

Algebraic Geometry, VT2025

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Chapter 1

Lecture 1

Let $k = \bar{k}$ (so k algebraically closed).

\mathbb{A}_k^n

Definition 1.0.1. Let $\mathbb{A}_k^n := \{(a_1, \dots, a_n) : a_i \in k\}$ which we call the **affine n -space over k** .

$V(S)$

If $S \subset k[x_1, \dots, x_n]$ then $V(S) = \{\underline{a} \in \mathbb{A}_k^n : f(\underline{a}) = 0\}$.

Properties of $V(-)$

Proposition 1.0.2.

- If $S \subset T \rightsquigarrow V(S) \supset V(T)$ (reverses inclusion).
- $V(S) = V(\langle S \rangle)$ where $\langle S \rangle = \langle S \rangle$, which is *the ideal generated by S* .
- $V(I_1) \cup V(I_2) = V(S_1 S_2)$.
- $\bigcap_{\alpha} V(I_{\alpha}) = V\left(\bigcup_{\alpha} I_{\alpha}\right)$.
- $V((1)) = \emptyset$ (notice that $(1) = k[x_1, \dots, x_n]$).
- $V(0) = \mathbb{A}_k^n$.

Proposition 1.0.3. The sets of the form $V(S)$ satisfies the axiom for closed sets in a topology on \mathbb{A}_k^n (Zariski-topology).

Affine algebraic variety $X \subset \mathbb{A}_k^n$

Definition 1.0.4. We say $X \subseteq \mathbb{A}_k^n$ is an **affine algebraic variety** if $X = V(S)$ for some $S \subseteq k[x_1, \dots, x_n]$.

Remark 1.0.5. $k[x_1, \dots, x_n]$ is *Noetherian* so

$$\begin{aligned} V(S) &= V((S)) \\ &= V((f_1, \dots, f_m)) \\ &= V(\{f_1, \dots, f_m\}) \\ &= V(f_1, \dots, f_m). \end{aligned}$$

$I(X)$

If $X \subset \mathbb{A}_k^n$ then $I(X) = \{f \in k[x_1, \dots, x_n] : f(\underline{a}) = 0, \forall \underline{a} \in X\}$. This is an ideal. If $f, g \in I(X)$ and $\underline{a} \in X$ then

$$\alpha f(\underline{a}) + \beta g(\underline{a}) = 0, \forall \alpha, \beta \in k[x_1, \dots, x_n].$$

It is a *radical* ideal.

Recall: If R is a commutative ring with 1 and $J \subseteq R$ then we define

$$\sqrt{J} = \{f \in R : f^k \in J \text{ for some } k \in \mathbb{Z}_{\geq 1}\}. \quad (1.0.1)$$

We call \sqrt{J} the **radical of J** and we say that an ideal $I \subseteq R$ is **radical** if $\sqrt{I} = I$.

Example 1.0.6. $(x^2) \subseteq k[x_1, \dots, x_n]$ is *not radical* since $x \notin (x^2)$ but $x \in \sqrt{(x^2)}$.

Example 1.0.7. $(x) \subseteq k[x_1, \dots, x_n]$ is radical.

We have that $\sqrt{I(X)} \supseteq I(X)$ always, so suppose that $f \in \sqrt{I(X)}$ then there exists $m > 0$ such that $f^m \in I(X)$ so for $\underline{a} \in X$ you have that

$$\begin{aligned} f^m(\underline{a}) &= 0 \\ \xRightarrow{\mathbb{A}_k^n \text{ ID}} f(\underline{a}) &= 0 \\ \Rightarrow f &\in I(X). \end{aligned}$$

$$\begin{aligned} \{\text{affine algebraic varieties in } \mathbb{A}_k^n\} &\longleftrightarrow \{\text{radical ideals in } k[x_1, \dots, x_n]\}, \\ X &\longmapsto I(X), \\ V(J) &\longleftarrow J. \end{aligned} \quad (1.0.2)$$

Theorem 1.0.8 (Hilbert Nullstellensatz). *If $k = \bar{k}$ then the operations given in (1.0.2) are mutually inverse operations. That is,*

$$\begin{aligned} I(V(J)) &= J, \\ V(I(X)) &= X. \end{aligned}$$

Remark 1.0.9. Notice that J is presumed radical here so that $I(V(J)) = \sqrt{J} = J$.

Remark 1.0.10. The only hard inclusion to show is that $I(V(J)) \subseteq J$, which is also the only one that requires that $k = \bar{k}$.

Example 1.0.11. Let $k = \mathbb{R}$. We then have that

$$\begin{aligned} V(x^2 + 1) &= \emptyset \\ &= V(1), \end{aligned}$$

but $(x^2 + 1) \subseteq \mathbb{R}[x]$ is radical. Two radical ideals give the same output.

Example 1.0.12 (Multiplicity lost). Let $k = \mathbb{C}$ and consider

$$p(x) = (x - a_1)^{m_1} \cdots (x - a_n)^{m_n}.$$

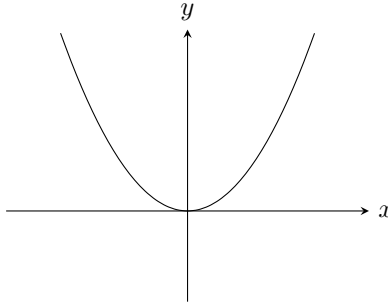
Then we claim that $(p(x))$ is not radical. We claim that

$$\sqrt{(p(x))} = ((x - a_1) \cdots (x - a_n))$$

and that $V(p(x)) = \{a_1, \dots, a_n\}$.

Example 1.0.13. Consider $k = \mathbb{C}$ and consider $y, y - x^2 \in \mathbb{C}[x, y]$. We then have that

$$\begin{aligned} V(y) \cap V(y - x^2) &= \underbrace{V(y, y - x^2)}_{=V(x^2, y)} \\ &= V(\sqrt{(y, y - x^2)}) \\ &= V((x, y)) \\ &= \{0, 0\} \end{aligned}$$



Properties of $I(-)$

Proposition 1.0.14.

- $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.
- $X_1 \subseteq X_2 \rightsquigarrow I(X_1) \supseteq I(X_2)$ (inclusion-reversing).

Proof. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$: We instead proceed as in [Gat21, Lemma 1.12]. A polynomial $f \in k[x_1, \dots, x_n]$ is contained in $I(X_1 \cup X_2) \iff f(x) = 0$ for all $x \in X_1$ and all $x \in X_2 \iff f \in I(X_1) \cap I(X_2)$.

$X_1 \subseteq X_2 \rightsquigarrow I(X_2) \subseteq I(X_1)$: Assume that $f \in I(X_2)$ so that $f(a) = 0$ for all $a \in X_2$. But $X_1 \subseteq X_2$ so that for all $a \in X_1$ we have $f(a) = 0$ hence $f \in I(X_1)$. \square

Lemma

Lemma 1.0.15. Let I_1, I_2 be ideals of R . Then

$$\begin{aligned} \sqrt{I_1 I_2} &= \sqrt{I_1 \cap I_2} \\ &= \sqrt{I_1} \cap \sqrt{I_2} \\ &= I_1 \cap I_2, \quad (\text{if } I_1, I_2 \text{ are both radical}). \end{aligned}$$

Proof.

$$\begin{aligned} I_1 I_2 &\subseteq I_1 \cap I_2 \\ \rightsquigarrow \sqrt{I_1 I_2} &\subseteq \sqrt{I_1 \cap I_2}. \end{aligned}$$

Let $f \in \sqrt{I_1 \cap I_2}$ so that there is some $m > 0$ such that $f^m \in I_1 \cap I_2$, hence $f^m \in I_1$ and $f^m \in I_2 \Rightarrow f \in \sqrt{I_1}$ and $f \in \sqrt{I_2}$ so that $f \in \sqrt{I_1} \cap \sqrt{I_2}$. But this means that there are $m_1, m_2 > 0$ such that $f^{m_1} \in I_1$ and $f^{m_2} \in I_2$ and hence $f^{m_1+m_2} \in I_1 I_2 \Rightarrow f \in \sqrt{I_1 I_2}$. Hence

$$\begin{aligned} \sqrt{I_1 \cap I_2} &\subseteq \sqrt{I_1 I_2} \\ \Rightarrow \sqrt{I_1 I_2} &= \sqrt{I_1 \cap I_2}. \end{aligned}$$

□

We want to provide affine varieties with a topology arising from algebra. If X is an affine variety then $I(X)$ is the associated ideal.

$A(X)$

Let $X \subset \mathbb{A}_k^n$ be an affine variety. Then

$$A(X) := k[x_1, \dots, x_n]/I(X)$$

is the **coordinate ring of the affine variety** X , and the elements of $A(X)$ are called **polynomial functions on X** .

Remark 1.0.16. If $f, g \in k[x_1, \dots, x_n]$ are such that $f + I(X) = g + I(X)$ then $f - g \in I(X)$ so that $f(a) - g(a) = 0$ for all $a \in X \Rightarrow f(a) = g(a)$ for all $a \in X$.

If $f \in A(X)$ it makes sense to consider $f(a)$ for $a \in X$.

If $S \subset A(X)$ then

$$\begin{aligned} V(S) &= V_X(S) \\ &= \{a \in X : f(a) = 0, \forall f \in S\}. \end{aligned}$$

Furthermore, if $S \subseteq T \rightsquigarrow V(T) \subseteq V(S)$.

Proposition 1.0.17.

•

$$\begin{aligned} V(S) &= V((S)) \\ &= V(\sqrt{S}). \end{aligned}$$

- $V(I_1) \cup V(I_2) = V(I_1 I_2)$.
- $\bigcap_{\alpha} V(I_{\alpha}) = V\left(\sum_{\alpha} I_{\alpha}\right)$.
- $V(1 + I(X)) = \emptyset$.
- $V(0 + I(X)) = X$.

Proposition 1.0.18. *The $V(S)$ satisfies the axioms for closed set of a topology in X .*

Proposition 1.0.19. *This topology is the topology induced by the Zariski-topology on \mathbb{A}_k^n . We call it the **Zariski topology** on X .*

Proof. $C \subseteq X$ is closed (in X) \Leftrightarrow there is a $C' \subseteq \mathbb{A}_k^n$ that is closed with respect to the Zariski-topology on \mathbb{A}_k^n , i.e. such that $C = C' \cap X$. By definition, this means that $C = V_X(J)$ for some $J \subseteq k[x_1, \dots, x_n]$. We have a quotient map $\pi : k[x_1, \dots, x_n] \twoheadrightarrow k[x_1, \dots, x_n]/I(X) = A(X)$.

Lemma 1.0.20. *Let $\varphi : R \rightarrow S$ be a ring homomorphism and let $J \trianglelefteq S$ be an radical ideal $\Rightarrow \varphi^{-1}(J) \trianglelefteq R$ is a radical ideal of R .*

By lemma 1.0.20 we see that $\pi^{-1}(J) \trianglelefteq k[x_1, \dots, x_n]$ is a radical ideal. Furthermore, we claim that $\pi^{-1}(J) \supseteq I(X)$. To see this, first notice that $\pi^{-1}(J) = \{f \in k[x_1, \dots, x_n] : \pi(f) \in J\}$. Let $f \in I(X) \rightsquigarrow f(x) = 0, \forall x \in X$. Then we see that

$$\begin{aligned}\pi(f) &= f + I(X) \\ &= 0 \in A(X).\end{aligned}$$

Therefore, since 0 is in every ideal of $A(X)$ we see that $f \in \pi^{-1}(J)$. Furthermore, we claim that $C = V_X(\pi^{-1}(J))$. Let $f \in \pi^{-1}(J)$ and let $a \in C = V(J)$. Then $\pi(f) = f + I(X)$ and evaluation at a gives us $f(a) = 0$ since $\pi(f) = f + I(X) \in J$. Notice here that $V_X(\pi^{-1}(J)) = \{a \in X : f(a) = 0, \forall f \in \pi^{-1}(J)\}$. Therefore, this shows that if $a \in V(J)$ then $a \in V(\pi^{-1}(J)) \Rightarrow V(J) \subseteq V(\pi^{-1}(J))$.

If we instead let $b \in V(\pi^{-1}(J))$ and let $f + I(X) \in J$, then since π is surjective there is some $g \in k[x_1, \dots, x_n]$ such that $\pi(g) = f + I(X)$ and hence $(f + I(X))(b) = 0$. Therefore it follows that $b \in V(J) \Rightarrow V(\pi^{-1}(J)) \subseteq V(J)$. We conclude that $V(\pi^{-1}(J)) = V(J)$.

Let $\mathfrak{a} = \pi^{-1}(J)$ and let $C' = V(\mathfrak{a})$ so that

$$\begin{aligned}C' \cap X &= V(\mathfrak{a}) \cap V(I(X)), && \text{since } X \text{ is an affine variety and by definition of } C' \\ &= V(\mathfrak{a} + I(X)), && \text{by 1.0.17} \\ &= V(\pi(\mathfrak{a})) \\ &= V(J) \\ &= C.\end{aligned}$$

□

Chapter 2

Lecture 2

Plan:

1. Connectedness and irreducibility for affine varieties.
2. Dimension theory.

2.0.1 Connectedness and Irreducibility for affine varieties

We recall the following definition.

Connectedness

Definition 2.0.1. A topological space (X, τ) is connected if $X = U_1 \cup U_2$ with $U_1 \cap U_2 = \emptyset$ and $U_i \in \tau$ (U_i open) implies that $U_1 = \emptyset$ or $U_2 = \emptyset$.

$I_Y(X)$

Let $Y \subset \mathbb{A}_k^n$ be an affine variety and let $X \subset Y$. Then we define

$$\begin{aligned} I(X) &:= I_Y(X) \\ &= \{f \in A(Y) : f(x) = 0, \forall x \in X\}. \end{aligned}$$

$A_Y(X)$

Definition 2.0.2. Let $Y \subset \mathbb{A}_k^n$ and let $X \subset Y$ be an affine subvariety of Y . Then

$$\begin{aligned} A(X) &:= A_Y(X) \\ &= A(Y)/I(X). \end{aligned}$$

Proposition 2.0.3. Let X be an affine variety. If X is not connected then $A(X) = A(X_1) \times A(X_2)$ for X_1, X_2 affine varieties.

Proof. If X is not connected $\rightsquigarrow \exists X_1, X_2$ such that $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \emptyset$ and $X_i \subseteq X$ are

clopen. Since X_i are closed they are on the form $V_X(S_i)$ for $S_i \subset A(X)$, hence it follows that they are affine varieties of X .

Therefore, we have that $X_i = V(I(X_i))$ [Gat21, Relative Nullstellensatz, Remark 1.18].

It follows that

$$V(I(X)) = V(I(X_1)) \cup V(I(X_2)) \quad (2.0.1)$$

By 1.0.2 with $S_1 = I(X_1)$ and $S_2 = I(X_2)$ we see that

$$\begin{aligned} V(I(X_1)) \cup V(I(X_2)) &= V(I(X_1)I(X_2)) \\ &= V(I(X_1) \cap I(X_2)), \end{aligned}$$

where we in the last step used [Gat21, Lemma 1.7.(b)].

Therefore, it follows from the above together with (2.0.1) that

$$\begin{aligned} I(V(I(X))) &= I(V(I(X_1)) \cup I(X_2)) \\ &= I(V(I(X_1) \cap I(X_2))). \end{aligned}$$

We have that $X_1 \cap X_2 = \emptyset$ and that

$$\begin{aligned} I(X) &= I(X_1 \cup X_2) \\ &= I(X_1) \cap I(X_2) \\ &= \emptyset. \end{aligned}$$

where we have used 1.0.14 and that $I(X) = \emptyset$.

Since

$$\begin{aligned} V(1) &= X_1 \cap X_2 \\ &= \emptyset, \end{aligned}$$

$V(I(X_i)) = X_i$ together with [Gat21, Lemma 1.7.(c)] we see that

$$\begin{aligned} X_1 \cap X_2 &= V(I(X_1)) \cap V(I(X_2)) \\ &= V(I(X_1) + I(X_2)) \\ &= V(1) \\ &\rightsquigarrow I(V(I(X_1) + I(X_2))) = I(V(1)) \\ &= A(X) \end{aligned}$$

We claim (without proof for now) that the only ideal S such that $I(V(S)) = A(X)$ is $A(X)$ itself. Therefore, we see that $I(X_1) + I(X_2) = A(X)$.

Chinese Remainder Theorem

Lemma 2.0.4 (Chinese Remainder Theorem). *Let A_1, \dots, A_k be ideals in a ring R . The map*

$$R \rightarrow R/A_1 \times R/A_2 \times \dots \times R/A_k$$

defined by

$$r \mapsto (r + A_1, r + A_2, \dots, r + A_k)$$

is a ring homomorphism with kernel $A_1 \cap A_2 \cap \dots \cap A_k$. If for each $i, j \in \{1, 2, \dots, k\}$ with $i \neq j$ the ideals A_i and A_j are comaximal, then this map is surjective and $A_1 \cap A_2 \cap \dots \cap A_k = A_1 A_2 \dots A_k$, so

$$\begin{aligned} R/(A_1 A_2 \dots A_k) &= R/(A_1 \cap A_2 \cap \dots \cap A_k) \\ &\cong R/A_1 \times R/A_2 \times \dots \times R/A_k. \end{aligned}$$

By 2.0.4 we then have that

$$\begin{aligned} A(X)/(I(X_1) \cap I(X_2)) &= A(X)/(0) \\ &= A(X) \\ &\cong A(X)/I(X_1) \times A(X)/I(X_2) \\ &= A(X_1) \times A(X_2) \end{aligned}$$

□

Irreducible topological space

Definition 2.0.5. Let (X, τ) be a topological space. We then say that X is **irreducible** if $X = C_1 \cup C_2$ for C_1, C_2 closed then either $C_1 = X$ or $C_2 = X$.

We say that X is **reducible** if it is not irreducible.

Proposition 2.0.6. *If (X, τ) is a topological space then*

$$\text{non-connected} \Leftrightarrow \text{irreducible}.$$

Example 2.0.7. Take \mathbb{A}_k^1 with k algebraically closed ($k = \bar{k}$). Then the closed sets are $\emptyset, \mathbb{A}_k^n$ and finite sets. We have that

$$\begin{aligned} |\mathbb{A}_k^n| &= |k| \\ &= +\infty \end{aligned}$$

(algebraically closed fields are infinite) and there can't be written as a union of two finite sets, hence is irreducible.

Noetherian topological space

Definition 2.0.8. A space (X, τ) is **Noetherian** if every descending chain

$$X \supsetneq C_1 \supsetneq C_2 \supsetneq \dots \supsetneq C_k \supsetneq \dots$$

stabilizes.

Proposition 2.0.9. *If X is Noetherian then $X = C_1 \cup C_2 \cup \dots \cup C_n$ with C_i closed and irreducible.*

Remark 2.0.10. The C_i above are then called **irreducible components**.

Proof below taken from [Gat21, Prop. 2.14].

Proof. For $X = \emptyset$ the statement is clear. Otherwise, assume that X is a Noetherian topological space for which this is false. Then clearly X is not irreducible $\Rightarrow X$ is reducible so $X = X_1 \cup X'_1$. Furthermore, the statement must be false for either X_1 or X'_1 since otherwise the statement would still hold. By proceeding this way one finds an infinite chain

$$X \supsetneq X_1 \supsetneq X_2 \supsetneq X_3 \supsetneq \dots$$

□

Proposition 2.0.11. *Affine varieties are Noetherian (2.0.8).*

Proof. We have that $k[x_1, \dots, x_n]$ is Noetherian.

Lemma 2.0.12. *Quotient of Noetherian spaces are Noetherian.*

By 2.0.12 we see that $A(X)$ is Noetherian. Given a descending chain of closed sets

$$X \supsetneq X_1 \supsetneq X_2 \supsetneq \dots$$

and each X_i is a subvariety of X hence $V(I(X_i)) = X_i$ therefore we get

$$X \supsetneq V(I(X_1)) \supsetneq V(I(X_2)) \supsetneq \dots$$

Applying $I(-)$ to this and noting that

1. $I(-)$ is inclusion-reversing.
2. $I(X) = 0$.

we get a an infinite increasing chain

$$0 \subsetneq I(X_1) \subsetneq I(X_2) \subsetneq \dots$$

of ideals, which are all in $A(X)$. Since $A(X)$ is Noetherian this is a contradiction

□

Remark 2.0.13. Notice that the inclusions $I(X_i) \subsetneq I(X_{i+1})$ must be strict since if there was some i such that $I(X_i) = I(X_{i+1})$ then

$$\begin{aligned} V(I(X_i)) &= V(I(X_{i+1})) \\ \Leftrightarrow X_i &= X_{i+1} \end{aligned}$$

Proposition 2.0.14. *X is an irreducible affine variety such that $X \neq \emptyset \Leftrightarrow A(X)$ is an integral domain.*

$$\{\text{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } k[x_1, \dots, x_n]\},$$

$$\text{point} \longleftrightarrow \text{maximal ideal},$$

$$\text{irreducible} \longleftrightarrow \text{prime}.$$

Remark 2.0.15. Proof below from [Gat21, Prop. 2.8]

Proof. Since $X \neq \emptyset$ we have that $A(X) \neq 0$.

\Rightarrow : Assume that $A(X)$ is not an integral domain, i.e. there are non-zero $f_1, f_2 \in A(X)$ such that $f_1 f_2 = 0$. Then we have that $X_1 = V(f_1)$ and $X_2 = V(f_2)$ are closed and not equal to X (since $f_i \neq 0$). Furthermore, we have that

$$\begin{aligned} X_1 \cup X_2 &= V(f_1) \cup V(f_2) \\ &= V(f_1 f_2) \\ &= V(0) \\ &= X. \end{aligned}$$

Hence X is reducible.

\Leftarrow : Assume that X is reducible so that $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$. We note that if $I(X_i) = 0$ then $V(I(X_i)) = X_i = X$ contradicting that $X_i \subsetneq X$, therefore $I(X_i) \neq 0$ so there is some $f_i \in I(X_i)$ for $i = 1, 2$. Then $(f_1 f_2)(x) = 0$ for all $x \in X_1 \cup X_2 = X$. Hence $f_1 f_2 = 0 \in A(X)$ so that $A(X)$ is not an integral domain. \square

2.0.2 Dimension theory

Dimension of a Noetherian topological space

Definition 2.0.16. Let X be a Noetherian topological space. Then we define

$$\dim X = \sup\{n \in \mathbb{N} : \exists \text{ chain } \emptyset \subsetneq C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_n \subset X\}$$

where C_i is closed and irreducible (2.0.5) for all $i = 0, \dots, n$.

Example 2.0.17 (\mathbb{A}^1). $\emptyset \subsetneq \{p\} \subsetneq \mathbb{A}^1$. Notice that $\{a_1, \dots, a_n\} = \{a_1\} \cup \{a_2, \dots, a_n\}$ for $n > 1$ where both $\{a_1\}$ and $\{a_2, \dots, a_n\}$ are closed (the finite sets are precisely the closed sets, besides \emptyset, \mathbb{A}^1).

Example 2.0.18. Take $Y = \{(a_1, a_2) \in \mathbb{A}^2 : a_1 a_2 = 0\} \subseteq \mathbb{A}^2$. This means that either $a_1 = 0$ or $a_2 = 0$ or both. Therefore, we have that $Y = (\text{x-axis}) \cup (\text{y-axis})$. We then have

$$\emptyset \subsetneq \{x\} \subsetneq (\text{x-axis})$$

.

Notice further that $Y = V(xy)$ so that Y is closed in \mathbb{A}^2 and has the induced subspace topology. Furthermore, we have that $\{x\}$ is closed in \mathbb{A}^2 and that $Y \cap \{x\} = \{x\} \Rightarrow \{x\}$ is closed. Also, we see that $\text{x-axis} = V(y)$ (so closed in \mathbb{A}^2) and therefore $V(y) \cap Y = V(y)$ so that indeed x-axis is closed.

Lastly, we have that $V(xy) = V(x) \cup V(y)$. If there was some Z closed and irreducible such that $V(y) \subsetneq Z \subsetneq V(xy)$ then we would have that

$$\begin{aligned} Z \cap V(xy) &= Z \cap (V(x) \cup V(y)) \\ &= (Z \cap V(x)) \cup (Z \cap V(y)) \\ &= (Z \cap V(x)) \cup V(y). \end{aligned}$$

But here $V(y)$ and $Z \cap V(x)$ are both closed in Y and so we have written Z as a union of two closed sets, and $Z \cap V(x) \neq \emptyset$, hence Z is not irreducible. This is a contradiction.

On the other hand, assume that there was some Z closed and irreducible such that

$$\emptyset \subsetneq \{p\} \subsetneq Z \subsetneq V(y) \subsetneq V(xy).$$

Then Z must be a set of points along the x -axis that are closed and irreducible in $V(xy)$. But we have that $V(y) \cong \mathbb{A}^1$ and furthermore since $Z \subsetneq V(y)$ and Z is closed in $V(xy)$ we have that $Z = Z \cap V(y)$ so that Z is closed in $V(y)$ as well. Therefore it follows (by $V(y) \cong \mathbb{A}^1$ and the Zariski-topology on \mathbb{A}^1) that Z must be a finite set of points (since $\emptyset \subsetneq Z \subsetneq V(y)$). But a finite set of points is not irreducible, so this contradicts the irreducibility of Z .

Even if X is Noetherian the dimension can be infinite.

Example 2.0.19 (\mathbb{N}). Take $X = \mathbb{N}$ with the topology such that the closed subsets of X (apart from \emptyset , X) are precisely $Y_n = \{0, 1, 2, \dots, n\}$ then we claim that X is Noetherian. Furthermore, then $C \subseteq X$ closed $\Leftrightarrow C = \{0, 1, \dots, m\}$ irreducible with $\{0, 1, \dots, m\} \subsetneq X$. Then we have

$$\emptyset \subsetneq \{0\} \subsetneq \{0, 1\} \subsetneq \{0, 1, 2\} \subsetneq \dots$$

so that $X = \mathbb{N}$ has dimension ∞ .

Remark 2.0.20. To see that $X = \mathbb{N}$ with the closed sets as above is indeed a topology one checks that

$$\bigcap_{\alpha \in A} Y_{n_\alpha} = Y_{\min\{n_\alpha : \alpha \in A\}} \quad (2.0.2)$$

so that arbitrary intersections of closed sets are indeed closed, and that

$$\bigcup_{i=1}^n Y_{n_i} = Y_{\max\{n_i : i=1, \dots, n\}} \quad (2.0.3)$$

so that finite unions of closed sets are closed.

Furthermore, X is Noetherian since it has no infinite descending sequences. To see this, just note that even in the “worst” case of

$$C_0 \supsetneq C_1 \supsetneq C_2 \supsetneq \dots$$

where $C_0 = \mathbb{N}$ we need $C_1 \subsetneq \mathbb{N}$ and so $C_1 = Y_{n_1}$ for some finite subset Y_{n_1} and therefore we see that it must stabilize in a finite number of steps.

Codimension $\text{codim}_X(Y)$

Definition 2.0.21. Let X be a non-empty topological space and let $Y \subset X$ be a non-empty irreducible closed subset. Then

$$\text{codim}_X(Y) := \sup\{n \in \mathbb{N} : \exists \text{ chain } Y \subsetneq Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n \subsetneq X\}$$

where Y_i are irreducible closed subsets of X .

If we have a chain

$$\emptyset \subsetneq C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_m = Y \subsetneq C_{m+1} \subsetneq \dots \subsetneq C_n = X \quad (2.0.4)$$

then

$$\begin{aligned} \dim X &\geq m + (n - m) \\ &\rightsquigarrow \dim X \geq \dim Y + \text{codim}_X(Y) \end{aligned}$$

and we claim (without proof) that we have equality when all the maximal chains have the same length.

Krull dimension

Definition 2.0.22. Let R be a Noetherian commutative ring with 1. We then define the **Krull dimension of R** as

$$\sup\{n \in \mathbb{N} : \exists R \supsetneq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \dots \supsetneq \mathfrak{p}_m \supset (0)\} \quad (2.0.5)$$

where \mathfrak{p}_i are prime-ideals.

Proposition 2.0.23. Let $X \subset \mathbb{A}_k^n$ be an affine variety. Then $\dim X =$ the Krull-dimension of $A(X)$ (2.0.22).

Height of \mathfrak{p}

Definition 2.0.24. Let $\mathfrak{p} \subseteq R$ for R ring be a prime ideal. Then the **height of \mathfrak{p}** as

$$\text{ht}(\mathfrak{p}) = \sup\{n \in \mathbb{N} : \exists \text{ a descending chain of prime ideals properly contained in } \mathfrak{p}\} \quad (2.0.6)$$

Proposition 2.0.25. $Y \subseteq X$ irreducible $\Leftrightarrow I(Y)$ prime.

Proposition 2.0.26. $\text{codim}_X(Y) = \text{ht}(I(Y))$.

Proposition 2.0.27. Affine varieties have finite dimension, and if $X \subset \mathbb{A}_k^n$ is an affine variety then $A(X)$ is a finitely generated k -algebra.

Noether Normalization

Theorem 2.0.28. Let R be a finitely generated k -algebra with generators x_1, \dots, x_r . Then there is an injective k -algebra morphism

$$\varphi : k[z_1, \dots, z_n] \rightarrow R$$

such that R is a finite ring extension of $k[z_1, \dots, z_n]$. If $|k| = \infty$ then $\varphi(z_i)$ can be taken to be a linear combination of the x_i .

Lemma 2.0.29. Let S/R be a finite ring extension. Then the Krull dimension of $S =$ the Krull dimension of R (2.0.22).

To show finiteness \rightsquigarrow is enough to compute the Krull dimension of $k[x_1, \dots, x_n] = n$.

Lemma 2.0.30. Let R be a ring of finite dimension, such that all the maximal chains of ideals have maximal length. Then

1. R/\mathfrak{p} have the same property.
2. $\dim R = \dim R/\mathfrak{p} + \text{ht}(\mathfrak{p})$.
3. $\dim R_{\mathfrak{p}} = \dim R$ if \mathfrak{p} is a maximal ideal.

Remark 2.0.31.

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_m \subsetneq \mathfrak{p}.$$

Remark 2.0.32. $R \rightarrow R_{\mathfrak{p}}$ and $\text{ht}(\mathfrak{p}) = \dim R_{\mathfrak{p}}$.

Prove: $k[x_1, \dots, x_n]$ satisfies the hypothesis. Let X be an irreducible and let $Y \subseteq X$. Then $\dim X = \dim Y + \text{codim}_X Y$. Need irreducibility, counterexample.

Chapter 3

Lecture 3

3.0.1 The sheaf of regular functions

Presheaf

Definition 3.0.1. A **presheaf** \mathcal{F} (of rings) on a topological space X consists of:

- Every open set $U \subset X$ has an associated ring $\mathcal{F}(U)$ (“ring of functions on U ”).
- For all inclusions $U \hookrightarrow V$ of open sets U, V we have a ring homomorphism $\rho_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$.

The data given above must satisfy the following:

- $\mathcal{F}(\emptyset) = 0$.
- $\rho_{U,U}$ is the identity map on $\mathcal{F}(U)$ for all open sets $U \subset X$.
- For $U \subset V \subset W$ with U, V, W open in X , we have that $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$.

The elements of $\mathcal{F}(U)$ we call the **sections** of \mathcal{F} over U .

Remark 3.0.2. The restriction maps $\rho_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ are written as $\mathcal{F}(V) \ni \varphi \mapsto \varphi|_U \in \mathcal{F}(U)$. So we are essentially taking sections of \mathcal{F} over V and restricting them to \mathcal{F} over U .

Sheaf

Definition 3.0.3. A presheaf \mathcal{F} (3.0.1) is called a **sheaf** of rings if it has the *gluing property*:

- If $U \subset X$ is an open set and $\{U_i\}_{i \in I}$ is an arbitrary open cover of U and $\varphi_i \in \mathcal{F}(U_i)$ are sections for all i such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ for all $i, j \in I$ \rightsquigarrow there exists a unique $\varphi \in \mathcal{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all $i \in I$.

