ell(\overline{b}) \\ &= \Theta(\overline{a}) \cdot \Theta(\overline{b}). \end{aligned}$$

Thus we conclude that Θ is a ring-homomorphism.

Since $\Theta = \bigoplus_{k=0}^{\infty}$ and since the image and kernel of Θ commute with the direct sum (and we already showed that θ_k are all automorphisms of K-vector spaces), we find that Θ is a ring automorphism of S_n that respects the K-vector space structure.

Remark 0.12. Another way to show that Θ is an automorphism would be to define $\Theta' := \bigoplus_{k=0}^{\infty} \theta_k'$ and check that it is the inverse to Θ.

Chapter 8

Remark 0.13. The exercise below is the same up to bold text and italic for some words.

4.1

A K-algebra R is **affine** if there exists elements $r_1, \ldots, r_s \in R$ so that the monomials $r_1^{m_1} \cdots r_s^{m_s}$ form a K-basis for R.

(1) Show that if R is an affine commutative K-algebra then it is a homomorphic image of a polynomial ring in *finitely* many variables over K.

(1): Consider the map $\varphi : K[x_1, \ldots, x_s] \to R$ defined as $x_1 \mapsto r_1, \ldots, x_s \mapsto r_s$ and extended by linearity to all of $K[x_1, \ldots, x_n]$. By assumption R is commutative so all $r_i^{m_i}$ commute for $i = 1, \ldots, s$. If we identify φ with its extension by linearity, we see that this is in fact a ring-homomorphism which respects the K-algebra structure on $K[x_1, \ldots, x_n]$ and R. Thus this map is clearly surjective, since any element $r \in R$ can be written as $r = \sum_{\alpha \in \mathbb{N}^s} c_\alpha r^\alpha$ where $r^\alpha = r_1^{\alpha_1} \cdots r_s^{\alpha_s}$.

Furthermore, if our interpretation of the exercise is correct: The monomials being a K-basis in fact means that the map as we defined it is *injective* as well (notice that $\{x^\alpha \ : \ \alpha \in \mathbb{N}^s\}$ is a basis for $K[x_1, \ldots, x_s]$ as a K-vector space).

Thus, we see from this that $K[x_1, \ldots, x_s]/ \ker \varphi \cong R$ as an isomorphism of K-algebras (if that is the correct word, i.e. a ring.homomorphism respecting the K-algebra structures). Since $\ker \varphi = 0$ we then have that $R \cong K[x_1, \ldots, x_n]$.

Remark 0.14. Notice that R being *commutative* is essential, otherwise φ would not be well-defined since we would have elements $p, q \in K[x_1, \ldots, x_n]$ so that $\varphi(p) = p'$ and $\varphi(q) = q'$ would be such that $p'q' \neq q'p'$ but

$$\begin{aligned} p'q' &= \varphi(p)\varphi(q) \\ &= \varphi(pq) \\ &= \varphi(qp) \\ &= \varphi(q)\varphi(p) \\ &= q'p', \end{aligned}$$

contradiction!

(2): By Hilberts basis theorem and induction, one shows that $K[x_1, \ldots, x_n]$ is Noetherian. Thus it follows that R is Noetherian - for example, any non-finitely generated K-submodule of R would correspond to a non-finitely generated K-vector space of $K[x_1, \ldots, x_n]$ contradicting that $K[x_1, \ldots, x_n]$ is Noetherian.

(3): Elements in $K\{x, y\}$ are *finite* combination of words in x, y with coefficients from K, where multiplication is defined as *concatenation*.

We claim that $(x) \subsetneq (x, xy) \subsetneq (x, xy, xy^2) \subsetneq \ldots \subsetneq (x, xy, xy^2, \ldots, xy^k) \subsetneq \ldots$ is an infinite ascending sequence of left ideals of $K\{x, y\}$ that does not stabilize.

4.6

Let M be a left A_n-module with a good filtration Γ. Show that the annihilator of $\mathsf{gr}^\Gamma M$ is a homogeneous ideal of S_n.

Remark 0.15. We will assume that M having a good filtration Γ is with respect to the Bernstein filtration $\mathcal{B}$ on A_n (cf. [Cou95, Chap. 8, §3]).

We want to show that $I = \mathrm{Ann}_{S_n}(\mathsf{gr}^\Gamma M) = \{s \in S_n \ : \ sr = 0, \forall r \in \mathsf{gr}^\Gamma M\}$ is a *homogeneous ideal* of S_n, which we interpret as being generated by *homogeneous elements* in S_n, i.e. things on the form $t_k \in \mathcal{B}_k/\mathcal{B}_{k-1}$ for some $k \in \mathbb{Z}_{\geqslant 0}$.

We may assume that $\mathsf{gr}^\Gamma M$ is generated by finitely many *homogeneous* elements $\mu_{k_1}(\gamma_1), \ldots, \mu_{k_s}(\gamma_s)$. If $f \in I$ then $f \cdot \mu_{k_i}(\gamma_i) = 0$ for all $i = 1, \ldots, s$. Assume that $f = \sum\limits_{d \in \mathbb{N}} f_d$ and that there is some $f_d \notin I$. If f_d annihilated every generator $\mu_{k_1}(\gamma_1), \ldots, \mu_{k_s}(\gamma_s)$ then since S_n is commutative ([Cou95, Chap. 7, Theorem 3.1]) we see that f_d would annihilate every element of I. Thus there must be some $j \in \{1, \ldots, s\}$ such that $f_d \cdot \mu_{k_j}(\gamma_j) \neq 0$. But then we see that since

$$f \cdot \mu_{k_j}(\gamma_j) = \bigoplus_{n \in \mathbb{N}} f_n \cdot \mu_{k_j}(\gamma_j)$$

we have that $f \cdot \mu_{k_j}(\gamma_j) \neq 0$ since given that $f_d = \sigma_d(g_d)$ for some $g_d \in \mathcal{B}_d$ is such that

$$\begin{aligned} f_d \cdot \mu_{k_j}(\gamma_j) &= \sigma_d(g_d) \cdot \mu_{k_j}(\gamma_j) \\ &= \mu_{k_j + d}(g_d \gamma_j) \\ &\neq 0. \end{aligned}$$

This contradicts f being an element of I. Thus there can be no homogeneous component f_d of $f \in I$ not in I. By [Gat21, Lemma 6.10.(a)] we have that I is homogeneous if and only if for each $f \in I$ with *homogeneous decomposition* $f = \sum\limits_{d \in \mathbb{N}} f_d$ we have that $f_d \in I$ for all $d \in \mathbb{N}$. Hence we conclude that I is homogeneous.

Chapter 9

$\mathcal{B}_{i+j} = \mathcal{B}_i \cdot \mathcal{B}_j$

We want to show that if $\mathcal{B} = \{\mathcal{B}_i\}_{i \in \mathbb{N}}$ is the Bernstein-filtration on the n:th Weyl algebra A_n, then $\mathcal{B}_{i+j} = \mathcal{B}_i \cdot \mathcal{B}_j$ for all non-negative integers i, j.

Notice that since this is a filtration, the inclusion $\mathcal{B}_i \cdot \mathcal{B}_j \subseteq \mathcal{B}_{i+j}$ is clear.

For the other direction, we aim to prove this by induction on $i \geqslant 0$. Notice that since $\mathcal{B}_0 = K$, we have that $\mathcal{B}_{0+j} \subseteq B_0 \mathcal{B}_j = K\mathcal{B}_j = \mathcal{B}_j$ (since $1 \in K$).

Assume that it holds for $i = n$, we then want to show that it holds for $i = n+1$.

Lemma 0.16. *We have that*

$$\partial_i^\beta x_j^\alpha = \begin{cases} \sum\limits_{k=0}^{\min(\alpha,\beta)} \binom{\beta}{k} (\alpha)_k x_j^{\alpha-k} \partial_i^{\beta-k}, & i = j \\ x_j^\alpha \partial_i^\beta, \quad i \neq j \end{cases}$$

where $(\alpha)_k := \alpha(\alpha-1)\cdots(\alpha-k+1)$ *and* $(\alpha)_0 := 1$.

Proof. The formula is clear for $i \neq j$ since we may commute $\partial_i^{\beta_i}$ with $x_j^{\alpha_j}$ due to the fact that $[\partial_i, x_j] = 0$ when $i \neq j \Leftrightarrow \partial_i x_j = x_j \partial_i$.

We prove this by induction on β. For $\beta = 1$ one may show (again by induction) that

$$\begin{aligned} \partial_i x_j^\alpha &= x_j^\alpha \partial_i + \alpha x_j^{\alpha-1} \\ &= \binom{1}{0} (\alpha)_0 x_j^\alpha \partial_i^1 + \binom{1}{1} \alpha x_j^{\alpha-1} \end{aligned}$$

so that the formula holds in this case.

Now assume it holds for $\beta = n$, we want to show that it holds for $\beta = n+1$.

We then get

$$\begin{aligned}
\partial_i^{\beta+1} x_j^{\alpha} &= \partial_i(\partial_i^{\beta} x_j^{\alpha}) \\
&= \partial_i \left(\sum_{k=0}^{\min(\alpha,\beta)} \binom{\beta}{k} (\alpha)_k x_j^{\alpha-k} \partial_i^{\beta-k} \right) \\
&= \sum_{k=0}^{\min(\alpha,\beta)} \binom{\beta}{k} (\alpha)_k x_j^{\alpha-k} \partial_i^{\beta-k+1} + \sum_{k=0}^{\min(\alpha,\beta)} \binom{\beta}{k} (\alpha)_k \cdot (\alpha-k) x_j^{\alpha-k-1} \partial_i^{\beta-k}
\end{aligned}$$

where we in the last step used that $\partial_i x_j^{\alpha-k} = x_j^{\alpha-k}\partial_i + (\alpha-k)x_j^{\alpha-k-1}$.

We notice that since $(\alpha)_k$ can be written as $(\alpha)_k = \prod_{j=0}^{k-1}(\alpha - j)$ we have that

$$\begin{aligned}
(\alpha)_{k+1} &= \prod_{j=0}^{k}(\alpha - j) \\
&= \prod_{j=0}^{k-1}(\alpha - j) \cdot (\alpha - k) \\
&= (\alpha)_k \cdot (\alpha - k).
\end{aligned}$$

We may thus write

$$\begin{aligned}
\partial_i^{\beta+1} x_j^{\alpha} &= \sum_{k=0}^{\min(\alpha,\beta)} \binom{\beta}{k} (\alpha)_k x_j^{\alpha-k} \partial_i^{\beta-k+1} + \sum_{k=0}^{\min(\alpha,\beta)} \binom{\beta}{k} (\alpha)_{k+1} x_j^{\alpha-k-1} \partial_i^{\beta-k} \\
&= \sum_{k=0}^{\min(\alpha,\beta)} \binom{\beta}{k} (\alpha)_k x_j^{\alpha-k} \partial_i^{\beta-k+1} + \sum_{k=1}^{\min(\alpha,\beta)+1} \binom{\beta}{k-1} (\alpha)_k x_j^{\alpha-k} \partial_i^{\beta-k+1}, \qquad (k \mapsto k-1) \\
&= x_j^{\alpha} \partial_i^{\beta+1} + \sum_{k=1}^{\min(\alpha,\beta)+1} \left(\binom{\beta}{k} - \binom{\beta}{k-1} \right) (\alpha)_k x_j^{\alpha-k} \partial_i^{\beta-k+1} \\
&= x_j^{\alpha} \partial_i^{\beta+1} + \sum_{k=1}^{\min(\alpha,\beta)+1} \binom{\beta+1}{k} (\alpha)_k x_j^{\alpha-k} \partial_i^{(\beta+1)-k} \\
&= \sum_{k=0}^{\min(\alpha,\beta)+1} \binom{\beta+1}{k} (\alpha)_k x_j^{\alpha-k} \partial_i^{(\beta+1)-k} \\
&= \sum_{k=0}^{\min(\alpha,\beta+1)} \binom{\beta+1}{k} (\alpha)_k x_j^{\alpha-k} \partial_i^{(\beta+1)-k}.
\end{aligned}$$

where we use the convention that $(\alpha)_{k+1} = 0$ and $\binom{\beta}{\beta+1} = 0$ to extend the upper index for the first sum (depending on if $\min(\alpha,\beta) = \alpha$ or $\min(\alpha,\beta) = \beta$). □

Using lemma 0.16 we see that

$$\begin{aligned}
x^{\alpha'}(\partial^{\beta'} x^{\alpha''})\partial^{\beta''} &= x^{\alpha'} \cdot (x^{\alpha''}\partial^{\beta'} + \text{lower degree terms})\partial^{\beta''} \\
&= x^{\alpha'+\alpha''}\partial^{\beta'+\beta''} + \text{lower degree terms}.
\end{aligned}$$

Let $x^\alpha \partial^\beta$ be an element in $\mathcal{B}_{n+j}$ such that $|\alpha| + |\beta| \leqslant n + j$. Set $s := \min(i, d)$ where $d := |\alpha| + |\beta|$. We claim that we can find $\alpha', \alpha'' \in \mathbb{N}^n$ and $\beta', \beta'' \in \mathbb{N}^n$ such that $|\alpha'| + |\beta'| \leqslant n$ and $|\alpha''| + |\beta''| \leqslant j$ such that $(\alpha' + \alpha'') + (\beta' + \beta'') = \alpha + \beta$.

We then have that

$$\begin{aligned} x^{\alpha'}(\partial^{\beta'} x^{\alpha''})\partial^{\beta''} &= x^{\alpha'} \cdot (x^{\alpha''}\partial^{\beta'} + \text{lower degree terms})\partial^{\beta''} \\ &= x^{\alpha'+\alpha''}\partial^{\beta'+\beta''} + \text{lower degree terms} \\ &= x^\alpha \partial^\beta + \text{lower order terms}. \end{aligned}$$

Since $x^{\alpha'}\partial^{\beta'} \in \mathcal{B}_n$ and $x^{\alpha''}\partial^{\beta''} \in \mathcal{B}_j$ we see that $x^\alpha\partial^\beta + \text{lower order terms} \in \mathcal{B}_n\mathcal{B}_j$ and then $x^\alpha\partial^\beta + \text{lower order terms} - \text{lower order terms} = x^\alpha\partial^\beta$ is in $\mathcal{B}_n \cdot \mathcal{B}_j$, which is what we wanted to show.

5.1

Show that if M is a finitely generated torsion A_n-module, then $d(M) \leqslant 2n - 1$.

<u>$M \neq 0$</u>: We proceed by induction on the number of generators of M. For the base case with one generator we have that M is cyclic. Then we get that $M \cong A_n/I$ as A_n-modules. We will assume without proof (atleast for now) that if we have such an isomorphism, specialized to the case for A_n, then the *dimension* of the two modules are the same. Then we see that $d(A_n/\ker\phi) = d(M)$. Now, since $\ker\phi \neq 0$, we may use [Cou95, Chap 10. Cor. 3.5] to see that $d(M) \leqslant 2n - 1$, which is what we wanted to show.

Assume that $d(M) \leqslant 2n-1$ for any finitely generated left torsion A_n-module M with $k-1$ generators. We want to show that it holds for k generators $m_1, \dots, m_k$. Let $M_1 = A_n\langle m_1\rangle$ and let $M' := M/M_1$. Consider the short exact sequence

$$0 \to M_1 \to M \to M' \to 0.$$

Notice that m_1 and $\overline{m_1}, \dots, \overline{m}_{k-1}$ are the generators for M_1 and M' as (left) A_n-modules. Furthermore, notice that M' is still a torsion module so that the inductive assumption applies, i.e. since M is torsion and $M' = \{m + M_1 \ : \ m \in M\}$ and the action is defined as $r \cdot (m + M_1) := rm + M_1$ we find that since any $m \in M$ has a torsion element r_m, we have that $r_m \cdot (m + M_1) = r_m m + M_1 = 0 + M_1$. Thus the inductive assumption applies so that $d(M_1), d(M') \leqslant 2n - 1$. We have that $M = \max\{d(M_1), d(M')\} \leqslant 2n - 1$ by [Cou95, Chap. 10, Theorem 3.2.(1)]. We conclude by induction that $d(M) \leqslant 2n-1$ for any finitely generated torsion submodule, which is what we wanted to show.