Remark 3.0.4. If we exchange everything above pertaining to rings to instead k -algebras, abelian groups etc with the corresponding morphisms, we get a presheaf/sheaf in the corresponding category instead.

Example 3.0.5. Let X, Y be topological spaces. Consider the presheaf defined by associating to each open set $U \subset X$ the set of continuous functions:

$$\mathcal{F}(U) := \{f : U \rightarrow Y \mid f \text{ is continuous}\}.$$

This forms a sheaf, called the *sheaf of continuous functions*. To see that it is a presheaf, notice that

$f \mapsto f|_U = f$ if $f \in \mathcal{F}(U)$. Also, we see that if $U \subset V \subset W$ then

$$\begin{aligned} (\rho_{V,U} \circ \rho_{W,V})(\varphi) &= \rho_{V,U}(\varphi|_V) \\ &= (\varphi|_V)|_U \\ &= \varphi|_U \\ &= \rho_{W,U}. \end{aligned}$$

To see that it is a sheaf, use the gluing lemma.

Example 3.0.6. Let X, Y be topological spaces and let $U \mapsto \mathcal{F}(U) := \{f : U \rightarrow Y : f \text{ is constant}\}$.

\mathcal{F}_p : The stalk of \mathcal{F} at p

Definition 3.0.7. Let X be a topological space, let \mathcal{F} be a sheaf (3.0.3) on X and let $p \in X$. Then

$$\mathcal{F}_p := \{(U, \varphi) : p \in U, \varphi \in \mathcal{F}(U)\} / \sim$$

where $(U, \varphi) \sim (V, \psi)$ if there is an open subset W with $p \in W \subset U \cap V$ such that $\varphi|_W = \psi|_W$ is called the **stalk of \mathcal{F} at p** .

Example 3.0.8. Let $X = \mathbb{R}^n$, $Y = \mathbb{R}$ and for $U \subset \mathbb{R}^n$ open let $\mathcal{F}(U) := \{f : U \rightarrow \mathbb{R} : f \in C^\infty(U)\}$. We claim this is a sheaf.

Germes of \mathcal{F} at a

Definition 3.0.9. The *elements* of \mathcal{F}_a (3.0.7) are called **germs** of \mathcal{F} at a .

Remark 3.0.10. In differential geometry, X, \mathcal{O}_X define $f : X \rightarrow Y$ is smooth/differentiable/holomorphic $\Leftrightarrow \forall s \in \mathcal{O}_Y(X)$ we have that $s \circ f \in \mathcal{O}_X(f^{-1}(v))$.

Regular function

Definition 3.0.11. Let $X \subset \mathbb{A}_k^n$ be an affine variety and let $U \subseteq X$ be open. Then a function $\varphi : U \rightarrow k$ is called **regular** if for every $a \in U$ there are polynomial functions $f, g \in A(X)$ with $f(a) \neq 0$ and

$$\varphi(x) = \frac{g(x)}{f(x)}$$

for all x in an open subset U_a such that $a \in U_a \subset U$. The set of all such functions φ will be denoted $\mathcal{O}_X(U)$. That is,

$$\mathcal{O}_X(U) := \{\varphi : U \rightarrow k : \varphi \text{ regular}\}. \quad (3.0.1)$$

Structure sheaf/Sheaf of regular functions for an affine variety X

Definition 3.0.12. Let X again be an affine variety. Then with $\mathcal{O}_X(U)$ defined as in 3.0.1 on open subsets $U \subset X$, form a *sheaf* of k -algebras of X . We then call \mathcal{O}_X the **sheaf of regular functions** on X which is its **structure sheaf**.

- $(\lambda \cdot f)(x) = \lambda \cdot f(x)$.
- $\rho_{U,V}(\varphi) = \varphi|_V$.

Lemma 3.0.13. Let X be affine, let $U \subseteq X$ be open and let $\varphi \in \mathcal{O}_X(U)$. Then

$$V(\varphi) = \{p \in U : \varphi(p) = 0\} \subseteq U$$

is a closed set in U with the subspace topology coming from X .

Proof.

$$\begin{aligned} p &\mapsto \varphi|_{V_p} = \frac{g_p(x)}{f_p(x)} \\ \Rightarrow \varphi|_{V_p}(x) = 0 &\Leftrightarrow g_p(x) = 0, \quad (\forall x \in U) \\ &\rightsquigarrow V(\varphi) = V(g_p) \cap U, \end{aligned}$$

and so since $V(g_p)$ is closed in X it follows that $V(\varphi)$ is closed in U . \square

Corollary 3.0.14. *If X is irreducible (2.0.5) and $U \subseteq V$ with $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$ then $\varphi_1|_U = \varphi_2|_U \Rightarrow \varphi_1 = \varphi_2$.*

Proof. Let $\psi = \varphi_1 - \varphi_2$. Since $\varphi_1|_U = \varphi_2|_U$ we have that $\psi|_U = 0$. Therefore, we have that $U \subseteq V(\psi) = \{v \in V : \psi(v) = 0\}$. Since $V(\psi)$ is closed, and we define the closure of U as the intersection of all closed sets containing U , we see that $\overline{U} \subseteq V(\psi)$.

Lemma 3.0.15. *If X is irreducible in the sense of 2.0.5 then every non-empty open subset of X is dense, i.e. if $V \subseteq X$ is open and non-empty then $\overline{V} = X$.*

Proof. If $\overline{V} \neq X$ then $X = \overline{V} \cup (X \setminus \overline{V})$ where since $(X \setminus \overline{V})^\circ = V$ is open $\Rightarrow X \setminus \overline{V}$ is closed, and both are proper. This contradicts X being irreducible in the sense of 2.0.5. \square

By assumption, V is open in X and so it follows that $\overline{V} = X$ by 3.0.15. Therefore, \overline{V} is irreducible.

Lemma 3.0.16 ([Gat21, Exercise 2.20.(b)]: \Leftarrow). *If \overline{V} is irreducible then V is irreducible.*

Proof. Assume to the contrary that V is not irreducible, so that $V = A \cup B$ where A, B are closed and proper subsets of V . Then we see that $\overline{V} = \overline{A} \cup \overline{B}$. We claim that $\overline{A}, \overline{B}$ are proper closed subsets of \overline{V} . Assume to the contrary that (without loss of generality) $\overline{A} = \overline{V}$. Then we have that $V \subseteq \overline{V} = \overline{A}$. That is, $V \subseteq \overline{A}$. Since A is closed in V we have that $A = V \cap C$ where C is closed in X , so that $A \subseteq C$ and so $\overline{A} = \overline{V} \subseteq C$. Therefore, we see that $V \subseteq C$ and so

$$\begin{aligned} A &= V \cap C \\ &= V, \end{aligned}$$

contradicting our assumption, i.e. that $A \subsetneq V$. \square

It follows by lemma 3.0.16 that V is irreducible. Since $U \subseteq V$ we then see that $\overline{U} = V$ by 3.0.15. Therefore, since $\overline{U} = V \subseteq V(\psi) \Rightarrow V = V(\psi)$. \square

$D(f)$: Distinguished open subset of f in X

Definition 3.0.17. Let $X \subseteq \mathbb{A}_k^n$ be an affine variety, and let $\bar{f} = \pi(f) \in A(X)$, where π is the projection $\pi : k[x_1, \dots, x_n] \twoheadrightarrow k[x_1, \dots, x_n]/I(X) = A(X)$. Then we call

$$D(f) := \{x \in X : \bar{f}(x) \neq 0\}$$

the **distinguished open subset** of f in X .

Remark 3.0.18. Notice that $D(f) = X \setminus V(f)$ where we by $V(f)$ mean $V(f) := V_{A(X)}(f)$.

Proposition 3.0.19.

1. $D(f) \cap D(g) = D(fg)$.
2. Any open subset $U \subset X$ is a finite union of distinguished open subsets (3.0.17).

Proof.

1: We have

$$\begin{aligned}
 D(f) \cap D(g) &= (X \setminus V(f)) \cap (X \setminus V(g)) \\
 &= X \setminus (V(f) \cup V(g)) \\
 &= X \setminus V(fg), \quad \text{by 1.0.2} \\
 &= D(fg).
 \end{aligned}$$

2: By definition, if $U \subset X$ is open it is the complement of an affine variety $V(S)$. But by Hilbert's Basis Theorem we have that S is finitely generated; hence $S = \langle f_1, \dots, f_n \rangle$, so that $V(S) = V(f_1, \dots, f_n)$.

Therefore, we have that

$$\begin{aligned}
 U &= X \setminus V(f_1, \dots, f_n) \\
 &= X \setminus (V(f_1) \cap \dots \cap V(f_n)) \\
 &= \bigcup_{i=1}^n (X \setminus V(f_i)) \\
 &= \bigcup_{i=1}^n D(f_i).
 \end{aligned}$$

□

Lemma 3.0.20.

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^n} \mid g \in A(X), n \in \mathbb{N} \right\}.$$

Proof. See [Gat21, Cor. 3.8]. The inclusion \supset follows from the fact that every function on the form $\frac{g}{f^n}$ for $g \in A(X)$ and $n \in \mathbb{N}$ is regular on $D(f)$.

\subseteq : Assume that $\varphi : D(f) \rightarrow k$ is a regular function (3.0.11). Then for every $a \in D(f)$ there is some open subset U_a containing a such that $\varphi(x) = \frac{g_a}{f_a}$ with $f_a(x) \neq 0$ for all $x \in U_a$ and $f_a, g_a \in A(X)$. By [Gat21, Remark 3.7.(b)] we have that

$$U_a = \bigcup_{i \in I} D(h_i), \quad (I \text{ some index set and } h_i \in A(X)).$$

After possibly shrinking U_a we may presume then presume that $U_a = D(h_a)$ with $h_a \in A(X)$. We may then replace f_a and g_a with $f_a h_a$ and $g_a h_a$ since this makes no difference to the quotient on $D(h_a)$. We relabel $f_a := f_a h_a$ and $g_a := g_a h_a$. Then we see that f_a and g_a vanish on the complement $V(h_a)$ of $D(h_a)$. Since $f_a = f_a h_a$ it follows that f_a vanishes on $V(h_a)$, but does not vanish on $D(h_a)$ (by definition, since f_a did not vanish on $U_a \supset D(h_a)$). It follows that h_a and f_a have the same zero locus, and we can therefore assume that $h_a = f_a$.

As a consequence, we have that

$$g_b f_a = g_a f_b, \quad (\forall a, b \in D(f)). \quad (3.0.2)$$

To see this, notice that on $D(f_a) \cap D(f_b)$ we have that

$$\begin{aligned} \varphi &= \frac{g_a}{f_a} \\ &= \frac{g_b}{f_b} \\ \Leftrightarrow g_a f_b &= f_a g_b, \quad (\text{on } D(f_a) \cap D(f_b)). \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} D(f_a) \cap D(f_b) \cup (V(f_a) \cup V(f_b)) &= D(f_a) \cap D(f_b) \cup (X \setminus D(f_a) \cap D(f_b)) \\ &= X. \end{aligned}$$

Therefore when x is not in $D(f_a) \cap D(f_b)$ then x is in either $V(f_a)$ or $V(f_b)$. If $x \in V(f_a)$ then $f_a(x) = 0$ and $g_a(x) = 0$ so that $g_a f_b = f_a g_b = 0$ and if $x \in V(f_b)$ then $f_b(x) = g_b(x) = 0$ so that $g_a f_b = g_b f_a$ (here we also used that $g_a := g_a h_a$ and $f_a = h_a$ and similarly for f_b, g_b and h_b). Hence they agree for all $a, b \in D(f)$. Furthermore, since $D(f_a) := D(h_a) \subset U_a \subset D(f)$ by construction, it follows that

$$D(f) = \bigcup_{a \in D(f)} D(f_a),$$

so that the right-hand side above is an open cover of $D(f)$ in X . Thus the complement is

$$\begin{aligned} V(f) &= \bigcap_{a \in D(f)} V(\{f_a\}) \\ &= V\left(\bigcup_{a \in D(f)} \{f_a\}\right) \quad (1.0.2) \\ &= V(\{f_a : a \in D(f)\}). \end{aligned}$$

Therefore, by the relative nullstellensatz, we have that

$$\begin{aligned} f \in I(V(f)) &= \sqrt{f} \\ &= \sqrt{\langle f_a : a \in D(f) \rangle}. \end{aligned}$$

In particular, since $f \in \sqrt{f}$ it follows that $f \in \sqrt{\langle f_a : a \in D(f) \rangle}$, which by definition means that there is some $n \in \mathbb{N}$ and $k_a \in A(X)$ such that

$$f^n = \sum_a k_a f_a \quad (3.0.3)$$

for finitely many $a \in D(f)$. Let $g := \sum_a k_a g_a$. The claim is then that $\varphi = \frac{g}{f^n}$ on $D(f)$: For all $b \in D(f)$ we have that $\varphi_b = \frac{g_b}{f_b}$ and

$$\begin{aligned} g f_b &= \sum_a k_a g_a f_b \\ &\stackrel{(3.0.2)}{=} \sum_a k_a g_b f_a \\ &\stackrel{(3.0.3)}{=} g_b f^n. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
gf_b - g_b f^n &= 0 \\
&\Leftrightarrow gf_b = g_b f^n \\
&\Leftrightarrow \frac{g}{f^n} = \frac{g_b}{f_b}, \quad (\text{since } f^n, f_b \text{ are invertible on } D(f_b) \subset D(f)) \\
&= \varphi_b.
\end{aligned}$$

locally on $D(f_b)$ for $b \in D(f)$. Now notice that $\{D(f_b) : b \in D(f)\}$ is an open cover of $D(f)$ and that φ_b agrees with φ_a on overlaps $D(f_a) \cap D(f_b)$ (it is always equal to $\frac{g}{f^n}$). Therefore by *the gluing condition* we get a unique map $\varphi = \frac{g}{f^n}$ on all of $D(f)$. \square

Theorem 3.0.21. *We have that $\mathcal{O}_X(D(f)) \cong A(X)_f$ as k -algebras.*

Remark 3.0.22. Here $A(x)_f$ is the *localization* of $A(x)$ at f , i.e.

$$A(X)_f = \left\{ \frac{g}{f^n} \mid g \in A(X), n \in \mathbb{N} \right\} / \sim$$

where

$$\frac{g}{f^n} \sim \frac{h}{f^m} \Leftrightarrow \exists \ell \in \mathbb{N} \text{ such that } f^\ell(f^m g - h f^n) \in I(X),$$

i.e. so that $f^\ell(f^m g - h f^n) = 0 + I(X) \in A(X)$.

Proof. We have $A(X)_f \rightarrow \mathcal{O}_X(D(f))$ defined explicitly by

$$A(X)_f \ni \frac{g}{f^n} \mapsto \frac{g}{f^n} \in \mathcal{O}_X(D(f)).$$

Well-defined: If

$$\begin{aligned}
A(X)_f \ni \frac{g}{f^n} &\sim \frac{g'}{f^m} \\
&\Leftrightarrow f^\ell(g f^m - g' f^n) = 0 \in A(X) \text{ for some } \ell \in \mathbb{N}.
\end{aligned}$$

This means that $f^\ell(g f^m - g' f^n) \in I(X)$ so that $f^\ell(g f^m - g' f^n)(x) = 0$ for all $x \in X$. On $D(f) \subset X$ we see that $f(p) \neq 0$ for $p \in D(f)$ and so therefore since $f(p) \neq 0$ we must have that $f(p)^\ell \neq 0$ (since k is a field) so that

$$\begin{aligned}
g(p)f^m(p) - g'(p)f^n(p) &= 0 \\
&\Leftrightarrow g(p)f^m(p) = g'(p)f^n(p) \\
&\Leftrightarrow \frac{g(p)}{f^m(p)} = \frac{g'(p)}{f^n(p)},
\end{aligned}$$

for all $p \in D(f)$. Therefore $\frac{g}{f^n} = \frac{g'}{f^m}$ on $D(f)$, so that the map defined above is well-defined.

Surjective: Follows directly from 3.0.20.

Injective: Assume that $\frac{g}{f^n} = 0$ in $\mathcal{O}_X(D(f))$. This implies that $g = 0$ on $D(f)$ and therefore $fg = 0$ on $D(f)$, and since $X \setminus D(f) = V(f)$ we see that $fg = 0$ on all of X . Hence $fg \in I(X)$ and so

$fg = 0 \in A(X)$. Then we have

$$f \cdot (g \cdot 1 - 0 \cdot f^n) = 0 \in A(X)$$

$$\frac{g}{f^n} = \frac{0}{1} \in A(X)_f.$$

□

Corollary 3.0.23 (From 3.0.21).

1. $\mathcal{O}_X(X) = \mathcal{O}_X(D(1)) \simeq A(X)_1 = A(X)$.
2. On distinguished open sets $\varphi \in \mathcal{O}_X(D(f))$, $\varphi = \frac{g}{f^n}$.

Lemma 3.0.24. $\mathcal{O}_{X,a} \cong A(X)_{\mathfrak{m}_a}$ where $a = (a_1, \dots, a_n)$ and $\mathfrak{m}_a = \langle x_1 - a_1, \dots, x_n - a_n \rangle$

Proof. We have $A(X)_{\mathfrak{m}_a} \rightarrow \mathcal{O}_{X,a}$ defined by

$$\frac{g}{f} \mapsto \left(D(f), \frac{g(x)}{f(x)} \right)$$

where $f \notin \mathfrak{m}_a$ and $a \in D(f)$. Let $(U, \varphi) \in \mathcal{O}_{X,a}$ with $\varphi \in \mathcal{O}_X(U)$ then $\exists V_a \ni a$ such that $\varphi|_{V_a}(x) = \frac{g_a(x)}{f_a(x)}$ and

$$(U, \varphi) = \left(D(f_a), \frac{g_a}{f_a} \right).$$

Then

$$(V_a, \varphi|_{V_a}) = \left(V_a \cap D(f_a), \frac{g_a}{f_a} \right)$$

$$= \left(D(f_a), \frac{g_a}{f_a} \right)$$

□

Lemma 3.0.25. Let $R' \supset R$ be an integral ring extension ($a \in R'$ is the zero of a monic polynomial in $R[x]$). Then the Krull dimension (2.0.22) of $R =$ the Krull dimension of R' .

Corollary 3.0.26. The Krull dimension of $k[x_1, \dots, x_n] = n$ and if \mathfrak{m} is a maximal ideal of $k[x_1, \dots, x_n]$ then the $\text{ht}(\mathfrak{m}) = n$ (2.0.24). Therefore, finitely generated k -algebras have finite Krull dimension.

Theorem 3.0.27 (Krull PIT). Let $a \in R$ and let \mathfrak{p} be the minimal prime-ideal in R that contains a . Then $\text{ht}(\mathfrak{p}) \leq 1$.

Corollary 3.0.28. If \mathfrak{m} is maximal then $\text{ht}(\mathfrak{m}) = 1$. So $\dim V(f) = n - 1$.

Proposition 3.0.29. Let R be a Noetherian domain. Then:

1. $\text{ht}(\mathfrak{m}) = 1 \Rightarrow \mathfrak{m} = (f)$.
2. R is a Unique Factorization Domain.

Chapter 4

Lecture 4: Morphisms

Recall: Sheaves on X , where X is a topological space. Important examples are \mathcal{O}_X (3.0.11), the sheaf of regular functions on an affine variety X , where $\mathcal{O}_X(X) = A(X)$ and **stalk** $\mathcal{O}_{X,a} = A(X)_{\mathfrak{m}_a}$.

Today:

- Ringed spaces (X, \mathcal{O}_X) .
 - Morphisms of ringed spaces.
 - Morphisms of (quasi-)affine varieties.
 - Homomorphisms of k -algebras.
- Abstract affine varieties.

4.0.1 Ringed space (X, \mathcal{O}_X)

Ringed spaces

Definition 4.0.1. A **ringed space** is a topological space X together with a *sheaf of rings* \mathcal{O}_X on X .

Remark 4.0.2. \mathcal{O}_X in 4.0.1 is called the **structure sheaf** of the ringed space.

Remark 4.0.3. The convention until lecture ~ 10 is that

$$\mathcal{O}_X \subset \text{Hom}(-, k)$$

so that

- $\mathcal{O}_X(U) \subset \text{Hom}(U, k)$.
- $\rho_{U,V}$ is a restriction of a function.

Example 4.0.4. If $X \subset \mathbb{A}_k^n$ then with \mathcal{O}_X sheaf of regular functions.

Example 4.0.5. $U \subset X$ open subset of affine variety then (U, \mathcal{O}_U) ringed space where $\mathcal{O}_U = \mathcal{O}_X|_U$ (“quasi-affine” variety).

Example 4.0.6. If X is a (smooth) manifold then \mathcal{O}_X is the *sheaf of C^∞ functions*, so that for $U \subset X$ open we have

$$\mathcal{O}_X(U) = \{f : U \rightarrow \mathbb{R} \mid f \text{ smooth}\}.$$

Morphism of ringed spaces

Definition 4.0.7. A morphism $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of ringed spaces is:

- A continuous map $f : X \rightarrow Y$ such that
 - For all open subsets $U \subset Y$ and for all $\varphi \in \mathcal{O}_Y(U)$ we have

$$f^*(\varphi) := \varphi \circ f \in \mathcal{O}_X(f^{-1}(U)).$$

Remark 4.0.8. f^* in 4.0.7 is called the **pullback**. Also notice that f is continuous so that $f^{-1}(U)$ is open.

Example 4.0.9. Let X, Y be affine varieties. Then the condition on $U \subset Y$ and $\varphi \in \mathcal{O}_Y(U)$ in 4.0.7 means that if φ is regular (3.0.11) $\Rightarrow f^*(\varphi)$ is regular.

Remark 4.0.10. It seems like it is fine if $X \subset \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$ (where m does not necessarily equal n) but one wishes the base field to be the same.

Warning: Non-standard definition. Without 4.0.3 need to give as data maps $f^* : \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$.

Proposition 4.0.11. *Morphism between affine varieties ([Gat21, Prop. 4.7]) Let $U \subset X$ be an open subset of an affine variety X and let $Y \subset \mathbb{A}_k^n$ be another affine variety. Then the morphisms $f : U \rightarrow Y$ are exactly the maps of the form*

$$\begin{aligned} f &= (\varphi_1, \dots, \varphi_n) : U \rightarrow Y \\ x &\mapsto (\varphi_1(x), \dots, \varphi_n(x)), \end{aligned}$$

with $\varphi_i \in \mathcal{O}_X(U)$. In particular, the morphisms from U to \mathbb{A}^1 are exactly the regular functions in $\mathcal{O}_X(U)$.

Remark 4.0.12. Proof below taken from [Gat21, Prop. 4.7]

Proof. First assume that $f : U \rightarrow Y$ is a morphism. For $i = 1, \dots, n$ we have that the coordinate-function $y_i : Y \rightarrow \mathbb{A}^1$ defined by $y_i(a_1, \dots, a_n) := a_i$ for $a \in Y \subset \mathbb{A}_k^n$ is regular (3.0.11), since we can write y_i as

$$\begin{aligned} y_i(x_1, \dots, x_n) &= x_i \\ &= \frac{x_i}{1}, \quad (\text{with } x_i, 1 \in A(Y)) \end{aligned}$$

and where $1 \neq 0$ if $Y \neq \emptyset$. Since f is a morphism we have that

$$\varphi_i := f^*(y_i) \in \mathcal{O}_X(f^{-1}(Y)) = \mathcal{O}_X(U), \quad (\text{by 4.0.7}).$$

But this is just the i^{th} coordinate function of f , so that $f = (\varphi_1, \dots, \varphi_n)$.

On the other hand, assume that $f = (\varphi_1, \dots, \varphi_n)$ where $\varphi_i \in \mathcal{O}_X(U)$ and $f(U) \subset Y$. We then claim that f is continuous: Let $Z \subset Y$ be closed $\Rightarrow Z = V(g_1, \dots, g_m)$ for $g_1, \dots, g_m \in A(Y)$, so we have that

$$\begin{aligned} f^{-1}(Z) &= \{x \in U : f(x) \in V(g_1, \dots, g_m)\} \\ &= \{x \in U : (\varphi_1(x), \dots, \varphi_n(x)) \in V(g_1, \dots, g_m)\}, \quad (\text{since } f = (\varphi_1, \dots, \varphi_n)) \\ &= \{x \in U : g_i(\varphi_1(x), \dots, \varphi_n(x)) = 0, \text{ for } i = 1, \dots, m\}. \end{aligned}$$

The functions $\eta_i : x \mapsto g_i(\varphi_1(x), \dots, \varphi_n(x))$ we claim are regular (3.0.11): We know that locally we have that $\varphi_i = \frac{g_i}{f_i}$ with $f_i \neq 0$. Therefore, when we give $\varphi_1, \dots, \varphi_n$ as arguments to $g_i \in A(Y)$ we

get back (after putting everything on the same denominator) a quotient of polynomials back. Then the denominator will still be non-zero and so indeed we have that $g_i(\varphi_1, \dots, \varphi_n) = \frac{g'_i}{f'_i}$ locally. Then since by 3.0.13 $V(\eta_i)$ are closed for $i = 1 \dots, n$ we see that since

$$\begin{aligned} f^{-1}(Z) &= V(\eta_1, \dots, \eta_n) \\ &= V(\eta_1) \cap \dots \cap V(\eta_n), \end{aligned}$$

we have that $f^{-1}(Z)$ is closed $\Rightarrow f$ is continuous ($f : X \rightarrow Y$ continuous $\Leftrightarrow U \subset Y$ closed then $f^{-1}(U)$ closed).

If $\varphi \in \mathcal{O}_Y(W)$ is a regular function on some open subset $W \subset Y$ then

$$\begin{aligned} f^*(\varphi) &= \varphi \circ f : f^{-1}(W) \rightarrow k \\ x &\mapsto \varphi(\varphi_1(x), \dots, \varphi_n(x)), \quad (\text{since } f = (\varphi_1, \dots, \varphi_n)). \end{aligned}$$

This is again a quotient of polynomials locally. Therefore, $f^* \circ \varphi$ is regular. Hence f is a morphism. \square

Remark 4.0.13. It seems like we want to assume that $Y \neq \emptyset$ in the proof given above. If not then perhaps another proof works for the case $Y = \emptyset$.

Corollary 4.0.14. *Given X, Y affine varieties we have a bijection.*

$$\begin{array}{ccc} \{\text{morphisms } X \rightarrow Y\} & \xleftrightarrow{1:1} & \{k\text{-algebra homomorphisms } A(Y) \rightarrow A(X)\} \\ f & \mapsto & f^* \end{array}$$

In particular, isomorphisms of affine varieties correspond to isomorphisms of k -algebras.

Proof. Any morphism $f : X \rightarrow Y$ determines a k -algebra homomorphism $f^* : \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(X)$ (note the contravariance). By 3.0.23 we get $f^* : A(Y) \rightarrow A(X)$.

On the other hand, let $g : A(Y) \rightarrow A(X)$ be a k -algebra homomorphism. We assume that $Y \subset \mathbb{A}_k^n$ and let y_1, \dots, y_n be the coordinate-functions. Viewing $y_i \in A(Y)$ we see that $\varphi_i := g(y_i) \in A(X) = \mathcal{O}_X(X)$ for $i = 1, \dots, n$. Let $f := (\varphi_1, \dots, \varphi_n) : X \rightarrow \mathbb{A}^n$ then for $h \in k[y_1, \dots, y_n]$ and all $x \in X$ we have

$$\begin{aligned} (f^*h)(x) &= h(f(x)) \\ &= h(\varphi_1(x), \dots, \varphi_n(x)) \\ &= g(h)(x), \quad (*) \end{aligned}$$

where $(*)$ follows from the fact that both $h(\varphi_1, \dots, \varphi_n)$ and $g(h)$ are k -algebra homomorphisms and we note that if

$$h = \sum_{\alpha} c_{\alpha} y_1^{\alpha_1} \dots y_n^{\alpha_n}$$

then

$$g(h) = \sum_{\alpha} c_{\alpha} g(y_1)^{\alpha_1} \dots g(y_n)^{\alpha_n}.$$

Since $\varphi_i = g(y_i)$ we see that they are equal. Furthermore, we see that $h(f(x)) = 0$ for all $h \in I(Y) = \{g \in A(Y) : g(a) = 0, \forall a \in Y\}$ since $f(x) \in Y$ for all $x \in X$. Since $(h \circ f)(x) = g(h)(x)$ we see that g vanishes on $h \in I(Y)$. We see that $f(x) \in V(I(Y)) = Y$ (by Nullstellensatz, since Y is an affine variety). Therefore $f : X \rightarrow Y$. Since $f = (\varphi_1, \dots, \varphi_n)$ it follows by 4.0.11 that f is a morphism. Furthermore, since $f^*(h) = g(h)$ for all $h \in k[y_1, \dots, y_n]$ we see that $f^* = g$. Furthermore, assume that f, g are mutual inverses. Then

$$\begin{aligned} (f \circ g)^* &= g^* \circ f^* \\ &= \text{id}^* \\ &= \text{id} \end{aligned}$$

and

$$\begin{aligned}(g \circ f)^* &= f^* \circ g^* \\ &= \text{id}^* \\ &= \text{id}\end{aligned}$$

so that f^* and g^* are mutual inverses. On the other hand, if $h : A(Y) \rightleftharpoons A(X) : g$ are two k -algebras that are mutual inverses, then they correspond to $f_h : X \rightleftharpoons Y : f_g$ such that $f_g^* = g, f_h^* = h$. Hence

$$\begin{aligned}f_g^* \circ f_h^* &= (f_h \circ f_g)^* \\ &= \text{id}_{A(X)}\end{aligned}$$

and

$$\begin{aligned}f_h^* \circ f_g^* &= (f_g \circ f_h)^* \\ &= \text{id}_{A(Y)}\end{aligned}$$

But what we have just shown (bijective property), and the fact that $\text{id}_X \rightsquigarrow \text{id}_X^* = \text{id}_{A(X)}$ and $\text{id}_Y \rightsquigarrow \text{id}_Y^* = \text{id}_{A(Y)}$ we see that $f_g \circ f_h = \text{id}_X$ and $f_h \circ f_g = \text{id}_Y$. \square

Example 4.0.15 ([Gat21, Ex. 4.9]; (Isomorphisms of affine varieties \neq bijective morphisms). Let $X = V(x_1^2 - x_2^3) \subset \mathbb{A}^2$ (**cubic curve** since in *affine setting* “defining polynomial” is of multidegree 3).

Consider the map $f : \mathbb{A}^1 \rightarrow X$ defined by $t \mapsto (t^3, t^2)$. This is a morphism by 4.0.11. By 4.0.14 we get a corresponding k -algebra homomorphism $f^* : A(X) = k[x_1, x_2]/(x_1^2 - x_2^3) \rightarrow A(\mathbb{A}^1) = k[t]$. To see how f^* looks explicitly; we then see that

$$\begin{aligned}f^*x_1 &= (x_1 \circ f)(t) \\ &= x_1(t^3, t^2) \\ &= t^3\end{aligned}$$

and

$$\begin{aligned}f^*x_2 &= (x_2 \circ f)(t) \\ &= x_2(t^3, t^2) \\ &= t^2\end{aligned}$$

Since f^* is a k -algebra homomorphism and \bar{x}_1, \bar{x}_2 generated $A(X)$ we see that this determines f completely.

We claim that $f^{-1} : X \rightarrow \mathbb{A}^1$ defined by

$$(x_1, x_2) \mapsto \begin{cases} \frac{x_1}{x_2}, & \text{if } x_2 \neq 0 \\ 0, & \text{if } x_2 = 0. \end{cases}$$

is an inverse to f .