<u>$M = 0$</u>: By [Cou95, Chap. 10, §2] the dimension of M is *independent* of good filtration, and since a property of a filtration $\{\Gamma_i\}_{i\in\mathbb{N}}$ of M is that $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq M = \{0\}$ it follows that $\Gamma_i = 0$ for all $i \in \mathbb{N}$. Thus

$$\begin{aligned} \dim_K(\Gamma_i/\Gamma_{i-1}) &= \dim_K(\{0\}) \\ &= 0 \end{aligned}$$

so that for $t \gg 0$ we have that

$$\chi(t, \Gamma, M) = 0.$$

Since $\chi(t, \Gamma, M) \in \mathbb{Q}[t]$ and is zero for infinitely many $t \in \mathbb{Z}_{>0}$ it follows that $\chi(t, \Gamma, M)$ is the zero-polynomial, hence $d(M) = 0$, which is clearly less than $2n - 1$ for $n \geqslant 1$.

5.4

Let D be a (non-commutative) domain. We say that D satisfies the Öre condition if $Da \cap Db \neq 0$, for any two non-zero elements $a, b \in D$. A divison ring Q is *left quotient ring* of D if:

1. D is a subring of Q.
2. Every non-zero element of D is invertible in Q.
3. Every element of Q is on the form $b^{-1}a$ where $a, b \in D$ and $b \neq 0$.

Show that if a domain has a left quotient ring, then it satisfies the Öre condition.

By property 3., we now that every element of Q is on the form $c^{-1}d$ with $c, d \in D$ and $c \neq 0$. In particular, it follows that ab^{-1} is on the form $c^{-1}d$, i.e. we have

$$\begin{aligned} ab^{-1} &= c^{-1}d \\ \Leftrightarrow a &= c^{-1}db \\ \Leftrightarrow ca &= db. \end{aligned}$$

Since $c, d \in D$ we find that the element $\zeta = ca = db$ is in $Da \cap Db$. Furthermore, by assumption $c \neq 0$. Since $a \neq 0$ and D is a domain it follows that $ac = \zeta \neq 0$. Thus $0 \neq \zeta \in Da \cap Db$ so that $Da \cap Db \neq 0$.

Chapter 10

4.4

Let $p \in K[x]$ be a non-constant polynomial. Is the module $K[X, p^{-1}]$ irreducible?

We interpret this as asking whether or not $K[X, p^{-1}]$ is irreducible as a $A_n(K)$ module (n coming from $X = (x_1, \ldots, x_n)$).

We claim that if we look at $K[X] \subsetneq K[X, p^{-1}]$ then it is clear that this (left-) action is well-defined (we have seen this in [Cou95] with x_i [as seen in $A_n(K)$] acting by left multiplication, and ∂_j acting by differentiation).

Notice that $x^\alpha (p^{-1})^\beta = \frac{x^\alpha}{p^\beta}$ with $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}$ are the K-basis for $K[X, p^{-1}]$. If we extend the action of $A_n(K)$ on $K[X]$ to $K[X, p^{-1}]$ by the application of the leibniz rule, then will assume without proof that this gives us a well-defined action (c.f. [Cou95, p. 92] for how to extend the action from $K[X]$ to $K(X)$).

To be a bit more precise, we mean that we define the action of ∂_j on anything on the form $\frac{x^\alpha}{p^\beta}$ as

$$\partial_j \left(\frac{x^\alpha}{p^\beta} \right) = \frac{\partial_j (x^\alpha) p^\beta - x^\alpha \partial_j (p^\beta)}{p^{2\beta}}$$

and then extend this by linearity to all of $K[X, p^{-1}]$.

With the above action in mind, and noticing that any element $f = \frac{f}{1} \in K[X, p^{-1}]$ is such that

$$\begin{aligned} \partial_j \left(\frac{f}{1} \right) &= \frac{\partial_j f \cdot 1 - f \cdot \partial_j (1)}{1^2} \\ &= \frac{\partial f}{\partial x_j} \end{aligned}$$

(using the leibniz rule) and that x_i still acts by multiplication, we see that the $A_n(K)(K[X]) \subseteq K[X] \subsetneq K[X, p^{-1}]$. Thus we have found a non-zero proper submodule of $K[X, p^{-1}]$. Hence $K[X, p^{-1}]$ is not an irreducible $A_n(K)$-module.

4.8

Let p and $M = A_n[s]p^s$ be as in [Cou95, Chap. 10, exc 4.7]. Let $t : M \to M$ be the map defined by $t(D(s) \cdot p^s) = D(s+1)p \cdot p^s$.

(1) Show that t is an endomorphism of M as an A_n-module but not as an $A_n[s]$-module.
(2) Show that $[t, s] = t$.
(3) Use (2) to show that M/tM is an $A_n[s]$-module, even though t is not $A_n[s]$-linear.

Remark 0.17. We will assume that $D(s)$ is an arbitrary polynomial in $A_n[s]$.

First, we may (by lazyness) assume from [Cou95, Chap. 10, exc. 4.7] that M is an A_n-module. Notice that since $a(s+1) = a$ given that a is an element of A_n not depending on s. Thus we have that

$$\begin{aligned} t(aD(s) \cdot p^s) &= aD(s+1)p \cdot p^s \\ &= a(D(s+1)p \cdot p^s) \\ &= a \cdot t(D(s) \cdot p^s). \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} t((D(s) + Q(s)) \cdot p^s) &= (D(s+1) + Q(s+1))p \cdot p^s \\ &= D(s+1)p \cdot p^s + Q(s+1)p \cdot p^s \\ &= t(D(s) \cdot p^s) + t(Q(s) \cdot p^s) \end{aligned}$$

for $D(s), Q(s) \in A_n[s]$. Take $Q(s) \in A_n[s]$ to be a non-constant polynomial. Then we see that

$$Q(s) \cdot t(D(s) \cdot p^s) = Q(s)(D(s+1)p \cdot p^s)$$

while

$$t(Q(s) \cdot D(s) \cdot p^s) = Q(s+1)D(s+1)p \cdot p^s.$$

For this to hold we would need $Q(s+1) = Q(s)$ since A_n is a domain so that $A_n[s]$ is a domain. It is then easy to find a non-constant polynomial that does not fulfill this criteria, for example s, since $s \neq s+1$ whenever K is of characteristic 0. Moreoever, since $p \in K[X] \subseteq A_n \subseteq A_n[s]$ we find that $D(s+1)p \in A_n[s]$ so that $\operatorname{Im}(t) \subseteq A_n[s] \cdot p^s$.

(2): Let $D(s) \cdot p^s \in A_n[s] \cdot p^s$. Then we have that

$$\begin{aligned} [t, s](D(s) \cdot p^s) &= (ts - st)(D(s) \cdot p^s) \\ &= (ts)(D(s) \cdot p^s) - (st)(D(s) \cdot p^s) \\ &= t(sD(s) \cdot p^s) - s(t(D(s) \cdot p^s)) \\ &= (s+1)(D(s+1)p \cdot p^s) - s(D(s+1)p \cdot p^s) \\ &= D(s+1)p \cdot p^s \\ &= t(D(s) \cdot p^s). \end{aligned}$$

Since $Q(s) \cdot p^s \in A_n[s] \cdot p^s$ were arbitrary, the statement follows.

(3): We want to show that $M/t(M)$ is an $A_n[s]$-module, even though t is not $A_n[s]$-linear. Notice that since t is an A_n-module homomorphism we have that $t(M)$ is a submodule of $A_n[s] \cdot p^s$. We may assume that $A_n[s] \cdot p^s$ is an $A_n[s]$-module, and t is a homomorphism, so that if $m, m' \in t(M)$ then $m + m' \in t(M)$. What is left is to show that if $r \in A_n[s]$ and $m \in t(M)$ then $rm \in t(M)$ ("submodule test"). Notice that $s \in A_n[s]$ so that given $m \in M$ then $m = t(m')$ for some $m' \in A_n[s] \cdot p^s$. Then we have that

$$\begin{aligned} s \cdot m &= s \cdot t(m') \\ &= t(sm') - t(m') \in t(M) \end{aligned}$$

where we have used (2). Assume that it holds for $k < n \in \mathbb{N}$ that $s^k \cdot m \in t(M)$, we then want to show that it holds for s^n: We have that

$$\begin{aligned} s \cdot (s^{n-1}) \cdot m &= s \cdot s^{n-1} \cdot t(m) \\ &= s(m'') \end{aligned}$$

where $m'' \in t(M)$ by induction. Thus $s(m'')$ is in $t(M)$ by the base case, and so by induction we conclude that $s^i \cdot m$ is in $t(M)$ for all $i \in \mathbb{N}$ and all $m \in t(M)$.

Since any element in $A_n[s]$ is on the form

$$\sum_{i=0}^{k} c_i s^i$$

with $c_i \in A_n$ and since t is A_n-linear, we find that

$$\begin{aligned} \left(\sum_{i=0}^{k} c_i s^i \right) \cdot m &= \left(\sum_{i=0}^{k} c_i s^i \right) \cdot t(m') \\ &= \sum_{i=0}^{k} c_i \cdot (s_i \cdot t(m')) \\ &= \sum_{i=0}^{k} c_i \cdot m_i'' \end{aligned}$$

where $m_i'' \in t(M)$ by the inductive proof. We may thus write $m_i'' = t(n_i)$ so that

$$\begin{aligned} \sum_{i=0}^{k} c_i \cdot m_i'' &= \sum_{i=0}^{k} c_i \cdot t(n_i) \\ &= \sum_{i=0}^{k} t(cn_i) \in t(M) \end{aligned}$$

since t was shown in (1) to be A_n-linear.

Chapter 11

Remark 0.18. By laziness/tiredness/lack, I choose somewhat simpler problems in this chapter.

4.3

Let J be a left ideal of A_n, and put $M = A_n/J$. Show that $\mathrm{Ch}(M) = \mathcal{Z}(\mathrm{gr}(J))$.

Recall that $\mathrm{Ch}(M) = \mathcal{Z}(\mathrm{rad}(\mathrm{ann}(M, \Gamma)))$. By [Cou95, Chap. 10, Theorem 1.1] the annihilator is independent of choice of *good* filtration Γ. We may hence choose the filtrations on $J \subset A_n$ and A_n/J induced from the Bernstein filtration $\mathcal{B} = \{\mathcal{B}_k\}_{k \in \mathbb{N}}$ on A_n. By [Cou95, Chap. 8, Prop. 3.1] and the fact that $\mathcal{B}_i\mathcal{B}_j = \mathcal{B}_{i+j}$ for all $i, j \in \mathbb{Z}_{\geqslant 0}$ we have that $\mathcal{B}$ is a *good* filtration for A_n. By [Cou95, Chap. 9, §3] the induced filtrations on J and $M = A_n/J$, which we will denote $\mathcal{B}'$ and $\mathcal{B}''$, respectively, are then also good. Hence we may choose $\Gamma := \mathcal{B}''$.

Recall from [Cou95, Chap. 7, Lemma 5.1] that we have an exact sequence of S_n-modules

$$0 \to \operatorname{gr}^{\mathcal{B}'} J \xrightarrow{\phi} \operatorname{gr}^{\mathcal{B}} A_n (= S_n) \xrightarrow{\pi} \operatorname{gr}^{\mathcal{B}''} M \to 0.$$

Remark 0.19. From now on, we let $\operatorname{gr}(J) := \operatorname{gr}^{\mathcal{B}'} J$.

Since ϕ is injective, and the sequence is *exact* at $\operatorname{gr}^{\mathcal{B}} A_n$ it follows that $\operatorname{Im}\phi \cong \operatorname{gr}^{\mathcal{B}'} J = \ker \pi$. Since π is surjective, we have that $S_n/\operatorname{gr}(J) \cong \operatorname{gr}^{\mathcal{B}''} M$ as S_n-modules.

Lemma 0.20. *Assume that R is a commutative ring and M, N are R-modules such that $M \cong N$ as R-modules. Then $Ann_R(M) = Ann_R(N)$.*

Proof. Let $\phi : M \xrightarrow{\sim} N$ be an R-module isomorphism and let $\tau : N \xrightarrow{\sim} M$ be its inverse.

$\underline{\operatorname{Ann}_R(M) \subseteq \operatorname{Ann}_R(N)}$: Assume that $r \in \operatorname{Ann}_R(M)$ so that $r \cdot m = 0$ for all $m \in M$. Then for any $n \in N$ there is an $m \in M$ such that $\phi(m) = n$. Then we have that

$$\begin{aligned} r \cdot n &= r \cdot \phi(m) \\ &= \phi(rm) \\ &= \phi(0) \\ &= 0. \end{aligned}$$

$\underline{\operatorname{Ann}_R(N) \subseteq \operatorname{Ann}_R(M)}$: We claim that it follows by applying the same logic as in the other inclusion to τ. ☐

Lemma 0.21. *If R is a commutative ring and J is an ideal of R, then $Ann_R(R/J) = J$.*

Proof. Note that $\operatorname{Ann}_R(R/J) = \{r \in R \ : \ ra \in J, \forall a \in R\}$. Since J is an ideal, it is clear that $J \subseteq \operatorname{Ann}_R(R/J)$. On the other hand, if we $r \in \operatorname{Ann}_R(R/J)$ then $r \cdot 1 = r \in J$. Hence $\operatorname{Ann}_R(R/J) \subseteq J$ so that $\operatorname{Ann}_R(R/J) = J$. ☐

By lemma 0.20 we see that $\operatorname{Ann}_{S_n}(S_n/\operatorname{gr}(J)) = \operatorname{Ann}_{S_n}(\operatorname{gr}^{\mathcal{B}''} M)$. By lemma 0.21 we have that (since we claim that $\operatorname{gr}(J)$ is an ideal of S_n) $\operatorname{Ann}_{S_n}(S_n/\operatorname{gr}(J)) = \operatorname{gr}(J)$ so that $\operatorname{Ann}_{S_n}(\operatorname{gr}^{\mathcal{B}''} M) = \operatorname{gr}(J)$.

Thus, we get that

$$\operatorname{rad}(\operatorname{ann}(M, \Gamma(= \mathcal{B}''))) = \operatorname{rad}(\operatorname{gr}(J)).$$

Hence

$$\begin{aligned} \operatorname{Ch}(M) &= \mathcal{Z}(\operatorname{rad}(\operatorname{Ann}(M, \Gamma))) \\ &= \mathcal{Z}(\operatorname{rad}(\operatorname{gr}(J))) \\ &= \mathcal{Z}(\operatorname{gr}(J)) \end{aligned}$$

where we in the last equality used [Gat21, Lemma 1.7.(a)] together with the identification $S_n \cong K[y_1, \dots, y_{2n}]$.

4.11

Let J be an ideal of S_n. Show that J^2 is always closed for the Poisson-bracket.

Let $f_1g_1, f_2g_2 \in J^2$ be arbitrary. We then have that

$$\begin{aligned}
\{f_1g_1, f_2g_2\} &= \sum_{i=1}^{n} \left(\frac{\partial(f_1g_1)}{\partial y_{n+i}} \cdot \frac{\partial(f_2g_2)}{\partial y_i} - \frac{\partial(f_2g_2)}{\partial y_{n+i}} \cdot \frac{\partial(f_1g_1)}{\partial y_i} \right) \\
&= \sum_{i=1}^{n} \left(\left(g_1 \frac{\partial f_1}{\partial y_{n+i}} + f_1 \frac{\partial g_1}{\partial y_{n+i}} \right) \left(g_2 \frac{\partial f_2}{\partial y_i} + f_2 \frac{\partial g_2}{\partial y_i} \right) - \left(g_2 \frac{\partial f_2}{\partial y_{n+i}} + f_2 \frac{\partial g_2}{\partial y_{n+i}} \right) \left(g_1 \frac{\partial f_1}{\partial y_i} + f_1 \frac{\partial g_1}{\partial y_i} \right) \right) \\
&= \sum_{i=1}^{n} g_1 \frac{\partial f_1}{\partial y_{n+i}} \left(g_2 \frac{\partial f_2}{\partial y_i} + f_2 \frac{\partial g_2}{\partial y_i} \right) + f_1 \frac{\partial g_1}{\partial y_{n+i}} \left(g_2 \frac{\partial f_2}{\partial y_i} + f_2 \frac{\partial g_2}{\partial y_i} \right) \\
&- g_2 \frac{\partial f_2}{\partial y_{n+i}} \left(g_1 \frac{\partial f_1}{\partial y_i} + f_1 \frac{\partial g_1}{\partial y_i} \right) - f_2 \frac{\partial g_2}{\partial y_{n+i}} \left(g_1 \frac{\partial f_1}{\partial y_i} + f_1 \frac{\partial g_1}{\partial y_i} \right)
\end{aligned}$$

But S_n is a commutative ring (it is isomorphic to $K[y_1, \ldots, y_{2n}]$) and the derivatives of f_i, g_j are in S_n. By inspection we see that $\{f_1g_1, f_2g_2\}$ is a sum of terms on the form $f_ig_ja, f_if_jb, g_ig_jc$ for $i, j \in \{1, 2\}$ with $a, b, c \in S_n$. Since J^2 is an ideal and f_i, g_i are in J it follows that f_if_j, g_ig_j, f_ig_j are all in J^2, and furthermore multiplication (from left or right) with elements $a, b, c \in S_n$ are still in J^2. Since ideals are closed under addition, sums of such elements are still in J^2. Thus we conclude that J^2 is closed under the Poisson-bracket by "simple elements" on the form $f_1g_1, f_2g_2 \in J^2$. Since taking a partial derivative of a polynomial in S_n is a linear operation, the Poisson-bracket is linear (not necessarily S_n-linear, we just mean linear in the sense that $\{a + b, c\} = \{a, c\} + \{b, c\}$ and similarly for the second argument) in both its arguments, and since we may think of J^2 as

$$J^2 := \left\{ \sum_{j=1}^{n} a_jb_j \;\middle|\; a_j, b_j \in J \right\}$$

we see that any two elements $p = \sum_{i=1}^{n} a_ib_i$ and $q = \sum_{j=1}^{m} a'_jb'_j$ in J^2 are such that $\{p, q\} \in J^2$.

Chapter 12

6.5

Let R be a K-algebra and let $k \geqslant 0$ be an integer. Show that if $N \to M$ is an injective map of left R-modules, then so is

$$R^k \otimes_R N \to R^k \otimes_R M.$$

Let $\phi : N \to M$ be our injective map. Upon tensoring with $R^k \otimes_R -$ we get the map

$$1 \otimes \phi : R^k \otimes_R N \to R^k \otimes_R M.$$

Proving this proposition amounts to proving that R^k is a **flat** R-module (see for example [DF04, p. 400]).

Mimicking the proof of [DF04, Cor. 10.5.42]), we notice that $R^k \otimes_R N \cong N^k$ and $R^k \otimes_R M \cong M^k$.

We may this consider

$$N^k \xrightarrow{\sim} R^k \otimes_R N \xrightarrow{1\otimes\phi} R^k \otimes_R M \xrightarrow{\sim} M^k. \tag{0.5}$$

Assume that $1 \otimes \psi$ is not injective so that $(1 \otimes \phi)(a) = (1 \otimes \phi)(b) = c$. Let $n \neq n' \in N^k$ such that $n \mapsto a, n' \mapsto b$. This is then sent to c, which is sent isomorphically to some element $d \in M^k$. Then the preimage (of the chain of maps in 0.5) of d will include atleast the two elements n, n', hence the composition of the maps in 0.5 will not be injective. The contrapositive of this is that if the composition in equation 0.5 is injective, then $1 \otimes \psi$ is injective. Writing out equation 0.5 more thoroughly, we have

$$N^k \cong \bigoplus_{i=1}^{k} (R \otimes_R N) \cong R^k \otimes_R N \xrightarrow{1\otimes\phi} R^k \otimes_R M \cong \bigoplus_{i=1}^{k} (R \otimes_R M) \cong M^k. \tag{0.6}$$

Explicitly, we want to claim that 0.6 looks like

$$(n_1, \ldots, n_k) \mapsto \bigoplus_{i=1}^{k} (e_i \otimes n_i) \mapsto \sum_{i=1}^{k} (e_i \otimes n_i) \xmapsto{1\otimes\phi} \sum_{i=1}^{k} (e_i \otimes \phi(n_i)) \mapsto \bigoplus_{i=1}^{k} (e_i \otimes \phi(n_i)) \mapsto (\phi(n_1), \ldots, \phi(n_k))$$

Let's call this composition $f : N^k \to M^k$. Then

$$\begin{aligned} f(n_1, \ldots, n_k) &= f(n'_1, \ldots, n'_k) \\ \Rightarrow (\phi(n_1), \ldots, \phi(n_k)) &= (\phi(n'_1), \ldots, \phi(n'_k)) \\ \Rightarrow \phi(n_i) &= \phi(n'_i) \qquad (\forall i \in \{1, \ldots, k\}) \\ \Rightarrow n_i &= n'_i \qquad (\forall i \in \{1, \ldots, k\}), \end{aligned}$$

where the last implication followed from the assumed injectivity of ϕ.

6.6

Let $\phi : \mathbb{Z}_2 \to \mathbb{Z}_4$ be the injective homomorphisms of $\mathbb{Z}$-modules defined by $\phi(1 + 2\mathbb{Z}) = 2 + 4\mathbb{Z}$. Show that:
(1) $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 \cong \mathbb{Z}_2$.
(2) $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_4 \cong \mathbb{Z}_2$.
(3) The map $\mathbb{Z}_2 \to \mathbb{Z}_2$ induced by tensoring ϕ by $\mathbb{Z}_2$ over $\mathbb{Z}$ is identically zero.
Conclude that the tensor product by $\mathbb{Z}_2$ does not preserve injectivity.

<u>(1)</u>: We have $M \otimes_R R/I \cong M/MI$. On applying this to $M = \mathbb{Z}/2\mathbb{Z}$, $R = \mathbb{Z}$ and $I = 2\mathbb{Z}$ we find that

$$\begin{aligned} \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 &\cong \mathbb{Z}_2/(\mathbb{Z}_2 \cdot 2\mathbb{Z}) \\ &= \mathbb{Z}_2 \end{aligned}$$

where we in the last step used that $2\mathbb{Z}$ annihilates the non-zero element $1 + 2\mathbb{Z}$ in $\mathbb{Z}_2$.

<u>(2)</u>: Recall (in the "simpler" commutative case) that $M \otimes_R R/I \cong M/MI$. On applying this to $M = \mathbb{Z}_2$, $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$ we find that

$$\begin{aligned} \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_4 &= \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} \\ &\cong (\mathbb{Z}/2\mathbb{Z}) / (\mathbb{Z}/2\mathbb{Z} \cdot 4\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z}, \end{aligned}$$

where we in the last step used that that $4\mathbb{Z} \subset \mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$.

<u>(3)</u>: Assume that we start with $\mathbb{Z}_2 \xrightarrow{\phi} \mathbb{Z}_4$. Upon tensoring with $\mathbb{Z}_2 \otimes_{\mathbb{Z}} -$ we get the map

$$\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 \xrightarrow{1\otimes\phi} \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_4.$$

Assuming the results of (1),(2), this is the same (up to isomorphism) as a map $\mathbb{Z}_2 \to \mathbb{Z}_2$.

We will assume that the isomorphism $M \otimes_R (R/I) \cong M/MI$ is defined explicitly as $m \otimes (r + I) \mapsto rm + MI$ with inverse $f : M/MI \to M \otimes_R R/I, \quad m + MI \mapsto m \otimes 1$ (cf. [DF04, p. 370]).

We then get

$$\mathbb{Z}_2 \xrightarrow{\sim} \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 \xrightarrow{1\otimes\phi} \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_4 \xrightarrow{\sim} \mathbb{Z}_2$$

is defined on $1 + 2\mathbb{Z}$ as

$$(1+2\mathbb{Z}) \mapsto (1+2\mathbb{Z}) \otimes (1+2\mathbb{Z}) \mapsto (1+2\mathbb{Z}) \otimes (2+4\mathbb{Z}) \mapsto 2+2\mathbb{Z} = 0+2\mathbb{Z}.$$

Notice that ϕ is injective, but upon tensoring with $\mathbb{Z}_2 \otimes_{\mathbb{Z}} -$ we get the map $1 \otimes \phi$ that sends $(1+2\mathbb{Z})\otimes(1+2\mathbb{Z})$ to

$$\begin{aligned}(1+2\mathbb{Z}) \otimes (2+4\mathbb{Z}) &= (1+2\mathbb{Z}) \otimes 2(1+4\mathbb{Z})\\ &= (2+2\mathbb{Z}) \otimes (1+4\mathbb{Z})\\ &= (0+2\mathbb{Z}) \otimes (1+4\mathbb{Z})\\ &= 0.\end{aligned}$$

$(1+2\mathbb{Z}) \otimes (1+4\mathbb{Z})$ is the multiplicative identity in the $\mathbb{Z}$-algebra $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2$ (we will not show this; we just state as a fact that if R is a commutative ring and S_1, S_2 are R algebras then $S_1 \otimes_R S_2$ is an R-algebra), hence is non-zero $\rightsquigarrow 1 \otimes \phi$ is not injective, i.e. $\mathbb{Z}_2$ is not a flat $\mathbb{Z}$-module.

Chapter 13

Note on [Cou95, Lemma 13.3.2]

We believe there is a typo and should be $k \gg 0$ and not $t \gg 0$ in the next to last sentence of the proof.

We aim to show that $\mathcal{B}_t(A_{n+m})\Gamma_k(M\hat{\otimes}N) = \Gamma_{k+t}(M\hat{\otimes}N)$. We claim that the inclusion $\subset$ follows from showing that $\{\Gamma_k(M\hat{\otimes}N)\}_{k\in\mathbb{N}}$ is a filtration, so it is enough to show the other inclusion. For this, we want to show that $\Gamma_i(M) \otimes \Gamma_j(N) \subset \mathcal{B}_t(A_{m+n})\Gamma_k(M\hat{\otimes}N)$ whenever $i + j = k + t$ (for some $k \gg 0$ and any non-negative integer t). Since by assumption $\Gamma(M)$ and $\Gamma(N)$ are *good* filtrations, there exists non-negative integers i_0, j_0 such that

$$\begin{aligned}\mathcal{B}_i\Gamma_k(M) &= \Gamma_{i+k}(M) \qquad (\forall k \geqslant j_0, \forall i \geqslant 0)\\ \mathcal{B}_i\Gamma_k(N) &= \Gamma_{i+k}(N) \qquad (\forall k \geqslant i_0, \forall i \geqslant 0).\end{aligned}$$

Set $K_0 := i_0 + j_0$. Assume that (for fixed $p + q = t$ and $k \geqslant K_0$) we have that $i + j = k + t$ and that we are given $\Gamma_i(M) \otimes \Gamma_j(N)$.

We divide by cases. Notice that since $i + j = k + t \geqslant k \geqslant K_0 = i_0 + j_0$ we have that either $j \geqslant j_0$ or $i \geqslant i_0$.

$\underline{j \geqslant j_0}$: Let $p = \max(0, t - (j - j_0))$ and note that $q = t - p$. Set $i' = i - p$ and $j' = j - q$. Then we see that

$$\begin{aligned}i' + j' &= (i-p) + (j-q)\\ &= (i+j) - (p+q)\\ &= k + t - t\\ &= k.\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}(i-i_0) + (j-j_0) &= (i+j) - (i_0+j_0)\\ &= k + t - (i_0 + j_0)\\ &\geqslant t \qquad (\text{since } k \geqslant i_0 + j_0)\\ \Rightarrow (i - i_0) &\geqslant t - (j - j_0)\end{aligned}$$

If $p = t - (j - j_0)$ then $(i - i_0) \geqslant p \Leftrightarrow i - p = i' \geqslant i_0$. Hence we have that

$$\begin{aligned}\mathcal{B}_p\Gamma_{i'} &= \Gamma_{i'+p}\\ &= \Gamma_i\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_q\Gamma_{j'} &= \Gamma_{j'+q}\\ &= \Gamma_j.\end{aligned}$$

If on the other hand $t - (j - j_0) < 0$ so that $p = 0$, then $q = t$. Notice that since $\mathcal{B}_0 = K$ it holds that $\mathcal{B}_p\Gamma_{i'} = \Gamma_{i'} = \Gamma_i$, and we still have that

$$\begin{aligned}j' &= j - q\\ &= j - t\\ &\geqslant j - (j - j_0)\\ &= j_0.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{B}_q\Gamma_{j'} &= \mathcal{B}_t\Gamma_{j-t}\\ &= \Gamma_j.\end{aligned}$$

$\underline{i \geqslant i_0}$: Let $q = \max(0, t - (i - i_0))$ and notice that $p = t - q$. Again, set $i' = i - p$ and $j' = j - q$. We again get that $i' + j' = k$ and we also see that $(j - j_0) \geqslant t - (i - i_0)$.

If $q = t - (i - i_0)$ then $(j - j_0) \geqslant q \Leftrightarrow j - q = j' \geqslant j_0$ and the proof proceeds as in the case $j \geqslant j_0$. If $q = 0$ then $p = t$ and we have that $i' = i - t \geqslant i - (i - i_0) \geqslant i_0$, and the proof proceeds as previously.

By [Cou95, Lemma 13.3.1] and by definition of $\Gamma_k(M\hat{\otimes}N)$ we have that

$$\mathcal{B}_t(A_{n+m})\Gamma_k(M\hat{\otimes}N) = \sum_{\substack{p+q=t\\ i+j=k}} \mathcal{B}_p(A_n)\Gamma_i(M) \otimes \mathcal{B}_q(A_m)\Gamma_j(N).$$

By the above reasoning we see that $\mathcal{B}_t(A_{n+m})\Gamma_k(M\hat{\otimes}N) \supset \Gamma_{k+t}(M\hat{\otimes}N)$.

0.1 Note on the proof of [Cou95, Lemma 13.3.3]

Let $X = \bigoplus_{i+j=k} \Gamma_i(M) \otimes \Gamma_j(N)$. We have the following diagram:

$$\begin{tikzcd}
 & X \arrow[d, two heads, "q"] \arrow[dr, "\eta"] & \\
X \arrow[r, two heads, "q"] \arrow[dr, "f=\mu_k\circ\alpha"'] & X/\bigoplus\ker\eta_{ij} \arrow[r, dotted, "\overline{\eta}"] \arrow[d, dotted, two heads, "\tilde{f}"] & \bigoplus_{i+j=k}\mathrm{gr}_i(M)\otimes\mathrm{gr}_j(N) \\
 & \mathrm{gr}_K(M\hat{\otimes}N) \arrow[ur, dotted, "\theta"'] &
\end{tikzcd}$$

where α takes elements in X to $\Gamma_k(M\hat{\otimes}N) = \sum_{i+j=k} \Gamma_i(M) \otimes \Gamma_j(N)$ and $\mu_k : \Gamma_k(M\hat{\otimes}N) \twoheadrightarrow \Gamma_k(M\hat{\otimes}N)/\Gamma_{k-1}(M\hat{\otimes}N)$.

If $u \otimes v \in X$ then

$$\begin{tikzcd}
 & u \otimes v \arrow[d, mapsto] \arrow[dr, mapsto] & \\
u \otimes v \arrow[r, mapsto] \arrow[dr, mapsto] & {[u \otimes v]} \arrow[r] \arrow[d, mapsto] & \mu_i(u) \otimes \mu_j(v) \\
 & u \otimes v + \Gamma_{k-1}(M \hat{\otimes} N) = \mu_k(u \otimes v) \arrow[ur, mapsto] &
\end{tikzcd}$$

5.2

Show that the multiplication map induces an isomorphism of A_{n+m}-modules $K[X, Y] \cong K[X]\hat{\otimes}K[Y]$.

From [Cou95, Cor. 13.1.2] we already know that the multiplication map $\phi : K[X] \times K[Y] \to K[X, Y]$ defined by $(a, b) \mapsto ab$ induces an isomorphism of K-algebras $\Phi : K[X]\hat{\otimes}K[Y] \cong K[X, Y]$ defined by $u \otimes v \mapsto uv$. We aim to show that this map respects the A_{n+m}-module structure on the domain and codomain of Φ.

Notice that the action of $A_{n+m} = A_n \otimes A_m$ on $K[X]\hat{\otimes}K[Y]$ is defined by

$$(a \otimes b)(u \otimes v) := au \otimes bv.$$

On the other hand, the action of A_{n+m} on $K[X, Y]$ is the "ordinary" one where the generators $x_1, \ldots, x_n, y_1, \ldots, y_m$ act by multiplication and $\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{y_1}, \ldots, \partial_{y_m}$ act by *differentiation*.