Assume $t \neq 0$. Then

$$\begin{aligned}(f^{-1} \circ f)(t) &= f^{-1}(t^3, t^2) \\ &= \frac{t^3}{t^2} \\ &= t.\end{aligned}$$

Assume $t = 0$. Then

$$\begin{aligned}(f^{-1} \circ f)(0) &= f^{-1}(0, 0) \\ &= 0.\end{aligned}$$

On the other hand, for $(x_1, x_2) \in X = V(x_1^2 - x_2^3)$ we have that for $x_2 \neq 0$:

$$\begin{aligned}(f \circ f^{-1})(x_1, x_2) &= f\left(\frac{x_1}{x_2}\right) \\ &= \left(\left(\frac{x_1}{x_2}\right)^2, \left(\frac{x_1}{x_2}\right)^3\right) \\ &= \left(\frac{x_1^2}{x_2^2}, \frac{x_1^3}{x_2^3}\right) \\ &= (x_1, x_2).\end{aligned}$$

On the other hand, if $(x_1, x_2) \in X$ with $x_2 = 0$ then since $x_2^3 = 0 = x_1^2$ we must have that $x_1 = 0$ so that

$$\begin{aligned}(f \circ f^{-1})(x_1, x_2) &= f(0) \\ &= (0, 0).\end{aligned}$$

If f was an isomorphism (so that f^{-1} was also a morphism) then by 4.0.14 f^* would have to be an isomorphism as well. But $t \notin \text{Im}(f^*)$ so this is false.

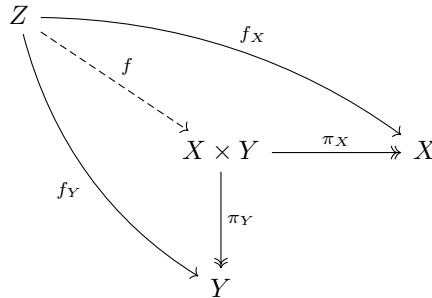
4.0.2 The product of varieties

Recall: $X = V(I) \subset \mathbb{A}^n$ and $Y = V(J) \subset \mathbb{A}^m \rightsquigarrow \underbrace{X \times Y}_{=V(I^e + J^e)} \subset \mathbb{A}^n \times \mathbb{A}^m$.

Proposition 4.0.16 (Universal property of products ([Gat21, Prop. 4.10])). *Let X and Y be affine varieties and let $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ be the projection morphisms from the product onto the two factors. Then for every affine variety Z and two morphisms $f_X : Z \rightarrow X$ and $f_Y : Z \rightarrow Y$ there is a unique morphism $f : Z \rightarrow X \times Y$ such that*

$$\begin{aligned}f_X &= \pi_X \circ f \\ f_Y &= \pi_Y \circ f.\end{aligned}$$

In other words, giving a morphism from an affine variety to a product $X \times Y$ is the same as giving a morphism to each of the factors X, Y . We illustrate with a diagram below.



(Abstract) affine variety

Definition 4.0.17. A **abstract affine variety** is a ringed space (4.0.1) isomorphic to an affine variety (1.0.4).

Proposition 4.0.18 ([Gat21, Prop. 4.17]). *Let X be an affine variety and let $f \in A(X) = \mathcal{O}_X(X)$. Then the distinguished open subset $D(f)$ (3.0.17) is an affine variety with coordinate ring*

$$A(D(f)) \cong A(X)_f.$$

Proof. We have that

$$Y := \{(x, t) \in X \times \mathbb{A}^1 : tf(x) = 1\} \quad (4.0.1)$$

is an affine variety since $Y = V_{X \times \mathbb{A}^1}(tf - 1)$ where $tf - 1 \in k[x_1, \dots, x_n, t]/I(X \times \mathbb{A}^1)$ if $X \subset \mathbb{A}_k^n$.

The affine variety Y is isomorphic to $D(f)$ by the projection morphism

$$\begin{aligned} g : Y &\rightarrow D(f) \\ (x, t) &\mapsto x \end{aligned}$$

with inverse

$$\begin{aligned} g^{-1} : D(f) &\rightarrow Y \\ x &\mapsto \left(x, \frac{1}{f(x)}\right) \end{aligned}$$

Now, $(D(f), \mathcal{O}_X|_{D(f)})$ is a *ringed space*, and $D(f)$ is isomorphic to the affine variety Y , so by 4.0.17 $(D(f), \mathcal{O}_X|_{D(f)})$ is an abstract affine variety. By 3.0.21 we have

$$\mathcal{O}_X(D(f)) \cong A(X)_f.$$

It remains to show that $A(D(f)) \cong \mathcal{O}_X(D(f))$. But for an affine variety Y we have that $\mathcal{O}_Y(Y) = A(Y)$. Then by [Gat21, Remark 3.16] we have that

$$\begin{aligned} \mathcal{O}_{D(f)}(D(f)) &:= \mathcal{O}_X(D(f)) \\ &= A(D(f)), \end{aligned}$$

so that

$$\begin{aligned} A(D(f)) &= \mathcal{O}_X(D(f)) \\ &\cong A(X)_f. \end{aligned}$$

□

Chapter 5

Lecture 5

5.0.1 Prevarieties

Example 5.0.1 (The need for a bigger category). $\mathbb{A}^2 \setminus \{(0,0)\}$ is not a variety. Regular functions (3.0.11) on $\mathbb{A}^2 \setminus \{(0,0)\} \cong k[x, y]$ if it was an affine variety it would be isomorphic to (V, \mathcal{O}_V) . $\mathcal{O}_V \simeq k[x, y] \Rightarrow V = \mathbb{A}^2$. $\mathbb{A}^2 \setminus \{(0,0)\} = D(x) \cup D(y)$

Prevariety

Definition 5.0.2. A **prevariety** is a ringed space (X, \mathcal{O}_X) such that

$$X = \bigcup_{i=1}^n U_i, \quad (5.0.1)$$

where U_i is open such that $(U_i, \mathcal{O}_X|_{U_i})$ are affine varieties.

Remark 5.0.3. The U_i in 5.0.2 are called **affine open sets of X** , \mathcal{O}_X the **structure sheaf** and $\mathcal{O}_X(U)$ are called **regular functions**. Notice that we don't require $U_i \cong V(J) \subset \mathbb{A}_k^n$ given that $X \subset \mathbb{A}_k^n$, only that $U_i \cong \mathbb{A}_k^{n_i}$ where n_i does not necessarily equal n .

Morphism of prevarieties

Definition 5.0.4. A morphism $f : X \rightarrow Y$ of prevarieties (5.0.2) is:

- Continuous map of topological spaces such that for all $V \subset Y$ open and $\varphi : V \rightarrow k$ regular we have that $f^*(\varphi) = \varphi \circ f : f^{-1}(V) \rightarrow k$ is regular, i.e. f^* induces a **morphism of sheaves** $f^* : \mathcal{O}_Y \rightarrow f^* \mathcal{O}_X$ on Y , where $f^* \mathcal{O}_X(V) := \mathcal{O}_X(f^{-1}(V))$ on $V \subset Y$ open.

Example 5.0.5. Let $U \subseteq X$ where (X, \mathcal{O}_X) is an affine variety $\Rightarrow (U, \mathcal{O}_X|_U)$ is a (pre-)variety (5.0.2).

$$\begin{aligned} U &= X \setminus V(f_1, \dots, f_n) \\ &= D(f_1) \cup \dots \cup D(f_n), \\ &\rightsquigarrow \mathcal{O}_X|_U|_{D(f_i)} = \mathcal{O}_X|_{D(f_i)} \end{aligned}$$

and $(D(f_i), \mathcal{O}_X|_{D(f_i)})$ affine.

5.0.2 Gluing

Gluing two prevarieties: Let X_1, X_2 be prevarieties, let $U_{1,2} \subset X_1$ and $U_{2,1} \subset X_2$ be open sets with an isomorphism of prevarieties $f : U_{1,2} \rightarrow U_{2,1}$. We can construct X as $X = (X_1 \sqcup X_2)/(a \sim f(a))$ with the quotient topology, and where

$$\begin{aligned}\mathcal{O}_X(V) &:= \{\varphi : V \rightarrow k \mid i_i^* \varphi \in \mathcal{O}_{X_i}(i_i^{-1}(V))\} \\ &= \{\varphi : V \rightarrow k \mid \varphi \circ i_i \in \mathcal{O}_{X_i}(i_i^{-1}(V))\}\end{aligned}$$

where

$$\begin{aligned}i_1 : X_1 &\rightarrow X \\ i_2 : X_2 &\rightarrow X\end{aligned}$$

that takes points in X_i to their equivalence class in X .

Since X_i are prevarieties we have

$$X_i = \bigcup_{j=1}^n V_{i,j}, \quad \text{for } i = 1, 2,$$

and

$$X = \bigcup_{i=1,2} i_i(V_{i,j})$$

and ...

Example 5.0.6. Let

$$\begin{aligned}X_1 &= X_2 \\ &= \mathbb{A}^1\end{aligned}$$

$$\begin{aligned}U_{1,2} &= U_{2,1} \\ &= \mathbb{A}^1 \setminus \{0\}.\end{aligned}$$

Remark 5.0.7. Notice that

$$\begin{aligned}D(x) &= \{a \in \mathbb{A}^1 : f(a) \neq 0, \forall f(x) \in \langle x \rangle\} \\ &= \mathbb{A}^1 \setminus \{0\}\end{aligned}$$

so that $\mathbb{A}^1 \setminus \{0\}$ is indeed *open*.

Bad news: Let $f : U_{1,2} \rightarrow U_{2,1}$ be defined by $x \mapsto x$ (i.e. basically the identity map). Then

as in 5.1 we get an affine line with two zero points (“not separated”).

Good news: $f : U_{1,2} \rightarrow U_{2,1}$ defined by $t \mapsto \frac{1}{t}$ with $X = \mathbb{P}^1$. By the gluing construction, we call a set $U \subset X$ is open if $i_1^{-1}(U)$ and $i_2^{-1}(U)$ is open. If we take $X_1 = \mathbb{A}^1 \subset X$ (i.e. $i_1(X_1)$) then $i_1^{-1}(i_1(X_1)) = X_1$ which is open in X_1 , and $i_2^{-1}(i_1(X_1)) = X_2 \setminus \{0\}$ since $i_1(0)$ does not get identified with 0 coming from X_2 and so $0 \in X_2$ is not in the preimage of $i_1(X_1)$ under i_2^{-1} ; but every other element $X_2 \ni s \neq 0$ is identified with an element of X_1 . $X_2 \setminus \{0\}$ is open in X_2 by what we said earlier, hence X_1 is open in X .

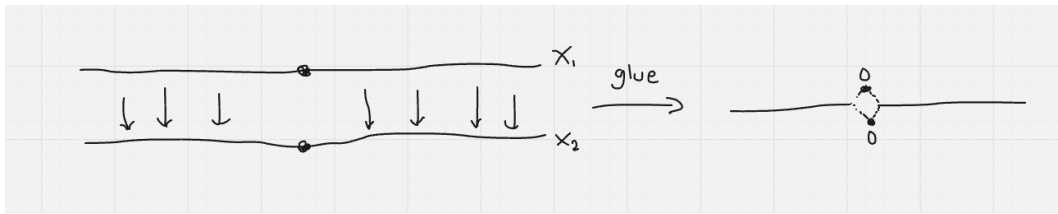


Figure 5.1:

We have that

$$\begin{aligned}
 X \setminus X_1 &:= X \setminus i_1(X_1) \\
 &= \{0_{X_2}\}, \quad (0_{X_2} \text{ being the zero in } X_2) \\
 &= X_2 \setminus U_{2,1}
 \end{aligned}$$

which corresponds to “ $\infty = \frac{1}{0}$ ” in X_1 (intuitively not strictly speaking, since we identify t in X_1 with $\frac{1}{t}$ in X_2 , and so “ $t = 0$ gets identified with $\frac{1}{0}$ in X_2 ”). Again, intuitively speaking, “from the perspective of X_1 it looks like $t = 0$ corresponds to $\infty = \frac{1}{0}$ in X_2 ”.

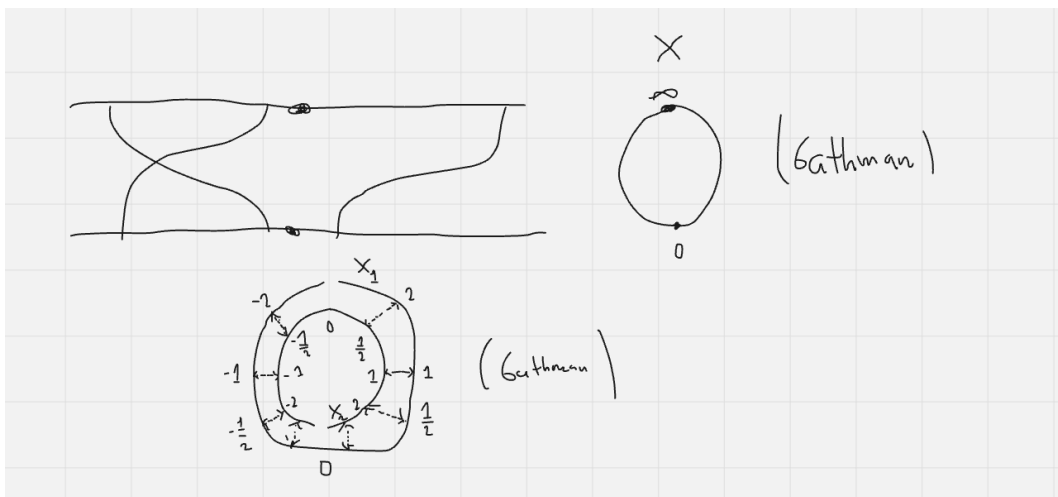


Figure 5.2:

You can extend/study topological properties for prevarieties.

Open subprevariety

Definition 5.0.8. Let X be a prevariety (5.0.2) and let $U \subset X$ be an open subset. Then U is again a prevariety $(U, \mathcal{O}_X|_U)$. If $X = \bigcup_{i=1}^n U_i$ is an open (finite) cover by affine varieties

$$\begin{aligned} U &= X \cap U \\ &= \left(\bigcup_{i=1}^n U_i \right) \cap U \\ &= \bigcup_{i=1}^n (U_i \cap U). \end{aligned}$$

We note that the $U_i \cap U$ is open in U_i (since U is open in X). Since every open subset of an affine variety is a prevariety ([Gat21, Example 5.3]), it follows that $U_i \cap U$ can be covered by a finite family of affine varieties, so it follows that so can U .

- $\mathcal{O}_X|_U(V) = \mathcal{O}_X(U \cap V)$.

We then call U a **open subprevariety**.

Closed subprevariety

Definition 5.0.9. Let $Y \subseteq X$ be closed. Since an open subset $U \subset Y$ need not be open in X , one can not set $\mathcal{O}_Y(U) = \mathcal{O}_X(U)$. Instead, one defines $\mathcal{O}_Y(U)$ to be the k -algebra of functions that are *locally restriction of functions on X* :

$$\mathcal{O}_Y(U) := \left\{ \varphi : U \rightarrow k \text{ such that for all } a \in U, \exists \text{ open } V \ni a \text{ in } X, \right. \\ \left. \text{and } \psi \in \mathcal{O}_X(V) \text{ with } \psi = \varphi \text{ on } U \cap V. \right\}$$

We then call Y a **closed subprevariety**.

Let $f : Z \rightarrow X$ be a morphism of prevarieties (5.0.4) such that $f(Z) \subseteq Y$ where $Y \subset X$ is a closed subprevariety (5.0.9). We can consider $f : Z \rightarrow Y$ a morphism of prevarieties. Closed subprevarieties satisfy the required **universal property** (Y, i) :

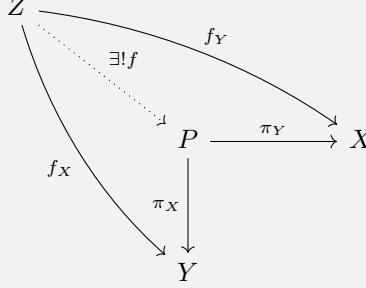
$$\begin{array}{ccc} Y & \xrightarrow{\quad} & X \\ & \nwarrow \text{!} & \nearrow \\ & Z & \end{array}$$

Proposition 5.0.10. Let $f : X \rightarrow Y$ be a morphism of prevarieties. Then the preimage of an open/closed subprevariety is an open/closed subprevariety.

Example 5.0.11. Images can be horrible: Let $X = V(x_2x_3 - 1) \cup \{(0, 0, 0)\} \subseteq \mathbb{A}^3$ and let $f : X \rightarrow \mathbb{A}^2$ be defined by $(x_1, x_2, x_3) \mapsto (x_1, x_2)$. Then we claim that the $\text{Im}(f) = \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\}) \cup \{(0, 0)\}$. It is clear that the x_1 coordinate is free to vary over \mathbb{A}^1 since the condition imposed on X by $x_2x_3 - 1$ *only pertains* to the last two coordinates. Since $(0, 0, 0) \in X$ we see that $(0, 0)$ is indeed in the image of f . Lastly, if (a_1, a_2, a_3) is such that $a_2 = 0$ then $a_2a_3 - 1 = -1$ and hence no element with $a_2 = 0$ is in X and hence not in the image of f . But as long as $a_2 \neq 0$ we can set $a_3 = \frac{1}{a_2}$ and we get that $(a_1, a_2, a_3) \in V(x_2x_3 - 1)$. That is why we have the factor $\mathbb{A}^1 \setminus \{0\}$ in $\text{Im}(f)$.

Products of prevarieties

Definition 5.0.12. Let X, Y be prevarieties. A **product** of X, Y is a prevariety (5.0.2) P together with morphisms $\pi_X : P \rightarrow X$ and $\pi_Y : P \rightarrow Y$ satisfying the following *universal property*: For any pair of morphisms $f_X : Z \rightarrow X$ and $f_Y : Z \rightarrow Y$ where Z is a prevariety, there is a *unique* morphism $f : Z \rightarrow P$ such that $\pi_X \circ f = f_X$ and $\pi_Y \circ f = f_Y$. Or in diagrammatic form:



Proposition 5.0.13. (*Existence of products*) Let X, Y be prevarieties. Then the product of X, Y always exists.

Proof. We proceed as in [Gat21, Prop. 5.15]: Let $X = \bigcup_{i=1}^n U_i$ and $Y = \bigcup_{j=1}^m V_j$ where U_i, V_j are affine varieties for all $i = 1, \dots, n$ and $j = 1, \dots, m$. Assuming the result that gives that if U_i, V_j are affine varieties then so are $U_i \times V_j$ ([Gat21, Example 1.5]; HW1). Therefore we have that for all $i = 1, \dots, n$ and all $j = 1, \dots, m$ we have that $U_i \times V_j$ is an affine variety. Then we glue any pairs $U_i \times V_j$ and $U_{i'} \times V_{j'}$ together by the identity morphism $\text{id} : (U_i \cap U_{i'}) \times (V_{j'} \cap V_{j'}) \rightarrow (U_i \cap U_{i'}) \times (V_{j'} \cap V_{j'})$. To see that $(U_i \cap U_{i'}) \times (V_{j'} \cap V_{j'}) \subseteq X \times Y$ is open, we notice that $U_i \cap U_{i'}$ is open in X and that $V_j \cap V_{j'}$ is open in Y (since they are finite intersections of open sets). We want to express $(U_i \cap U_{i'}) \times (V_{j'} \cap V_{j'})$ as the complement of a closed subset in $X \times Y$, i.e. we want to show that $(U_i \cap U_{i'}) \times (V_{j'} \cap V_{j'}) = X \times Y \setminus (V(I) \cap X \times Y)$ with $I \leq k[x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}]$ if $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$. By assumption we have

$$U_i = \bigcup_{\alpha} D(f_{\alpha})$$

$$U_{i'} = \bigcup_{\beta} D(g_{\beta})$$

and so

$$\begin{aligned} U_i \cap U_{i'} &= \left(\bigcup_{\alpha} D(f_{\alpha}) \right) \cap \left(\bigcup_{\beta} D(g_{\beta}) \right) \\ &= \bigcup_{\alpha, \beta} D(f_{\alpha}) \cap D(g_{\beta}) \\ &= \bigcup_{\alpha, \beta} D(f_{\alpha} g_{\beta}) \end{aligned}$$

and similarly we have that

$$\begin{aligned} V_j \cap V_{j'} &= \dots \\ &= \bigcup_{\gamma, \ell} D(f_{\gamma} g_{\ell}). \end{aligned}$$

We can view all $f_\alpha, g_\beta, f_\gamma, g_\ell$ as polynomials in $k[x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}]$ and then we have that

$$\begin{aligned} (U_i \cap U_{i'}) \times (V_j \cap V_{j'}) &= \left(\bigcup_{\alpha, \beta} D(f_\alpha g_\beta) \right) \times \left(\bigcup_{\gamma, \ell} D(f_\gamma g_\ell) \right) \\ &= \bigcup_{\alpha, \beta, \gamma, \ell} D((f_\alpha g_\beta)(f_\gamma g_\ell)) \\ &= (X \times Y) \setminus (V(I) \cap X \times Y) \end{aligned}$$

where $I = \langle f_\alpha g_\beta f_\gamma g_\ell : \forall \alpha, \beta, \gamma, \ell \rangle \subset k[x_1, \dots, x_n, x_{n+1}, \dots, x_m]$. We left out some details in the above proof that needs verifying, e.g. that $D(f) \times D(g) = D(fg)$ (where we in the right-hand side take by f mean f as in $k[x_1, \dots, x_n, x_{n+1}, \dots, x_m]$ while on the left-hand side we take f as sitting in say $k[x_1, \dots, x_n]$ and g sitting inside $k[x_{n+1}, \dots, x_{n+m}]$).

Moving on with the proof: the gluing happens as said earlier along the identity map on $(U_i \cap U_{i'}) \times (V_j \cap V_{j'})$. Notice that $X_{i,j} := U_i \times V_j$ so the index set is $I \times J$ (which is finite since I, J are finite; this is analogously to our earlier gluing construction). Furthermore, we let $\text{id}_{(i,j)} : U_i \times V_j \supseteq (U_i \cap U_{i'}) \times (V_j \cap V_{j'}) \rightarrow U_i \cap U_{i'} \times (V_j \cap V_{j'}) \subseteq U_{i'} \times V_{j'}$. Then we see that $\text{id}_{(i,j)}^{-1} = \text{id}_{i',j'}$

- Since all gluing morphisms are id one sees that $f_{i,j}^{-1} = f_{j,i}$.
- We note that $\text{id}_{(i,j)(i',j')}^{-1}((U_{i'} \cap U_{i''}) \times (V_{j'} \cap V_{j''})) = (U_i \cap U_{i'} \cap U_{i''}) \times (V_j \cap V_{j'} \cap V_{j''}) \subset (U_i \cap U_{i''}) \times (V_j \cap V_{j''})$ and so $\text{id}_{(i',j')(i'',j'')} \circ \text{id}_{(i,j)(i',j')} = \text{id}_{(i,j)(i'',j'')}$ holds on $(U_i \cap U_{i''}) \times (V_j \cap V_{j''})$. So the second condition in [Gat21, Construction 5.6.(b)] is satisfied.

By [Gat21, Lemma 4.6] one can glue all projections $U_i \times V_j \rightarrow U_i \subset X$ and $U_i \times V_j \rightarrow V_j \subset Y$ to $\pi_X : P \rightarrow X$ and $\pi_Y : P \rightarrow Y$.

We now want to check the universal property in 5.0.12: Let $f_X : Z \rightarrow X$ and $f_Y : Z \rightarrow Y$ be any pair of morphisms from a prevariety Z to X, Y , then the way to get that $\pi_X \circ f = f_X$ and $\pi_Y \circ f = f_Y$ is by letting $f := (f_X, f_Y) : Z \rightarrow P$ with $f(z) = (f_X(z), f_Y(z))$ (we think of P as $X \times Y$ [atleast as sets]).

We want to check that $f := (f_X, f_Y)$ is in fact a morphism (recall: that it is a morphism of prevarieties just means that it is a morphism of ringed spaces (4.0.7)).

By [Gat21, Lemma 4.6] it is enough to check that f restricted to an open cover of Z is such that the family of restriction-maps are morphisms. Let $Z_{ij} = f_X^{-1}(U_i) \cap f_Y^{-1}(V_j)$ and this indeed open in Z since f_X, f_Y are by assumption a morphism so is in particular continuous, and U_i, V_j are open in X, Y respectively so that $f_X^{-1}(U_i), f_Y^{-1}(V_j)$ are open and so are then their intersection. Furthermore we see that $Z = \bigcup_{i,j} Z_{ij}$ so that $\{Z_{i,j}\}_{i,j}$ is a (finite) open cover of Z . We note that Z_{ij} is an open subprevariety

(5.0.8) and so can be covered by a finite family of affine open subsets. This gives us a new open cover of Z which we can label as $\{Z_{ij}^{\alpha_k}\}$. Then we see that any $Z_{ij}^{\alpha_k}$ is such that $f(Z_{ij}^{\alpha_k}) \subseteq U_i, f(Z_{ij}^{\alpha_k}) \subseteq V_j$ (since $Z_{ij}^{\alpha_k} \subset f_X^{-1}(U_i) \cap f_Y^{-1}(V_j)$); the $U_i \times V_j$ are affine sets as we argued for earlier.

Then we get morphisms $f_X|_{Z_{ij}^{\alpha_k}} : Z_{ij}^{\alpha_k} \rightarrow U_i$ and $f_Y|_{Z_{ij}^{\alpha_k}} : Z_{ij}^{\alpha_k} \rightarrow V_j$ (restrictions of morphisms to open subsets are morphisms by [Gat21, Remark 4.5.(b)]) which by 4.0.16 gives us morphisms $f_{ij}^{\alpha_k} : Z_{ij}^{\alpha_k} \rightarrow U_i \times V_j$ where $f_{ij}^{\alpha_k}$ is the restriction of $f = (f_X, f_Y)$ to each $Z_{ij}^{\alpha_k}$. But each such restriction is now a morphism, so by [Gat21, Lemma 4.6] $f : Z \rightarrow X \times Y$ is a morphism.

For *existence* of a product, see [Gat21, Prop. 5.15]. □

$\mathcal{O}_{X \times Y}$

- $\mathcal{O}_{X \times Y} = \mathcal{O}_X \otimes_k \mathcal{O}_Y$.
- $\mathcal{O}_{X \times Y}(V) = \mathcal{O}_X(V) \otimes_k \mathcal{O}_Y(V)$.

Variety

Definition 5.0.14. Let X be a prevariety. We define

$$\Delta_X := \{(x, x) : x \in X\} \subseteq X \times X.$$

Then we say that X is a **variety** if Δ_X is *closed* in $X \times X$.

Proposition 5.0.15 ([Gat21, Lemma 5.18]).

- (a) *Affine varieties are varieties (5.0.14).*
- (b) *Open and closed subprevarieties (5.0.8, 5.0.9) of varieties, are varieties.*

Proof.

(a): If $X \subset \mathbb{A}^n$ is an affine variety then $\Delta_X = V(x_1 - y_1, \dots, x_n - y_n) \subset X \times X$ where x_1, \dots, x_n and y_1, \dots, y_n are the coordinates of the two factors (i.e. in $(x, x) \in X \times X$), respectively. Hence Δ_X is closed.

(b): Let $Y \subset X$ be open or closed where X is a variety.

Claim: There is an inclusion morphism $i : Y \times Y \rightarrow X \times X$.

Proof. Assuming that $\iota : Y \hookrightarrow X$ is a morphism together with π_Y^1 that sends $Y \times Y \ni (y_1, y_2) \mapsto y_1$ and π_Y^2 that takes $Y \times Y \ni (y_1, y_2) \mapsto y_2$, consider the diagram

$$\begin{array}{ccccc}
 Y \times Y & & & & \\
 \swarrow \scriptstyle \exists! i & \searrow \scriptstyle \iota \circ \pi_Y^1 & & \searrow & \\
 & X \times X & \xrightarrow{\pi_X} & & X \\
 \searrow \scriptstyle \iota \circ \pi_Y^2 & \downarrow \scriptstyle \pi_X & & & \\
 & X & & &
 \end{array}$$

□

and use the universal property 5.0.12 to see that $i : Y \times Y \rightarrow X \times X$ defined by $Y \times Y \ni (y, y) \mapsto (y, y) \in X \times X$ is a morphism. Assuming this result, it follows in particular that i is continuous. Since X is a variety, it follows by definition (5.0.14) that Δ_X is closed and so $i^{-1}(\Delta_X) = \Delta_Y$ is closed $\Rightarrow Y$ is a variety. □

Proposition 5.0.16 (Properties of varieties ([Gat21, Prop. 5.20])). *Let $f, g : X \rightrightarrows Y$ be morphisms of prevarieties, and assume that Y is a variety. Then*

- (a) *The graph $\Gamma_f := \{(x, f(x)) : x \in X\} \subset X \times Y$ is closed in $X \times Y$.*
- (b) *The set $\{x : f(x) = g(x)\}$ is closed in X .*

Chapter 6

Lecture 6

6.0.1 Projective varieties - Topology

\mathbb{P}^n : Projective n -space

$$\begin{aligned}\mathbb{P}^n &= \{1\text{-dim linear subspaces of } k^{n+1}\} \\ &= k^{n+1} \setminus \{(0, \dots, 0)\} / \sim\end{aligned}$$

where

$$\begin{aligned}a \sim b &\Leftrightarrow \text{span}_k(\bar{a}) = \text{span}_k(\bar{b}) \\ \Leftrightarrow \bar{b} &= \lambda \bar{a}, \quad (\bar{a}, \bar{b} \in k^{n+1} \setminus \{(0, \dots, 0)\}, \lambda \in k^\times = k \setminus \{0\}).\end{aligned}$$

Three ways of topologizing \mathbb{P}^n :

1. Quotient topology (Zariski/Euclidian).
2. \mathbb{P}^n glued by several \mathbb{A}^n .
3. Analogous (I guess to affine n -space) theory of projective varieties; closed set = projective variety.

Let $f_i : \mathbb{A}^n \rightarrow \mathbb{P}^n$ be defined by $(x_1, \dots, x_n) \mapsto [x_1 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_n]$ for $i = 0, \dots, n$ with $f_0(x_1, \dots, x_n) = [1 : x_1 : x_2 : \dots : x_n]$. Then

$$\mathbb{P}^n = \underbrace{f_0(\mathbb{A}^n)}_{x_0 \neq 0} \cup \dots \cup \underbrace{f_n(\mathbb{A}^n)}_{x_n \neq 0}.$$

$$\mathbb{P}^n = \underbrace{\mathbb{A}^n}_{x_0 \neq 0} \cup \underbrace{\mathbb{P}^{n-1}}_{x_0=0}. \quad f_i \text{ injective with image } x_i \neq 0; (x_0 : \dots : x_n) \mapsto \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}\right) \text{ and } U_i = \{x_i \neq 0\}.$$

Today: Topology on \mathbb{P}^n by projective varieties. Next time we look at \mathbb{P}^n as ringed space/variety.

Problem: Can't evaluate polynomials at points in \mathbb{P}^n .

Example 6.0.1. $f(x_0, x_1) = x_0^2 + x_0x_1$ homogeneous of degree 2. We have that

$$\begin{aligned} f(1, 1) &= 2, \\ f(2, 2) &= 4 + 4 \\ &= 8 \\ \Rightarrow f(1, 1) &\neq f(2, 2), \end{aligned}$$

but $f(2, 2) = 2^2 \cdot f(1, 1)$.

Homogeneous polynomial of degree d

Definition 6.0.2. A polynomial $f(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ is **homogeneous of degree d** if it is a *sum of monomials*, each of degree d . If k is infinite (so in particular, if k is algebraically closed) this is equivalent to $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$. In particular, if $f(\bar{a}) = 0 \Leftrightarrow f(\lambda \bar{a})$ for all $\lambda \in k^\times$. Hence vanishing loci of homogeneous polynomials make

Graded ring

Definition 6.0.3. A **graded ring** R is a ring R together with abelian subgroups $R_d \subset R$ for all $d \in \mathbb{N}$ such that

- $R = \bigoplus_{d \in \mathbb{N}} R_d$; that is, for every $f \in R$ there is a decomposition of f as $f = \sum_{d \in \mathbb{N}} f_d$ such that $f_d \in R_d$ for all $d \in \mathbb{N}$ and for only *finitely many* $d \in \mathbb{N}$ we have that $f_d \neq 0$.
- For all $d, e \in \mathbb{N}$ and $f \in R_d, g \in R_e$ we have that $fg \in R_{d+e}$.