Thus, what we want to show is that for all $a \otimes b \in A_n \otimes A_m$ and all pure tensors $u \otimes v$ (this is enough we claim since we can extend by linearity) we have that

$$a(u) \otimes b(v) = a(u)b(v),$$

under the isomorphism $A_n \otimes A_m \cong A_{n+m}$. That is, what we really want to show is that

$$a(u)b(v) = ab(uv).$$

If we start by element $x_i \otimes 1$ and arbitrary $u \otimes v$ we have that

$$(x_i \otimes 1)(u \otimes v) = x_i u \otimes v$$

which is sent to $(x_i u)v$. It is clear that this is the same as $x_i(uv)$ since multiplication is associative in $K[X, Y]$.

If we instead take $\partial_{x_i} \otimes 1$ and arbitrary $u \otimes v$ we on the one hand get $(\partial_{x_i} \otimes 1)(u \otimes v) = \partial_{x_i}(u) \otimes v \mapsto \partial_{x_i}(u)v$ and on the other hand get

$$\begin{aligned} \partial_{x_i}(uv) &= \partial_{x_i}(u)v + \underbrace{\partial_{x_i}(v)}_{=0} u \\ &= \partial_{x_i}(u)v \end{aligned}$$

so that they coincide.

By associativity and commutativity of $K[X,Y]$ together with similar reasoning as above, we see that Φ is A_{n+m}-linear with respect to $1 \otimes y_j$ and $1 \otimes \partial_{y_j}$ as well.

By additivity of Φ together with the fact that we have well-defined actions on the domain and codomain of Φ, and that we have shown that Φ is A_{n+m}-linear with respect to generators, it follows that Φ is an A_{n+m}-module isomorphism.

5.8

Let $1 \leqslant i \leqslant n$. Suppose that d_i is an operator of A_n of degree ≥ 1 which is a linear combination of monomials in x_i and ∂_i. Let J be the ideal generated by $d_1, \ldots, d_n$. Show that A_n/J is a holonomic module over A_n.
Hint: $A_n/J \cong (A_1/A_1 d_1)\hat{\otimes} \cdots \hat{\otimes}(A_1/A_1 d_n)$.

By applying a generalized version of [Cou95, Lemma 13.2.2] to arbitrary finite tensor products, we see that

$$\begin{aligned}(A_1/A_1 d_1)\hat{\otimes} \cdots \hat{\otimes}(A_1/A_1 d_n) &\cong A_n/(A_n d_1 + \ldots + A_n d_n)\\ &= A_n/J,\end{aligned}$$

as A_n-modules.

By applying (again) a generalized version to arbitrary (finite) tensor products of [Cou95, Theorem 13.4.1.(1)] we see that

$$d\left((A_1/A_1 d_1)\hat{\otimes} \cdots \hat{\otimes}(A_1/A_1 d_n)\right) = d\left(A_1/A_1 d_1\right) + \ldots + d(A_1/A_1 d_n).$$

Assuming the result of [Cou95, Exc. 9.5.2] we have that $d\left(A_1/A_1 d_i\right) = 1$ for $i = 1, \ldots, n$ so that $d\left((A_1/A_1 d_1)\hat{\otimes} \cdots \hat{\otimes}(A_1/A_1 d_n)\right) = n$.

Assuming the dimension of a finitely generated A_n-module M with respect to some good filtration Γ is preserved under isomorphisms of A_n-modules, it follows that $d(A_n/J) = n$, so that A_n/J is indeed holonomic.

Chapter 14

4.1

Let R, S, T be rings and $\phi : R \to S$ and $\psi : S \to T$ be ring homomorphisms. If M is a left R-module, show that

$$T \otimes_\psi S \otimes_\phi M \cong T \otimes_{\psi\phi} M$$

as left T-modules.

Notice that T has a right R-module structure by $t \cdot r := t \cdot \psi\phi(r)$, so that $T \otimes_{\psi\phi} M$ is well-defined.

We notice that any pure tensor $t \otimes s \otimes m = t \cdot \psi(s) \otimes 1 \otimes m$. Let $\Phi : T \otimes_\psi S \otimes_\phi M \to T \otimes_{\psi\phi} M$ be defined by $t \cdot \psi(s) \otimes 1 \otimes m \mapsto t \cdot \psi(s) \otimes m$ on pure tensors. Since $1 \in S$ it is clear that this map is surjective.

It is clear that this map is T-linear (since in both the domain and codomain, we just have the canonical [left] T-module structure). If we extend Φ by linearity to finite sums of pure tensors, we get a T-linear surjective homomorphism. It remains to show that Φ is in fact injective. Instead of showing this directly (which is sometimes hard, since one has to identify which elements in the tensor products are *equal*), we will try to find a (two-sided) *inverse* to Φ. Let $\Psi : T \otimes_{\psi\phi} M \to T \otimes_\psi S \otimes_\phi M, \quad t \otimes m \mapsto t \otimes 1 \otimes m$.

Then we have that

$$
\begin{aligned}
(\Psi \circ \Phi)(t \otimes s \otimes m) &= \Psi(t \cdot \psi(s) \otimes m) \\
&= t \cdot \psi(s) \otimes 1 \otimes m \\
&= t \otimes s \otimes m
\end{aligned}
$$

and

$$
\begin{aligned}
(\Phi \circ \Psi)(t \otimes m) &= \Phi(t \otimes 1 \otimes m) \\
&= t \cdot \psi(1) \otimes m \\
&= t \otimes m \qquad \text{(since } \psi \text{ is a ring homomorphism).}
\end{aligned}
$$

It is clear that Ψ is T-linear, almost from the definition, so we do not check this in detail. We extend Ψ by linearity to finite sums of pure tensors, and it follows that Ψ is in fact a two-sided T-module homomorphism to the T-module homomorphism Φ, hence Φ is a T-module isomorphism.

We check that Φ and Ψ are well-defined.

<u>Φ is well-defined</u>: Consider the following diagram:

$$
\begin{tikzcd}
(s,m) \arrow[ddrr, mapsto] & S \times M \arrow[rrr, "\iota"] \arrow[drr, "\eta"'] & & & S \otimes_\phi M \arrow[dl, "\zeta"] \\
& & & T \otimes_{\psi\phi} M & \\
& & \psi(s) \otimes m & &
\end{tikzcd}
$$

We have that

$$
\begin{aligned}
\eta(sr, m) &= \eta(s\phi(r), m) \\
&= \psi(s\phi(r)) \otimes m \\
&= \psi(s)\psi(\phi(r)) \otimes m \\
&= \phi(s) \otimes rm \\
&= \eta(s, rm)
\end{aligned}
$$

so that the map η is R-balanced with respect to ϕ, hence we get a map $\zeta : S \otimes_\phi M \to T \otimes_{\psi\phi} M$ such that $\zeta(s \otimes m) = \psi(s) \otimes m$.

Now consider the following diagram:

$$
\begin{tikzcd}
(t, s \otimes m) \arrow[ddrr, mapsto] & T \times (S \otimes_\phi M) \arrow[rrr, "\iota"] \arrow[drr, "\xi"'] & & & T \otimes_\psi (S \otimes_\phi M) \arrow[dl, dotted, "\Phi"] \\
& & & T \otimes_{\psi\phi} M & \\
& & t \cdot \zeta(s \otimes m) & &
\end{tikzcd}
$$

Since the action of t on $T \otimes_{\psi\phi} M$ is well-defined by assumption, and we have shown that ζ is well-defined in the previous step, then we note that if $(t, s \otimes m) = (t', s' \otimes m')$ then $t = t'$ and $s \otimes m = s' \otimes m'$, hence $\zeta(s \otimes m) = \zeta(s' \otimes m')$ and so

$$\begin{aligned} t \cdot \zeta(s \otimes m) &= t' \cdot \zeta(s \otimes m) \\ &= t' \cdot \zeta(s' \otimes m') \end{aligned}$$

so that ξ is wel-defined. Furthermore, we claim that ξ is S-balanced:

$$\begin{aligned} \xi(ts', s \otimes m) &= \xi(t\psi(s'), s \otimes m) \\ &= t\psi(s') \cdot \psi(s) \otimes m \\ &= t \cdot \psi(s's) \otimes m \\ &= \xi(t, s' \cdot (s \otimes m)). \end{aligned}$$

Thus we get a get a *well-defined* map Φ such that $\Phi(t \otimes (s \otimes m)) = t \cdot \psi(s) \otimes m$, which is what we wanted to show.

<u>Ψ is well-defined</u>: Consider the following diagram:

$$\begin{tikzcd}
(t,m) \arrow[ddr, mapsto] & T \times M \arrow[rr, "\iota"] \arrow[dr, "\tau"'] & & T \otimes_{\psi\phi} M \arrow[dl, dotted, "\Psi"] \\
& & T \otimes_{\psi} (S \otimes_{\phi} M) & \\
& t \otimes 1 \otimes m & &
\end{tikzcd}$$

We need to check that τ is R-balanced: We have that

$$\begin{aligned} \tau(t \cdot r, m) &= \tau(t \cdot \psi\phi(r), m) \\ &= t \cdot \psi\phi(r) \otimes 1 \otimes m \\ &= t \otimes \phi(r) \otimes m \\ &= t \otimes 1 \otimes rm \\ &= \tau(t, rm). \end{aligned}$$

Thus τ gives us a *well-defined* map $\Psi : T \otimes_{\psi\phi} M \to T \otimes_{\psi} S \otimes_{\phi} M$ defined by $\Psi(t \otimes m) = t \otimes 1 \otimes m$, which is what we wanted to show.

4.5

Let $F : \mathbb{C} \to \mathbb{C}$ be the map defined by $y = F(x) = x^m$, where $m \geq 2$ is an integer. Let δ be the Dirac microfunction; see [Cou95, Chap. 6, §3]. The purpose of this exercise is to show that the inverse image $F^*(A_1(\mathbb{C})\delta)$ is isomorphic to the $A_1(\mathbb{C})$-module generated by δ^m, the m-th derivative of δ.

(1) Show that x^m and $x\partial_x + m$ annihilate $1 \otimes \delta$.

(2) Show by induction that there exists non-zero complex numbers c_{pq} such that

$$\partial_x^{mp+q}(1 \otimes \delta) = c_{pq} x^{m-q} \otimes \partial_y^{p+1}\delta$$

for $q = 0, 1, \ldots, m$.

(3) Using (2), show that $1 \otimes \partial_y^{p+1}\delta = \frac{1}{c_{pm}} \partial_x^{(p+1)m}(1 \otimes \delta)$.

(1): We will let $A_1(\mathbb{C}) = A_1$ for brevity. Then consider that $F^*(A_1\delta) = K[X] \otimes_{K[Y]} A_1\delta$.

The action of x^m on $1 \otimes \delta$ is by

$$\begin{aligned} x^m(1 \otimes \delta) &= x^m \otimes \delta \\ &= 1 \otimes y\delta \\ &= 0 \qquad \text{(c.f. [Cou95, Chap. 6, §3])} \end{aligned}$$

since the left $K[Y]$-module structure on $A_1\delta$ is induced by the comomorphism $F^\sharp : K[Y] \to K[X], \quad p(y) \mapsto p(x^m)$, and (as far as we can tell) $y\delta = 0$.

We then have that

$$\begin{aligned} (x\partial_x + m)(1 \otimes \delta) &= x\partial_x(1 \otimes \delta) + m(1 \otimes \delta) \\ &= x(\underbrace{\partial_x(1) \otimes \delta}_{=0} + mx^{m-1} \otimes \partial_y\delta) + m \otimes \delta \\ &= mx^m \otimes \partial_y\delta + m \otimes \delta \\ &= m \otimes y\partial_y\delta + m \otimes \delta \\ &= m \otimes (\partial_y y - 1)\delta + m \otimes \delta \\ &= m \otimes \partial_y \cdot \underbrace{(y\delta)}_{=0} - m \otimes \delta + m \otimes \delta \\ &= 0. \end{aligned}$$

(2): We proceed by induction on p.

Base-case $p = 0$: Then we have ∂_x^q. We need to know what $\partial_x^q(1 \otimes \delta)$ is for general $q \in \mathbb{N}$. We check for $q = 0, 1, 2$ and see if we can find a pattern, utilizing that $y\partial_y = \partial_y y - 1$ and the defined action of ∂_x on general elements $u \otimes v$, together with $y\delta = 0$.

We will not write out all the calculations. We find that

$$\begin{aligned} \partial_x(1 \otimes \delta) &= mx^{m-1} \otimes \partial_y\delta \\ \partial_x^2(1 \otimes \delta) &= -(m^2 + m)x^{m-2} \otimes \partial_y\delta \\ &= -(m(m+1))x^{m-2} \otimes \partial_y\delta \\ \partial_x^3(1 \otimes \delta) &= (m^3 + 3m^2 + 2m)x^{m-3} \otimes \partial_y\delta \\ &= m(m+1)(m+2)x^{m-3} \otimes \partial_y\delta. \end{aligned}$$

We will try to show by induction that the general formula (for $1 \leqslant q \leqslant m$) is

$$\partial_x^q(1 \otimes \delta) = (-1)^{q-1} \left(\prod_{i=0}^{q-1} (m+i) x^{m-q} \right) \otimes \partial_y\delta.$$

By looking at $\partial_x(1 \otimes \delta)$ above, we see that the formula holds for the base case.

Assume it holds for all q such that $1 \leqslant q < m$. We want to show that it follows that it holds for $q + 1$.

We have that

$$\begin{aligned}
\partial_x^{q+1}(1\otimes\delta) &= \partial_x(\partial_x^q(1\otimes\delta)) \\
&= \partial_x\left((-1)^{q-1}\left(\prod_{i=0}^{q-1}(m+i)x^{m-q}\right)\otimes\partial_y\delta\right) \\
&= (-1)^{q-1}\partial_x\left(\left(\prod_{i=0}^{q-1}(m+i)x^{m-q}\right)\otimes\partial_y\delta\right) \\
&= (-1)^{q-1}\left(\left((m-q)\prod_{i=0}^{q-1}(m+i)x^{m-q-1}\right)\otimes\partial_y\delta+\left(m\prod_{i=0}^{q-1}(m+i)x^{2m-q-1}\right)\otimes\partial_y^2\delta\right) \\
&= (-1)^{q-1}\left(\left((m-q)\prod_{i=0}^{q-1}(m+i)x^{m-q-1}\right)\otimes\partial_y\delta+\left(m\prod_{i=0}^{q-1}(m+i)x^{m-q-1}\right)\otimes y\partial_y^2\delta\right) \\
&= (-1)^{q-1}\left(\left((m-q)\prod_{i=0}^{q-1}(m+i)x^{m-q-1}\right)\otimes\partial_y\delta+\left(m\prod_{i=0}^{q-1}(m+i)x^{m-q-1}\right)\otimes(\partial_y^2y-2\partial_y)\delta\right) \\
&= (-1)^{q-1}\left(\left((m-q)\prod_{i=0}^{q-1}(m+i)x^{m-q-1}\right)\otimes\partial_y\delta-\left(2m\prod_{i=0}^{q-1}(m+i)x^{m-q-1}\right)\otimes\partial_y\delta\right) \quad (\text{since } \partial_y^2y\delta=0) \\
&= (-1)^{q-1}\left((-m-q)\prod_{i=0}^{q-1}(m+i)x^{m-q-1}\otimes\partial_y\delta\right) \\
&= (-1)^{q}\left(\prod_{i=0}^{q}(m+i)x^{m-(q+1)}\otimes\partial_y\delta\right),
\end{aligned}$$

which is what we wanted to show.

Thus we see that for $p=0$ we may take $c_{pq}:=(-1)^{q-1}\prod_{i=0}^{q-1}(m+i)$ which is a complex number, whenever $q=1,\dots,m$. When $q=0$ then we have that

$$\partial_x^0(1\otimes\delta)=1\otimes\delta$$

(assuming the convention $\partial_x^0=1$). Consider that

$$\begin{aligned}
c_{00}x^m\otimes\partial_y\delta &= c_{00}\otimes y\partial_y\delta \\
&= c_{00}\otimes(\partial_y y-1)\delta \\
&= -c_{00}\otimes\delta,
\end{aligned}$$

where we used that $y\partial_y=\partial_y y-1$ and $\partial_y y\delta=0$. If we take $c_{00}=-1\in\mathbb{C}\setminus\{0\}$ then we get that

$$\partial_x^0(1\otimes\delta)=c_{00}x^m\otimes\partial_y\delta,$$

so that this also holds for $q=0$.

This proves the base case. Now assume it holds for $p=n$ that there is a non-zero complex number c_{pq} such that

$$\partial_x^{mp+q}(1\otimes\delta)=c_{pq}x^{m-q}\otimes\partial_y^{p+1}\delta.$$

We then want to show that it holds for $p=n+1$. We have

$$\begin{aligned}
\partial_x^{m(p+1)+q}(1\otimes\delta) &= \partial_x^m(\partial_x^{mp+q}(1\otimes\delta)) \\
&= \partial_x^m(c_{pq}x^{m-q}\otimes\partial_y^{p+1}\delta).
\end{aligned}$$

To go further, we would like to find an expression for $\partial_x^k(c_{pq}x^{m-q} \otimes \partial_y^{p+1}\delta)$.

Before going further, we note that one can show that $[\partial_y^k, y] = k\partial_y^{k-1}$. Since $y\partial_y^k = \partial_y^k y - [\partial_y^k, y]$ we have that $y\partial_y^k = \partial_y^k y - k\partial_y^{k-1}$.

We see that we have

$$\begin{aligned}
\partial_x(c_{pq}x^{m-q} \otimes \partial_y^{p+1}\delta) &= (m-q)c_{pq}x^{m-q-1} \otimes \partial_y^{p+1}\delta + mc_{pq}x^{2m-q-1} \otimes \partial_y^{p+2}\delta \\
&= (m-q)c_{pq}x^{m-q-1} \otimes \partial_y^{p+1}\delta + mc_{pq}x^{m-q-1} \otimes y\partial_y^{p+2}\delta \\
&= (m-q)c_{pq}x^{m-q-1} \otimes \partial_y^{p+1}\delta + mc_{pq}x^{m-q-1} \otimes (\partial_y^{p+2}y - (p+2)\partial_y^{p+1})\delta \\
&= (m-q)c_{pq}x^{m-q-1} \otimes \partial_y^{p+1}\delta - (p+2)mc_{pq}x^{m-q-1} \otimes \partial_y^{p+1}\delta \\
&= c_{pq}x^{m-q-1}((m-q) - m(p+2)) \otimes \partial_y^{p+1}\delta \\
&= c_{pq}x^{m-q-1}(-m(1+p) - q) \otimes \partial_y^{p+1}\delta \\
&= (-m(1+p) - q)c_{pq} \cdot x^{m-q-1} \otimes \partial_y^{p+1}\delta.
\end{aligned}$$

After calculating $\partial_x^2(c_{pq}x^{m-q} \otimes \partial_y^{p+1}\delta)$ (we don't write out the calculations), we hypothesize that the general formula for $1 \leqslant k \leqslant m-q$ and $q \leqslant m$ is

$$\partial_x^k(c_{pq}x^{m-q} \otimes \partial_y^{p+1}\delta) = c_{pq} \cdot ((-1)^k \prod_{i=0}^{k-1}(m(1+p)+q+i)x^{m-q-k}) \otimes \partial_y^{p+1}\delta$$

We see that it holds for the base case $n = 1$ by the calculation above. Assume it holds for $n < k$. Then

$$\begin{aligned}
\partial_x^{n+1}(c_{pq}x^{m-q} \otimes \partial_y^{p+1}\delta) &= \partial_x\partial_x^n(c_{pq}x^{m-q} \otimes \partial_y^{p+1}\delta) \\
&= c_{pq}(-1)^n \prod_{i=0}^{n-1}(m(1+p)+q+i)\partial_x(x^{m-q-n} \otimes \partial_y^{p+1}\delta) \\
&= c_{pq}(-1)^n \prod_{i=0}^{n-1}((m(1+p)+q+i)((m-q-n)x^{m-q-n-1} \otimes \partial_y^{p+1}\delta \\
&\quad +mx^{m-q-n-1} \otimes y\partial_y^{p+2}\delta)) \\
&= c_{pq}(-1)^n \prod_{i=0}^{n-1}((m(1+p)+q+i)(m-q-n)x^{m-q-n-1} \otimes \partial_y^{p+1}\delta \\
&\quad -(p+2)mx^{m-q-n-1} \otimes \partial_y^{p+1}\delta) \\
&= c_{pq}(-1)^n \prod_{i=0}^{n-1}((m(1+p)+q+i)((m-q-n-2m-mp)x^{m-q-n-1} \otimes \partial_y^{p+1}\delta) \\
&= c_{pq}(-1)^{n+1} \prod_{i=0}^{n-1}((m(1+p)+q+i)(m(p+1)+q+n)x^{m-q-n-1} \otimes \partial_y^{p+1}\delta) \\
&= c_{pq}(-1)^{n+1} \prod_{i=0}^{n}((m(1+p)+q+i)x^{m-q-(n+1)} \otimes \partial_y^{p+1}\delta),
\end{aligned}$$

which is what we wanted to show.

Upon applying this, we get that

$$\begin{aligned}
\partial_x^{m(p+1)+q}(1\otimes\delta) &= \partial_x^m(c_{pq}x^{m-q}\otimes\partial_y^{p+1}\delta)\\
&= \partial_x^q\partial_x^{m-q}(c_{pq}x^{m-q}\otimes\partial_y^{p+1}\delta)\\
&= \partial_x^q\left(c_{pq}(-1)^{m-q}\prod_{i=0}^{m-q-1}(m(1+p)+q+i)\otimes\partial_y^{p+1}\delta)\right)\\
&= c_{pq}(-1)^{m-q}\prod_{i=0}^{m-q-1}(m(1+p)+q+i)\partial_x^q(1\otimes\partial_y^{p+1}\delta)
\end{aligned}$$

We want to find a formula for $\partial_x^q(1\otimes\partial_y^{p+1}\delta)$. Notice that

$$\partial_x(1\otimes\partial_y^{p+1}\delta) = mx^{m-1}\otimes\partial_y^{p+2}\delta$$

and (we don't write out the calculations) that

$$\begin{aligned}
\partial_x^2(1\otimes\partial_y^{p+1}\delta) &= -(m^2(p+2)+m)x^{m-2}\otimes\partial_y^{p+2}\delta\\
&= -m(m(p+2)+1)x^{m-2}\otimes\partial_y^{p+2}\delta
\end{aligned}$$

We then have that

$$\begin{aligned}
\partial_x^3(1\otimes\partial_y^{p+1}\delta) &= \partial_x(\partial_x^2(1\otimes\partial_y^{p+1}\delta))\\
&= \partial_x(-(m^2(p+2)+m)x^{m-2}\otimes\partial_y^{p+2}\delta)\\
&= -(m^2(p+2)+m)\partial_x(x^{m-2}\otimes\partial_y^{p+2}\delta)\\
&= -(m^2(p+2)+m)((m-2)x^{m-3}\otimes\partial_y^{p+2}\delta - mx^{m-3}(p+3)\otimes\partial_y^{p+2}\delta)\\
&= -(m^2(p+2)+m)(((m-2)-m(p+3))x^{m-3}\otimes\partial_y^{p+2}\delta)\\
&= -(m^2(p+2)+m)((m(1-p-3)-2)x^{m-3}\otimes\partial_y^{p+2}\delta)\\
&= -(m^2(p+2)+m)((-m(p+2)-2)x^{m-3}\otimes\partial_y^{p+2}\delta)\\
&= m(m(p+2)+1)(m(p+2)+2)x^{m-3}\otimes\partial_y^{p+2}\delta.
\end{aligned}$$

We hypothesize that

$$\partial_x^k(1\otimes\partial_y^{p+1}\delta) = m(-1)^{k-1}\prod_{i=1}^{k-1}(m(p+2)+i)x^{m-k}\otimes\partial_y^{p+2}\delta$$

for $1\leqslant k\leqslant q$, with the convention that $\prod_{i=1}^{0}=1$. By comparison with $\partial_x(1\otimes\partial_y^{p+1}\delta)$ we see that the base case hold.

Assuming it holds for $k < q$, we want to show that it holds for $k+1$. We have

$$\begin{aligned}
\partial_x^{k+1}(1 \otimes \partial_y^{p+1}\delta) &= \partial_x(\partial_x^k(1 \otimes \partial_y^{p+1})) \\
&= \partial_x(m(-1)^{k-1}\prod_{i=1}^{k-1}(m(p+2)+i)x^{m-k} \otimes \partial_y^{p+2}\delta) \\
&= m(-1)^{k-1}\prod_{i=1}^{k-1}(m(p+2)+i)\partial_x(x^{m-k} \otimes \partial_y^{p+2}\delta) \\
&= m(-1)^{k-1}\prod_{i=1}^{k-1}(m(p+2)+i)((m-k)x^{m-k-1} \otimes \partial_y^{p+2}\delta - m(p+3)x^{m-k-1} \otimes \partial_y^{p+2}\delta) \\
&= m(-1)^{k-1}\prod_{i=1}^{k-1}(m(p+2)+i)(((m-k)-m(p+3))x^{m-k-1} \otimes \partial_y^{p+2}\delta) \\
&= m(-1)^{k}\prod_{i=1}^{k-1}(m(p+2)+i)(m(p+2)+k)x^{m-k-1} \otimes \partial_y^{p+2}\delta) \\
&= m(-1)^{k}\prod_{i=1}^{k}(m(p+2)+i)x^{m-(k+1)} \otimes \partial_y^{p+2}\delta) \\
&= m(-1)^{(k+1)-1}\prod_{i=1}^{(k+1)-1}(m(p+2)+i)x^{m-(k+1)} \otimes \partial_y^{p+2}\delta)
\end{aligned}$$

which is what we wanted to show.

Putting this all together, we have that

$$\begin{aligned}
\partial_x^{m(p+1)+q}(1 \otimes \delta) &= \partial_x^m(c_{pq}x^{m-q} \otimes \partial_y^{p+1}\delta) \\
&= c_{pq}(-1)^{m-q}\prod_{i=0}^{m-q-1}(m(1+p)+q+i)\partial_x^q(1 \otimes \partial_y^{p+1}\delta) \\
&= \underbrace{c_{pq}(-1)^{m-q}\prod_{i=0}^{m-q-1}(m(1+p)+q+i)m(-1)^{q-1}\prod_{\ell=1}^{q-1}(m(p+2)+\ell)}_{\in\mathbb{C}}x^{m-q} \otimes \partial_y^{(p+1)+1}\delta,
\end{aligned}$$

where the underbraced number is in $\mathbb{C}$ and nonzero since $m \geqslant 2$, $c_{pq} \in \mathbb{C} \setminus \{0\}$ is nonzero and all numbers in the product besides m are non-negative, together with the convention that $\prod_{i=1}^{0} = 1$. We don't prove this rigorously. Taking the underscored elements as our true c_{pq}, we are done.

(3): Let $q = m$. Then by the formula from (2) we have that

$$\begin{aligned}
&\partial_x^{mp+m}(1 \otimes \delta) = c_{pm} \otimes \partial_y^{p+1}\delta \\
\Leftrightarrow\ &\partial_x^{(p+1)m}(1 \otimes \delta) = c_{pm} \cdot (1 \otimes \partial_y^{p+1}\delta).
\end{aligned}$$

Since c_{pm} is a *non-zero* complex number, $\frac{1}{c_{pm}}$ exists. Since both sides are (identified elements) in the inverse image $F^\star(A_1(\mathbb{C})\delta)$ and the action of $\mathbb{C}[X]$ from the left is *well-defined*, we have that on acting on the left by $\frac{1}{c_{pm}} \in \mathbb{C} \subseteq \mathbb{C}[X]$ we get

$$\frac{\partial_x^{(p+1)m}}{c_{pm}}(1 \otimes \delta) = (1 \otimes \partial_y^{p+1}\delta),$$

which is what we wanted to show.

(4): Note that by (3) we can get any $1 \otimes \partial_y^\ell \delta$ for $\ell \in \mathbb{N}_{\geqslant 1}$ by letting $A_1(\mathbb{C})$ act on the left on $1 \otimes \delta$. Since $\mathbb{C}[X] \subset A_1(\mathbb{C})$ we can also get any elements in the first tensor factor $\mathbb{C}[X]$ by the left-action. But this means that for an arbitrary element $p(x) \otimes \sum_\alpha \partial_y^\alpha \delta$ in $F^\star(A_1(\mathbb{C})\delta)$ we can write it as a finite linear combination $\sum_{i=1}^m c_i(1 \otimes \delta)$ with $c_i \in A_1(\mathbb{C})$.

Remark 0.22. Maybe add something about why y^β does not show up in the second tensor factor - using that $y\partial_y^\alpha = \partial_y^\alpha y - \alpha\partial_y^{\alpha-1}$ so that the first term annihilates, but applied to $y^\beta \partial^\alpha$, i.e. $y^\beta \partial_y^\alpha \delta = y^{\beta-1}(-\alpha\partial_y^{\alpha-1}) = (-\alpha)y^{\beta-2}(-(\alpha-1)\partial_y^{\alpha-2})$ etc, and that $\{y^\beta \partial_y^\alpha \ : \ \beta, \alpha \in \mathbb{N}\}$ is a canonical basis for $A_1(\mathbb{C})$.

Chapter 15

4.1

Let $\iota : K \to K^2, \quad x_1 \mapsto (x_1, 0)$ be the standard embedding. Compute the inverse image under ι of the following A_2-modules:
(1) $A_2/A_2\partial_2$.
(2) $K[x_1, x_2]$.
(3) A_2/A_2x_2.
(4) $A_2/A_2x_2\partial_2$.
(5) $A_2/A_2\partial_2^3$

Notice that ι induces the comorphism

$$\iota^\sharp : K[x_1, x_2] \to K[x_1], \qquad f(x_1, x_2) \mapsto f(x_1, 0).$$

For a general A_2-module, we thus have that $\iota^\star M = K[x_1] \otimes_{K[x_1,x_2]} M$.

(1): We get

$$\iota^\star M = K[x_1] \otimes_{K[x_1,x_2]} A_2/A_2\partial_2$$

where ∂_2 denotes ∂_{x_2}.

By [Cou95, Chap. 15, §1] and with $M := A_2/A_2\partial_2$ we have that $\iota^\star M \cong M/(x_2)M$ as A_1-modules. Since A_2 has a canonical K-basis $\{x_1^{\alpha_1} x_2^{\alpha_2} \partial_1^{\beta} \partial_2^{\beta_2} \ : \ \alpha_i, \beta_i \in \mathbb{N} \text{ for } i = 1, 2\}$ we may write any element in M as a K-linear combinations of elements on the form $\{x_1^{\alpha_1} x_2^{\alpha_2} \partial_1^{\beta_2} \ : \ \alpha_1, \alpha_2, \beta_1 \in \mathbb{N}\}$. Since x_2M then has basis $\{x_1^{\alpha_1} x_2^{\alpha_2} \partial_1^{\beta_1} \ : \ \alpha_1, \alpha_2, \beta_1 \in \mathbb{N} \text{ and } \alpha_2 > 0\}$ we see claim that it follows that $M/(x_2)M$ has a K-basis on the form $\{x_1^{\alpha_1} \partial_1^{\beta_1} \ : \ \alpha_1, \beta_1 \in \mathbb{N}\}$. The $x_1, x_2, \partial_1, \partial_2$ and their exponents stay K-linearly independent in the first quotient, i.e. M and then x_1 and ∂_1 (and their exponents) stay linearly independent in the quotient $M/(x_2)M$. Now, we have an obvious K-linear isomorphism from $M/(x_2)M$ to A_1 by sending (going back to $\overline{x_1}, \overline{\partial_1}$-notation; see remark below) $\Phi : M/(x_2)M \to A_1$ defined on the K-linear basis $\overline{x_1}^{\alpha_1} \overline{\partial_1}^{\beta_1}$ of $M/(x_2)M$ by sending them to $x_1^{\alpha_1} \partial_1^{\beta_1}$ and extending by K-linearity to all of $M/(x_2)M$. We would like to show that this is in fact an A_1-module isomorphism.

Remark 0.23. Note that we write $x_1, x_2, \partial_1, \partial_2$ instead of $\overline{x_1}, \overline{x_2}, \overline{\partial_1}, \overline{\partial_2}$ above (except at the end of the paragraph), but it is clear from the context that it is the second notation that is intended since we are working in the quotient M of A_2 and $M/(x_2)M$.