Example 6.0.4. Let $R = k[x_0, \dots, x_n]$. Then $R_d = \{f \in R : f \text{ homogeneous of degree } d\}$.

Homogeneous ideal

Definition 6.0.5. An ideal $J \subset R$ is a **homogeneous ideal** if it can be *generated* by homogeneous elements in R . By **homogeneous elements** we here mean that $u \in R$ is homogeneous if $u \in R_d$ for some $d \in \mathbb{N}$.

Lemma 6.0.6.

- (a) $J \subset R$ homogeneous ideal $\Leftrightarrow \forall f \in J$ if $f = \sum_{d \in \mathbb{N}} f_d$ then $f_d \in J$ for all $d \in \mathbb{N}$.
- (b) If J_1, J_2 are homogeneous ideals $\rightsquigarrow J_1 + J_2, J_1 J_2, J_1 \cap J_2$ and $\sqrt{J_i}$ are homogeneous ideals.
- (c) If J is homogeneous then the quotient R/J is a graded ring (6.0.3) with homogeneous decomposition $R/J = \bigoplus_{d \in \mathbb{N}} R_d / (R_d \cap J)$.

Proof. (a)(\Rightarrow): Let $J = \langle h^{(i)} : i \in I \rangle$ where $h^{(i)} \in R$ are homogeneous ideals of degree d . Let $f \in J$. Then by definition $f = \sum_{i \in I} g^{(i)} h^{(i)}$ where $g^{(i)}$ is not necessarily homogeneous of degree d , and where only finitely many $g^{(i)}$ are non-zero. Let

$$g^{(i)} = \sum_{e \in \mathbb{N}} g_e^{(i)}$$

be the *homogeneous decomposition* (6.0.3) of $g^{(i)}$ in $R = \bigoplus_{e \in \mathbb{N}} R_e$. We then have that

$$\begin{aligned} f &= \sum_{i \in I} g^{(i)} h^{(i)} \\ &= \sum_{i \in I} \left(\sum_{e \in \mathbb{N}} g_e^{(i)} \right) h^{(i)} \\ \Rightarrow f_d &= \sum_{\substack{i \in I, e \in \mathbb{N} \\ e + \deg(h^{(i)}) = d}} g^{(i)} h^{(i)} \in J. \end{aligned}$$

where f_d denotes the d -homogeneous part of f , for each $d \in \mathbb{N}$. But this is an R -linear combination of elements of J (the $h^{(i)}$ are in J). Since only finitely many $g^{(i)}$ are non-zero and for each $g^{(i)}$ there can only be finitely many non-zero $g_e^{(i)}$ this is a finite R -linear combination of elements $h^{(i)}$ and hence is in J . Hence $f_d \in J$ for all $d \in \mathbb{N}$.

(a)(\Leftarrow): Assume that for all $f \in J$ we have that $f = \sum_{d \in \mathbb{N}} f_d$ with each $f_d \in J$. Then we **claim** that $J = \langle h_d : h \in J, d \in \mathbb{N} \rangle$.

Proof of claim: Note that for all $h \in J$ we have that $h = \sum_{d \in \mathbb{N}} h_d$ with only finitely many h_d non-zero since R is a graded ring. Then clearly $J \subseteq \langle h_d : h \in J, d \in \mathbb{N} \rangle$ since $h_d \in J$ for all $d \in \mathbb{N}$. On the other hand, for the other direction, we just note that each h_d already is in J and J is an ideal so finite R -linear combination of such elements are still in J , therefore $J \supseteq \langle h_d : h \in J, d \in \mathbb{N} \rangle$. ■

Therefore, we conclude that $J = \langle h_d : h \in J, d \in \mathbb{N} \rangle$, and so indeed J is generated by homogeneous elements h_d .

(b)(\Rightarrow): Assume that J_1, J_2 are generated by homogeneous elements. Assuming the result $J_1 + J_2 = \langle J_1 \cup J_2 \rangle$ we see that $J_1 \cup J_2$ is a union of homogeneous elements, it is homogeneous.

Furthermore, notice that

$$J_1 J_2 = \left\{ \sum_{\text{finite}} j_1 j_2 \mid j_1 \in J_1, j_2 \in J_2 \right\}$$

and that since j_1, j_2 are homogeneous, we have that $j_1 \in R_{d_1}$ and $j_2 \in R_{d_2}$ so that $j_1 j_2 \in R_{d_1 + d_2}$ (6.0.3). Therefore we conclude that $J_1 J_2 = \langle j_1 j_2 : j_1 \in J_1, j_2 \in J_2 \rangle$ where each $j_1 j_2$ is homogeneous.

For $J_1 \cap J_2$ we use the equivalent characterization from (a), i.e. that $J_1 \cap J_2$ is homogeneous if for all $f \in J_1 \cap J_2$ we have that $f = \sum_{d \in \mathbb{N}} f_d$ with $f_d \in J$ for all d . First since f is in both J_1 and J_2 then

applying (a) to J_1, J_2 respectively shows that since f has a unique decomposition $f = \sum_{d \in \mathbb{N}} f_d$ with $f_d \in J_1$ and $f_d \in J_2 \Rightarrow f_d \in J_1 \cap J_2$ for all $d \in \mathbb{N}$ (where we assumed uniqueness of representation as a direct sum). Therefore, again by (a) we see that $J_1 \cap J_2$ is homogeneous (we here implicitly also use the result that says that the intersection of two ideals is again an ideal).

Lastly, we want to show that $\sqrt{J_i}$ is a homogeneous ideal if J_i is a homogeneous ideal. We use the equivalent condition coming from (a). Let $f \in \sqrt{J_i}$. We will use induction over the degree d of f . We write $f = f_0 + \dots + f_d$ which is its homogeneous decomposition (which exists since $R \ni f$ is graded). We then have

$$\begin{aligned} f^n &= (f_0 + \dots + f_d)^n \\ &= f_d^n + \text{terms of lower degree} \in J_i \end{aligned} \tag{6.0.1}$$

for some $n \in \mathbb{N}$. But then by (a) (assuming that f^n then has the homogeneous decomposition in 6.0.1) we see that $f_d^n \in J_i$. But then we have that $f_d \in \sqrt{J_i}$ so that $f - f_d \in \sqrt{J_i}$. Then there is some m so

that $(f - f_d)^m = f_d^m + \text{terms of lower degree}$ is in J_i . Then similarly it follows that $f_{d'}$ is in $\sqrt{J_i}$. By repeating the process a finite number of times we see that all f_i in $f = f_0 + \dots + f_d$ are in $\sqrt{J_i}$ and so by (a) it follows that $\sqrt{J_i}$ is homogeneous.

(c): Notice that we have a surjective group-homomorphism $R \twoheadrightarrow R/J$. If we restrict to R_d (which is an abelian subgroup of $(R, +)$) we get a group-homomorphism $R_d \rightarrow R/J$. The kernel of this is precisely $J \cap R_d$. The kernel is a normal subgroup by the first isomorphism theorem so $R_d/(R_d \cap J)$ is well-defined. We also get a unique injective group-homomorphism $\tilde{g} : R_d/(R_d \cap J) \rightarrow R/J$. Then we can take $\tilde{g}(R_d/(R_d \cap J)) := R_d/(R_d \cap J)$ as an isomorphic copy of $R_d/(R_d \cap J)$ in R/J (notice that the image of a group-homomorphism is a subgroup of the target of the group homomorphism).

Let $f \in R$ be arbitrary with homogeneous decomposition $f = \sum_{d \in \mathbb{N}} f_d$. Then project it down to R/J and use the homomorphism property to see that $\bar{f} = \sum_{d \in \mathbb{N}} \bar{f}_d$. We then consider that the injective homomorphism from $R_d/(R_d \cap J) \rightarrow R/J$ is injective, taking $\bar{f}_d \mapsto \bar{f}_d$ where we just quotient out the elements in J . But they have already been quotient out earlier by $R \rightarrow R/J$ since if $f \in J$ then $\pi\left(\sum_{d \in \mathbb{N}} f_d\right) = \sum_{d \in \mathbb{N}} \bar{f}_d$ by the homomorphism property where $\bar{f}_d = \pi(f_d)$. Now notice that by (a) if f is in J then f_d is in J for all d so that $\bar{f}_d = 0$. Therefore all non-zero elements $\bar{f}_d \in R/J$ are already in $R_d/(R_d \cap J)$ and so indeed $\bar{f}_d \in R_d/(R_d \cap J)$ (also notice that $0 \in R_d/(R_d \cap J)$ since it is a subgroup). Lastly, we claim that this decomposition is unique: If $\sum_{d \in \mathbb{N}} \bar{f}_d$ and $\sum_{d \in \mathbb{N}} \bar{g}_d$ are two decompositions of the same element in R/J then $\sum_{d \in \mathbb{N}} (\bar{f}_d - \bar{g}_d) = 0 \in R/J$ so that $\sum_{d \in \mathbb{N}} (f_d - g_d) \in J$ and hence $f_d - g_d \in J$ for all $d \in \mathbb{N}$ (by (a)) so that $\bar{f}_d - \bar{g}_d = 0 \in R/J \Leftrightarrow \bar{f}_d = \bar{g}_d \in R/J$. \square

6.0.2 Projective varieties

Projective varieties

Definition 6.0.7.

- Let $S \subseteq k[x_0, \dots, x_n]$. Then we define

$$V_p(S) := \{x \in \mathbb{P}^n : f(x) = 0, \text{ for all homogenous } f \in S\}.$$

Subsets $X \subset \mathbb{P}^n$ so that $X = V_p(S)$ for some $S \subseteq k[x_0, \dots, x_n]$ are called **projective varieties**.

- If $X \subset \mathbb{P}^n$ then

$$I_p(X) := \langle f \in k[x_0, \dots, x_n] : f \text{ homogenous and } f(x) = 0, \forall x \in X \rangle.$$

Exc: If $J = (h_1, \dots, h_n)$ and h_i homogenous then $V(J) = V(\{h_1, \dots, h_n\})$.

Example 6.0.8. $V(0) = \mathbb{P}^n$, $V((1)) = \emptyset$, $V(\underbrace{(x_0, \dots, x_n)}_{\text{irrelevant ideal}}) = \emptyset$.

Remark 6.0.9. If $J \subset k[x_0, \dots, x_n]$ and $J \neq (1)$ then $J \subset (x_0, \dots, x_n)$.

Proposition 6.0.10 (Projective Nullstellensatz ([Gat21, Prop. 6.20])).

- For any projective variety $X \subset \mathbb{P}^n$ we have $V_p(I_p(X)) = X$.
- For any homogeneous ideal $J \subseteq k[x_0, \dots, x_n]$ such that $\sqrt{J} \neq (x_0, \dots, x_n)$ we have $I_p(V_p(J)) = \sqrt{J}$.

There is an inclusion reversing correspondence

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{Projective Varieties} \\ X \subset \mathbb{P}^n \end{array} \right\} & \begin{array}{c} \xleftarrow{V_p(J) \leftarrow J} \\ \xrightarrow{X \mapsto I_p(X)} \end{array} & \left\{ \begin{array}{c} \text{homogenous radical ideals} \\ J \subset k[x_0, \dots, x_n] \\ J \neq (x_0, \dots, x_n) \end{array} \right\} \end{array}$$

Proof.

(a): According to [Gat21] the proof is the same as for the (affine) Nullstellensatz (cf. [Gat21, Prop. 1.10]).

(b): The inclusion $I_p(V_p(J)) \supset \sqrt{J}$ follows according to [Gat21] in the same way as for the affine case. It remains to show that $I_p(V_p(J)) \subseteq \sqrt{J}$:

We have

$$\begin{aligned} I_p(V_p(J)) &= \langle f \in k[x_0, \dots, x_n] : f \text{ homogeneous and } f(x) = 0, \forall x \in V_p(J) \rangle \\ &= \langle f \in k[x_0, \dots, x_n] : f \text{ homogeneous and } f(x) = 0, \forall x \in V_a(J) \setminus \{(0, \dots, 0)\} \rangle \end{aligned}$$

where the last equality follows from the fact that if $f(x) = 0$ then

$$\begin{aligned} f(\lambda x) &= \lambda^d f(x) \\ &= 0, \quad (\lambda \in k^\times) \end{aligned}$$

for f homogeneous of degree d . If $f \in V_a(J)$ then $f(x) = 0$ for all $f \in S \rightsquigarrow f(\lambda x) = 0$ for all $f \in S$ and $\lambda \in k^\times$ and so in particular it holds that for all *homogenous* $f \in J$ we have that $f(\lambda x) = 0$ and so $x \in V_p(J)$. On the other hand if $x \in V_p(J)$ so that $f(x) = 0$ for all homogeneous elements $f \in J$ then note that by 6.0.6.(a) we have that for arbitrary $f \in J$ we can rewrite it as $f = \sum_{d \in \mathbb{N}} f_d$ with f_d homogeneous and $f_d \in J$ but then surely $f(x) = 0$. Therefore it follows that $x \in V_a(J)$.

We note that $V_a(J)$ is closed, and that if $f(x) = 0$ for all $x \in V_a(J) \setminus \{0\}$ then $f(0) = 0$ as well since if not then

$$\begin{aligned} f(0) &\neq 0 \\ \Rightarrow f(\lambda \cdot 0) &= \lambda^d f(0), \quad (f \text{ homogeneous}) \\ \Rightarrow \lambda^d &= 1, \forall \lambda \in k^\times, \quad (\text{contradiction!}). \end{aligned}$$

Hence using the above reasoning, that $V_a(J)$ is closed so that $\overline{V_a(J) \setminus \{0\}} \subseteq \overline{V_a(J)} = V_a(J)$ and that only the homogeneous polynomials f are interesting, we can rewrite $I_p(V_p(J))$ as

$$I_p(V_p(J)) = \langle f \in k[x_0, \dots, x_n] : f \text{ homogeneous and } f(x) = 0, \forall x \in \overline{V_a(J) \setminus \{0\}} \rangle. \quad (6.0.2)$$

If $V_a(J) = \{0\}$ then $V_a(J) \setminus \{0\} = \emptyset$ and then

$$\begin{aligned} I_a(V_a(J)) &= I_a(\{0\}) \\ &= (x_0, \dots, x_n) \\ &= \sqrt{J}, \quad (\text{by affine nullstellensatz}). \end{aligned}$$

It follows that $V_a(J) \neq \emptyset$. So either $V_a(J)$ is empty or $V_a(J)$ is a **cone** (see [Gat21, Remark 6.17]). In the first case with $V_a(J) = \emptyset$ we clearly have $\overline{V_a(J) \setminus \{0\}} = V_a(J)$. On the other hand, if $V_a(J)$ is a cone and we see that since $V_a(J)$ is closed we have $\overline{V_a(J) \setminus \{0\}} \subseteq \overline{V_a(J)} = V_a(J)$. We claim that $0 \in \overline{V_a(J) \setminus \{0\}}$ so that $\overline{V_a(J) \setminus \{0\}} = V_a(J)$. To see this, recall that the principal open sets $D(f) = \{x \in \mathbb{A}^{n+1} : f(x) \neq 0\}$ forms a basis for the Zariski-topology on \mathbb{A}^{n+1} . Take any open subset

$D(f) \ni \{0\}$. Then we want to show that $D(f)$ contains a point from $V_a(J) \setminus \{0\}$. Let $x \in V_a(J) \setminus \{0\}$ and let $g(\lambda) := f(\lambda x)$. Then $g(\lambda) \in k[\lambda]$ (x is fixed) so have only finitely many zeroes. Therefore there must be some $\lambda \neq 0$ in k^\times such that $g(\lambda) \neq 0 \Rightarrow f(\lambda x) \neq 0$. But $\lambda x \in V_a(J) \setminus \{0\}$ since $x \neq 0$ and $\lambda \neq 0$, therefore $\lambda x \in D(f)$. We conclude that $\overline{V_a(J) \setminus \{0\}} = V_a(J)$. Furthermore, by [Gat21, Remark 6.17.(b)] we have that $I_p(X)$ when X is a cone is homogeneous. Therefore, by equation (6.0.2) we have that

$$\begin{aligned} I_p(V_p(J)) &= \langle f \in k[x_0, \dots, x_n] : f(x) = 0, \forall x \in V_p(J) \rangle \\ &= I_a(V_a(J)) \\ &= \sqrt{J}, \quad (\text{affine nullstellensatz}). \end{aligned}$$

□

Remark 6.0.11. I've reordered the parts covered in the lecture below, to make it more consistent (in my opinion) with [Gat21].

Homogeneous coordinate ring

Definition 6.0.12. If $X \subset \mathbb{P}^n$ is a projective variety then we define the **homogeneous coordinate ring of X** as $S(X) := k[x_0, \dots, x_n]/I_p(X)$.

Warning: Elements of $S(X)$ are *not* functions on X , not even the homogeneous elements.

But: $V_{p,X}(J) \subset X$ makes sense for $J \subset S(X)$. We also have that $I_{p,X}(Y) \subset S(X)$ for $Y \subset X$.

Zariski-topology

Definition 6.0.13. The **Zariski topology** on a projective variety X is the topology whose closed sets are exactly the subvarieties of X , i.e. subsets of the form $V_p(S)$ for some set $S \subset S(X)$ of homogeneous elements.

Dehomogenization

Definition 6.0.14. For a homogeneous polynomial $f \in k[x_0, \dots, x_n]$, the **dehomogenization** of f is defined to be the polynomial $f^i := f(x_0 = 1) \in k[x_1, \dots, x_n]$ that one obtains from f by setting $x_0 = 1$. By claim in [Gat21, Construction 6.25] it is in general an inhomogeneous polynomial, hence the i in f^i . Furthermore we claim that $(-)^i : k[x_0, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$ is a ring homomorphism so that $(fg)^i = f^i g^i$ and $(f + g)^i = f^i + g^i$ for all $f, g \in k[x_0, \dots, x_n]$. It is clear that it is surjective (since $k[x_1, \dots, x_n] \hookrightarrow k[x_0, \dots, x_n]$) so that assuming the lemma that gives that the image of an ideal under a surjective ring homomorphism is an ideal, we have that for a homogeneous ideal $J \leq k[x_0, \dots, x_n]$ the dehomogenization $J^i = \{f^i : f \in J\}$ is an ideal in $k[x_1, \dots, x_n]$.

Homogenization

Definition 6.0.15. Let $f = \sum_{i_1, \dots, i_n \in \mathbb{N}} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} \in k[x_1, \dots, x_n]$ be a non-zero polynomial of degree d (not necessarily homogeneous). We then define its **homogenization** as

$$\begin{aligned} f^h &:= x_0^d f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= \sum_{i_1, \dots, i_n \in \mathbb{N}} a_{i_1, \dots, i_n} x_0^{d-i_1-\dots-i_n} x_1^{i_1} \cdots x_n^{i_n} \in k[x_0, \dots, x_n]. \end{aligned}$$

We see that this is indeed a homogeneous polynomial (of degree $\deg(f^h) = d$) since it is the sum of monomials with degree $(d - i_1 - \dots - i_n) + i_1 + \dots + i_n = d$.

For all $f, g \in k[x_1, \dots, x_n]$ of degree d, e respectively we have that fg is of degree $d + e$ (by general algebra theory) so that

$$\begin{aligned} (fg)^h &= x_0^{d+e} (fg)\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= x_0^{d+e} f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \cdot g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= \left(x_0^d f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)\right) \left(x_0^e g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)\right) \\ &= f^h \cdot g^h. \end{aligned}$$

On the other hand, $(f + g)^h$ is not generally equal to $f^h + g^h$. To apply $(-)^h$ to ideals $J \trianglelefteq k[x_1, \dots, x_n]$ we define $J^h := \langle f^h : f \in J \rangle \trianglelefteq k[x_0, \dots, x_n]$.

Remark 6.0.16. To give an example (not given in lecture) for why $f^h + g^h \neq (f + g)^h$ generally: Let $f = x + 1$ and let $g = x^2 + 1$. Then

$$\begin{aligned} f^h &= yf\left(\frac{x}{y}\right) \\ &= y\left(\frac{x}{y} + 1\right) \\ &= x + y \end{aligned}$$

and

$$\begin{aligned} g^h &= y^2 g\left(\frac{x}{y}\right) \\ &= y^2 \left(\frac{x^2}{y^2} + 1\right) \\ &= x^2 + y^2 \end{aligned}$$

so that

$$f^h + g^h = x^2 + y^2 + x + y$$

which is not homogeneous, hence can not equal $(f + g)^h$ which is homogeneous.

Compare $\mathbb{A}^n \subset \mathbb{P}^n$.

Recall: $f_0 : \mathbb{A}^n \rightarrow U_0 \subset \mathbb{P}^n$ where $U_0 = \{[1 : x_1 : \dots : x_n]\}$. Here we let f_0 be defined by $f_0(x_1, \dots, x_n) = [1 : x_1 : \dots : x_n]$.

Proposition 6.0.17. f_0 is a homeomorphism.

Proof. We claim that $f_0^{-1} : U_0 \rightarrow \mathbb{A}^n$ defined by $[x_0 : x_1 : \dots : x_n] \mapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$ is an inverse to f_0 . \square

Remark 6.0.18 (\mathbb{A}^n as an open subset of \mathbb{P}^n). f_0 is injective, so we can embed \mathbb{A}^n into \mathbb{P}^n through the identification of \mathbb{A}^n with U_0 . NOT FINISHED

Claim: $X = V_a(J) \subset \mathbb{A}^n \subset \mathbb{P}^n$ and its closure $\overline{X} = V_p(J^h)$ in \mathbb{P}^n , then $X = \overline{X} \cap U_0$. Conversely if $X = V_p(J)$ then $X \cap U_0 = V_a(J^i) \subset \mathbb{A}^n$.

Proof. To show that $\overline{X} = V_p(J^h)$: Clearly $V_p(J^h)$ is closed in \mathbb{P}^n and contains $X = V_a(J)$. To see that $V_p(J^h)$ contains $V_a(J)$ let $x \in V_a(J)$ so that $f(x) = 0$ for all $f \in J$. Then recall that for x to be in $V_p(J^h)$ we need x to be zero on f^h for all $f \in J$. But we have that $f^h(x_0, \dots, x_n) = x_0^d f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$ and since $V_a(J) \subset \mathbb{A}^n$ this corresponds to points $[1 : x_1 : \dots : x_n]$ so that we get

$$\begin{aligned} f^h(x) &= f^h(1, x_1, \dots, x_n) \\ &= 1^d f\left(\frac{x_1}{1}, \dots, \frac{x_n}{1}\right) \\ &= f(x_1, \dots, x_n) \\ &= 0. \end{aligned}$$

therefore indeed $x \in V_p(J^h)$ so that $V_a(J) \subseteq V_p(J^h)$.

We want to show that $V_p(J^h)$ is the smallest closed set that contains $X = V_a(J)$. So let $Y \supset X$ be any closed set. We then need to show that $Y \supseteq V_p(J^h)$. Y is closed so $Y = V_p(J')$ for some $J' \trianglelefteq k[x_0, \dots, x_n]$ that is homogeneous.

Lemma 6.0.19. Any homogeneous polynomial $g \in J'$ can be written as $x_0^d f^h$ for some $d \in \mathbb{N}$ and some $f \in k[x_0, \dots, x_n]$.

Proof. Assume that $\deg(g) = d$. Then g has the form

$$\begin{aligned} g &= \sum_{\substack{i_0, \dots, i_n \in \mathbb{N} \\ i_0 + \dots + i_n = d}} a_{i_0, \dots, i_n} x_0^{i_0} \dots x_n^{i_n} \\ &= x_0^d \sum_{\substack{i_0, \dots, i_n \in \mathbb{N} \\ i_0 + \dots + i_n = d}} a_{i_0, \dots, i_n} x_0^{i_0 - d} \dots x_n^{i_n} \\ &= x_0^d \sum_{\substack{i_0, \dots, i_n \in \mathbb{N} \\ i_0 + \dots + i_n = d}} a_{i_0, \dots, i_n} \left(\frac{x_1}{x_0}\right)^{i_1} \dots \left(\frac{x_n}{x_0}\right)^{i_n} \\ &= x_0^d f^h(x_1, \dots, x_n), \quad \text{for some } f \in k[x_1, \dots, x_n] \end{aligned}$$

where we used that $i_0 + \dots + i_n = d \Leftrightarrow -(i_1 + \dots + i_n) = i_0 - d$ so that

$$\begin{aligned} x_0^{-i_1} \dots x_0^{-i_n} &= x_0^{-i_1 - \dots - i_n} \\ &= x_0^{i_0 - d} \end{aligned}$$

so that when we multiply in x_0^d we get back g . \square

We note that $X \subset Y$ so that if $g = x_0^d f^h \in J'$ and $x \in X \subset Y = V_p(J')$ then $x_0^d f^h(x) = 0$ for all $x \in X$. But we note that $X = V_a(J) \subset \mathbb{A}^n \subset \mathbb{P}^n$ where $x_0 \neq 0 \Rightarrow x_0^d \neq 0$. Hence we need $f^h(x) = 0$. To see this note that if $f^h(x_0, \dots, x_n)$ then since f^h is homogeneous we have that $f^h(\lambda x_0, \dots, \lambda x_n) = 0$. So in particular we have that with $\lambda = \frac{1}{x_0}$ we get that

$$\begin{aligned} f^h(\lambda x_0, \dots, \lambda x_n) &= f^h\left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= 0. \end{aligned}$$

but $\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in \mathbb{A}^n$ is precisely the set of **affine coordinates** for the subset $\mathbb{A}^n \subset \mathbb{P}^n$. I.e. we are using that $[x_0 : \dots : x_n] = \left[1 : \frac{x_1}{x_0} : \dots : \frac{x_n}{x_0}\right]$ on \mathbb{P}^n . Therefore we conclude that $f = 0$ on $X \subset \mathbb{A}^n$

$$\begin{aligned} \rightsquigarrow f &\in I_a(X) = I_a(V_a(J)) \\ &= \sqrt{J}, \quad (\text{affine nullstellensatz}) \\ \Rightarrow f^m &\in J, \text{ for some } m \in \mathbb{N}. \end{aligned}$$

We claim that $(f^h)^m = (f^m)^h$. To see this, we can use that $(fh)^h = f^h g^h$ repeatedly to get that

$$\begin{aligned} (f^m)^h &= \underbrace{f^h \dots f^h}_{m \text{ times}} \\ &= (f^h)^m. \end{aligned}$$

Recall that J^h is the ideal generated by the homogenizations of all polynomials in J , so it follows that $(f^h)^m \in J^h$. Therefore $f^h \in \sqrt{J^h}$. Since $x_0^d \in k[x_0, \dots, x_n]$ and $\sqrt{J^h}$ is an ideal it follows that $g = x_0^d f^h \in \sqrt{J^h}$. Hence we have shown that $J' \subset \sqrt{J^h}$. Then by the inclusion-reversing property we have that $Y = V_p(J') \supset V_p(\sqrt{J^h})$, but $V_p(J^h) \subseteq V_p(\sqrt{J^h})$ (if $x \in V_p(J^h)$ then $f(x) = 0$ for all $f \in J^h$. If then $g \in \sqrt{J^h}$ then there is some m such that $g^m \in J^h$ and so $g^m(x) = 0$ which implies that $g(x) = 0$).

(On the other hand if $x \in V_p(\sqrt{J^h})$ then $f(x) = 0$ for all $f \in \sqrt{J^h}$ but $J^h \subseteq \sqrt{J^h}$ so that $g(x) = 0$ for all $g \in J^h \Rightarrow x \in V_p(J^h) \rightsquigarrow V_p(J^h) = V_p(\sqrt{J^h})$.)

□

Chapter 7

Lecture 7

Let $U_i \subseteq \mathbb{P}^n$ be defined as $U_i = \{[x_0 : \dots : x_n] : x_i \neq 0\} \cong \mathbb{A}^n$. The topology induced by the Zariski-topology on \mathbb{P}^n is exactly the Zariski-topology on \mathbb{A}^n .

- $V_p(J) \cap U_i = V_a(J^i)$ where J^i means that for $J \leq k[x_0, \dots, x_n]$ we set $x_i = 1$ and so $V_a(J^i)$ is the zero-locus of $J^i = \{f(x_0, \dots, 1, \dots, x_n) : f \in J\} \leq k[x_1, \dots, x_n]$ (after possibly reordering the indeterminates x_i).
- $X \cap U_i = V_a(I) \Rightarrow X = V_p(I^h)$.

7.0.1 Projective closure

Let $C = V_a(I)$ in $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$. We want \overline{C} in \mathbb{P}^n . We have that $\overline{C} = V_p(I^h)$ if C is a hypersurface.

- If $I = (f)$ then

$$\begin{aligned}\overline{C} &= V_p(I^h) \\ &= V_p(f^h).\end{aligned}\tag{7.0.1}$$

Example 7.0.1. Let $J = (x_1, x_2 - x_1^2) \subseteq k[x_1, x_2]$. Then

$$V_a(J) = \{x \in \mathbb{A}^2 : f(x) = 0, \forall f \in (x_1, x_2 - x_1^2)\}.$$

We see that since if $x \in V_a(J)$ we need

$$x_1 = 0\tag{7.0.2}$$

and so from

$$x_2 - x_1^2 = 0\tag{7.0.3}$$

where now since $x_1 = 0$ it follows that $x_2 = 0$. Therefore $V_a(J) = \{(0, 0)\} \subset \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$. Now recall the map $f : \mathbb{A}^n \rightarrow \mathbb{P}^n$ defined by $(x_1, \dots, x_n) \mapsto [1 : x_1 : \dots : x_n]$ ([Gat21, Remark 6.3]). We then see that $V_a(J)$ corresponds to $[1 : 0 : 0] \in \mathbb{P}^2$.

Furthermore, $\overline{V_a(J)} = V_a(J) \subset \mathbb{A}^n$ but if we compute the *homogenization* (6.0.15) of $x_1, x_2 - x_1^2$ and let $g = x_1$ we get $g^h = x_1$ and if we let $f(x_1, x_2) = x_2 - x_1^2$ where we note that $\deg(f) = 2$ we find that

$$\begin{aligned}f^h(x_1, x_2) &= x_0^2 f\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \\ &= x_0^2 \left(\frac{x_2}{x_0} - \frac{x_1^2}{x_0^2}\right) \\ &= x_0 x_2 - x_1^2.\end{aligned}$$

Then we see that

$$\begin{aligned}\overline{C} &= V_p(J) \\ &= V_p(g^h, f^h) \\ &= V_p(x_1, x_2x_0 - x_1^2).\end{aligned}$$

For $x \in \mathbb{P}^2$ to be in $V_p(x_1, x_2x_0 - x_1^2)$ we investigate its behavior on affine coordinates by looking at patches U_i for $i = 0, 1, 2$. For U_0 we have that $x_0 \neq 0$ so we set $x_0 = 1$. Then from $x_1 = 0$ we see that we get the equations $x_1 = 0$ and $x_2 - x_1 = 0 \Leftrightarrow x_2 = x_1$ so that we get the point $[1 : 0 : 0]$. If we look at U_1 where $x_1 \neq 0$ we see that this does not work since $x_1 = 0$. For U_2 we have that $x_2 \neq 0$ and $x_1 = 0$ so we get the equation $x_0x_2 = 0$ where $x_2 \neq 0 \Rightarrow x_0 = 0$. Hence we get the point $[0 : 0 : 1]$. Since $U_0 \cup U_1 \cup U_2 = \mathbb{P}^n$ we are done and so $V_p(J^h) = \{[1 : 0 : 0], [0 : 0 : 1]\}$ so that $V_p(J^h) \neq f(V_a(J))$ where f “embeds” \mathbb{A}^n into \mathbb{P}^n .

Example 7.0.2. Let $C = V(xy - 1)$ and look at $\overline{C} \subset \mathbb{P}^2$. We then use what we showed in the earlier lecture (see also [Gat21, Prop. 6.32]), i.e. that if $X = V_a(J)$ then $\overline{X} = V_p(J^h)$ in \mathbb{P}^n . So we need to compute the *homogenization* of $f = xy - 1$, where $\deg(f) = 2$. This gives us

$$\begin{aligned}f^h &= z^2 f\left(\frac{x}{z}, \frac{y}{z}\right) \\ &= z^2 \left(\frac{xy}{z^2} - 1\right) \\ &= xy - z^2.\end{aligned}$$

We then want to compute $V_p(xy - z^2)$. We divide into the cases U_0, U_1, U_2 as before (with the ordering $x_0 := x, x_1 := y, x_2 := z$):

U_0 : On U_0 , we have that $x \neq 0$, so we can set $x = 1$ and then we see that $y - z^2 = 0 \Leftrightarrow y = z^2$. If $z = 0$ then $y = 0$ which gives us the point $[1 : 0 : 0] \in \mathbb{P}^2$.

U_1 : Set $y = 1$ then similarly we get $x = z^2$ and so that when $z = 0$ we get the point $[0 : 1 : 0]$ and otherwise $x = z^2$.

U_2 : Set $z = 1$ and which gives $xy = 1$ so that neither x nor y can be zero.

Example 7.0.3. Let $C = V_a(x^2 + y^2 - 1)$ which describes the circle say when $k = \mathbb{R}$. Then if we let $f = x^2 + y^2 - 1$ we have that $\overline{C} = V_p(f^h)$. We compute f^h (where we note that $\deg(f) = 2$):

$$\begin{aligned}f^h(x, y) &= z^2 \left(\frac{x}{z}, \frac{y}{z}\right) \\ &= z^2 \left(\frac{x^2}{z^2} + \frac{y^2}{z^2} - 1\right) \\ &= x^2 + y^2 - z^2.\end{aligned}$$

Therefore we have $V_p((x^2 + y^2 - 1)^h) = V_p(x^2 + y^2 - z^2)$. We proceed with the same procedure as in the previous example:

U_0 : Let $x = 1$ then we have the equation

$$\begin{aligned}1 + y^2 - z^2 &= 0 \\ \Leftrightarrow 1 + y^2 &= z^2,\end{aligned}$$

which has the equation of a hyperbola in the y, z -plane.

U_1 : Let $y = 1$ then we get

$$\begin{aligned} 1 + x^2 - z^2 &= 0 \\ \Leftrightarrow 1 + x^2 &= z^2, \end{aligned}$$

which has the equation of a hyperbola in the x, z -plane.

U_2 : Let $z = 1$; then we get the equation

$$\begin{aligned} x^2 + y^2 - 1 &= 0 \\ \Leftrightarrow x^2 + y^2 &= 1, \end{aligned}$$

which I believed is called the **affine circle** (if we don't want to commit to a base field).

Remark 7.0.4. Notice in the above example that on U_0 and U_1 , if we take the base field $k = \mathbb{R}$ we see that for $1 + y^2 = z^2$ or $1 + x^2 = z^2$ with $z = 0$ we don't have any solutions, while if $k = \mathbb{C}$ then $[1 : i : 0]$ and $[1 : -i : 0]$ are solutions on U_0 and similarly $[i : 1 : 0]$ and $[-i : 1 : 0]$ are solutions on U_1 .

Example 7.0.5. Let $C = V_a(y^2 - x(x-1)(x-2))$. We have

$$\begin{aligned} y^2 - x(x-1)(x-2) &= y^2 - (x^2 - x)(x-2) \\ &= y^2 - (x^3 - 2x^2 - x^2 + 2x) \\ &= y^2 - x^3 + 3x^2 - 2x. \end{aligned}$$

Hence

$$\overline{C} = V_p((y^2 - x^3 + 3x^2 - 2x)^h).$$

Let $f = y^2 - x^3 + 3x^2 - 2x$. Then notice that $\deg(f) = 3$ and so

$$\begin{aligned} f^h(x, y) &= z^3 f\left(\frac{x}{z}, \frac{y}{z}\right) \\ &= z^3 \left(\frac{y^2}{z^2} - \frac{x^3}{z^3} + \frac{3x^2}{z^2} - 2\frac{x}{z} \right) \\ &= y^2 z - x^3 + 3x^2 z - 2xz^2. \end{aligned}$$

We proceed by the same procedure:

U_0 : Let $x = 1$. Then we get $y^2 z - 1 + 3z - 2z^2 = 0$. Here we see that $z \neq 0$ since otherwise we have $-1 = 0$. If on the other hand $y = 0$ we get the equation $3z - 2z^2 = 0$ which has two solutions if k is algebraically closed.

U_1 : Let $y = 1$. Then we get

$$z - x^3 + 3x^2 z - 2xz^2$$

If $x = 0$ we get

$$z = 0 \tag{7.0.4}$$

and if $z = 0$ we get $x^3 = 0 \Rightarrow x = 0$.

U_2 : Let $z = 1$. This gives us

$$\begin{aligned} y^2 - x^3 + 3x^2 - 2x &= 0 \\ \Leftrightarrow y^2 &= x^3 - 3x^2 + 2x. \end{aligned}$$

If $y = 0$ and k is algebraically closed we get three solutions (in affine space, that is).

7.0.2 Regular functions on \mathbb{P}^n

Regular functions on \mathbb{P}^n

Definition 7.0.6. Let $U \subseteq \mathbb{P}^n$ be open. Then a **regular function** on U is a map $\varphi : U \rightarrow k$ (k the base field) such that for all $a \in U$ there exists an open neighborhood $V_a \subseteq U$ with $a \in V_a$ so that there are *homogeneous* polynomials $f, g \in k[x_0, \dots, x_n]$ of the same degree such that

$$\varphi(x_0 : \dots : x_n) = \frac{g(x_0, \dots, x_n)}{f(x_0, \dots, x_n)}$$

with $f(x_0, \dots, x_n) \neq 0$ on V_a .

Remark 7.0.7. With respect to the above definition: If $a = [a_0 : \dots : a_n]$ and $b = [b_0 : \dots : b_n]$ such that $a_i = \lambda b_i$ for all i and $\lambda \in k^\times$ then we claim that on $U_a \subset U$, with $\deg(f) = \deg(g) = d$ we have

$$\begin{aligned} \varphi(a) &= \frac{g(a)}{f(a)} \\ &= \frac{g(\lambda b)}{f(\lambda b)} \\ &= \frac{\lambda^d g(b)}{\lambda^d f(b)} \\ &= \frac{g(b)}{f(b)}, \end{aligned}$$

so that φ is a *well-defined* function \rightsquigarrow sheaf \mathcal{O}_X .

Remark 7.0.8. Below we by $(x_0 : \dots, x_n)$ mean the same thing as when we write $[x_0 : \dots : x_n]$. Henceforth if not otherwise specified they will mean the same thing.

Proposition 7.0.9 (Projective varieties are prevarieties). *The pair (X, \mathcal{O}_X) with X a projective variety (6.0.7), is a prevariety (5.0.2).*

Proof.

Lemma 7.0.10. *Let $X \subset \mathbb{P}^n$ be a projective variety. Then*

$$U_i := \{(x_0 : \dots : x_n) \in X \mid x_i \neq 0\}$$

for $i = 0, \dots, n$ are affine varieties.

Proof. We show this for $i = 0$. Since X is a projective variety it follows by definition 6.0.7 that $X = V_p(J)$ for some homogeneous polynomial $J \in k[x_0, \dots, x_n]$ (we are using that if $S \subseteq k[x_0, \dots, x_n]$ such that $X = V_p(S)$ then the subset $S' \subset S$ of homogeneous polynomials are such that $V_p(S) = V_p(S')$ and so $J := \langle S' \rangle$ is homogeneous). If we let

$$\begin{aligned} Y &= V_a(J^i) \\ &= \{(x_1, \dots, x_n) \in \mathbb{A}^n \mid f^i(x) = 0, \forall f \in J\}, \end{aligned}$$

then we claim that $F : Y \rightarrow U_0$ defined explicitly as $(x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n)$ is an isomorphism with the inverse $F^{-1} : U_0 \rightarrow Y$ defined by

$$(x_0 : x_1 : \dots : x_n) \mapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right).$$

Since one checks that F_0 is a (set)inverse it follows that it is a F is a set-bijection. To say a bit more about this notice that U_0 is defined as a subset of X , i.e. U_0 are all the elements of X such that $x_0 \neq 0$. So assume that $(1 : x_1 : \dots : x_n) \in U_0$. Then by definition we see that $(1 : x_1 : \dots : x_n)$ is such that $f(1, x_1, \dots, x_n) = 0$ for all $f \in J$ but $f(1, x_1, \dots, x_n) = f^i(x_1, \dots, x_n)$ so that $f^i(x_1, \dots, x_n) = 0$. Therefore this corresponds to $(x_1, \dots, x_n) \in \mathbb{A}^n$ and so (x_1, \dots, x_n) maps to $[1 : x_1 : \dots : x_n]$ and so F is *surjective*. By the fact that $[1 : x_1 : \dots : x_n] = [1 : y_1 : \dots : y_n] \Rightarrow \exists \lambda \in k^\times$ so that $\lambda(1, x_1, \dots, x_n) = (1, y_1, \dots, y_n)$ we see by comparing the first coefficients that $\lambda = 1$ so that $x_i = y_i$, hence the map is *injective* and so bijective.

Furthermore we claim that F is continuous: Let $V_p(J') \cap U_0$ be a closed set in U_0 with $J' \leq k[x_0, \dots, x_n]$ homogeneous. Then we claim that $F^{-1}(V_p(J') \cap U_0) = V_a(J'^i)$.

\subseteq : Let $x = (x_1, \dots, x_n) \in F^{-1}(V_p(J') \cap U_0)$. Then we have that $F(x) = (x_0 : x_1 : \dots : x_n) \in U_0$ so that $x_0 \neq 0$ so we can write this as $(1 : x_1 : \dots : x_n)$. Furthermore we have that $(1 : x_1 : \dots : x_n) \in V_p(J')$ so that $f(1, x_1, \dots, x_n) = 0$. But $f(1, x_1, \dots, x_n) = f^i(x_1, \dots, x_n)$. so that $f^i(x_1, \dots, x_n) = 0$. Therefore $x \in V_a(J'^i)$.

\supseteq : Let $x \in V_a(J'^i)$. Then we have that $f^i(x_1, \dots, x_n) = 0$ for all $f \in J'^i$. But $f^i(x_1, \dots, x_n) = f(1, x_1, \dots, x_n)$. We note that $F(x_1, \dots, x_n) = (1 : x_1 : \dots : x_n)$ so that $(1 : x_1 : \dots : x_n) \in U_0$ and furthermore $f(1, x_1, \dots, x_n) = 0$ as we argued (for all $f \in J'$) so that $(1 : x_1 : \dots : x_n) \in V_p(J')$ so indeed $x \in F^{-1}(V_p(J') \cap U_0)$.

Therefore since $F^{-1}(V_p(J') \cap U_0)$ has the form $V_a(J'^i)$ we see that it is closed. Hence F the preimage of each closed set is closed $\Leftrightarrow F$ is continuous.

It remains to show that F^{-1} is continuous. We recall that if F is a continuous bijection (between topological spaces) that is a closed map (i.e. takes closed sets to closed sets) then F is a homeomorphism (so that F^{-1} is continuous). So it is enough to show that F takes closed sets to closed sets.

So let $V_a(J') \subset Y$. Then we claim that $F(V_a(J')) = V_p(J'^h) \cap U_0$:

\subseteq : Let $y \in F(V_a(J'))$. Then $y = F(x)$ for some $x \in V_a(J')$ so that $f(x) = 0$ for all $f \in J'$. Under F x is then sent to $(1 : x_1 : \dots : x_n) \in U_0$, therefore $y = (1 : x_1 : \dots : x_n) \in U_0$. Let $f^h \in J'^h$. Then we have that

$$\begin{aligned} f^h(y) &= f^h(1, x_1, \dots, x_n) \\ &= f(x_1, \dots, x_n) \\ &= 0 \end{aligned}$$

so that indeed $y \in V_p(J'^h) \Rightarrow y \in V_p(J'^h) \cap U_0$.

\supseteq : Let $x \in V_p(J'^h) \cap U_0$. Then since $x \in U_0 \rightsquigarrow x = (1 : x_1 : \dots : x_n)$. Since $x \in V_p(J'^h)$ we have that $f^h(x) = 0$ for all $f \in J'$. But then we have

$$\begin{aligned} f^h(1, x_1, \dots, x_n) &= f(x_1, \dots, x_n) \\ &= 0, \end{aligned}$$

and since f^h is homogeneous the same holds for $\lambda(1, x_1, \dots, x_n)$ where $\lambda \in k^\times$ is arbitrary. But then we see that $x = (x_1, \dots, x_n)$ is in $V_a(J')$ and so

$$\begin{aligned} F((x_1, \dots, x_n)) &= (1 : x_1 : \dots : x_n) \\ &= y, \end{aligned}$$

hence $y \in F(V_a(J'))$.

We conclude that F is a bijective continuous map with continuous inverse (i.e. a homeomorphism).

Lastly we want to check F and F^{-1} takes regular functions to regular functions (3.0.11). A regular function on an open subset U_0 is locally of the form $\frac{g(x_0, \dots, x_n)}{f(x_0, \dots, x_n)}$ (with $f(x_0, \dots, x_n) \neq 0$) where $f, g \in k[x_0, \dots, x_n]$ are two *homogeneous* polynomials of the same degree d . We then see that for $(x_1, \dots, x_n) \in V_a \subset Y$ we have

$$\begin{aligned} (F^* \circ \varphi)(x_1, \dots, x_n) &= \varphi \circ F(x_1, \dots, x_n) \\ &= \varphi(1 : x_1, \dots, x_n) \\ &= \frac{g(1, x_1, \dots, x_n)}{f(1, x_1, \dots, x_n)} \\ &= \frac{g^i(x_1, \dots, x_n)}{f^i(x_1, \dots, x_n)}. \end{aligned}$$

For F^{-1} we see that if $V \ni x = (x_0 : \dots : x_n)$ is an open subset of U_0 that contains x , and $\varphi : V \rightarrow k$ then we have

$$\begin{aligned} ((F^{-1})^* \circ \varphi)(x) &= (\varphi \circ F^{-1})(x_0 : \dots : x_n) \\ &= \varphi\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= \frac{g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)}{f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)}, \end{aligned}$$

If we multiply both the numerator and the denominator by x_0^d where $d = \max(\deg(f), \deg(g))$ then it is clear that if (without loss of generality) $\max(\deg(f), \deg(g)) = \deg(g)$ we get that $x_0^d g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) = g^h(x_1, \dots, x_n)$ is homogeneous. If $\deg(f) = \deg(g)$ then $x_0^d f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) = f^h(x_1, \dots, x_n)$ is also homogeneous and of the same degree as g . If $\deg(f) = n < d = \deg(g)$. Then

$$\begin{aligned} x_0^d f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) &= x_0^d \sum_{i_0, \dots, i_n \in \mathbb{N}} a_{i_1, \dots, i_n} \left(\frac{x_1}{x_0}\right)^{i_1} \dots \left(\frac{x_n}{x_0}\right)^{i_n} \\ &= \sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} x_0^{i_1 + \dots + i_n - d} x_1^{i_1} \dots x_n^{i_n}. \end{aligned}$$

We see that $x_0^d f$ is a sum of monomials each of degree d , so that indeed f is of degree d and is homogeneous.

Therefore, we conclude that F is an isomorphism of ringed spaces by 4.0.7. Therefore U_0 is an affine subset of X . We claim that investigating the proof one sees that each part follows through when looking at U_i for $i = 1, \dots, n$ instead of U_0 . \square

We see that $\bigcup_{i=1}^n U_i = X$ so that X is covered by affine varieties. By definition (5.0.2) we then have that X is a prevariety. \square

Word: $X \times Y$, where X, Y projective then $X \times Y$ projective. However, we claim that $\mathbb{P}^1 \times \mathbb{P}^1 \not\hookrightarrow \mathbb{P}^2$.

We have the **Segre Embedding** $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ where $N = (n+1)(m+1) - 1$.

Lemma 7.0.11. *Let $X \subseteq \mathbb{P}^n$ be a projective variety and let $f_0, \dots, f_n \in S(X)$ be homogeneous polynomials of the same degree. Let $U = X \setminus V_p(f_0, \dots, f_n)$. Then $f : U \rightarrow \mathbb{P}^m$ defined by $[x_0 : \dots : x_n] \mapsto [f_0(x_0, \dots, x_n) : \dots : f_n(x_0, \dots, x_n)]$ is a morphism (of ringed spaces; see definition 4.0.7).*

Proof. We first show that f is well-defined as a function: First we claim that the image of f can never be zero. To see this we note that U is defined as the complement of the projective locus of f_0, \dots, f_n so that $f_i(x) \neq 0$ for $i = 0, \dots, n$ on U . By using that f_i are all homogeneous, we also see that if $\lambda \in k^\times$ then

$$\begin{aligned} (f_0(\lambda x_0, \dots, \lambda x_n) : \dots : f_n(\lambda x_0, \dots, \lambda x_n)) &= (\lambda^d f_0(x_0, \dots, x_n) : \dots : \lambda^d f_n(x_0, \dots, x_n)) \\ &= (f_0(x_0, \dots, x_n) : \dots : f_n(x_0, \dots, x_n)), \end{aligned}$$

Therefore we have that $(x_0 : \dots : x_n) = (y_0 : \dots : y_n) \Leftrightarrow \lambda x_i = y_i$ for all $i = 0, \dots, n$ then $f(x_0 : \dots : x_n) = f(y_0 : \dots : y_n)$ so that f is indeed well-defined.

To show that f is a morphism we will use the gluing property ([Gat21, Lemma 4.6]). Let $\{V_i \mid i = 0, \dots, n\}$ with $V_i := \{(y_0 : \dots : y_n) \mid y_i \neq 0\}$ be the **affine open cover** of \mathbb{P}^n . Then

$$\begin{aligned} U_i &:= f^{-1}(V_i) \\ &= \{x = (x_0, \dots, x_n) \in U = X \setminus V_p(f_0, \dots, f_n) \mid (f_0(x_0, \dots, x_n) : \dots : f_n(x_0, \dots, x_n)) \in V_i\} \\ &= \{x \in U \mid f_i(x_0, \dots, x_n) \neq 0\}. \end{aligned}$$

If we take the union of all such U_i then surely the set of $x \in \mathbb{P}^n$ that are non-zero on all f_i are included (since they must be in each U_i). Since $U_i = D(f_i) \cap U$ they are open in U . If we look at the restriction of f to each U_i we want to show that $f|_{U_i}$ is continuous for $i = 0, \dots, n$. Let $A \subset Y$ be open. Then we look at the restricted maps $f|_{U_i} : U_i \rightarrow V_i \cong \mathbb{A}^n$. Then we see that $f|_{U_i}$ has coordinates on the form $\frac{f_j}{f_i}$ with $f_i \neq 0$ for all $i = 1, \dots, n, i \neq j$ (it has affine coordinates $\frac{y_j}{y_i}$ with $y_i \neq 0$). By 7.0.6 these are regular functions on U_i . We want to claim that $f|_{U_i}$ is a morphism by 4.0.11. It is clear from the earlier argument that the target $V_i \cong \mathbb{A}^n$ can be viewed as affine. We want to see that we can view U_i as an open subset of an affine variety. That U_i is affine means that U_i is (up to isomorphism) a subset of some set of the form $\{x \in \mathbb{P}^n \mid x_i \neq 0\}$ for some $i = 0, \dots, n$.

If we restrict f to U_i so that we get $f|_{U_i} : U_i \rightarrow \mathbb{P}^n$ □

Example 7.0.12 ([Gat21, Example 7.5.(b)]). Let $a = (1 : 0 : \dots : 0)$ and let

$$\begin{aligned} L &= V_p(x_0) \\ &\cong \mathbb{P}^{n-1}. \end{aligned}$$

Then the map $f : \mathbb{P}^n \setminus \{a\} \rightarrow \mathbb{P}^{n-1}$ defined by $(x_0 : \dots : x_n) \mapsto (x_1 : \dots : x_n)$ that forgets the first homogeneous coordinate, is a morphism by 7.0.11. To see how we use 7.0.11 we let $X = \mathbb{P}^n$ and then we take $U = \mathbb{P}^n \setminus V_p(f_1, \dots, f_n)$ with $f_i = x_i$ for $i = 1, \dots, n$.

We note that

$$\begin{aligned} V_p(f_1, \dots, f_n) &= \{x \in \mathbb{P}^n \mid x_1 = \dots = x_n = 0\} \\ &= \{a\}, \end{aligned}$$

so that $f : \mathbb{P}^n \setminus V_p(f_1, \dots, f_n) \rightarrow \mathbb{P}^{n-1}$ is the same map as $f : \mathbb{P}^n \setminus \{a\} \rightarrow \mathbb{P}^{n-1}$. If we let $(x_0 : \dots : x_n) \in \mathbb{P}^n \setminus \{a\}$ then the *unique* line through a and x is given by

$$\{(s : tx_1 : \dots : tx_n) \mid (s : t) \in \mathbb{P}^1\}.$$

To see this note that each point in \mathbb{P}^n represents a line in k^{n+1} through the origin. So if we have *two* points in \mathbb{P}^n , they represent two lines in k^{n+1} that must be linearly independent if they are distinct. Therefore, they together span a plane in k^{n+1} . A line in \mathbb{P}^n between two points is then (when viewed in k^{n+1}) the plane spanned by the two lines (i.e. the lines through the origin in k^{n+1} that lie on this plane). When we **projectivize** this plane we get a line in \mathbb{P}^n . The plane spanned by points say P, Q is then $\text{span}_k(P, Q) = \{s \cdot P + t \cdot Q \mid s, t \in k\}$. When projectivizing we get

$$[\text{span}_k(P, Q)] := \{s \cdot [P] + t \cdot [Q] \mid (s : t) \in \mathbb{P}^1\}.$$

Remark 7.0.13. I don't entirely understand this.

In our case with $a = (1 : a_1 : \dots : a_n)$ and $(x_0 : \dots : x_n) \in \mathbb{P}^n \setminus \{a\}$ we get

$$\begin{aligned} [\text{span}_k(a, x)] &= \{s \cdot a + t \cdot x \mid (s : t) \in \mathbb{P}^1\} \\ &= \{(s : 0 : \dots : 0) + (tx_0 : \dots : tx_n) \mid (s : t) \in \mathbb{P}^1\} \\ &= \{(s + tx_0 : tx_1 : \dots : tx_n) \mid (s : t) \in \mathbb{P}^1\}. \\ &= \{(s' : tx_1 : \dots : tx_n) \mid (s' : t) \in \mathbb{P}^1\}. \end{aligned}$$

Its intersection with the line L (see figure below) will be when $s = 0$ so that we get the point

$$(0 : tx_1 : \dots : tx_n) = (0 : x_1 : \dots : x_n).$$

It is clear that this is a point in L and we also see that under the identification $L \cong \mathbb{P}^{n-1}$ this is $f(x) = (x_1 : \dots : x_n)$. We call f the **projection** from a to the linear subspace L .

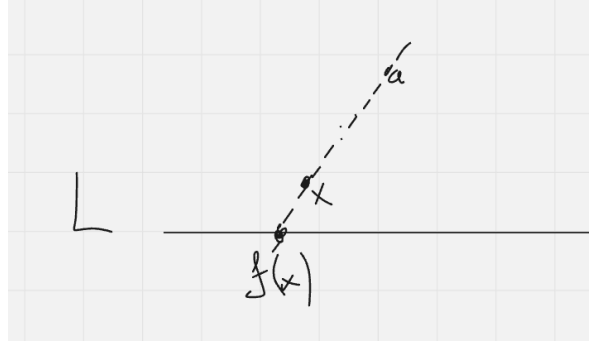


Figure 7.1:

Example 7.0.14. Continuing on from our previous example (7.0.12), we claim that the projection morphism $f : \mathbb{P}^n \setminus \{a\} \rightarrow \mathbb{P}^{n-1}$ can not be extended to a morphism from \mathbb{P}^n . One “intuitive” explanation is that the line through a, x (and $f(x)$) does not have a well-defined limit as x approaches a .

If we instead let $X = V(x_0x_2 - x_1^2)$ and then consider the map $f : X \rightarrow \mathbb{P}^1$ defined by

$$(x_0 : x_1 : x_2) \mapsto \begin{cases} (x_1 : x_2), & \text{if } (x_0 : x_1 : x_2) \neq (1 : 0 : 0) \\ (x_0 : x_1), & \text{if } (x_0 : x_1 : x_2) \neq (0 : 0 : 1) \end{cases}$$

then if $(x_0 : x_1 : x_2) = (y_0 : y_1 : y_2) \in X$ then we have

$$\begin{aligned} x_0x_2 &= x_1^2 \\ y_0y_2 &= y_1^2 \end{aligned}$$

so that $(x_1 : x_2) = (x_0 : x_1)$ whenever both these points in \mathbb{P}^1 are defined (by defined we mean whenever $(x_0 : x_1 : x_2) \neq (1 : 0 : 0), (0 : 0 : 1)$). To show that this is true, we note that if $x_0 = 0$ then $x_1^2 = 0$ and so then $x_2 = 1$ but this is not an allowed point. So assume that $x_0 \neq 0$. Then we have

$$\begin{aligned} x_0x_2 - x_1^2 &= 0 \\ \Leftrightarrow x_0x_2 &= x_1^2 \\ \Leftrightarrow x_2 &= \frac{x_1^2}{x_0} \end{aligned}$$

Let $\lambda = \frac{x_1}{x_0}$. Then we see that

$$\frac{x_1}{x_0}(x_0, x_1) = \left(x_1, \frac{x_1^2}{x_0}\right) \quad (7.0.5)$$

$$= (x_1, x_2) \quad (7.0.6)$$

Segre embedding

Definition 7.0.15. Consider $\mathbb{P}_{x_i}^n$ with homogeneous coordinates x_0, \dots, x_n and $\mathbb{P}_{y_j}^m$ with homogeneous coordinates y_0, \dots, y_m . Set $N = (n+1)(m+1) - 1$ and let $\mathbb{P}_{z_{i,j}}^N$ be projective N -space with homogeneous coordinates $z_{i,j}$ for $0 \leq i \leq n$ and $0 \leq j \leq m$ (there are $N+1 = (n+1)(m+1)$ coordinates describing \mathbb{P}^N so that this works out). Then there is a well-defined set-theoretic map

$$f : \mathbb{P}_{x_i}^n \times \mathbb{P}_{y_j}^m \rightarrow \mathbb{P}_{z_{i,j}}^N$$

$$(x, y) \mapsto (f_{ij}(x, y))$$

with $f_{ij}(x, y) = x_i y_j$, i.e. $z_{ij} = x_i y_j$ in \mathbb{P}^N under this map. We call f a **Segre embedding** and we call the z_{ij} the **Segre coordinates**.

Proposition 7.0.16 ([Gat21, Prop. 7.10.(a)]). *Let f be the map in 7.0.15. Then the image $X = f(\mathbb{P}_{x_i}^n \times \mathbb{P}_{y_j}^m)$ is a projective variety given by*

$$X = V_p(z_{i,j}z_{k,l} - z_{i,l}z_{k,j} \mid 0 \leq i, k \leq n, 0 \leq j, l \leq m)$$

Proof. Let $[z_{i,j}] := (z_{0,0} : \dots : z_{n,m})$. If $f(p) = [z_{i,j}]$ then

$$z_{i,j} = x_i y_j$$

$$z_{k,l} = x_k y_l$$

$$z_{i,l} = x_i y_l$$

$$z_{k,j} = x_k y_j$$

so that

$$z_{i,j}z_{k,l} = x_i y_j x_k y_l$$

and

$$z_{i,l}z_{k,j} = x_i y_l x_k y_j$$

so that $z_{i,j}z_{k,l} - z_{i,l}z_{k,j} = 0$.

On the other hand, if $[z_{i,j}] \in \mathbb{P}^N$ with homogeneous coordinates $z_{0,0}, \dots, z_{n,m}$ such that $[z_{i,j}]$ satisfy the equations $z_{i,j}z_{k,l} - z_{i,l}z_{k,j} = 0$ for all i, j, k, l . Since $[z_{i,j}]$ is projective atleast some $z_{i,j} \neq 0$. Without loss of generality we let $z_{0,0} = 1$. Then by the equations we have that (with $(k, l) = (0, 0)$)

$$z_{i,j} \underbrace{z_{0,0}}_{=1} - z_{i,0}z_{0,j} = 0 \quad (7.0.7)$$

$$\Leftrightarrow z_{i,j} = z_{i,0}z_{0,j}. \quad (7.0.8)$$

Let $x_i = z_{i,0}$ and $y_j = z_{0,j}$. Then we see that $z_{i,j} = x_i y_j$ so that

$$\begin{aligned} x_0 &= y_0 \\ &= z_{0,0} \\ &= 1, \end{aligned}$$

we get a point $p = (x, y) \in \mathbb{P}^n \times \mathbb{P}^m$ such that $f(p) = [z_{i,j}] \in f(\mathbb{P}^n \times \mathbb{P}^m)$.

We conclude that

$$f\left(\mathbb{P}_{x_i}^n \times \mathbb{P}_{y_j}^m\right) = V_p(z_{i,j}z_{k,l} - z_{i,l}z_{k,j} \mid 0 \leq i, k \leq n, 0 \leq j, l \leq m).$$

□

Chapter 8

Lecture 8

8.0.1 Projective varieties are separated

We want to verify that projective varieties (6.0.7) are separated (this is the same as saying that they are varieties (5.0.14), and not just prevarieties (5.0.2). This amounts to showing that a projective variety $X \subset \mathbb{P}^n$ is such that $\Delta_X \subset X \times X$ is closed.

Proposition 8.0.1. *Every projective variety is a variety (5.0.14).*

Proof. We have already seen that a projective variety is a prevariety (7.0.9). If $X \subset \mathbb{P}^n$ is a projective variety, then it is a closed subset of \mathbb{P}^n . Therefore by [Gat21, Lemma 5.12.(b)] it is enough to show that \mathbb{P}^n is a variety (since then X is a closed subprevariety, so is a variety by the lemma). That is, we want to show that $\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$.

We have that

$$\Delta_{\mathbb{P}^n} = \{((x_0 : \dots : x_n), (y_0 : \dots : y_n)) \in \mathbb{P}^n \times \mathbb{P}^n \mid x_i y_j - x_j y_i = 0, \forall i, j\}. \quad (8.0.1)$$

One may ask why these equations mean that $x = (x_1 : \dots : x_n)$ and $y = (y_1 : \dots : y_n)$ are proportional. To see this we note that since we are in projective space there must be some j such that $x_j \neq 0$. Then we get that

$$\begin{aligned} x_i y_j - x_j y_i &= 0 \\ \Leftrightarrow x_i y_j &= x_j y_i \\ \Leftrightarrow \left(\frac{y_j}{x_j}\right) x_i &= y_i \quad (\forall 0 \leq i \leq n). \end{aligned} \quad (8.0.2)$$

If $y_j = 0$ then $y_i = 0$ for all $i = 0, \dots, n$ which is impossible, hence $y_j \neq 0$ and then we can take $\lambda = \frac{y_j}{x_j}$ to see that $x = \lambda y$. On the other hand, if $x = \lambda y$ then $x_i = \lambda y_i$ and $x_j = \lambda y_j$ so the equations $x_i y_j - x_j y_i$ becomes $y_i y_j - y_j y_i = 0$ which is trivially true. Therefore $\Delta_{\mathbb{P}^n}$ is exactly as described in 8.0.1. If we let $z_{i,j} = x_i y_j$ then we see that

$$\Delta_{\mathbb{P}^n} = V_p(z_{i,j} - z_{j,i} \mid \forall i, j).$$

where each $z_{i,j} - z_{j,i}$ is homogeneous of degree 2 in **Segre coordinates**. Assuming the result of [Gat21, Prop 7.10.(b)] we have that $\mathbb{P}^n \times \mathbb{P}^n$ can be embedded into $\mathbb{P}^N = \mathbb{P}^{n^2+2n}$ as a closed subset. But since then $f(\Delta_{\mathbb{P}^n})$ is closed in \mathbb{P}^{n^2+2n} and $f(\mathbb{P}^n \times \mathbb{P}^n) = X$ is closed in \mathbb{P}^{n^2+2n} and since $f(\Delta_{\mathbb{P}^n})$ is contained in X it follows that $f(\Delta_{\mathbb{P}^n})$ is closed in X (with the subspace topology induced from

\mathbb{P}^{n^2+2n}). Since f that embeds $\mathbb{P}^n \times \mathbb{P}^n$ into \mathbb{P}^{n^2+2n} is an isomorphism it follows that it is continuous. Therefore $f^{-1}(f(\Delta_{\mathbb{P}^n})) = \Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$. \square

Notice that separated \Leftrightarrow “Hausdorff”.