If we use the action from [Cou95, Chap. 15, §1] of A_1 on $M/(Y)M$ (with $(Y) = (x_2)$) we see that since we may write the K-basis elements of $M/(Y)M$ as $x_1^{\alpha_1}\overline{\partial_1^{\beta_1}}$ we then have

$$\begin{aligned}
\partial_1(x_1^{\alpha_1}\overline{\partial_1^{\beta_1}}) &= \frac{\partial(x_1^{\alpha_1})}{\partial x_1}\overline{\partial_1^{\beta_1}} + x_1^{\alpha_1}\overline{\partial_1^{\beta_1+1}} \\
&= \alpha_1 x^{\alpha_1-1}\overline{\partial_1^{\beta_1}} + x_1^{\alpha_1}\overline{\partial_1^{\beta_1+1}} \\
&= \alpha_1\overline{x_1^{\alpha_1-1}\partial_1^{\beta_1}} + \overline{x_1^{\alpha_1}\partial_1^{\beta_1+1}}
\end{aligned}$$

which under Φ is sent to

$$x_1^{\alpha_1}\partial_1^{\beta_1+1} + \alpha_1 x_1^{\alpha_1-1}\partial_1^{\beta_1}.$$

But notice that in A_1, by using the relations, we have that

$$\partial_1 x_1^{\alpha_1} = x_1^{\alpha_1}\partial_1 + \alpha_1 x_1^{\alpha_1-1}.$$

Thus

$$\begin{aligned}
\partial_1 \cdot (x_1^{\alpha_1}\partial_1^{\beta_1}) &= (\partial_1 x_1^{\alpha_1}) \cdot \partial_1^{\beta_1} \\
&= (x_1^{\alpha_1}\partial_1 + \alpha_1 x_1^{\alpha_1-1}) \cdot \partial_1^{\beta_1} \\
&= x_1^{\alpha_1}\partial_1^{\beta_1+1} + \alpha_1 x_1^{\alpha_1-1}\partial_1^{\beta_1}.
\end{aligned}$$

Thus we see that

$$\Phi(\partial_1(x_1^{\alpha_1}\overline{\partial_1^{\beta_1}})) = \partial_1(\Phi(x_1^{\alpha_1}\overline{\partial_1}^{\beta_1}))$$

and furthermore $x_1^{\alpha_1}\overline{\partial_1}^{\beta_1} = \overline{x_1^{\alpha_1}\partial_1^{\beta_1}}$. Since ∂_1:s action on the generators of $M/(Y)M$ (as a K-module so certainly as a $A_1 \supset K$-module) agrees with the action of ∂_1:s action on the image of said generators under Φ, it follows that Φ is in fact a A_1-module isomorphism.

<u>(2)</u>: We have that $\phi : \iota^\star M = K[x_1] \otimes_{K[x_1,x_2]} \otimes K[x_1, x_2] \cong K[x_1]$ is an isomorphism of $K[x_1, x_2]$-modules, by $p(x) \otimes g(x_1, x_2) \mapsto p(x_1) \cdot g(x_1, x_2) = p(x_1) \cdot g(x_1, 0)$. Hence it is also an isomorphism of $K[x_1]$-modules (it naturally respects the action of the subring $K[x_1] \subset K[x_1, x_2]$). If we can show that this map respects the action of ∂_1 on the tensor product, then ϕ becomes an A_1-module isomorphism. We have that the action of ∂_1 on $p(x_1) \otimes g(x_1, x_2)$ is defined as

$$\partial_1(p(x_1) \otimes g(x_1, x_2)) = \partial_1(p(x_1)) \otimes g(x_1, x_2) + p(x_1) \otimes \partial_1 g(x_1, x_2)$$

where we used the action defined in [Cou95, Chap. 14, eq. (2.1)] with $\iota = (\iota_1, \iota_2)$ such that $\iota_1 : K^2 \to K, \quad (x, y) \mapsto x$ and $\iota_2 : K^2 \to K, \quad (x, y) \mapsto 0$. Furthermore, we have

$$\partial_1(p(x_1)g(x_1, 0)) = \partial_1(p(x_1))g(x_1, 0) + \partial_1(g(x_1, 0))p(x_1)$$

Upon applying ϕ to $\partial_1(p(x_1) \otimes g(x_1, x_2))$ we get back the second expression above, i.e. $\partial_1(\phi(p(x_1) \otimes g(x_1, x_2))$, hence ϕ is indeed an A_1-module isomorphism.

<u>(3)</u>: By [Cou95, Chap. 15, §1] we have that $\iota^\star M \cong M/(x_2)M$ as A_1-modules, where $M = A_2/A_2x_2$.

Consider the element $m = \overline{\partial_2} \in M$. If we let left-multiply by x_2 we get $\overline{x_2\partial_2}$. By the relation $[\partial_2, x_2] = 1$ we see that $x_2\partial_2 = \partial_2 x_2 - 1$ in A_2. But $\partial_2 x_2$ is in A_2x_2, so that $\overline{x_2\partial_2} = \overline{-1} \Leftrightarrow \overline{1} = -\overline{x_2\partial_2}$ is in $(x_2)M$, since $-x_2 \in (x_2)$. But then we have that

$$\overline{-x_2\partial_2 m} = \overline{m}$$

for all $\overline{m} \in M$. But $\overline{-x_2\partial_2 m} = -x_2 \cdot \overline{\partial_2 m}$ which is an element in $(x_2)M$. Thus we find that $M \subset (x_2)M$ and so since $(x_2)M \subset M$ we have that $M = (x_2)M$, so that $M/(x_2)M = M/M = (0)$. Therefore $\iota^\star M \cong (0)$ as an A_1-module.

<u>(4)</u>: Again, by [Cou95, Chap. 15, §1] we have that

$$\iota^\star M \cong M/(x_2)M$$

for $M = A_2/A_2x_2\partial_2$.

By [Cou95, Chap. 13, Cor. 2.3] we have that $M \cong (A_1(x_2,\partial_2)/A_1(x_2,\partial_2)x_2\partial_2) \otimes_K A_1(x_1,\partial_1)$ as left A_1-modules. Denote $A_1(x_2,\partial_2)/A_1(x_2,\partial_2)x_2\partial_2$ by N, so that we have $N \otimes_K A_1(x_1,\partial_1) \cong M$ as left A_1-modules.

Furthermore, we claim that under this isomorphism, we may identify $(x_2)N$ with $(x_2)N \otimes_K A_1$, i.e. the A_1-submodule $(x_2)N \subset N$ is isomorphic as a left A_1-module to $(x_2)N \otimes_K A_1$.

Consider the short exact sequence of A_1-modules

$$0 \to (x_2)N \hookrightarrow N \twoheadrightarrow N/(x_2)N \to 0.$$

Upon tensoring with $-\otimes_K A_1(x_1,\partial_1)$ we claim that we get a short exact sequence of (left) A_1-modules

$$0 \to (x_2)N \otimes_K A_1(x_1,\partial_1) \to N \otimes_K A_1(x_1,\partial_1) \to N/(x_2)N \otimes_K A_1(x_1,\partial_1) \to 0.$$

Thus we have that $(N \otimes_K A_1(x_1,\partial_1))/((x_2)N \otimes_K A_1(x_1,\partial_1)) \cong N/(x_2)N \otimes_K A_1(x_1,\partial_1)$ as A_1-modules. Thus we claim that it follows that

$$M/(x_2)M \cong N/(x_2)N \otimes_K A_1(x_1,\partial_1)$$

as (left) A_1-modules.

Furthermore, we have that

$$N/(x_2)N = (A_1(x_2,\partial_2)/A_1(x_2,\partial_2)x_2\partial_2) \Big/ ((x_2)A_1(x_2,\partial_2)/A_1(x_2,\partial_2)x_2\partial_2).$$

First, we note that $\partial_2^d(x_2\partial_2)$ is in $A_1(x_2,\partial_2)x_2\partial_2$, so that

$$\overline{\partial_2^d(x_2\partial_2)} = \overline{0}$$

in N. But we also have that

$$\begin{aligned}\partial_2^d(x_2\partial_2) &= (x_2\partial_2^d - d\partial_2^{d-1})\partial_2 \\ &= x_2\partial_2^{d+1} - d\partial_2^d\end{aligned}$$

by lemma 0.16. Consider the quotient map $\pi : N \twoheadrightarrow N/(x_2)N$. There $\overline{\partial_2^d(x_2\partial_2)} = \overline{0}$ is sent to $\overline{0}$ since this is a homomorphism. But π kills $\overline{x_2\partial_2^{d+1}}$ and hence $-d\overline{\partial_2^d} = \overline{0}$ in $N/(x_2)N$. This is a left K-module and so we may left-multiply both sides of the equation by $\frac{1}{d}$ which gives us that $\overline{\partial_2^d} = 0$ in $N/(x_2)N$ for $d \geqslant 1$. On the other hand we also have that $\overline{x_2^d}$ for $d \geqslant 1$ is zero in $N/(x_2)N$. But $N/(x_2)N$ is induced from $A_1(x_2,\partial_2)$ which has a K-basis $x_2^a\partial_2^b$ and they are all killed in $N/(x_2)N$. Thus what remains is K, which is untouched by any quotients on the way from $A_1(x_2,\partial_2)$ to $N/(x_2)N$. Thus $N/(x_2)N = K$. Therefore we have that

$$\begin{aligned}\iota^\star M &\cong M/(x_2)M \\ &\cong N/(x_2)N \otimes_K A_1(x_1,\partial_1) \\ &= K \otimes_K A_1(x_1,\partial_1) \\ &\cong A_1(x_1,\partial_1)\end{aligned}$$

The first two isomorphisms we have already argued for are (left) A_1-module isomorphisms, and the last one is explicitly defined by $k \otimes a \mapsto ka$. Since $k \in K$ is in the center of $A_1(x_1, \partial_1)$ we claim that this is fact also a (left) $A_1(x_1, \partial_1)$-module isomorphism. Going back and being explicit about all the isomorphisms, we see that they in fact all act by $A_1(x_1, \partial_1)$ and not by $A_1(x_2, \partial_2)$, so that the isomorphisms are all compatible action-wise. This concludes this problem.

(5): Again, using [Cou95, Chap. 13, Cor. 2.3] we have that

$$M \cong (A_1(x_2.\partial_2)/A_1(x_2, \partial_2)\partial_2^3) \otimes_K A_1(x_1, \partial_1)$$

as left A_1-modules.

By similar seasoning as in (4), and with $N = A_1(x_2, \partial_2)/A_1(x_2, \partial_2)\partial_2^3$ we get that

$$\begin{aligned} \iota^\star M &\cong M/(x_2)M \\ &\cong N/(x_2)N \otimes_K A_1(x_1, \partial_1). \end{aligned}$$

It remains to compute $N/(x_2)N$. Note that what survives in N (from A_1) is elements in the PBW-basis on the form $x_2^a\partial_2^b$ with $a \geqslant 0$ and $b \leqslant 2$. Upon going to the quotient $N/(x_2)N$ we also kill (x_2). Hence we is left is $1, \partial_2, \partial_2^2$ and K. This is isomorphic to K^3 since we claim that $1, \partial_2, \partial_2^2$ are linearly independent in $N/(x_2)N$. Thus we see that

$$\begin{aligned} N/(x_2)N \otimes_K A_1(x_1, \partial_1) &\cong K^3 \otimes_K A_1(x_1, \partial_1) \\ &\cong A_1(x_1, \partial_1)^{\oplus 3}. \end{aligned}$$

4.6

Show that if $\iota : X \to X \times Y$ is the standard embedding, then

$$\iota^\star(A_{n+m}) = K[\partial_y]\hat{\otimes}A_n,$$

where $K[\partial_y]$ is the left A_m-module $K[\partial_{y_1}, \dots, \partial_{y_m}]$.

By [Cou95, Chap. 15, §1] we have that $\iota^\star(A_{n+m}) \cong A_{n+m}/(Y)A_{n+m}$ where $(Y) = (y_1, \dots, y_m)$ as left A_n-modules. By [Cou95, Chap. 13, Cor. 2.3] we have that

$$A_{m+n}/(Y)A_{m+n} \cong (A_m/(Y)) \otimes_K A_n$$

as an isomorphism of (left) A_n-modules (since it is a left isomorphism of A_{m+n}-modules and $A_n \subset A_{m+n}$. But $A_m/(Y) \cong K[\partial_{y_1}, \dots, \partial_{y_m}]$).

Tensoring this with $- \otimes_K A_n$ and noting that $- \otimes_K A_n$ is functorial (hence preserves isomorphisms), we get that $A_m/(Y) \otimes_K A_n \to K[\partial_y] \otimes_K A_n$ is an isomorphism. This is an A_n-module isomorphism with multiplication in the right-tensor factor of the domain and codomain. This A_n-module structure is compatible with the A_n-module isomorphism $A_{m+n}/(Y)A_{m+n} \cong (A_m/(Y)) \otimes_K A_n$ and so we see that

$$\begin{aligned} \iota^\star M &\cong K[\partial_y] \otimes_K A_n \\ &= K[\partial_y]\hat{\otimes}_K A_n, \end{aligned}$$

as left A_n-modules.

Chapter 16

4.1

Let R be a K-algebra and and τ_1, τ_2 be transpositions of R. Show that $\tau_1(\tau_2)^{-1}$ is an automorphism of R.

Notice that since τ_2 is a transposition, we have that $\tau_2^{-1} = \tau_2$, hence $\tau_1(\tau_2)^{-1} = \tau_1\tau_2$. Then we see that $\tau_2\tau_1$ is a two-sided inverse to $\tau_1\tau_2$, since $\tau_1^2 = \tau_2^2 = \mathrm{id}_R$ as a K-linear map. It remains to show that it respects multiplication in R. We have that, for $r, s \in R$

$$\begin{aligned}\tau_1\tau_2(rs) &= \tau_1(\tau_2(s)\tau_2(s)) \\ &= \tau_1(\tau_2(r)) \cdot \tau_1(\tau_2(s))\end{aligned}$$

where we used that $\tau_i(rs) = \tau_i(s)\tau_i(r)$ for $i = 1, 2$.

4.4

Let $F : X \to Y$ be a polynomial map and M a left A_n-module. Then show that

$$F_\star M \cong (M^t \otimes_{A_n} D_{X\to Y})^t.$$

By [Cou95, Lemma 16.2.2] we have that

$$\begin{aligned}(M^t \otimes_{A_n} D_{X\to Y})^t &\cong (D_{X\to Y})^t \otimes_{A_n} (M^t)^t \\ &= D_{Y\leftarrow X} \otimes_{A^n} M,\end{aligned}$$

where we used that $(M^t)^t = M$ (see [Cou95, Chap. 16, §2]) and that $D_{Y\leftarrow X} := (D_{X\to Y})^t$ ([Cou95, Chap. 16, §3]). But for left A_n-modules M, $F_\star M$ is defined as $D_{Y\leftarrow X} \otimes_{A_n} M$ ([Cou95, Chap. 16, §3]), which is what we wanted to show.

4.5

Let $F : X \to Y$ be a polynomial map. Define $G : X \times Y \to X \times Y$ by $G(X, Y) = (X, Y + F(X))$. Let M be a left A_{n+m}-module. Show that

$$G_\star G^\star M = M = G^\star G_\star M.$$

By [Cou95, Theorem 15.3.1] we have that if M is a left A_{m+n}-module then $G^\star M \cong M_\sigma$. Notice that $G_\star M$ is a left A_{n+m}-module isomorphic to $M_{\sigma^{-1}}$ by [Cou95, Prop. 16.3.2]. Since the action in M_σ is defined as $a \bullet m = \sigma(a) \cdot m$ for $m \in M$ and $a \in A_{n+m}$, we have that the action in $(M_\sigma)_{\sigma^{-1}}$ is defined as

$$\begin{aligned}a \diamond m &= \sigma^{-1}(a) \bullet m \\ &= \sigma(\sigma^{-1}(a)) \cdot m \\ &= a \cdot m,\end{aligned}$$

i.e. we get back M. We similarly get $(M_{\sigma^{-1}})_\sigma = M$.

Since both $G^\star$ and $G_\star$ are defined in terms of tensors, we claim that they must be functorial, hence preserves isomorphisms. Thus, we have

$$\begin{aligned} G^\star M &\cong M_\sigma \\ \Rightarrow G_\star(G^\star M) &\cong G_\star M_\sigma \\ &\cong (M_\sigma)_{\sigma^{-1}} \\ &= M. \end{aligned}$$

The other identity follows similarly.

Remark 0.24. We have only shown isomorphism and not identity, but we believe [Cou95, Chap. 16, Exc. 4.5] are using $=$ in a "sloppy" way.

Chapter 17

Preliminary version of Kashiwara's theorem

Lemma 0.25. *Let M be a left A_{n+1}-module, and let H be the hyperplane $y = 0$, and denote by $\iota : X \to X \times K$ the standard embedding. The A_{n+1}-modules $\iota_\star(\ker_M y)$ and $\Gamma_H M$ are isomorphic.*

Remark 0.26. By "standard embedding" we mean here that ι is defined explicitly as $x \mapsto (x, 0)$.

Denote $\ker_M y = M_0$. Since this is a left A_n-module, by [Cou95, Chap. 17, §1] we have that

$$\iota_\star(M_0) \cong K[\partial_y]\hat{\otimes} M_0.$$

We wish to show that the right-hand side above is isomorphic to

$$\Gamma_H M = \{u \in M \ : \ \exists k \geqslant 1 \text{ such that } y^k u = 0\}.$$

Consider the following diagram:

$$\begin{tikzcd} K[\partial_y] \times M_0 \arrow[rr, "i"] \arrow[dr, "\varphi"'] & & K[\partial_y]\hat{\otimes} M_0 \arrow[dl, dotted, "\phi"] \\ & \Gamma_H M & \end{tikzcd}$$

with $\varphi(f, u) = fu$. Notice that $M_0 \subset \Gamma_H M$. Hence any $u \in M_0$ is in $\Gamma_H M$. Furthermore, we have that if $f = \sum_{j=0}^{d} a_j \partial_y^j$ then

$$\begin{aligned} y^{d+1} f u &= \sum_{j=0}^{d} a_j y^{d+1} \partial_y^k u \\ &= 0 \end{aligned}$$

by [Cou95, Chap. 17. eq. (2.3)].

That φ is K-bilinear is a direct consequence of $\Gamma_H M$ being a A_{n+1}-submodule (so indeed a K-submodule) of M.

Thus, by the universal property of the tensor product, we get a map ϕ defined on pure tensors $f \otimes u$ as $\phi(f \otimes u) = fu$. By [Cou95, Lemma 17.2.2.(3)] the image of ϕ is $A_{n+1}M_0$, and its kernel is zero by the same lemma, since the sum is direct. To be a bit more precise here: Consider an arbitrary element $\sum_{j=0}^{d} p_j(\partial_y) \otimes u_j$.

We may rewrite each $p_j \otimes u_j$ on the form

$$\sum_{\ell(j)} \partial_y^{\ell(j)} \otimes u'_j$$

by moving over the coefficients of p_j into the second tensor factor. Thus we have

$$\begin{aligned}\sum_{j=0}^{d} p_j \otimes u_j &= \sum_{j=0}^{d} \left(\sum_{\ell(j)} \partial_y^{\ell(j)} \otimes u'_j \right) \\ &= \sum_{k=0}^{d'} \partial_y^k \otimes u''_k\end{aligned}$$

where $d' = \max\{\deg(p_0), \ldots, \deg(p_d)\}$. If this is sent to 0 this means that

$$\sum_{k=0}^{d'} \partial_y^k u''_k = 0$$

which implies that (since the sum is direct) $u''_k = 0$ for $k = 0, \ldots, d'$, which in turn implies that $\sum_{j=0}^{d} p_j \otimes u_j = 0$.

In fact, we claim that the induced map ϕ is A_{n+1}-bilinear (note that M is a left A_{n+1}-module and $\Gamma_H M$ is a submodule of M): The structure of the action is by $A_{n+1} := A_1 \hat{\otimes} A_n$ (since M_0 is an A_n-module but not an A_{n+1}-module [it is not closed under the action of ∂_y)]). We put an action of of A_1 on $K[\partial_y]$ by multiplication by ∂_y and $yf(\partial_y) = -f'(\partial_y)$ (as far as we can tell, this is the structure of the A_n-module action on $K[\partial]$ given in [Cou95, Chapter 5, §1]).

Then since the ∂_{x_i}, x_i commutes with ∂_y^k and $a_j \in K$ it is easy to see that ϕ is linear in the second tensor factor. It also direct that with respect to ∂_y it is linear in the first factor. Lastly, for y, we have that

$$\begin{aligned}\phi(yf \otimes u) &= \phi(-f' \otimes u) \\ &= -f'u\end{aligned}$$

while

$$\begin{aligned}y \cdot fu &= y \cdot \left(\sum_{j=0}^{d} a_j \partial_y^j u \right) \\ &= -\sum_{j=1}^{d} a_j \partial_y^{j-1} u\end{aligned}$$

where we used that $[y, \partial_y^j] = -j\partial_y^{j-1}$ so that

$$y\partial_y^j = -j\partial_y^{j-1} + \partial_y^j y$$

and that the second term in the right-hand side annihilates every u in M_0. But $-\sum_{j=1}^{d} a_j \partial_y^{j-1} u = -f'(\partial_y)u$ so indeed this is linear with respect to y.

If we can show that $\Gamma_H M \subset A_{n+1} M_0$, then we are done, since then ϕ is a bijective A_{n+1}-module isomorphism.

Let $u \in \Gamma_h M$ such that $y^k u = 0$. We want to show that $u \in A_{n+1}M_0$. We proceed by induction: if $k = 1$ then $yu = 0$ so indeed $u \in M_0$ by definition of M_0. Assume it holds for $k-1$ that if $y^{k-1}u = 0$ for $u \in \Gamma_H M$ then $u \in A_{n+1}M_0$. Assume that $y^k u = 0$ for $u \in \Gamma_H M$. Then $\partial_y(y^k u) = 0$. Consider that

$$\begin{aligned}
[\partial_y, y^k] &= -[y^k, \partial_y] \\
&= ky^{k-1} \\
\Leftrightarrow \partial_y y^k &= y^k \partial_y + ky^{k-1} \\
\Rightarrow y^{k-1}(y\partial_y u + ku) &= 0.
\end{aligned}$$

Since $u \in \Gamma_H M$ which is a left A_{n+1}-submodule we have that $y\partial_y u + ku \in \Gamma_H M$ and so by the inductive assumption it follows that $y\partial_y u + ku \in A_{n+1}M_0$. But since $y^k u = y^{k-1}(yu) = 0$ we have that (again by the inductive assumption) yu is in $A_{n+1}M_0$, so also $-\partial_y yu$ is in $A_{n+1}M_0$ since $-\partial_y \in A_{n+1}$. Since $A_{n+1}M_0$ is an A_{n+1}-submodule of M (by definition) we see that

$$ku + y\partial_y u - \partial_y yu \in A_{n+1}M_0.$$

But this is the same as

$$\begin{aligned}
ku + [y, \partial_y]u &= ku - u \\
&= (k-1)u \in A_{n+1}M_0 \\
&\Rightarrow u \in A_{n+1}M_0.
\end{aligned}$$

By induction, it follows that $\Gamma_H M \subset A_{n+1}M_0$ so that $\Gamma_H M = A_{n+1}M_0$. Thus

$$\Gamma_H M \cong K[\partial_y]\hat{\otimes} M_0$$

as A_{n+1}-modules.

Problems

Problem 3.3

Let $\iota : X \to X \times K$ be the standard embedding and let M be a left A_{n+1}-module with support on the hyperplane H of equation $y = 0$. Show that $\ker_M y$ is isomorphic, as a left A_n-module, to

$$\mathrm{Hom}_{A_{n+1}}(D_{X\times K \leftarrow X}, M)$$

By results from [Cou95, Chapter 17, §1] we have that

$$D_{X\times K\leftarrow X} \cong A_{n+1}/A_{n+1}y \tag{0.7}$$

This means that $\mathrm{Hom}_{A_{n+1}}(D_{X\times K\leftarrow X}, M) \cong \mathrm{Hom}_{A_{n+1}}(A_{n+1}/A_{n+1}y, M)$ by pre-composition with the isomorphism in equation 0.7.

Notice that any A_{n+1}-module homomorphism $\psi : A_{n+1}/A_{n+1}y \to M$ is determined by where it sends $\overline{1}$. Let $u \in \ker_M y$. Then if we define $\psi(1) = u$, and $\psi(r) = ru$ by extension, then we see that if $a = b$ then $\psi(a - b) = (a - b)u = 0$ since $a - b \in A_{n+1}y$ and u is annihilated by y. Thus ψ is well-defined. Denote each such ψ with ψ_u. Then we find that

$$\begin{aligned}
\psi_u(r \cdot a) &= rau \\
&= r \cdot (au) \\
&= r \cdot \psi_u(a).
\end{aligned}$$

Thus, each $u \in \ker_M y$ determines an A_{n+1}-module homomorphism from $A_{n+1}/A_{n+1}y$ to M. Observe that if $v \notin \ker_M y$ then if $a = b$ so that $a - b \in A_{n+1}y$, i.e. ay for $a \in A_{n+1}$, then $ayv = a(yv)$ needs to equal zero for this to be well-defined. Now take any element $a \in A_{n+1}$ and let $b = a + y$. Then we have that $a = b$ in the quotient ring, and $a - b = y \equiv 0$, but this would force $yv = 0$, which is impossible since $v \notin \ker_M y$. Therefore, there is a bijective map $\Psi : \mathrm{Hom}(A_{n+1}/A_{n+1}y, M) \to \ker_M y$ defined by $\Psi(\psi_u) = \psi(\overline{1})$.

It is then enough to show that the map Ψ is A_n-linear. The A_n-module structure on $A_{n+1}/A_{n+1}y$ is by right-multiplication. Define a *left* A_n-module structure on $\mathrm{Hom}_{A_{n+1}}(A_{n+1}/A_{n+1}y, M)$ by $r \cdot \psi(t) := \psi(t \cdot r)$. That this indeed an action follows from the fact that it is the induced action from $A_{n+1}/A_{n+1}y$ (it is routine to check that it fulfills the criteria for being a left action on $\mathrm{Hom}(A_{n+1}/A_{n+1}y, M)$).

Claim: With the action defined as above, we have that the map $\Psi : \mathrm{Hom}(A_{n+1}/A_{n+1}y, M) \to \ker_M y, \quad \psi \mapsto \psi(\overline{1})$ is A_n-linear.

Proof. Observe that for $r \in A_n$, and $\psi(\overline{1}) = u \in \ker_M y$, we have

$$\begin{aligned}\Psi(r \cdot \psi) &= \psi(r) \\ &= ru \\ &= r \cdot \psi(\overline{1}) \\ &= r \cdot \Psi(\psi).\end{aligned}$$

□

Remark 0.27. It does not seem that we used the fact (perhaps implicitly, when defining $D_{X \times K \leftarrow X}$) M was supported on the hyperplane of equation $y = 0$.

Problem 3.6

Let M be an irreducible A_n-module. For a non-zero element u in M let $J(u) = \{f \in K[X] : fu = 0\}$. Show that:
(1) If $0 \neq v \in M$ then $\mathrm{rad}J(v) = \mathrm{rad}J(u)$.
(2) $\mathrm{rad}J(u)$ is a prime ideal of $K[X]$.
(3) If M is supported on the hypersurface $x_n = 0$ then $\mathrm{rad}J(u)$ is contained in the ideal of $K[X]$ generated by x_n.

Remark 0.28. We interpret *irreducible* module to mean *simple* module, i.e. there does not exist any proper non-zero submodules of M.

(1): Let $U := \mathrm{rad}J(u)$ and $V := \mathrm{rad}J(v)$. Note that $U = \{f \in K[X] : f^n u = 0 \text{ for some } n \geqslant 1\}$ and that V is defined similarly.

By [Cou95, Problem 5.4.3] we have that M is cyclic, i.e. $M = A_n\langle m\rangle$ for some $m \in M$. In particular, since M is *irreducible*, we must have that $A_n \cdot u = A_n \cdot v = M$, since otherwise (due to $u, v \neq 0$) we get non-trivial proper submodules of M. This means that there is some $r_u \in A_n$ and $r_v \in A_n$ such that $v = r_u u$ and $u = r_v v$.

$U \subseteq V$: Assume that $f \in K[X]$ such that $f^n u = 0$ for some $n \geqslant 1$. Consider $v = r_u u$. We then have

$$\begin{aligned}f^n r_u u &= (r_u f^n + [f^n, r_u])u \\ &= [f^n, r_u]u.\end{aligned}$$

Let $\mathsf{ad}_f(a) = [f, a]$. Then $\mathsf{ad}_f(\cdot) = [f, \cdot]$ is a derivation on A_n, i.e.

$$\begin{aligned}
\mathsf{ad}_f(ab) &= [f, ab] \\
&= f(ab) - (ab)f \\
&= (fa - af)b + a(fb - bf) \\
&= [f, a]b + a[f, b] \\
&= \mathsf{ad}_f(a)b + a\mathsf{ad}_f(b),
\end{aligned}$$

for any $a, b \in A_n$.

Lemma 0.29. *For f in $K[X]$ and $a \in A_n$, we have that*

$$f^n a = \sum_{i=0}^{n} \binom{n}{j} (\mathsf{ad}_f)^i(a) f^{n-j}$$

Proof. For the base case $n = 1$, we have that $fa = af + \mathsf{ad}_f(a)$, which agrees with the formula. Assume this holds for $n = k$, we want to show that it holds for $n = k + 1$. We then have that

$$\begin{aligned}
f^{k+1}a &= f(f^k a) \\
&= f\left(\sum_{i=0}^{k} \binom{k}{i} (\mathsf{ad}_f)^i(a) f^{k-i}\right) \quad \text{(by the inductive step)} \\
&= \sum_{i=0}^{k} \binom{k}{i} f(\mathsf{ad}_f)^i(a) f^{k-i}
\end{aligned}$$

Notice that

$$\begin{aligned}
f(\mathsf{ad}_f)^i(a) - (\mathsf{ad}_f)^i(a)f &= [f, (\mathsf{ad}_f)^i(a)] \\
\Rightarrow f(ad_f)^i(a) &= (ad_f)^i(a)f + [f, (\mathsf{ad}_f)^i(a)] \\
&= (ad_f)^i(a)f + (\mathsf{ad}_f)^{i+1}(a).
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
\sum_{i=0}^{k} \binom{k}{i} f(\mathsf{ad}_f)^i(a) f^{k-i} &= \sum_{i=0}^{k} \binom{k}{i} (\mathsf{ad}_f)^i(a) f^{k+1-i} + \sum_{i=0}^{k} \binom{k}{i} (\mathsf{ad}_f)^{i+1}(a) f^{k-i} \\
&= \sum_{i=0}^{k} \binom{k}{i} (\mathsf{ad}_f)^i(a) f^{k+1-i} + \sum_{j=1}^{k+1} \binom{k}{j-1} (\mathsf{ad}_f)^j f^{k-j+1} \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} (\mathsf{ad}_f)^i(a) f^{k+1-i}
\end{aligned}$$

where we in th next to last step changed the index in the second sum to $j = i + 1$, and used that $\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}$. □

We see, that with respect to the order defined in [Cou95, Chapter 3, §1], and with $\mathrm{ord}(r_v) = m$, if we set $N := n + m + 1$, then we claim that by lemma 0.29 we have that $f^N v = f^N r_u u = 0$. To see this, note that

$$f^N r_u u = \sum_{i=0}^{n+m+1} \binom{n+m+1}{i} (\mathsf{ad}_f)^i(r_u) f^{n+m+1-i} u.$$

If $0 \leqslant i \leqslant m+1$ then this is zero since then $f^{n+m+1-i}u = 0$. If $i > m+1$ then $(\mathsf{ad}_f)^i(r_u) = 0$ since we assumed that r_u was of order m, which (as far as we can tell) means that (inductively) $(\mathsf{ad}_f)^i(r_u)$ is of order at most D^{m-i}. So for $i = m$, $(\mathsf{ad}_f)^i(r_u)$ is of order zero, which means that $(ad_f)^{m+1}(r_u) = 0$, by definition.

$\underline{V \subseteq U}$: Proved similarly as the other case.

We conclude that indeed $U = V$.