Closed map

Definition 8.0.2. Let $f : X \rightarrow Y$ be a map between topological spaces. Then we say that f is **closed** if whenever $C \subseteq X$ is closed $\Rightarrow f(C) \subseteq Y$ is closed.

Complete variety

Definition 8.0.3. A variety X is called **complete** if for all varieties Y we have that

$$\pi : X \times Y \rightarrow Y$$

is closed (8.0.2).

Proposition 8.0.4. $\pi : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m$ is closed.

Proof. Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ be a closed subset. Then we may write $Z = V_p(f_1, \dots, f_r)$ where the f_i are all (bi)-homogeneous of the same degree d in Segre coordinates of $\mathbb{P}^n \times \mathbb{P}^m$. Fix a point $a \in \mathbb{P}^m$. We will determine whether a is in $\pi(Z)$. Let $g_i := f_i(-, a) \in k[x_0, \dots, x_n]$ for $i = 1, \dots, r$. Then we have

$$\begin{aligned} a \notin \pi(Z) &\Leftrightarrow \text{there is no } x \in \mathbb{P}^n \text{ with } (x, a) \in Z \\ &\Leftrightarrow V_p(g_1, \dots, g_r) = \emptyset \\ &\Leftrightarrow \sqrt{\langle g_1, \dots, g_r \rangle} = \langle 1 \rangle \text{ or } \sqrt{\langle g_1, \dots, g_r \rangle} = \langle x_0, \dots, x_n \rangle \\ &\Leftrightarrow \text{there are } \ell_i \in \mathbb{N} \text{ such that } x_i^{\ell_i} \in \langle g_1, \dots, g_r \rangle \text{ for all } i \\ &\Leftrightarrow k[x_0, \dots, x_n]_\ell \subset \langle g_1, \dots, g_r \rangle \quad (\text{for } \ell = \ell_0 + \dots + \ell_n). \end{aligned}$$

The last condition above can only hold if $\ell \geq d \Leftrightarrow k[x_0, \dots, x_n]_\ell = \langle g_1, \dots, g_r \rangle_\ell$. We have

$$\langle g_1, \dots, g_r \rangle = \{h_1 g_1 + \dots + h_r g_r : h_i \in k[x_0, \dots, x_n]\}.$$

Thus we have

$$\begin{aligned} \langle g_1, \dots, g_r \rangle_\ell &= \left\{ \sum_i h_i g_i \mid h_i \in k[x_0, \dots, x_n] \text{ and } \deg h_i + \deg g_i = \ell \right\} \\ &= \left\{ \sum_i h_i g_i \mid h_i \in k[x_0, \dots, x_n] \text{ and } \deg h_i = \ell - d \right\} \quad (\text{since } \deg g_i = d \text{ for all } i) \\ &= \left\{ \sum_i h_i g_i \mid h_i \in k[x_0, \dots, x_n]_{\ell-d} \right\}. \end{aligned}$$

Thus the last equality above implies that claiming that $\langle g_1, \dots, g_r \rangle_\ell = k[x_0, \dots, x_n]_\ell$ is the same as saying that the map

$$F_\ell : (k[x_0, \dots, x_n]_{\ell-d})^r \rightarrow k[x_0, \dots, x_n]_\ell, \quad (h_1, \dots, h_r) \mapsto h_1 g_1 + \dots + h_r g_r$$

is surjective. F_ℓ is a k -linear map between k -vector spaces, and so that this map is surjective is the same as saying that the *rank* of F_ℓ is the same as the dimension of $k[x_0, \dots, x_n]_\ell$ for some $\ell \geq d$. This

is $\binom{\ell+n}{\ell}$. If we represent F_ℓ in some basis this is equivalent to there being some minor of size $\binom{\ell+n}{\ell}$ that do not vanish.

Here we claim there is perhaps some details left out in the proof given in [Gat21] that one have to check, i.e. that one can find a $\ell \geq d$ such that F_ℓ can be surjective (note that it can not have rank greater than the minimum of the number of row/columns). One may check that if $s_\ell := \binom{n+\ell}{\ell}$ then $\lim_{\ell \rightarrow \infty} \frac{s_\ell}{s_{\ell-d}} = 1$ and so for $r \geq 2$ there is some ℓ_0 such that $\ell_0 s_{\ell_0-d} \geq s_{\ell_0}$. Furthermore one may show that the matrix F_ℓ is generally a matrix with elements polynomials in a_0, \dots, a_m where $a = (a_0, \dots, a_m)$, and so is its minors. Thus the condition that some minor in F_ℓ for some ℓ does not vanish can be written as a (possibly infinite) union on the form $\bigcup_{i,\ell} D(f_{i,\ell}(a))$ with $f_{i,\ell}(a)$ polynomials in the coefficients of the components of a . Now the claim is that in fact the construction of all these minors are independent on our a in the sense that they are formal polynomials in y_0, \dots, y_m . Hence $a \in U$ iff $a \notin \pi(Z)$. Thus the complement of $\pi(Z)$ in \mathbb{P}^m is U which is open so $\pi(Z)$ is closed. \square

Corollary 8.0.5. *If Y is a variety, then $\pi : \mathbb{P}^n \times Y \rightarrow Y$ is closed. That is, \mathbb{P}^n is complete (8.0.3).*

Proof. We first show the statement for an affine variety $Y \subset \mathbb{A}^m$. Let $Z \subset \mathbb{P}^n \times Y$ be closed, and let \overline{Z} be its closure in $\mathbb{P}^n \times \mathbb{P}^m$ note that $\mathbb{P}^n \times Y$ can then be embedded into $\mathbb{P}^n \times \mathbb{P}^m$ (see 7.0.15). If $\pi : \mathbb{P}^n \rightarrow \mathbb{P}^m \rightarrow \mathbb{P}^m$ is the projection map then by 8.0.4 we have that $\pi(\overline{Z})$ is closed.

Lemma 8.0.6. *For any function $f : X \rightarrow Y$ and subsets $A \subset X, B \subset Y$ we have*

$$f(A) \cap B = f(A \cap f^{-1}(B)).$$

Lemma 8.0.7. *Let $U \subseteq S \subseteq X$ where X is a space and let U be closed in S . Then it follows that the closure of U in S is equal to $\overline{U} \cap S$.*

Proof. See proof. \square

Since Z is closed in $\mathbb{P}^n \times Y$ and it follows from 8.0.7 that

$$Z = \overline{Z} \cap \mathbb{P}^n \times Y,$$

so we have that

$$\begin{aligned} \pi(Z) &= \pi(\overline{Z} \cap (\mathbb{P}^n \times Y)) \\ &= \pi(\overline{Z} \cap \pi^{-1}(Y)) \\ &= \pi(\overline{Z}) \cap Y, \quad \text{by 8.0.20} \end{aligned}$$

which is closed in Y . By assumption Y is a variety (so in particular a prevariety (5.0.2)), so we can cover it by affine open sets $Y = \bigcup_{i=1}^n Y_i$. Then we get an associated universal projection $\pi_i : \mathbb{P}^n \times Y_i \rightarrow Y_i$ for each $i = 1, \dots, n$. By the above affine case each of these projections are closed. Note that since $Z \subset \mathbb{P}^n \times Y$ is closed it follows that $Z \cap (\mathbb{P}^n \times Y_i)$ is closed in $\mathbb{P}^n \times Y_i$ with the subspace topology from $\mathbb{P}^n \times Y$.

Claim: $\pi(Z) \cap Y_i = \pi_i(Z \cap (\mathbb{P}^n \times Y_i))$.

Proof.

$$\begin{aligned} \pi_i(Z \cap (\mathbb{P}^n \times Y_i)) &= \pi(Z \cap (\mathbb{P}^n \times Y_i)) \\ &= \pi(Z) \cap Y_i \end{aligned}$$

where we used that $\pi^{-1}(Y_i) = \mathbb{P}^n \times Y_i$.

By the above claim it follows that $\pi(Z) \cap Y_i$ is closed for each Y_i .

Lemma 8.0.8. *Let X be a space and $\{U_i\}_{i \in I}$ an open cover of X . Then a subspace $V \subset X$ is closed in $X \Leftrightarrow V \cap U_i$ is closed for all $i \in I$.*

By 8.0.8 we see that $\pi(Z)$ is closed in Y . □

□

Example 8.0.9 ([Gat21, Example 7.21]). \mathbb{A}^1 is not complete: The image $\pi(Z)$ of the closed subset $Z = V(x_1x_2 - 1) \subset \mathbb{A}^1 \times \mathbb{A}^1$ under the projection to the second factor is the subset $\mathbb{A}^1 \setminus \{0\} \subset \mathbb{A}^1$, which is not closed.

Proposition 8.0.10. *If $f : X \rightarrow Y$ is a morphism of varieties and if X is complete, then $f(X)$ is closed and complete.*

Proof.

Closed: We denote by $\pi : X \times Y \rightarrow Y$ to Y , which is a closed map since X is by assumption complete (8.0.3). We have seen that $\Gamma_f := \{(x, f(x)) \in X \times Y \mid x \in X\}$ is closed (see [Gat21, Prop. 5.20.(a)]). Therefore $\pi(\Gamma_f) = f(X)$ is closed as well.

Complete: We get $\psi : X \times Y' \rightarrow f(X) \times Y'$ by the universal property of the product $f(X) \times Y'$:

$$\begin{array}{ccccc}
 X \times Y' & \xrightarrow{\quad \pi_{Y'} \quad} & & & Y' \\
 & \searrow \exists! \psi & & \nearrow \pi'_{Y'} & \\
 & & f(X) \times Y' & \xrightarrow{\quad \pi'_{Y'} \quad} & Y' \\
 & & \downarrow \pi'_{f(X)} & & \\
 & & f(X) & &
 \end{array}$$

$f \circ \pi_X$ (curved arrow from $X \times Y'$ to $f(X)$)

It follows that ψ is defined as $(x, y) \mapsto (f(x), y)$ and that ψ is a morphism (so in particular, continuous) and that by construction ψ is surjective, so that $\psi(\psi^{-1}(Z)) = Z$ for any $Z \subset X \times Y'$. Therefore, for any closed subset $Z \subset X \times Y'$, we have

$$\begin{aligned}
 \pi'_{Y'}(Z) &= \underbrace{(\pi'_{Y'} \circ \psi)}_{=\pi_{Y'}}(\psi^{-1}(Z)) \\
 &= \pi_{Y'}(\psi^{-1}(Z)).
 \end{aligned}$$

Since $\psi^{-1}(Z)$ is closed (by continuity of ψ) and $\pi_{Y'}$ is a closed map by completeness of X it follows that $\pi_{Y'}(\psi^{-1}(Z))$ is closed in Y' . Thus $\pi'_{Y'}$ is a closed map. Since Z and Y' were arbitrary, $f(X)$ is complete. □

Corollary 8.0.11. *If X is connected and complete variety. Then $\mathcal{O}_X(X) \cong k$.*

Proof. A global regular function $\varphi \in \mathcal{O}_X(X)$ determines a map $\varphi : X \rightarrow \mathbb{A}^1$: Notice that X is a variety, and so we may restrict any global regular function $\varphi \in \mathcal{O}_X(X)$ to affines in the cover of X , yielding $\varphi_i : U_i \rightarrow \mathbb{A}^1$. Naturally $\varphi_i : U_i \rightarrow \mathbb{A}^1$ is now still a regular function, but now its domain and codomain are affine varieties (with domain *open*), and since it is regular it is in fact a morphism by [Gat21, Prop. 4.7]. Then by [Gat21, Prop. 4.6] we see that since it is a morphism on an open cover φ itself is a morphism.

We can now extend $\varphi : X \rightarrow \mathbb{A}^1$ to a $\varphi : X \rightarrow \mathbb{A}^1 \cup \{\infty\} := \mathbb{P}^1$ where $\varphi(X) \subset \mathbb{P}^1$ does not contain ∞ . By 8.0.10 we see that $\varphi(X)$ is closed since X is complete by assumption. Assuming the result that gives that the only closed sets of \mathbb{P}^1 are the finite sets, we see that $\varphi(X)$ must be finite. Since φ is continuous, then assuming the result of [Gat21, Exc. 2.22.(a)] we find that $\varphi(X)$ is connected since X is connected. Assuming the only connected closed finite sets in \mathbb{P}^1 are points we find that $\varphi(X)$ must be a point. \square

8.0.2 Grassmanians

We have $\mathbb{P}^n \Leftrightarrow \{1\text{-dim vector spaces of } k^{n+1}\}$.

Grassmanian $\mathbb{G}(k, n)$

Let $n \in \mathbb{N}_{>0}$ and let $\ell \in \mathbb{N}$ with $0 \leq \ell \leq n$. Then we denote by $\mathbb{G}(\ell, n)$ the *set of all ℓ -dimensional linear subspaces of k^n* . We call $\mathbb{G}(\ell, n)$ the **Grassmanian** of ℓ -planes in k^n .

The goal is to show that $\mathbb{G}(\ell, n)$ is a projective variety (6.0.7).

Alternating (ℓ -fold) multilinear map $f : V^\ell \rightarrow W$

Definition 8.0.12. Let V be a vector space over k and let $\ell \in \mathbb{N}$. A (ℓ -fold) multilinear map $f : V^\ell \rightarrow W$ where W is another vector space is said to be **alternating** if $f(v_1, \dots, v_\ell) = 0$ for all $v_1, \dots, v_\ell \in V$ such that $v_i = v_j$ for some $i \neq j$.

Remark 8.0.13 ([Gat21, Remark 8.4]). One can show that an alternating map (8.0.12) $f : V^\ell \rightarrow W$ is such that $f(v_{\sigma(1)}, \dots, v_{\sigma(\ell)}) = \text{sgn}(\sigma) \cdot f(v_1, \dots, v_\ell)$ with $\sigma \in S_\ell$.

Alternating tensor product

Definition 8.0.14. Let V be a vector space over k and let $\ell \in \mathbb{N}$. An **ℓ -fold alternating tensor product** of V is a vector space denoted $\Lambda^\ell V$ together with an alternating ℓ -fold multilinear map (8.0.12) $\tau : V^\ell \rightarrow \Lambda^\ell V$ that satisfies the following *universal property*: For every ℓ -fold alternating multilinear map $f : V^\ell \rightarrow W$ to another vector space W there exists a unique linear map $g : \Lambda^\ell V \rightarrow W$ with $f = g \circ \tau$, i.e. such that the diagram below commutes.

$$\begin{array}{ccc} V^\ell & \xrightarrow{f} & W \\ \downarrow \tau & \searrow \exists! g & \\ \Lambda^\ell V & & \end{array}$$

Remark 8.0.15. One writes $\tau(v_1, \dots, v_\ell) = v_1 \wedge \dots \wedge v_\ell \in \Lambda^\ell V$, and $\Lambda^\ell V$ is defined as

$$\Lambda^\ell V := \underbrace{(V \otimes \dots \otimes V)}_{\ell \text{ times}} / \langle v_1 \otimes \dots \otimes v_\ell \mid v_i = v_j \text{ for some } i \neq j \rangle.$$

Example 8.0.16. $\dim V = n$ with $V = \langle e_1, \dots, e_n \rangle$. Then $\mathcal{E} = \{e_{i_1} \wedge \dots \wedge e_{i_n} \mid i_1 < i_2 < \dots < i_n\}$ gives a basis for $\Lambda^\ell V$ and so one can show (we claim) that

$$\dim \Lambda^\ell V = \binom{n}{\ell}.$$

so that

$$\begin{aligned}\Lambda^0 V &= \Lambda^n V \\ &= k\end{aligned}$$

and

$$\Lambda^1 V = V.$$

Example 8.0.17. $\Lambda^2 k^3 \cong k^3$. Let $v = \sum a_i e_i$ and $w = \sum b_i e_i$. Then

$$v \wedge w = (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 + (a_1 b_3 - a_3 b_1) e_1 \wedge e_3 + (a_2 b_3 - a_3 b_2) e_2 \wedge e_3$$

which is the same as the “minors” of the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

where we by minors mean that we take the determinant of the three 2×2 -matrices sitting inside this matrix.

We have that $v \wedge w = \sum \text{minors } v_1 \wedge \cdots \wedge v_\ell$.

Lemma 8.0.18. *Let $v_1, \dots, v_\ell \in k^n$. Then*

- $v_1 \wedge \cdots \wedge v_\ell \Leftrightarrow$ the v_i are linearly dependent.
- If v_1, \dots, v_ℓ and w_1, \dots, w_ℓ are two sets of linearly independent vectors such that $\text{span}(v_1, \dots, v_\ell) = \text{span}(w_1, \dots, w_\ell) \Leftrightarrow v_1 \wedge \cdots \wedge v_\ell = \lambda w_1 \wedge \cdots \wedge w_\ell$.

Construction: Plücker Embedding

Let $0 \leq \ell \leq n$ and consider the map $f : \mathbb{G}(\ell, n) \rightarrow \mathbb{P}^{\binom{n}{\ell}-1}$ given by sending $\text{span}(v_1, \dots, v_\ell) \in \mathbb{G}(\ell, n)$ to $v_1 \wedge \cdots \wedge v_\ell \in \Lambda^\ell V \cong k^{\binom{n}{\ell}} \hookrightarrow \mathbb{P}^{\binom{n}{\ell}-1}$. Since $\text{span}(v_1, \dots, v_\ell) \in \mathbb{G}(\ell, n)$ we have that the vectors v_1, \dots, v_ℓ are linearly independent so that $v_1 \wedge \cdots \wedge v_\ell \neq 0$ by 8.0.18. Furthermore, if $\text{span}(v_1, \dots, v_\ell) = \text{span}(w_1, \dots, w_\ell)$ then $v_1 \wedge \cdots \wedge v_\ell = \lambda w_1 \wedge \cdots \wedge w_\ell$. The map is also injective by 8.0.18 since $v_1 \wedge \cdots \wedge v_\ell = w_1 \wedge \cdots \wedge w_\ell \implies \text{span}(v_1, \dots, v_\ell) = \text{span}(w_1, \dots, w_\ell)$. We call this the **Plücker embedding** of $\mathbb{G}(\ell, n)$.

Proposition 8.0.19. $\mathbb{G}(\ell, n)$ is a closed subset of $\mathbb{P}^{\binom{n}{\ell}-1}$.

Proof. Notice that $\mathbb{G}(\ell, n) = \{*\}$ and $\mathbb{G}(\ell, n) = \emptyset$ for $\ell > n$ so assume that $\ell < n$. Let $\omega \in \mathbb{P}^{\binom{n}{\ell}-1}$. Then $\omega \in \text{pl}(\mathbb{G}(\ell, n)) \Leftrightarrow \omega = \lambda v_1 \wedge \cdots \wedge v_\ell$. By [Gat21, Lemma 8.14] this holds if and only if $\text{rank}(f) = n - \ell$ where $f : \Lambda^\ell k^n \rightarrow \Lambda^{\ell+1} k^n$ is defined by $v \mapsto v \wedge \omega$.

Lemma 8.0.20. *A linear map f is of rank $< r$ if and only if each $r \times r$ minor of f vanishes (for some matrix-representation of f).*

Proof. See [htt] for an argument. □

We know by [Gat21, Lemma 8.14] that the rank of f is always atleast $n - \ell$. Let A be a matrix-representation of f . This means that if we can show that every $(n - \ell + 1) \times (n - \ell + 1)$ minor vanishes, it follows from 8.0.20 that f has rank less than $n - \ell + 1$ so must be of rank $n - \ell \Leftrightarrow \omega = v_1 \wedge \cdots \wedge v_\ell$. But the condition that certain minors vanishes are polynomial conditions in the entries of the matrix.

Now fix a basis $\{e_1, \dots, e_n\}$ for k^n and the basis

$$\mathcal{E} = \{e_I := e_{i_1} \wedge \dots \wedge e_{i_\ell} \mid I = (i_1, \dots, i_\ell), 1 \leq i_1 < i_2 < \dots < i_\ell \leq n\}$$

for $\Lambda^\ell k^n$ and

$$J = \{e_J := e_{i_1} \wedge \dots \wedge e_{i_\ell} \wedge e_{i_{\ell+1}} \mid J = (i_1, \dots, i_{\ell+1}), 1 \leq i_1 < i_2 < \dots < i_{\ell+1} \leq n\} \quad (8.0.3)$$

for $\Lambda^{\ell+1} k^n$.

Then we can write $\omega \in \Lambda^\ell k^n$ as $\omega = \sum_I \omega_I e_I$. The map $f : k^n \rightarrow \Lambda^{\ell+1} k^n$ is then defined as $v \mapsto v \wedge \omega$ with $v = \sum_i v_i e_i$. Therefore we get that

$$\begin{aligned} v \wedge \omega &= \left(\sum_i v_i e_i \right) \wedge \left(\sum_I \omega_I e_I \right) \\ &= \sum_i \sum_I v_i \omega_I (e_i \wedge e_I), \end{aligned}$$

where we in the last equality used that the **wedge product** $-\wedge-$ is *linear* in each argument. Also, $i \in I \Leftrightarrow e_i \wedge e_I = 0$. For (i, I) with $i \notin I$, we have that (up to sign-changes) $e_i \wedge e_I$ is a basis element of $\Lambda^{\ell+1} k^n$. Hence $e_i \wedge e_I = \text{sgn}(i, I) e_J$ where $J = \{i\} \cup I$ and $\text{sgn}(i, I)$ is the sign required to reorder (i, i_1, \dots, i_n) into the standard form corresponding to e_J in 8.0.3. We then see that

$$\begin{aligned} f(e_i) &= e_i \wedge \omega \\ &= \sum_{I, i \notin I} \omega_I (e_i \wedge e_I) \\ &= \sum_{I, i \notin I} \omega_I \text{sgn}(i, I) e_J. \end{aligned}$$

By linear algebra theory (see e.g. [DF04, Chap. 11.2]) we then have that f represented with respect to the bases \mathcal{E} and J of k^n and $\Lambda^{\ell+1} k^n$ respectively, is $f = (a_{Ji})$ where the rows of f are the same as $\dim \Lambda^{\ell+1} k^n = \binom{n}{\ell+1}$ and the number of columns are the same as the $\dim k^n = n$. Hence each row correspond to some choice of J as in 8.0.3. We then have, with $J = \{i\} \cup I$

$$a_{Ji} = \omega_I \text{sgn}(i, I)$$

if $i \notin J$ and $a_{Ji} = 0$ otherwise. Therefore, f in the “standard” bases given consists of the coordinates of ω . So the condition of the vanishing of certain minors of f in this matrix representation is the same as the condition of certain polynomials in the coordinates of ω vanishing. \square

Chapter 9

Lecture 9

Remark 9.0.1. For reference for the arguments below, see [Gat21, Remark 8.9, Construction 8.18].

9.0.1 Alternating tensor products and determinant

First, let $0 \leq k \leq n$ and let $v_1, \dots, v_k \in k^n$. Take the *standard basis* $\{e_i : i = 1, \dots, n\}$ of k^n and write each v_i in this basis, i.e.

$$v_i = \sum_j a_{ij} e_j.$$

For *strictly increasing indices* $1 \leq i_1 < \dots < i_k \leq n$ we want to determine the coefficient of the basis vector $e_{i_1} \wedge \dots \wedge e_{i_k}$ of $\Lambda^k k^n$ in the tensor product $v_1 \wedge \dots \wedge v_k$. By *multilinearity* of the wedge-product \wedge we have

$$\begin{aligned} v_1 \wedge \dots \wedge v_k &= \left(\sum_{j_1=1}^n a_{1j_1} e_{j_1} \right) \wedge \left(\sum_{j_2=1}^n a_{2j_2} e_{j_2} \right) \wedge \dots \wedge \left(\sum_{j_k=1}^n a_{kj_k} e_{j_k} \right) \\ &= \sum_{\substack{j_1, \dots, j_k \\ 1 \leq j_1, \dots, j_k \leq n}} a_{1j_1} \dots a_{kj_k} e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k}. \end{aligned} \tag{9.0.1}$$

Here we don't necessarily have that $1 \leq j_1 < j_2 < \dots < j_k \leq n$. So if we write $i := (j_1, \dots, j_k)$, then we can sort this list by increasing indices and get a list $i := (i_1, \dots, i_k)$ where $i_\ell \in \{j_1, \dots, j_k\}$ for $1 \leq \ell \leq k$.

For each $1 \leq m \leq n$ we associate i_m in the list i with the corresponding element j_ℓ in the list j : We define τ by $\ell \xrightarrow{\tau} m \Leftrightarrow j_m = i_\ell$. But here note that $j_s \neq j_p$ for all $p \neq s$ by definition of the wedge product, i.e. each element (non-strictly) between 1 and k is represented by some j_ℓ . Therefore τ is surjective and it is injective since i is just j sorted. But then τ is by definition a bijection, so is a permutation. If we let $\tau \in S_k$ act on j componentwise we get that

$$\begin{aligned} \sigma(j) &= \sigma(j_1, \dots, j_k) \\ &= (j_{\sigma(1)}, \dots, j_{\sigma(k)}) \\ &= (i_1, \dots, i_k). \end{aligned}$$

The last equality follows from the fact that σ takes 1 to m if and only if $j_m = i_1$, similarly with 2 etc.

Furthermore, since $\tau \in S_k$ was defined as $\ell \mapsto m \Leftrightarrow j_m = i_\ell$ we see that $j_m = i_\ell$ then we see that $j_{\tau(\ell)} = i_\ell$. But then we have that $j_m = i_{\tau^{-1}(m)}$.

But then we have that

$$\begin{aligned}\tau^{-1}(i_1, \dots, i_k) &= (i_{\tau^{-1}(1)}, \dots, i_{\tau^{-1}(k)}) \\ &= (j_1, \dots, j_k).\end{aligned}$$

Therefore, we have that

$$\begin{aligned}e_{j_1} \wedge \dots \wedge e_{j_k} &= e_{i_{\tau^{-1}(1)}} \wedge \dots \wedge e_{i_{\tau^{-1}(k)}} \\ &= \operatorname{sgn}(\tau^{-1}) e_{i_1} \wedge \dots \wedge e_{i_k}.\end{aligned}$$

But then by (9.0.1) we have

$$\begin{aligned}v_1 \wedge \dots \wedge v_k &= \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) a_{1, i_{\sigma(1)}} \dots a_{k, i_{\sigma(k)}} \right) e_{i_1} \wedge \dots \wedge e_{i_k} \\ &= \sum_{1 \leq i_1 < \dots < i_k \leq n} \det((a_{p, i_q})_{p, q=1}^k) e_{i_1} \wedge \dots \wedge e_{i_k}\end{aligned}\tag{9.0.2}$$

where

$$(a_{p, i_q})_{p, q=1}^k := \begin{pmatrix} a_{1, i_1} & a_{1, i_2} & \dots & a_{1, i_k} \\ a_{2, i_1} & a_{2, i_2} & \dots & a_{2, i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k, i_1} & a_{k, i_2} & \dots & a_{k, i_k} \end{pmatrix}.$$

9.0.2 Affine cover of the Grassmannian

Let $U_0 \subset \mathbb{G}(k, n) \subset \mathbb{P}^{\binom{n}{k}-1}$ be the affine open subset where $e_1 \wedge \dots \wedge e_k \neq 0$. For $L(v_1, \dots, v_k)$ sent to $v_1 \wedge \dots \wedge v_k$ we have that the coefficient of $e_1 \wedge \dots \wedge e_k$ is the determinant of a matrix A as below:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k} \\ a_{2,1} & a_{2,2} & \dots & a_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \dots & a_{k,k} \end{pmatrix}\tag{9.0.3}$$

so that

$$\det(A) = \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k} \\ a_{2,1} & a_{2,2} & \dots & a_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \dots & a_{k,k} \end{vmatrix}.\tag{9.0.4}$$

So this means that we require that this scalar value in the field k is non-zero. But this determinant is precisely some plucker-coordinate x_I for $I = (1, \dots, k)$ i.e. $x_{1,2,\dots,k}$. Therefore we see that this can be described as $U_0 = D_+(x_I) \subset \mathbb{P}^{\binom{n}{k}-1}$ and so is indeed open. If we have a *linear subspace* $L \in \mathbb{G}(k, n)$ then L is in U_0 if and only if it is in the row-span of a $k \times n$ of the form $M = (A \mid B)$ for an invertible $k \times k$ matrix A (where A is as in (9.0.3)) and an arbitrary $k \times (n - k)$ matrix B :

The \Rightarrow direction is clear from what we wrote above, since we can just take $M = (A \mid B)$ as the matrix with row i being v_i coming from $L = \operatorname{Lin}(v_1, \dots, v_k)$.

On the other hand, for the \Leftarrow direction: If we have that $L = \operatorname{Lin}(v_1, \dots, v_k)$ is the *row-span* of a matrix of the form $M = (A \mid B)$ with A invertible then clearly the coefficient of $e_1 \wedge \dots \wedge e_k$ is non-zero since it is precisely the determinant of A , and A is by assumption invertible $\Leftrightarrow \det(A) \neq 0$.

If $M = (A \mid B)$ then if we multiply M from the right by A^{-1} we get

$$\begin{aligned} A^{-1}M &= A^{-1}(A \mid B) \\ &= (A^{-1}A \mid A^{-1}B) \\ &= (I_k \mid A^{-1}B), \end{aligned}$$

just by thinking about ordinary multiplication laws for matrices together with the fact that $A^{-1}A = I_k$.

We note that if P is an arbitrary $k \times k$ matrix then PM is the matrix where row ℓ for $1 \leq \ell \leq k$ is row ℓ of P multiplied with the columns of M . So in particular, matrix-element (ℓ, j) of PM is row ℓ of P multiplied with column j of M , i.e.

$$(PM)_{\ell, j} = \sum_{i=1}^k P_{\ell, i} M_{i, j}.$$

But then we can write row ℓ of PM as

$$(PM)_{\ell} = \left(\sum_{i=1}^k P_{\ell, i} M_{i, 1} \quad \sum_{i=1}^k P_{\ell, i} M_{i, 2} \quad \cdots \quad \sum_{i=1}^k P_{\ell, i} M_{i, k} \right)$$

and this in turn gives us that

$$PM = \begin{pmatrix} \sum_{i=1}^k P_{1, i} M_{i, 1} & \sum_{i=1}^k P_{1, i} M_{i, 2} & \cdots & \sum_{i=1}^k P_{1, i} M_{i, k} \\ \sum_{i=1}^k P_{2, i} M_{i, 1} & \sum_{i=1}^k P_{2, i} M_{i, 2} & \cdots & \sum_{i=1}^k P_{2, i} M_{i, k} \\ \vdots & \ddots & \ddots & \vdots \\ \sum_{i=1}^k P_{k, i} M_{i, 1} & \sum_{i=1}^k P_{k, i} M_{i, 2} & \cdots & \sum_{i=1}^k P_{k, i} M_{i, k} \end{pmatrix}$$

If now go back to think of matrix M as consisting of the rows v_1, \dots, v_k coming from $L = \text{Lin}(v_1, \dots, v_k)$ with $v_i \in k^n$ and so that $v_{i, j}$ is the j^{th} column of the vector v_i we see that we can rewrite this as

$$\begin{aligned} PM &= \begin{pmatrix} \sum_{i=1}^k P_{1, i} v_{i, 1} & \sum_{i=1}^k P_{1, i} v_{i, 2} & \cdots & \sum_{i=1}^k P_{1, i} v_{i, k} \\ \sum_{i=1}^k P_{2, i} v_{i, 1} & \sum_{i=1}^k P_{2, i} v_{i, 2} & \cdots & \sum_{i=1}^k P_{2, i} v_{i, k} \\ \vdots & \ddots & \ddots & \vdots \\ \sum_{i=1}^k P_{k, i} v_{i, 1} & \sum_{i=1}^k P_{k, i} v_{i, 2} & \cdots & \sum_{i=1}^k P_{k, i} v_{i, k} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^k P_{1, i} v_i \\ \sum_{i=1}^k P_{2, i} v_i \\ \vdots \\ \sum_{i=1}^k P_{k, i} v_i \end{pmatrix} \end{aligned}$$

That is, each row in PM is a linear combination of the rows in M .