(2): Recall the denotation $U = \mathrm{rad}J(u)$. Since $K[X]$ is a commutative ring, it is closed under multiplication by elements from $K[X]$. If $r_1, r_2 \in U$ such that there are positive integers α_1, α_2 so that $r_1\alpha_1$ and $r_2^{\alpha_2}$ annihilate u, if we choose $N = \alpha_1 + \alpha_2 + 1$ then by the binomial formula we have that

$$(r_1 - r_2)^N = \sum_{i=0}^{\alpha_1+\alpha_2+1} \binom{\alpha_1 + \alpha_2 + 1}{k} r_1^{\alpha_1+\alpha_2+1-i}(-r_2)^i.$$

If $0 \leqslant i \leqslant \alpha_2 + 1$ then this indeed annihilates u since $K[X]$ is commutative. If $i > \alpha_2 + 1$ then $(-r_2)^i = (-1)^i(r_2)^i$ annihilates u, hence so does the product. Thus we find that $r_1 - r_2 \in U$. Since also $0 \in U$, it is indeed an ideal (Ideal test).

Assume that $fg \in U$ so that there is some n such that $(fg)^n u = 0$. Assume that neither f nor g is in U. Then $g^n u \neq 0$ and $f^n u \neq 0$ but $g^n(f^n u) = 0$ and $f^n(g^n u) = 0$. This means that $f \in \mathrm{rad}J(g^n u)$ and $g \in \mathrm{rad}J(f^n u)$, but by (1), this is the same as $f, g \in U$, contradiction! Therefore either f or g is in U.

(3): This does not seem to be true (atleast if interpreted as *left* irreducible A_n-module): Consider the following (proposed) counterexample: Take $N = A_n / \sum_{i=1}^n A_n x_i$, with the induced A_n-action on N from the canonical A_n-module structure on A_n.

Then any element in N may be written as $\sum_{\alpha\in\mathbb{N}^n} c_\alpha \partial^\alpha$ with $c_\alpha \in K$.

$\underline{N \text{ is irreducible as an } A_n\text{-module}}$: This follows from the fact that M_σ is irreducible if and only if M is irreducible ([Cou95, Prop. 2.2.1.(1)]) and that $K[\partial]$ is the Fourier transform of $K[X]$ ([Cou95, Prop. 2.2.2]), and $K[X] \cong A_n / \sum_{i=1}^n A_n \partial_i$ is an irreducible A_n-module, i.e. $M := A_n / \sum_{i=1}^n A_n \partial_i$ is an irreducible A_n-module, and $\mathcal{F} : A_n \to A_n$ together with the above then gives that $M_{\mathcal{F}} = A_n / \sum_{i=1}^n A_n x_i$ is irreducible, since $K[X]$ is, and isomorphism of A_n-modules preserves irreducibility, since the image of a non-zero proper submodule give a non-zero proper submodule under an isomorphism.

$\underline{N \text{ is supported on the hyperplane } x_n = 0}$:

Lemma 0.30. $[x_n, \partial_n^m] = -m\partial_n^{m-1}$.

Proof. For the base case $n = 1$ we have that $[\partial_n, x_n] = 1$ by the Weyl-relations, therefore $[x_n, \partial_n] = -1 = -1\partial_n^0$, so that the equation hold. Assume it holds for $n = m$. Then

$$\begin{aligned} [x_n, \partial_n^{m+1}] &= [x_n, \partial_n]\partial_n^m + \partial_n[x_n, \partial_n^m] \\ &= -\partial_n^m + \partial_n(-m\partial_n^{m-1}) \\ &= -\partial_n^m - m\partial_n^m \\ &= -(m+1)\partial_n^m, \end{aligned}$$

so that it holds for $n = m + 1$, where we used that $[A, BC] = [A, B]C + B[A, C]$. □

Therefore, by the lemma, we have that

$$\begin{aligned} &[x_n, \partial_n^m] = -m\partial_n^{m-1} \\ \Leftrightarrow\ &x_n\partial_n^m = \partial_n^m x_n - m\partial_n^{m-1}. \end{aligned}$$

Therefore, in N, we have that $x_n\partial_n^m = -m\partial_n^{m-1}$ (since $\partial_n^m x_n \equiv 0$ in N). Hence, for x_n^{m+1} we have that $x_n^{m+1}\partial_n^m \equiv 0$ in N. For $j \neq n$ we have that $x_n\partial_j^m = \partial_j^m x_n \equiv 0 \in N$. It follows that powers of x_n annihilates everything in N, i.e. $\Gamma_H N = N$, so that N is supported on the hyperplane $x_n = 0$.

We may show similarly as for $\Gamma_H N = N$ that powers of x_i for $i = 1, \ldots, n$ annihilates N, so that for any $u \neq 0$ in N we have that $(x_1, \ldots, x_n) \subset \operatorname{rad} J(u)$. Therefore, since $(x_n) \not\supset (x_1, \ldots, x_n)$ we can not have that $\operatorname{rad} J(u) \subset (x_n)$.

Chapter 18

18.1: Notes on [Cou95, Theorem 18.3.1]

We want to check that the natural isomorphisms $\eta :_{\mathrm{id}_{\mathcal{M}^n}} \Rightarrow \mathcal{K} \circ \iota_*$ and $\varepsilon : \iota_* \circ \mathcal{K} \Rightarrow \mathrm{id}_{\mathcal{M}^{n+1}(H)}$ are indeed natural maps, i.e. that their associated squares commutes with respect to maps in their respective categories.

We will presume that ι_* acts on maps $f : N \to N'$ by $\iota_* f : D \otimes_{A_n} N \to D \otimes_{A_n} N', \quad d \otimes n \mapsto d \otimes f(n)$ and that $\mathcal{K}$ acts on maps $g : M \to M'$ by

$$\mathcal{K}(g) : \ker_M(y) \to \ker_{M'}(y),$$

which is explicitly defined by restricting $g : M \to M'$ to $g|_{\ker_M(y)}$. Observe that if $u \in \ker_M(y)$ then since g is A_{n+1}-linear we have that $0 = g(yu) = y \cdot g(u) = 0$ so that the image of the restricted g maps to $\ker_{M'}(y)$.

Define $\eta_N : N \to \mathcal{K}(\iota_* N)$ by

$$N \mapsto \ker_{D_{X\times K\leftarrow X}\otimes_{A_n} N}(y).$$

If we let $e \in D := D_{X\times K\leftarrow X}$ be the element corresponding to $\overline{1} \in A_{n+1}/A_{n+1}y$ under the isomorphism $\iota_* N \cong A_{n+1}/A_{n+1}y$ where we recall that $\iota_* N = D \otimes_{A_n} N$ then we define η_N explicitly as $n \mapsto e \otimes n$. We see that $e \otimes n$ is in the kernel $\ker_{D\otimes_{A_n} N}(y)$ since $y \cdot (e \otimes n) = ye \otimes n = 0 \otimes n = 0$.

Let $f : N \to N'$ be a morphism in the category $\mathcal{M}^n$. Consider the following diagram:

$$\begin{tikzcd}
N \arrow[r, "\eta_N"] \arrow[d, "f"'] & \mathcal{K}(\iota_* N) \arrow[d, "\mathcal{K}(\iota_* f)"] \\
N' \arrow[r, "\eta_{N'}"'] & \mathcal{K}(\iota_* N')
\end{tikzcd}$$

We see that

$$(\eta_{N'} \circ f)(n) = e \otimes f(n)$$

and that

$$\begin{aligned}
(\mathcal{K}(\iota_* f) \circ \eta_N)(n) &= \mathcal{K}(\iota_* f)(e \otimes n) \\
&= (\mathrm{id}_D \otimes f)|_{\ker_{D\otimes_{A_n} N}(y)}(e \otimes n) \\
&= (\mathrm{id} \otimes f)(e \otimes n) \qquad (\text{since } e \otimes n \in \ker_{D\otimes_{A_n} N}(y)) \\
&= e \otimes f(n).
\end{aligned}$$

Define $\varepsilon_M : \iota_*(\mathcal{K}(M)) \hookrightarrow M$ as the map $\phi(d \otimes m) = d \cdot m$ as in [Cou95, Theorem 17.2.4]. Here we identify $d \in D$ with some $p \in A_{n+1}$ such that p is sent to d under the map $\pi : A_{n+1} \twoheadrightarrow D$ (with kernel $A_{n+1}y$). Because $m \in \ker_M(y)$ we claim this is well-defined. Furthermore, observe that for $g : M \to M'$ in $\mathcal{M}^{n+1}(H)$ we have that

$$\begin{aligned}\iota_*(\mathcal{K}(g)) &= \iota_*(g|_{\ker_M(y)}) \\ &= \mathrm{id}_D \otimes g|_{\ker_M(y)},\end{aligned}$$

where

$$\mathrm{id}_D \otimes g|_{\ker_M(y)} : D \otimes_{A_n} \ker_M(y) \to D \otimes_{A_n} \ker_{M'}(y).$$

We get the following diagram:

$$\begin{tikzcd} \iota_*(\mathcal{K}(M)) \arrow[r, "\varepsilon_M"] \arrow[d, "\iota_*(\mathcal{K}(g))"'] & M \arrow[d, "g"] \\ \iota_*(\mathcal{K}(M')) \arrow[r, "\varepsilon_{M'}"'] & M' \end{tikzcd}$$

We see that

$$\begin{aligned}(g \circ \varepsilon_m)(d \otimes m) &= g(d \cdot m) \\ &= d \cdot g(m),\end{aligned}$$

where we used that g is A_{n+1}-linear, and that

$$\begin{aligned}(\varepsilon_{M'} \circ \iota_*(\mathcal{K}(g)))(d \otimes m) &= (\varepsilon_{M'} \circ (\mathrm{id}_D \otimes g|_{\ker_M(y)}))(d \otimes m) \\ &= \varepsilon_{M'}(d \otimes g(m)) \\ &= d \cdot g(m),\end{aligned}$$

so that they agree.

0.2 18.2: Short note on [Cou95, Cor. 18.3.2]

In the proof of [Cou95, Cor. 18.3.2] there are (to us) one important detail not worked out. In particular, it seems that one may have to use the following lemma for the proof to go through:

Lemma 0.31. *Assume that $F : \mathcal{C} \to \mathcal{D}$ with quasi-inverse $G : \mathcal{D} \to \mathcal{C}$ defines an equivalence of categories, and that $\mathcal{C}'$ and $\mathcal{D}'$ are* full *subcategories of $\mathcal{C}$ and $\mathcal{D}$ respectively, such that $F(\mathcal{C}') \subseteq \mathcal{D}'$ and $G(\mathcal{D}') \subseteq \mathcal{C}'$. Then F restricts to an equivalence of categories $F|_{\mathcal{C}'} : \mathcal{C}' \to \mathcal{D}'$ with quasi-inverse $G|_{\mathcal{D}'} : \mathcal{D}' \to \mathcal{C}'$.*

One may then apply this to see that $(\iota_2)_*|_{\mathcal{M}^{m+n-1}(H_1 \cap \ldots \cap H_{m-1})} : \mathcal{M}^{m+n-1}(H_1 \cap \ldots \cap H_{m-1}) \to \mathcal{M}^{m+n}(H_1 \cap \ldots \cap H_m)$ is an equivalence of categories, with

$$\mathcal{C}' := \mathcal{M}^{m+n-1}(H_1 \cap \ldots \cap H_{m-1})$$

and

$$\mathcal{D}' := \mathcal{M}^{m+n}(H_1 \cap \ldots \cap H_m)$$

in the notation of lemma 0.31. For this application to work out, one have to show that $\mathcal{K}$ restricted to $\mathcal{D}'$ is such that its image is in $\mathcal{C}'$. Here $\mathcal{K}(M)$ will be defined as $\ker_M(y_m)$ which is indeed in $\mathcal{M}^{m+n-1}$, and furthermore since $M \in \mathcal{D}'$ M is supported in $H_1 \cap \ldots \cap H_m$ and this is transported to the A_{m+n-1}-module $\ker_M(y_m)$ so that indeed $\ker_M(y_m)$ is supported on $H_1 \cap \ldots \cap H_{m-1}$, so that $\mathcal{K}(M) \subseteq \mathcal{C}'$.

18.3: Excercises

Problem 4.2

Let $p \in K[X]$ be a non-zero polynomial and let M be a holonomic A_n-module. Show that $M[p^{-1}] = K[X, p^{-1}] \otimes_{K[X]} M$ is a holonomic A_n-module.

It seems to us that this is just a straightforward application of [Cou95, Theorem 12.5.4]. $\square$

4.5

As in §3, let us identify Y with the linear subspace of equations $y_1 = \cdots = y_m = 0$ in $X \times Y$. Let $\mathcal{H}^{n+m}(Y)$ be the category of holonomic left A_{n+m}-modules with support on Y. Show that this category is equivalent to $\mathcal{H}^n$.

Recall that by [Cou95, Cor. 18.3.2] we have that the functor

$$\iota_* : \mathcal{M}^n \to \mathcal{M}^{m+n}(Y)$$

defines an equivalence of categories. If we restrict $\mathcal{M}^n$ to $\mathcal{H}^n$ then by [Cou95, Theorem 18.2.3] we have that $\iota_*(\mathcal{H}^n) \subseteq \mathcal{H}^{m+n}$ but also by [Cou95, Cor. 18.3.2] $\iota_*(\mathcal{H}^n) \subseteq \mathcal{M}^{n+m}(Y)$ and so in fact $\iota_*(\mathcal{H}^n) \subseteq \mathcal{H}^{n+m}(Y)$.

On the other hand, let $\mathcal{K} : \mathcal{H}^{n+m}(Y) \to \mathcal{H}^n$ be defined similarly to 18.1, but instead of sending M to $\ker_M(y)$ it sends M to $\ker_M(Y) := \bigcap_i \ker_M(y_i)$, and let $M \in \mathcal{H}^{m+n}(Y)$ be arbitrary. Observe that for $M \in \mathcal{H}^{n+m}(Y)$ we have that $\ker_M(Y)$ is an A_n-submodule of M since for any $a \in A_n$ (in the generators $x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n}$) we have that

$$\begin{aligned} y_i \cdot (a \cdot m) &= a \cdot (y_i m) \\ &= 0, \end{aligned}$$

since y_i commutes with a. Hence $\ker_M(Y)$ is closed under the action from A_n.

We also want to show that $\ker_M(Y)$ is a *holonomic* A_n-module.

Claim: $\Gamma_H M \cong K[\partial_y] \hat{\otimes} \ker_M(Y)$.

This is a direct application of the fact that $\iota_* \mathcal{K}(M) \cong M$, how ι_* is defined for left A_{n+m}-modules ([Cou95, Chap 17, §1]) and that $\Gamma_H M = M$ whenever M has support in H.

Since $\Gamma_H M$ is an A_{m+n}-submodule of the holonomic A_{m+n}-module M, it is holonomic by [Cou95, Prop. 10.1.1], and hence finitely generated with say generators $g_1, \ldots, g_t$. Then preimages of these generators under the (A_{m+n}-module) isomorphism $K[\partial_y] \hat{\otimes} \ker_M(Y) \cong \Gamma_H M$ gives generators for $K[\partial_y] \hat{\otimes} \ker_M(Y)$ as an $A_{n+m} = A_m \hat{\otimes} A_n$-module. By picking out the parts in the second tensor-factor of the preimages of these generators, we find that $\ker_M(Y)$ is finitely generated, since only A_n in $A_{m+n} = A_m \hat{\otimes} A_n$ acts in the right tensor factor of $K[\partial_y] \hat{\otimes} \ker_M(Y)$.

By [Cou95, Theorem 13.4.1.(1)] it follows that

$$\begin{aligned} d(K[\partial_y] \hat{\otimes} \ker_M(Y)) &= d(K[\partial_y]) + d(\ker_M(Y)) \\ &= m + n. \end{aligned}$$

We note that $K[\partial_y]$ is the Fourier-transform of the holonomic A_m-module $K[Y]$, and so (since if we understand it correctly, the Fourier-transform gives an A_m-module isomorphism between $K[Y]$ and $K[\partial_y]$) has the same dimension m (see [Cou95, Chap. 9, §2]), i.e. $d(K[\partial_y]) = d(K[\partial_{y_1}, \ldots, \partial_{y_m}]) = m$.

This forces the equality $d(\ker_M(Y)) = n$, so that indeed $\ker_M(Y)$ is a holonomic A_n-module $\Rightarrow$ $\mathcal{K}(M) \subseteq \mathcal{H}^n$.

By 0.31 it follows that ι_* restricts to define an equivalence $\mathcal{H}^n \to \mathcal{H}^{m+n}(Y)$, which is what we wanted to show. $\square$