Therefore, it follows that the row span of PM is in the row-span of M . If we now assume that P is invertible, then we can see by the same argument that $PP^{-1}M = M$ is in the row span of PM . It follows that the row-span of M is the same as the row-span of PM whenever P is invertible. On applying this to our case, we find that the row-span of $(I_k | C)$ is the same as the row-span of $(A | B)$ with $C = A^{-1}B$ given that A is invertible.

We define a map $f : \mathbb{A}^{k(n-k)} = M_{k \times (n-k)}(k) \rightarrow U_0$ explicitly by $C \mapsto \text{row span of } (I_k | C)$.

This map is surjective by what we said previously. To see that it is injective: Let $(I_k | C)$ have the same row span as $(I_k | C')$. We want to show that it follows that $C = C'$:

Assume that $C \neq C'$ so that there is atleast some pair (i, j) such that $c_{i,j} \neq c'_{i,j}$. Then since each row of $M = (I_k | C)$, $M' = (I_k | C')$ is of the form $(e_i, c_{i,1}, \dots, c_{i,n-k})$ we see that it is impossible to get the row $(e_i, c'_{i,1}, \dots, c'_{i,j}, \dots, c'_{i,n-k})$ as a linear combination of the rows of C , since the e_i, e_j can not cancel each other. So we have shown

$$\begin{aligned} C \neq C' &\Rightarrow \text{row span of } C \neq \text{row span of } C' \\ &\Leftrightarrow \text{row span of } C = \text{row span of } C' \Rightarrow C = C', \end{aligned}$$

so that f is indeed injective.

If we take all the maximal minors of $(I_k | C)$ (i.e. the determinants of all its $k \times k$ -matrices) then we see that they are polynomial functions in the elements of C , so that f is a morphism: Since L is precisely determined by the row span of $(I_k | C)$ we can view the row span of $(I_k | C)$ as the same as some $L = \text{Lin}(v_1, \dots, v_k)$ with non-zero coefficient in front of $e_1 \wedge \dots \wedge e_k$. Then this means precisely that the condition imposed on the image of f is that some polynomial entries of C are non-zero. On the other hand, the (i, j) element of C can be reconstructed from $f(C)$ (perhaps up to sign, why?) by taking the maximal minor of $(I_k | C)$ where we take all columns of I_k minus the i^{th} column and the j^{th} column of C , i.e. the $k \times 1$ vector $(c_{1,j}, c_{2,j}, \dots, c_{k,j})$.

To see this, we consider the matrix

$$M^{(i,j)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & c_{1,j} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & c_{2,j} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & c_{i,j} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & c_{k,j} & 0 & \cdots & 1 \end{pmatrix}.$$

We see that expanding along the first row gives us $(-1)^\ell c_{i,j}$ ((where ℓ is appropriately chosen)) since when we compute $c_{1,j} \cdot (\det(\text{minor}))$ the minor has the first column all zeroes, and so the determinant of this minor must be zero. But then f^{-1} is again a morphism since it can be defined as some polynomial in the elements of $(I_k | C)$. Therefore we have that $U_0 \cong \mathbb{A}^{k(n-k)}$ as varieties with the isomorphism (of varieties) given by f . So U_0 is an *affine space*. We claim then that $\mathbb{G}(k, n)$ can be covered by such affine patches U_i .

This leads to the statement [Gat21, Cor. 8.19].

9.0.3 Rational maps

Rational map

Definition 9.0.2. A **rational map** $f : X \dashrightarrow Y$ between two *irreducible* varieties (2.0.5) X, Y , is a morphism $f : U \rightarrow Y$ from a non-empty open subset $U \subset X$ to Y . We say that two such rational maps $f_1, f_2 : X \dashrightarrow Y$ defined on U_1 and U_2 respectively, are the same if $f_1 = f_2$ on a non-empty open subset $W \subset U_1 \cap U_2$.

Dominant rational map

A rational map (9.0.2) $f : X \dashrightarrow Y$ is **dominant** if the image of f contains a non-empty open subset of Y (note that if Y is irreducible then it in fact contains a *dense* subset of Y).

Remark 9.0.3. Since a subset $A \subset X$ of a space X is dense \Leftrightarrow for every $x \in X$ and every neighborhood U of x we have that $A \cap U \neq \emptyset$, this means that f is dominant if $f(U) \cap V \neq \emptyset$ for every neighborhood V of every point y of Y .

Remark 9.0.4. The above remark then in turn means that the image of f contains a non-empty open subset U of Y .

In the case of a dominant map

$$f : X \dashrightarrow Y \quad (f : U \rightarrow Y)$$

and

$$g : Y \dashrightarrow Z \quad (g : V \rightarrow Z)$$

another rational map \rightsquigarrow we can construct $g \circ f : X \dashrightarrow Z$ as a rational map on the non-empty open subset $f^{-1}(U' \cap V)$. Here we have that U' is the open non-empty subset of Y contained in $f(U)$, and so indeed $U' \subset U' \cap V$. We assume the result which gives that for an irreducible variety Y , any two non-empty open subsets of Y intersect non-trivially $\rightsquigarrow U' \cap V$ is non-empty and open. Therefore $f^{-1}(U' \cap V)$ is open, and since $U' \cap V \subset U' \subset f(U)$ we see that for every $v \in U' \cap V$ there is some $u \in U$ such that $f(u) = v$. Therefore $f^{-1}(U' \cap V)$ is a non-empty open subset of U . By [Gat21, Remark 4.5] we then have that $g \circ f_{f^{-1}(U' \cap V)} : f^{-1}(U' \cap V) \rightarrow Z$ is again a morphism.

Birational map

Definition 9.0.5. A rational map $f : X \dashrightarrow Y$ is called **birational** if there is some rational map $g : Y \dashrightarrow X$ such that

$$\begin{aligned} g \circ f &= \text{id}_X \\ f \circ g &= \text{id}_Y. \end{aligned}$$

Birational varieties

Definition 9.0.6. We say that X and Y are **birational** if there is some birational map (9.0.5) $f : X \dashrightarrow Y$.

Example 9.0.7. Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be defined by

$$[x_0 : x_1 : x_2] \mapsto [x_0x_1 : x_0x_2 : x_1x_2]$$

is not defined on $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$.

Notice that

$$\begin{aligned}
 f(f([x_0 : x_1 : x_2])) &= f([x_0x_1 : x_0x_2 : x_1x_2]) \\
 &= [(x_0x_1)(x_0x_2) : (x_0x_1)(x_1x_2) : (x_0x_2)(x_1x_2)] \\
 &= [x_0^2x_1x_2 : x_0x_2x_1^2 : x_0x_1x_2^2] \\
 &= x_0x_1x_2[x_0 : x_1 : x_2],
 \end{aligned}$$

so that $f \circ f|_U = \text{id}_U$ with $U = \mathbb{P}^n \setminus V(x_1, x_2, x_3)$.

9.0.4 Blow-up

Blow-up

Definition 9.0.8. Let $X \subset \mathbb{A}^n$ be an affine variety and $f_1, \dots, f_r \in A(X)$ with not all $f_i = 0$, and let $U := X \setminus V(f_1, \dots, f_r)$.

Let $f : U \rightarrow \mathbb{P}^{r-1}$ be defined by $x \mapsto [f_1(x) : \dots : f_r(x)]$. We claim that this is a morphism. Consider $\Gamma_f = \{(x, f(x)) \mid x \in U\} \subset U \times \mathbb{P}^{r-1}$. By [Gat21, Prop. 5.20.(a)] it is closed in $U \times \mathbb{P}^{r-1}$. This makes sense since U is an open subset of a variety (and varieties are prevarieties) so that U is a open subprevariety, which is again a prevariety. And so our assuming our f is a morphism of varieties it is indeed a morphism of prevarieties. Note that Γ_f is in general *not* closed in $X \times \mathbb{P}^{r-1}$. The *closure* of Γ_f in $X \times \mathbb{P}^{r-1}$ is called the **blow-up of X at f_1, \dots, f_r** , and is usually denoted by \tilde{X} . One has a natural projection $\pi_X|_{\tilde{X}} : \tilde{X} \rightarrow X$ by projection to the first factor. Sometimes we will also say that $\pi_X|_{\tilde{X}} : \tilde{X} \rightarrow X$ is the **blow-up of X at f_1, \dots, f_r** .

We then have a commuting diagram

$$\begin{array}{ccc}
 \tilde{X} := \overline{\Gamma_f} & \hookrightarrow & X \times \mathbb{P}^{r-1} \\
 & \searrow \pi_X|_{\tilde{X}} & \downarrow \pi_X \\
 & & X
 \end{array}$$

Consider $\pi|_{\tilde{X} \cap U \times \mathbb{P}^{r-1}} : U \times \mathbb{P}^{r-1} \rightarrow U$. We claim that $\tilde{X} \cap U \times \mathbb{P}^{r-1}$ is open in \tilde{X} . Note here that U is open in X and that $\pi|_{\tilde{X}} : \tilde{X} \subseteq U \times \mathbb{P}^{r-1} \rightarrow X$ is *continuous*. Therefore $\pi^{-1}|_{\tilde{X}}(U)$ is open in \tilde{X} . But $\pi|_{\tilde{X}}^{-1}(U) = \tilde{X} \cap (U \times \mathbb{P}^{r-1})$.

Remark 9.0.9. In the construction 9.0.8 one notes that Γ_f is isomorphic to U by $\pi|_{\Gamma_f} : \Gamma_f \rightarrow U$. One often uses this isomorphism to identify Γ_f with U .

Lemma 9.0.10. Let $A \subset X$ and let \tilde{A} be the closure of A in X . Then A is dense in \tilde{A} .

Proof. This follows from the fact that $\text{cl}(A) = \bar{A}$ is then the whole space. □

Therefore by the above lemma we have that Γ_f is dense in \tilde{X} so by identification of Γ_f with U we can view U as dense in \tilde{X} . We then have that the complement of U can be viewed as $\pi^{-1}(V(f_1, \dots, f_r))$.

To see this, notice that assuming that π is an isomorphism, we have that

$$\begin{aligned}
 \tilde{X} \setminus U &:= \tilde{X} \setminus \pi^{-1}(U) \\
 &= \tilde{X} \setminus \pi^{-1}(X \setminus V(f_1, \dots, f_r)) \\
 &= \tilde{X} \setminus (\pi^{-1}(X) \setminus \pi^{-1}(V(f_1, \dots, f_r))) \\
 &= \tilde{X} \setminus (\tilde{X} \setminus \pi^{-1}(V(f_1, \dots, f_r))) \\
 &= \pi^{-1}(V(f_1, \dots, f_r))
 \end{aligned}$$

where we call $\pi^{-1}(V(f_1, \dots, f_r))$ the **exceptional set** of the blow-up.

Furthermore, we claim that $\pi|_{\tilde{X} \cap (U \times \mathbb{P}^{r-1})}$ is an *isomorphism*. So $g : U \rightarrow V \subseteq \tilde{X}$ is a birational inverse so this gives a birational equivalence.

9.0.5 Strict transform

Strict transform

Definition 9.0.11 (Strict transform and blow-up of subvarieties). Let $Y \subset X$ be a subvariety of an affine variety $X \subset \mathbb{A}^n$. Analogously to 9.0.8, we can **blow-up** Y **at** f_1, \dots, f_r as well. By construction we then have that $\tilde{Y} \subset Y \times \mathbb{P}^{r-1} \subset \tilde{X} \times \mathbb{P}^{r-1}$ is then a closed subvariety of \tilde{X} . In fact, we claim that it is the closure of $Y \cap U \subset \tilde{X}$ (using the isomorphism $\Gamma_f \cong U$ to identify $Y \cap U$ with a subset of \tilde{X} ; cf. 9.0.9). If we consider \tilde{Y} as a subset of \tilde{X} in the way described above, then \tilde{Y} is then in this context called the **strict transform** of Y in the blow-up of X .

Remark 9.0.12. In relation to the strict transform (9.0.11), if $X = X_1 \cup \dots \cup X_m$ is the irreducible decomposition of X , then $\tilde{X}_i \subset \tilde{X}$ for $i = 1, \dots, m$.

Moreover, assuming the result above which said that $\overline{Y \cap U} = \tilde{Y}$, we have that

$$\begin{aligned}
 \tilde{X}_1 \cup \dots \cup \tilde{X}_n &= (\overline{X_1 \cap U}) \cup \dots \cup (\overline{X_n \cap U}) \\
 &= \overline{(X_1 \cap U) \cup \dots \cup (X_n \cap U)} \\
 &= \overline{(X_1 \cup \dots \cup X_n) \cap U} \\
 &= \overline{X \cap U} \\
 &= \overline{U} \\
 &= \overline{\Gamma_f}, \text{ under the identification } \Gamma_f = U \text{ (I guess)} \\
 &= \tilde{X}
 \end{aligned}$$

Example 9.0.13 (Trivial example). Let $r = 1$ in 9.0.8, so that we only blow-up at one function $f_1 \in A(X)$. Then $\mathbb{P}^{r-1} = \mathbb{P}^0$ but $\mathbb{P}^0 = \{*\}$. We then have that $\tilde{X} \subset X \times \{*\} \cong X$ and by our earlier remark we may assume $\Gamma_f \cong U$. So \tilde{X} is just the closure of U in X under this identification. If we assume that X is irreducible we get two cases ($f_1 \neq 0$ or $f_1 = 0$):

- (a) Assume If $f_1 \neq 0$ then $U = X \setminus V(f_1)$ is a non-empty subset of X (since only $V(0) = X$). Then since U is a non-empty open subset, its closure is all of X by [Gat21, Remark 2.16.(b)] so that

$$\begin{aligned}
 \overline{U} &= X \\
 &= \tilde{X}.
 \end{aligned}$$

- (b) if $f_1 = 0$ then $U := X \setminus V(0)$ but $V(0) = X$ so that $U = \emptyset \rightsquigarrow \overline{U} = \emptyset$ but under the identification $\overline{U} = \tilde{X}$ we then have $\tilde{X} = \emptyset$.

Example 9.0.14. Let $X = \mathbb{A}^2$ and let $f_1 = x_1$ and $f_2 = x_2$. Then

$$\begin{aligned} U &= \mathbb{A}^2 \setminus V(f_1, f_2) \\ &= \mathbb{A}^2 \setminus V(x_1, x_2) \\ &= \mathbb{A}^2 \setminus \{(0, 0)\}. \end{aligned}$$

Then

$$\begin{aligned} \Gamma_f &= \{ \underbrace{((x_1, x_2), [s : t])}_{\in U \times \mathbb{P}^{r-1}} : [x_1 : x_2] = [s : t] \} \\ &= \{ ((x_1, x_2), [s : t]) : sx_2 - tx_1 = 0 \} \end{aligned}$$

where the last equality follows from the fact that $(x_1, x_2) = \lambda(s, t)$ for $\lambda \in k^\times$ is equivalent to these two vectors (viewed as elements of $\mathbb{A}^2 \setminus \{(0, 0)\}$) being *linearly dependent*

$$\begin{aligned} &\Leftrightarrow \det \begin{pmatrix} s & t \\ x_1 & x_2 \end{pmatrix} = 0 \\ &\Leftrightarrow sx_2 - tx_1 = 0. \\ &\rightsquigarrow \tilde{X} = \overline{\Gamma_f} \\ &= \{ \underbrace{((x_1, x_2), [s : t])}_{\in X \times \mathbb{P}^{r-1}} : sx_2 - tx_1 = 0 \}. \end{aligned}$$

Furthermore, we have that the *exceptional set* is

$$\begin{aligned} \pi^{-1}(V(x_1, x_2)) &= \pi^{-1}(\{(0, 0)\}) \\ &= \{((0, 0), [s : t]) : s \cdot 0 - t \cdot 0 = 0\}. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{L} &= \{(a, b) \in \mathbb{A}^2 \mid ax + by = 0\} \setminus \{(0, 0)\} \\ &= \{(a, b) \in \mathbb{A}^2 \mid ax = -by\} \setminus \{(0, 0)\}. \end{aligned}$$

Then

$$\begin{aligned} \pi^{-1}(\mathcal{L}) &= \{((x, y), [s : t]) \in \mathcal{L} \times \mathbb{P}^{r-1} : tx - ys = 0\} \\ &\cong \mathbb{P}^1 \end{aligned}$$

We claim that $\overline{\mathcal{L}} \cap \pi^{-1}(\{(0, 0)\}) = \{(0, 0) \times [-b : a]\}$ and that $U_s = \{((x, y), [1 : t]) \mid y = tx\} \simeq \mathbb{A}^2 \Rightarrow$ irreducible and 2-dimensional.

Lemma 9.0.15. *The blow-up \tilde{X} of an affine variety X at $f_1, \dots, f_r \in A(X)$ depends only on the ideal $\langle f_1, \dots, f_r \rangle \trianglelefteq A(X)$.*

More precisely, if $f'_1, \dots, f'_s \in A(X)$ with $\langle f_1, \dots, f_r \rangle = \langle f'_1, \dots, f'_s \rangle \trianglelefteq A(X)$ and $\pi : \tilde{X} \rightarrow X$ and $\pi' : \tilde{X}' \rightarrow X$ are the corresponding blow-ups, there is an isomorphism $F : \tilde{X} \rightarrow \tilde{X}'$ with $\pi' \circ F = \pi$. That is, we get a commutative diagram as below:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{F} & \tilde{X}' \\ & \searrow \pi \quad \swarrow \pi' & \\ & X & \end{array}$$

Proof. By assumption, we have

$$f_i = \sum_{j=1}^s g_{i,j} f'_j, \quad \text{for } i = 1, \dots, r \quad (9.0.5)$$

and

$$f'_j = \sum_{k=1}^r h_{j,k} f_k, \quad \text{for all } j = 1, \dots, s, \quad (9.0.6)$$

in $A(X)$ for some $g_{i,j}, h_{j,k} \in A(X)$.

We then claim that $F : \tilde{X} \rightarrow \tilde{X}'$ defined explicitly by

$$(x, y) \mapsto (x, y') := \left(x, \left[\sum_{k=1}^r h_{1,k}(x) y_k : \dots : \sum_{k=1}^r h_{s,k}(x) y_k \right] \right) \quad (9.0.7)$$

is an isomorphism between $\tilde{X} \subset X \times \mathbb{P}^{r-1}$ and $\tilde{X}' \subset X \times \mathbb{P}^{s-1}$. The argument goes as follows:

- The homogeneous coordinates of y' are not all *simultaneously* zero: If we recall the construction of the blow-up at f_1, \dots, f_r as the closure (in $X \times \mathbb{P}^{r-1}$) of the graph

$$\Gamma_f = \{(x, y) \in U \times \mathbb{P}^{r-1} : x \in U\}$$

where f is defined as $x \mapsto [f_1(x) : \dots : f_r(x)]$ we see that $[y_1 : \dots : y_r] = [f_1 : \dots : f_r]$ on $U \setminus V(f_1, \dots, f_r) \subset \tilde{X} \subset X \times \mathbb{P}^{r-1}$. This means that (in affine coordinates) (y_1, \dots, y_r) is linearly dependent with (f_1, \dots, f_r) for each point $x \in U$. We then have that

$$\begin{aligned} f_i &= \sum_{j=1}^s g_{i,j} f'_j \quad (\text{by 9.0.5}) \\ &= \sum_{j=1}^s g_{i,j} \left(\sum_{k=1}^r h_{j,k} f_k \right) \quad (\text{by 9.0.6}) \\ &= \sum_{j=1}^s \left(\sum_{k=1}^r g_{i,j} h_{j,k} f_k \right) \end{aligned} \quad (9.0.8)$$

Now, since (by linear dependence) there is some $\lambda \in k^\times$ such that $\lambda y_i = f_i$ for all $i = 1, \dots, r$ we have that

$$\begin{aligned} \lambda y_i &= f_i \\ &= \sum_{j=1}^s \left(\sum_{k=1}^r g_{i,j} h_{j,k} \underbrace{f_k}_{=\lambda y_k} \right), \quad (\text{by 9.0.8}) \end{aligned}$$

$$\begin{aligned} &= \sum_{j=1}^s \left(\sum_{k=1}^r g_{i,j} h_{j,k} \lambda y_k \right) \\ \Leftrightarrow y_i &= \sum_{j=1}^s \left(\sum_{k=1}^r g_{i,j} h_{j,k} y_k \right) \quad (\text{by dividing both sides by } \lambda \neq 0) \end{aligned} \quad (9.0.9)$$

$$\Leftrightarrow y_i = \sum_{j=1}^s \left(\sum_{k=1}^r g_{i,j} y'_j \right) \quad (\text{by 9.0.7}) \quad (9.0.10)$$

(The following argument was given by Victor Groth): We then claim that the relations above still hold in \tilde{X} . We define a zero locus in $X \times \mathbb{P}^{r-1}$ by the equations on the form as in (9.0.9) as

$$Z = V \left(y_i - \sum_{j=1}^s \left(\sum_{k=1}^r g_{i,j} h_{j,k} y_k \right) : \forall i = 1, \dots, r \right).$$

Then we see that $\Gamma_f \subset Z$ and Z is closed so that the relations still hold on the closure of Γ_f , which is \tilde{X} , i.e. the relations given by Z still hold in the blow-up \tilde{X} .

Remark 9.0.16. Note here that confusion may arise from the fact that there does not seem to be any affine coordinates in the defining equations of Z , but this is just because we have “hidden” the affine coordinates in the polynomials $g_{i,j}, h_{j,k} = g_{i,j}(x)h_{j,k}(x)$ where $x = \pi((x, y))$ for $(x, y) \in X \times \mathbb{P}^{r-1}$. Here we can think of

$$y_i - \sum_{j=1}^s \left(\sum_{k=1}^r g_{i,j} h_{j,k} y_k \right) \in k[x_1, \dots, x_n, y_1, \dots, y_r].$$

Moving on, this means that if we had that the each argument $y'_i = \sum_{k=1}^r h_{i,k}(x)y_k$ in the image of F as defined explicitly in (9.0.7) was equal to *zero* then by equation (9.0.10) we would have that $y_i = 0$ for $i = 1, \dots, r$ which is a contradiction since we are in \mathbb{P}^{r-1} . Hence the y'_i are not all simultaneously zero.

- We claim that the image of F by construction lie in \tilde{X}' : We have

$$\begin{aligned} F(x, y) &= \left(x, \left[\sum_{k=1}^r h_{1,k}(x)y_k : \dots : \sum_{k=1}^r h_{s,k}(x)y_k \right] \right) \\ &= \left(x, \left[\sum_{k=1}^r h_{1,k}(x)\lambda f_k : \dots : \sum_{k=1}^r h_{s,k}(x)\lambda f_k(x) \right] \right) \\ &= \left(x, \left[\sum_{k=1}^r h_{1,k}(x)f_k(x) : \dots : \sum_{k=1}^r h_{s,k}(x)f_k(x) \right] \right) \\ &= (x, [f'_1(x) : \dots : f'_s(x)]) \in \tilde{X}', \quad (\text{by 9.0.6}), \end{aligned}$$

where we in the next to last equality used that we are in projective coordinates. This holds on the open subset $U = \Gamma_f$; that is, we have that $F(\Gamma_f) \subset \tilde{X}'$. But from what we saw earlier the relations defining our equations still hold on \tilde{X} , and so $F(\tilde{X}) \subset \tilde{X}'$ (roughly). One could perhaps also argue that F is defined by polynomial equations and so that F is continuous, and for F continuous we have that $F(\tilde{X}) \subset \overline{F(\Gamma_f)} \subset \tilde{X}'$ since $F(\Gamma_f) \subset \tilde{X}'$ (I am unsure, but this is perhaps the implicit argument given in Gathmann).

- Without showing this explicitly, [Gat21] argues that one can define F^{-1} similarly to find that F is an isomorphism.
- That $\pi' \circ F = \pi$ follows directly from the fact that F acts as the identity on the *affine coordinates*.

□

Advantages:

- If X is affine we can speak about blowing-up a subvariety $Y \subseteq X$.

- If X is any variety so that $X = \bigcup_i U_i$ and $Y \subseteq X$ is affine then $Y = \bigcup_i (Y \cap U_i)$. Then

$$\tilde{U}_i = \text{Bl}_{Y_i} U_i$$

. By gluing $\rightsquigarrow \tilde{X} = B|_Y X$.

- if X is a variety and $a \in X$ a point then $\pi : \tilde{X} = \text{Bl}_a X \rightarrow X$ with $\pi^{-1}(a) \subseteq \{a\} \times \mathbb{P}^{n-1}$.

9.0.6 Exceptional locus as a subvariety of \mathbb{P}^{n-1}

$V_p(F_i)$ and $C_a X = V_a(F_i)$ tangent cone at X . If we look back at example we have

$$\pi_{\mathcal{L}}^{-1}(0, 0) = [-b : a] \subseteq \mathbb{P}^1,$$

with $V_p(ax_0 + bx_1)$, and

$$\begin{aligned} TC_a(\mathcal{L}) &= V_a(ax + by) \\ &= \mathcal{L}. \end{aligned}$$

Example 9.0.17. Let $X = V_a(x_2 + x_1^2) \subseteq \mathbb{A}^2$. Then if we blow up at the polynomials x_1, x_2 we get

$$\begin{aligned} \tilde{X} &= \overline{\{(x, [s : t]) \in U \times \mathbb{P}^1 : x \in U\}} \\ &= \overline{\{(x_1, -x_1^2, [s : t]) \in U \times \mathbb{P}^1 : x \in U\}}, \quad (\text{since } (x_1, x_2) \in U = X \setminus V(x_1, x_2)), \\ &\subseteq \tilde{\mathbb{A}}^2 \end{aligned}$$

where $U = V_a(x_2 + x_1^2) \setminus V_a(x_1, x_2)$ and $\tilde{\mathbb{A}}^2$ is the blow up of \mathbb{A}^2 at $f_1 = x_1, f_2 = x_2$. By [Gat21, Lemma 9.14] we have that $x_1 t - x_2 s = 0$. But the last condition means that

$$\begin{aligned} \begin{vmatrix} x_1 & x_2 \\ s & t \end{vmatrix} &= x_1 t - x_2 s \\ &= 0. \end{aligned}$$

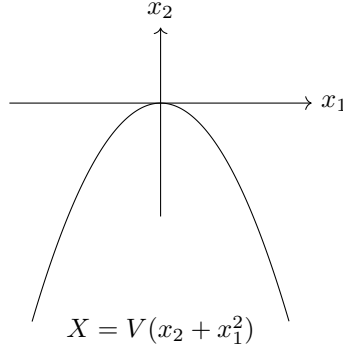
so that (x_1, x_2) is linearly dependent with $(s, t) \rightsquigarrow \exists \lambda \in k^\times$ such that $\lambda x_1 = s$ and $\lambda x_2 = t$.

We have the defining equation $x_2 + x_1^2 = 0$ so that

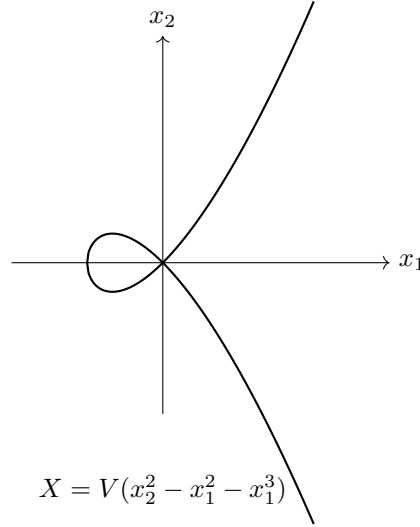
$$\begin{aligned} 0 &= x_2 + x_1^2 \\ &= \lambda(x_2 + x_1^2) \\ &= \lambda x_2 + (\lambda x_1)x_1 \\ &= t + s x_1. \end{aligned}$$

If we look at $\pi^{-1}(0) := E$'s intersection with \tilde{X} then we see that on $E \cap \tilde{X}$ we have that $(x_1, x_2) = (0, 0)$ so that we must have $t = 0$. Therefore $s \neq 0$ and so we get the point $[1 : 0]$ (since $E \cap \tilde{X} \subset \tilde{X} \subset X \times \mathbb{P}^1$ and $X = V_a(x_2 + x_1^2)$).

Pictorially, we have



Example 9.0.18. Let $X = V(x_2^2 - x_1^2 - x_1^3)$, or pictorially, as (if we take the ground field as $k = \mathbb{C}$ and look at its *real part*)



We look at \tilde{X} as a subset of $\tilde{\mathbb{A}}^2$ (we blow up at zero) and then by [Gat21, Lemma 9.14] we see (as in the previous example) that

$$\lambda x_1 = s$$

$$\lambda x_2 = t.$$

Furthermore, on $\tilde{X} \cap E \subset \tilde{X} \subset X \times \mathbb{P}^1$ we still have the defining equation

$$\begin{aligned} x_2^2 - x_1^2 - x_1^3 &= 0 \\ \Rightarrow \lambda^2(x_2^2 - x_1^2 - x_1^3) &= (\lambda x_2)^2 - (\lambda x_1)^2 - x_1(\lambda x_1)^2 \\ &= t^2 - s^2 - x_1 s^2 \\ &= 0. \end{aligned}$$

But on $\tilde{X} \cap E$ we have that $x_1 = x_2 = 0$ so this becomes $t^2 - s^2 = 0 \Leftrightarrow (t + s)(s - t) = 0$. This gives us the points $[1 : 1]$ and $[1 : -1]$.

Proposition 9.0.19. [Gat21, Prop. 9.23] Let $\pi : \tilde{X} \rightarrow X$ be the blow-up of an irreducible affine variety X at $f_1, \dots, f_r \in A(X)$. Then every irreducible component of the exceptional set $\pi^{-1}(V(f_1, \dots, f_r))$ has codimension 1 in \tilde{X} .

Remark 9.0.20. Because of the above proposition, $E = \pi^{-1}(V(f_1, \dots, f_r))$ is often called the **exceptional hypersurface** of the blow-up.

Proof. We prove the statement on all non-empty affine subsets $U_i \subset \tilde{X} \subset X \times \mathbb{P}^{r-1}$ where the i^{th} projective coordinate y_i is non-zero, since these open subsets cover \tilde{X} . We note that for $a \in U_i$ the condition $f_i(a) = 0$ implies $f_j(a) = 0$ for all $j = 1, \dots, r$ by [Gat21, Lemma 9.14]. But note that if $a \in U_i$ and $a \in E$ so that $a \in E \cap U_i$ then by definition of E we must have that $f_i(a) = 0$ for $i = 1, \dots, r$. Furthermore by [Gat21, Lemma 9.14] we have that $y_j f_i(a) = y_i f_j(a)$. But on U_i we can set $y_i = 1$ which gives us the equation

$$y_j f_i(a) = f_j(a).$$

Therefore, we have that $f_i(a) = 0 \Leftrightarrow f_1(a) = \dots = f_r(a) = 0$. Therefore we have that

$$\begin{aligned} U_i \cap E &= \{x \in U_i : f_i(x) = 0\} \\ &= U_i \cap V_a(f_i). \end{aligned}$$

Since the U_i cover \tilde{X} , we have that the $E \cap U_i = U_i \cap V_a(f_i)$ cover E . If $U_i \neq \emptyset$ then f_i is not identically zero on U_i : If f_i was identically zero on U_i then $f_j = 0$ for all $j = 1, \dots, r$ from what we saw above. Then $E \cap U_i = U_i$ so that $U_i \subset E$. Furthermore, since π is *continuous*, and E is defined as the preimage of a closed set, it is closed in \tilde{X} . Therefore we have that

$$\begin{aligned} \overline{U_i} &= \tilde{X} \\ &\subset \overline{E} \\ &= E, \end{aligned}$$

where we have used that U_i is a non-empty open subset of \tilde{X} and that if X is irreducible (which is true by assumption) then \tilde{X} is irreducible ([Gat21, Remark 9.11]). Therefore the closure of U_i must be \tilde{X} since U_i is then *dense* in \tilde{X} . But $\Gamma_f = \tilde{X} \setminus E$ so this leads to the conclusion that $\Gamma_f = \emptyset$ and so $\overline{\Gamma_f} = \tilde{X} = \emptyset$.

Since $\overline{U_i} = \tilde{X}$ is irreducible, it follows by [Gat21, Exercise 2.20.(b)] that U_i is irreducible. Then we see that from [Gat21, Remark 2.28.(c)] that every irreducible component of $V(f_i) \subset U_i$ has codimension 1 in U_i . By [Gat21, Exercise 2.34.(a)] we have that $\dim \tilde{X} = \dim U_i$. This means that each irreducible component of $V(f_i)$ having codimension 1 in $U_i \Leftrightarrow \dim U_i - 1 \Leftrightarrow \dim \tilde{X} - 1$.

We have that $E = \bigcup_i (E \cap U_i)$. Let $Z \subset E$ be an *irreducible component*. Then $Z \cap U_i \neq \emptyset$ for some $i \in \{1, \dots, r\}$. Since U_i is open in \tilde{X} we have that $Z \cap U_i$ is open in the subspace topology on Z inherited from \tilde{X} .

We claim that $Z \cap U_i$ is an irreducible component of $E \cap U_i = V_{U_i}(f_i)$: First, consider that since $Z \cap U_i$ is a non-empty open subset of Z which is irreducible, it follows that $Z \cap U_i$ is irreducible. Assume on the contrary that there is some larger closed subset W such that $Z \cap U_i \subsetneq W \subset E \cap U_i$. Then we have that

$$\overline{Z \cap U_i}^Z = Z \subset \overline{Z \cap U_i}^{\tilde{X}} \subset \overline{Z}^{\tilde{X}} = Z$$

since Z is closed in E and E is closed in $\tilde{X} \Rightarrow Z$ is closed in \tilde{X} . But this shows that $\overline{Z \cap U_i}^{\tilde{X}} = Z$.

But we had $Z \cap U_i \subsetneq W$ hence we have that

$$\overline{Z \cap U_i}^{\tilde{X}} = Z \subset \overline{W}^{\tilde{X}} \subset E$$

but $\overline{W}^{\tilde{X}}$ is now a closed subset of E , and it is irreducible since W irreducible $\Leftrightarrow \overline{W}^{\tilde{X}}$ is irreducible. But Z is an irreducible component and so can not be contained in some larger irreducible closed subset of E , therefore this forces $W = Z \cap U_i$.

Hence by what we have shown earlier we have that $Z \cap U_i$ has dimension 1 in U_i , i.e. that $\dim(Z \cap U_i) = \dim(U_i) - 1$. But notice that $\dim(Z \cap U_i) = \dim(Z)$ since $Z \cap U_i$ is a non-empty open subset of an irreducible space Z , and also that U_i is a non-empty irreducible subset of \tilde{X} . Thus $\dim(U_i) = \dim \tilde{X}$.

We have thus shown that

$$\begin{aligned} \dim(Z) &= \dim(Z \cap U_i) \\ &= \dim(U_i) - 1 \\ &= \dim \tilde{X} - 1. \end{aligned}$$

And so Z has codimension 1 in \tilde{X} . Since Z was arbitrary, we are done, we claim. \square

Corollary 9.0.21 ([Gat21, Cor. 9.24]). *Let a be a point on a variety X . Then the dimension $C_a X$ of the tangent cone of X at a is the local dimension $\text{codim}_a X$ of X at a .*

Chapter 10

Lecture 10: Smoothness

10.0.1 Smoothness

$\{a\} \times \mathbb{P}^{r-1} \supset E = \text{exceptional divisor of dimension } r-1$. Where $E = \pi^{-1}(a) \subset Bl_a X$.

$$\begin{array}{ccc}
 \pi^{-1}(a) & & \underbrace{Bl_a X}_{\dim r} \subset \mathbb{A}^n \times \mathbb{P}^{n-1} \\
 \downarrow & & \downarrow \text{birational} \\
 a & \in & X \subset \mathbb{A}^n
 \end{array}$$

Remark 10.0.1. Note that above we blow up $X = \mathbb{A}^n$ at the point $\{a\} \subset X$ which means (maybe more familiarly) that we blow up at n polynomials $x_1 - a_1, \dots, x_n - a_n$. Then our familiar U (9.0.8) becomes

$$\begin{aligned}
 U &= \mathbb{A}^n \setminus V(x_1 - a_1, \dots, x_n - a_n) \\
 &= \mathbb{A}^n \setminus \{a\}.
 \end{aligned}$$

Then naturally $Bl_a X \subset X \times \mathbb{P}^{n-1} = \mathbb{A}^n \times \mathbb{P}^{n-1}$.

Cones ([Gat21, Def. 6.16])

Definition 10.0.2. Let $\pi : \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ be defined by $(x_0, \dots, x_n) \mapsto [x_0 : \dots : x_n]$.

- (a) An affine variety $X \subset \mathbb{A}^n$ is called a **cone** if $0 \in X$, and $\lambda x \in X$ for all $\lambda \in k$ and $x \in X$. Or in words, we may say that a cone X consists of the origin together with a union of lines through the origin.
- (b) If $X \subset \mathbb{P}^n$ is a projective variety (6.0.7) we call

$$\begin{aligned}
 C(X) &:= \{0\} \cup \pi^{-1}(X) \\
 &= \{0\} \cup \{(x_0, \dots, x_n) : [x_0 : \dots : x_n] \in X\} \subset \mathbb{A}^{n+1}
 \end{aligned}$$

the **cone over** X (check that this is also a cone in the sense of (a)).

Tangent cone of X at a ; $C_a X$ ([Gat21, Construction 6.20])

Definition 10.0.3. Let a be a point on a variety X , and consider the blow-up of X at a (i.e. as above, at $x_1 - a_1, \dots, x_n - a_n$). This gives us the blow-up $\pi : \tilde{X} \rightarrow X$ with *exceptional set* $\pi^{-1}(\{a\})$, which we then claim is a *projective variety* (e.g. by choosing an affine open neighborhood U of a in X which is then an affine variety $U \subset \mathbb{A}^n$ with $a = (a_1, \dots, a_n)$ that we can blow up at $x_1 - a_1, \dots, x_n - a_n$). The exceptional set is then contained in the projective space $\{a\} \times \mathbb{P}^{n-1} \subset U \times \mathbb{P}^{n-1}$.

The cone (as in 10.0.2.(b)) over the exceptional set $\pi^{-1}(\{a\})$ is called the **tangent cone of X at a** , denoted as $C_a X$.

Remark 10.0.4. To say a bit more why $\pi^{-1}(\{a\})$ can be viewed as a *projective variety*, so that the cone $C(E) = C(\pi^{-1}\{a\})$ makes sense: On an affine chart $U \subset \mathbb{A}^n$ (which is an affine variety) of a we can define the blow-up $\tilde{U} \hookrightarrow U \times \mathbb{P}^{n-1}$ with the blow-up map $\pi_{\tilde{U}} : \tilde{U} \rightarrow U$. Then we have that

$$\begin{aligned} \pi_{\tilde{U}}^{-1}(\{a\}) &= \{(x, y) \in \tilde{U} : \pi_{\tilde{U}}(x, y) = a\} \\ &= \tilde{U} \cap (\{a\} \times \mathbb{P}^{n-1}) \\ &= E. \end{aligned}$$

But then we see that E is closed in $\{a\} \times \mathbb{P}^{n-1}$ with the induced subspace topology from $U \times \mathbb{P}^{n-1}$, and $\{a\} \times \mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}$. Therefore we can view \tilde{U} 's image under this isomorphism as a closed subvariety of \mathbb{P}^{n-1} , and so is itself a projective variety (we claim).

Remark 10.0.5. [Gat21] remarks that the tangent cone $C_a X$ is well-defined up to isomorphism by (9.0.15). In the special case of an affine patch $X \subset \mathbb{A}^n$ where $a \in X$ is the origin we consider $C_a X \subset C(\mathbb{P}^{n-1}) = \mathbb{A}^n$ as a closed subvariety of the same ambient affine space as for X by blowing up at x_1, \dots, x_n . I think that what is meant here is that blowing up \mathbb{A}^n at 0 gives the exceptional set $\pi^{-1}(\{0\}) \cong \mathbb{P}^{n-1}$ and so if we blow up X at 0 then we can perhaps view this as a subset of the blow up at \mathbb{A}^n , and perhaps then that

$$\begin{aligned} C_a X &= C_0 X \\ &= C(\pi_X^{-1}(0)) \\ &\subset C(\pi^{-1}(0)) \\ &= C(\mathbb{P}^{n-1}) \\ &= \mathbb{A}^n. \end{aligned}$$

Note here that

$$\begin{aligned} \pi_X^{-1}(0) &= \tilde{X} \cap (\{0\} \times \mathbb{P}^{n-1}) \\ &\subset \pi^{-1}(0) \\ &= \{0\} \times \mathbb{P}^{n-1}. \end{aligned}$$

We have

$$\begin{aligned} \dim C_a X &= \dim E + 1 \\ &= \underbrace{\dim_a X}_{\text{not in [Gat21]}} \\ &= \text{codim}_a X. \end{aligned}$$

[Gat21, Exc. 9.22]:

(b) $C_0X = V(f_{\text{in}} : f \in I(X))$.

(b) If $I(X) = V(f)$ then $C_0X = V(f_{\text{in}})$.

Warning:

1. If $I(X) = (f_1, \dots, f_n)$ then $C_0X = V((f_1)_{\text{in}}, \dots, (f_n)_{\text{in}})$ need not hold.
2. If $X = V(I)$ then $C_0X = V((f_{\text{in}} : f \in I))$ need not hold.

If

$$f = f_{i_0} + f_{i_1} + \dots + f_{i_d} \quad (10.0.1)$$

where f is a polynomial and the f_{i_j} are its **homogeneous components** of degree i_j and where the right-hand side in (10.0.1) is the homogeneous decomposition of f .

$$f_{\text{in}} = f_{i_j}$$

where i_j is *minimal* among i_0, \dots, i_d such that $i_j \neq 0$.

Remark 10.0.6. Note that $f_0 = 0$ in (10.0.1) if $0 \in V(f)$.

Example 10.0.7. Let $X = V(f)$ with $f = y^2 - x^3 - x^2$ (cf. 9.0.18). Then $C_aX = V(y^2 - x^2)$ by the above.

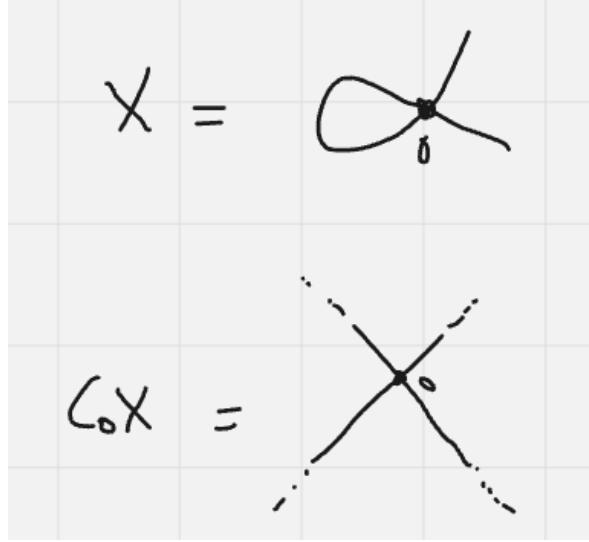


Figure 10.1: Example 10.0.7 sketch

Tangent space of X at a ; T_aX

Let a be a point on a variety X . By choosing an affine neighborhood of a we may assume that $X \subset \mathbb{A}^n$ and that $a = 0$. Then

$$T_aX := V(f_1 : f \in I(X))$$

is the **tangent space** of X at a , where $f_1 \in k[x_1, \dots, x_n]$ is the *linear term* of a polynomial $f \in k[x_1, \dots, x_n]$.

Note that 2. in the **Warning** above still holds but not 1; i.e. if $I(X) = (f_1, \dots, f_n)$ then we have that $T_aX = V((f_1)_1, \dots, (f_n)_1)$ (when $a = 0$).

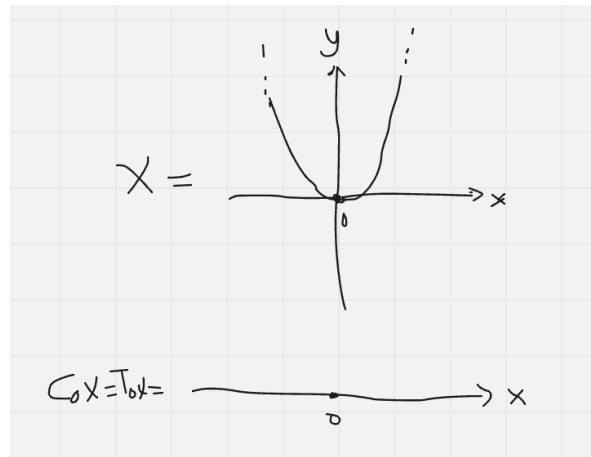


Figure 10.2: Example 10.0.9

Example 10.0.8. Note that if $X = V(f) \subset \mathbb{A}^n$ with where f does not have any linear term, then

$$\begin{aligned} \rightsquigarrow T_0(X) &= V(f_1) \\ &= V(0) \\ &= \mathbb{A}^n. \end{aligned}$$

In particular, this holds for 10.0.7 since if $f = y^2 - x^3 - x^2$ then $f_1 = 0$.

Example 10.0.9. Let $X = V(y - x^2)$. Then

$$\begin{aligned} C_0 X &= T_0 X \\ &= V(y). \end{aligned}$$

Or pictorially:

10.0.2 Properties of the tangent space and the cone

Proposition 10.0.10.

- (i) $C_0 X \subseteq T_0 X$.
- (ii) $T_0 X$ linear subspace of $\mathbb{A}^n = k^n$.
- (iii) $\dim T_0 X \geq \dim C_0 X = \dim_0 X$.

Remark 10.0.11. In relation to (i), (ii) above we have that $\text{span}(C_0 X) \subseteq T_0 X$.

Example 10.0.12. Let $X = V(y^2 - x^3)$. Then

$$\begin{aligned} C_0 X &= V(y^2) \\ &= V(y) \end{aligned}$$

and

$$\begin{aligned} T_0 X &= V(0) \\ &= \mathbb{A}^2. \end{aligned}$$

Pictorially:

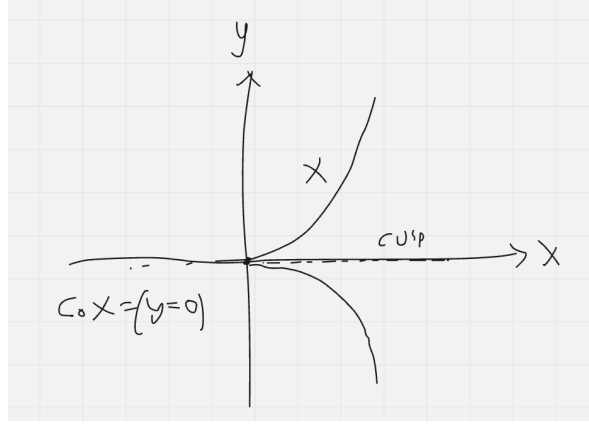


Figure 10.3: Example 10.0.12

10.0.3 Intrinsic definition of tangent space

Lemma 10.0.13 ([Gat21, Lemma 10.4]). *Let $X \subset \mathbb{A}^n$ be an affine variety containing the origin $a = 0$, whose ideal is then $I(0) = \langle \bar{x}_1, \dots, \bar{x}_n \rangle \trianglelefteq A(X)$. Then there is a natural vector space isomorphism*

$$\begin{aligned} I(a)/I(a)^2 &\cong \text{Hom}_k(T_a X, k) \\ &:= (T_a X)^\vee. \end{aligned}$$

Remark 10.0.14. Or in the notation of the lecture, $\mathfrak{m} = (x_1, \dots, x_n) \subset A(X)$ and

$$\mathfrak{m}/\mathfrak{m}^2 \cong \text{Hom}_k(T_0 X, k)$$

and

$$(\mathfrak{m}/\mathfrak{m}^2)^\vee = T_0 X,$$

where the last equality is different from what we said above (atleast naively). One may here presume that it follows from the above lemma that

$$\begin{aligned} ((T_0 X)^\vee)^\vee &= T_0 X \\ &= (\mathfrak{m}/\mathfrak{m}^2)^\vee. \end{aligned}$$

The reason for this (we claim) is that for finite-dimensional vector spaces V we have that V is (naturally) isomorphic to its double dual $V^{\vee\vee}$ (see for example [Rie16, p. 23]; note that there V^* is the dual instead of V^\vee).

Recall: $\mathcal{O}_{X,0} = A(X)_\mathfrak{m}$ is the *stalk* (3.0.7) at 0, and is a local ring, i.e. there is a unique maximal ideal in $\mathcal{O}_{X,0}$.

Let $S = (A(X) \setminus \mathfrak{m})$ be our *multiplicative set*. Then

$$\mathcal{O}_{X,0} = S^{-1}A(X)$$

and

$$S^{-1}\mathfrak{m} = \mathfrak{m}\mathcal{O}_{X,0}$$

is the unique maximal ideal in $\mathcal{O}_{X,0}$.

- $(T_a X)^\vee = (\mathfrak{m}_a/\mathfrak{m}_a^2)^\vee$ with $\mathfrak{m}_a \subset \mathcal{O}_{X,a}$ unique maximal ideal.

Remark 10.0.15. $\mathfrak{m}_a/\mathfrak{m}_a^2$ is an $\mathcal{O}_{X,a}/\mathfrak{m}_a \cong k$ -module.

Commutative algebra fact: $\mathfrak{m} \subset A \rightsquigarrow \mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}A\mathfrak{m}/(\mathfrak{m}A\mathfrak{m})^2$.

We prove 10.0.13:

Proof. □

10.0.4 Smoothness

Recall that $C_0 X \subset T_0 X$.

Smoothness

Definition 10.0.16. If X is a (pre-)variety and $x \in X$, then X is **smooth at** x if $C_a X = T_a X$ (i.e. the inclusion of $C_a X$ in $T_a X$ becomes an equality).

Remark 10.0.17. X is smooth at $x \Leftrightarrow \dim T_x X = \dim_x X$.

Below, we give a definition from commutative algebra:

Regular ring

Definition 10.0.18. A local ring R (with unique maximal ideal $\mathfrak{m} \triangleleft R$) is **regular** if it is Noetherian (i.e. satisfies the ascending chain condition) and there exists a **regular sequence** f_1, \dots, f_r such that $\mathfrak{m} = \langle f_1, \dots, f_r \rangle \Leftrightarrow \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \text{kr dim } R$ (where kr dim is the *krull-dimension* (2.0.22)).

Remark 10.0.19. By **regular sequence** in (10.0.18), we mean that f_1, \dots, f_r is such that r_1 is not a zero divisor in R and r_2 is not a zero divisor in $R/\langle r_1 \rangle$, r_3 is not a zero divisor in $R/\langle r_1, r_2 \rangle$ and so on.

We have that X is smooth at $x \Leftrightarrow \mathcal{O}_{X,x}$ is regular (10.0.18).

Fact: If R is regular $\Rightarrow R$ is an integral domain.

If X is smooth at $x \Rightarrow X$ is irreducible locally around x .

10.0.5 Jacobi criterion

Proposition 10.0.20 ([Gat21, p. 10.11]). Let $a \in X$ be a point of an affine variety $X \subset \mathbb{A}^n$ and let $I(X) = \langle f_1, \dots, f_r \rangle$. Then X is smooth at $a \Leftrightarrow$ the rank of the Jacobian matrix

$$\left(\frac{\partial f_j}{\partial x_i}(a) \right)_{i=1, \dots, n, j=1, \dots, r}$$

is at least $n - \text{codim}_X \{a\}$. And in this case, the rank of the matrix is in fact equal to $n - \text{codim}_X \{a\}$.

Remark 10.0.21. We will denote this matrix as $J(f)_a$ when there is a single function f involved and $J(\langle f_1, \dots, f_r \rangle)_a$ otherwise.

A more useful version is the following one.

Corollary 10.0.22 ([Gat21, Cor. 10.14]). Let $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ and let $a \in X := V(f_1, \dots, f_r) \subset \mathbb{A}^n$.

1. If $\text{rk } J(\langle f_1, \dots, f_r \rangle)_a \geq n - \text{codim}_X \{a\}$ then X is smooth at a .

2. If $\text{rk } J(\langle f_1, \dots, f_r \rangle)_a = r$ (so that the Jacobian has maximal row rank) then X is smooth at a .

We have

$$\begin{aligned} \dim_a X &= \min\{\dim U : a \in U \text{ and } U \subset X \text{ open}\} \\ &= \text{codim}_X \{a\} \\ &= \max_i \{\dim X_i : a \in X_i\} \end{aligned}$$

where the X_i in the last equality is the irreducible components of $X = \bigcup_i X_i$.

Example 10.0.23. Let $X = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2$ so that $r = 1$. We then have that

$$J(y^2 - x^3 - x^2) = (-3x^2 - 2x, 2y).$$

Then $\text{rk } J(y^2 - x^3 - x^2) = 1 \Leftrightarrow J(y^2 - x^3 - x^2) \neq 0$.

$\text{rk } J = 0 \Leftrightarrow J = 0 \Leftrightarrow x = y = 0$ (given points on X).

Example 10.0.24. Let $X = V(x^2 - y^2 z) \subset \mathbb{A}^3, r = 1$. Then we have that

$$J(x^2 - y^2 z) = (2x, -2yz, -y^2), \quad (\text{“ Whitney umbrella ”}).$$

We see that $\text{rk } J = 0 \Leftrightarrow x = y = 0$ gives whole line of singularities (z can then vary freely).

Corollary 10.0.25. *The set of smooth points of variety is open. That is, if X is a variety, then*

$$X^{\text{sm}} = \{x \in X : X \text{ is smooth at } x\} \subset X$$

is Zariski-open in X .

Proof. Let a in X be smooth. We claim that since smoothness is local in the sense that $a \in X$ is smooth $\Leftrightarrow \mathcal{O}_{X,a}$ is a regular local ring, and that for any $U \subset X$ open containing a we have $\mathcal{O}_{U,a} \cong \mathcal{O}_{X,a}$ a ring isomorphism, then $\mathcal{O}_{U,a}$ is regular local $\Leftrightarrow \mathcal{O}_{X,a}$ is regular local, hence we may take $X := U$ for affine open U in the cover of X (as a prevariety). The reason regular local is preserve is that there is a one-to-one correspondence between the prime ideals in R and S for any ring isomorphism $f : R \rightarrow S$ so that f preserves locality, and if r_1, \dots, r_n generate \mathfrak{m}_R then $f(r_1), \dots, f(r_n)$ generates \mathfrak{m}_S , so they have the same number of generators. Furthermore, f preserves dimension of the rings, so $\dim R = \#$ of generators of \mathfrak{m}_R , and $\dim S = \dim R$ and so

$$\dim S = \# \text{ of generators of } \mathfrak{m}_R = \# \text{ of generators of } \mathfrak{m}_S$$

so that S is regular.

So we may presume we are working in an affine space. Then we have that $\mathcal{O}_{X,a} = A(X)_{I(a)}$ where $I(a) \trianglelefteq A(X)$ is maximal. Assume that there was atleast two irreducible components X_i, X_j of X meeting a . We will then presume that $\mathcal{O}_{X,a}$ is an integral domain (we do not have time to check the proof, which is e.g. in [Gat13, Prop. 11.40]). We claim that this is a contradiction to X_i, X_j meeting a : We have that $I(X_i), I(X_j)$ correspond to minimal prime ideals in $A(X)$ (they are for example prime since $A(X_i) = A(X)/I(X_i)$ and irreducibility gives $A(X_i)$ integral domain so $I(X_i)$ is prime). Since $\{a\} \subset X_i, X_j$ we have $I(a) \supset I(X_i), I(X_j)$ and so $I(X_k) \cap I(a) = \emptyset$, thus $I(X_k)^e \trianglelefteq A(X)_{I(a)}$ for $k = i, j$. By the bijection of prime ideals $\mathfrak{p} \trianglelefteq A(X)$ such that $\mathfrak{p} \cap I(a) = \emptyset$ and prime ideals of the localization $S^{-1}A(X)$ we see that $I(X_k)^e$ is not the zero ideal, and $I(X_i)^e \neq I(X_j)^e$. Now if $I(X_k)^e$ was not a prime ideal in $A(X)_{I(a)}$ then there is some prime-ideal $\mathfrak{q} \subsetneq I(X_k)^e$. But then by contracting back to $A(X)$ we get a prime ideal $\mathfrak{q}^c \subsetneq (I(X_k)^e)^c = I(X_k)$ (contraction is injective), contradicting that $I(X_k)$ is minimal. Thus $I(X_k)^e$ must be minimal. But $I(X_k)$ is not the zero ideal and $A(X)_{I(a)}$ is an integral domain so its only minimal prime ideal is the zero-ideal. We have thus reached a contradiction.

Hence we may assume that X is an affine irreducible open subset of a . Thus since for any point $a \in X$, we have

$$\text{codim}_X \{a\} = \sup\{\dim X_i : a \in X_i\} = \dim X$$

so that the codimension of any point $a \in X$ is the same (c.f. [Gat21, Prop. 2.28.(b)]).

By the 10.0.20 see that the set of points smooth points a of X is points such that

$$\text{rk} \left(\frac{\partial f_i}{\partial x_j}(a) \right)_{i,j} \geq n - \dim X \quad (\text{for } I(X) = \langle f_1, \dots, f_r \rangle)$$

But this is equivalent to atleast one minor of size $n - \dim X$ the Jacobian matrix not vanishing in a when evaluated at a , which is an open condition. \square

Remark 10.0.26. In our understanding, what the proof shows above is essentially that we can find an open *neighborhood* around each point a that is smooth, that is, consisting of only smooth points in X , which is the same as saying that X^{sm} is open in X (this is why we wanted to find an irreducible affine component part for a , i.e. such that its local dimension was constant).

10.0.6 Implicit function theorem

Let $x = V(f_1, \dots, f_r) \subset \mathbb{A}^n$ with $f_1, \dots, f_r \in C^1(\mathbb{R}^n, \mathbb{R})$. Then $\text{rk } J \geq r \Rightarrow X$ is smooth of dimension $(n - r)$ at a .

The **implicit function theorem** then says (roughly) that if $\text{rk}(J(a)) = r$ and reorder words so that x_1, \dots, x_n columns gives an invertible matrix $J = \left(\underbrace{\text{inv}}_{r \times r} \mid \underbrace{\phantom{\text{inv}}}_{r \times (n-r)} \right)$. Then in a *euclidian* neighborhood

of $a \in X$ we can write

$$\begin{array}{ccc} x_1 = g_1(x_{r+1}, \dots, x_n) & & \\ \vdots & & \\ x_r = g_r(x_{r+1}, \dots, x_n). & & \\ \\ \begin{array}{ccc} a \in U \subset X & \subset & \mathbb{R}^n \\ \cong \downarrow & & \downarrow \pi_{r+1, \dots, n} \\ V & \subset & \mathbb{R}^{n-r} \end{array} \end{array}$$

In the diagram above we have that $U \subset X$ is euclidian open, $X \subset \mathbb{R}^n$ is euclidian closed and $V \subset \mathbb{R}^{n-r}$ is euclidian open.

We note however that there is no algebraic analogue of the implicit function theorem in general. For example, the polynomial equation $f(x_1, x_2) = x_2 - x_1^2 = 0$ cannot be solved for x_1 by a *regular* function locally around the point $(1, 1)$ although

$$\begin{aligned} J(x_2 - x_1^2)_{(1,1)} &= (-2x_1(1, 1), 1(1, 1)) \\ &= (-2, 1) \\ &\neq 0. \end{aligned}$$

It can only be solved by a continuously differentiable function $x_1 = \sqrt{x_2}$.

This problem was fixed in algebraic geometry by the étale topology ~ 1960 .

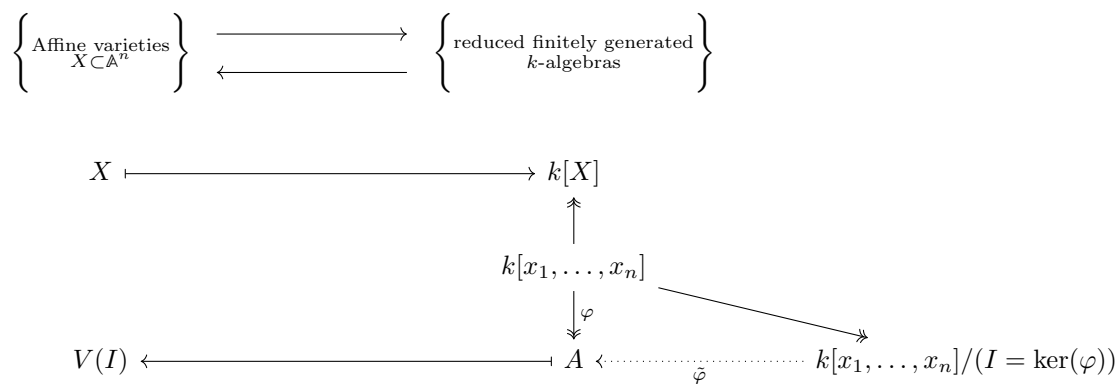
$$\begin{array}{ccc}
 a \in U \subset X & \subset & \mathbb{A}^n \\
 \downarrow \text{étale surjection} & & \downarrow \pi_{r+1, \dots, n} \\
 V & \subset & \mathbb{A}^{n-r}
 \end{array}$$

Above we have that U is Zariski-open in X and that V is Zariski-open in \mathbb{A}^{n-r} .

Over \mathbb{C} we have that $U^{\text{Eucl}} \rightarrow V^{\text{Eucl}}$ is a local homeomorphism (my interpretation is that the étale surjection becomes a local homeomorphism in this case).

Chapter 11

Lecture 11: Schemes I



Today:

$$\left\{ \begin{array}{l} \text{Affine schemes} \\ X \end{array} \right\} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \left\{ \begin{array}{l} \text{Commutative} \\ \text{rings} \end{array} \right\}$$

$$X = (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$$

$$\mathrm{Spec} R \longleftarrow R$$

As a set, we have that

$$\mathrm{Spec} R = \{\mathfrak{p} \trianglelefteq R : \mathfrak{p} \text{ prime ideal}\}. \quad (11.0.1)$$

Furthermore, we have

$$\mathrm{Spm}(R) = \{\mathfrak{m} \trianglelefteq R : \mathfrak{m} \text{ maximal ideal}\}. \quad (11.0.2)$$

Since every maximal ideal is a prime ideal, it follows that $\mathrm{Spm}(R) \subset \mathrm{Spec} R$.

Reason: If $\varphi : R \rightarrow S$ is a ring homomorphism then if $\mathfrak{p} \trianglelefteq S$ is a prime ideal then $\varphi^{-1}(\mathfrak{p})$ is a prime ideal. This does not necessarily hold for maximal ideals.

Remark 11.0.1. My understanding is that this is important for *functoriality*: We have that

$$\varphi : R \rightarrow S \rightsquigarrow \text{Spec}(\varphi) : \text{Spec}(S) \rightarrow \text{Spec} R$$

where $\text{Spec}(\varphi)$ is defined by $\mathfrak{p} \trianglelefteq S \xrightarrow{\text{Spec}(\varphi)} \varphi^{-1}(\mathfrak{p})$.

Example 11.0.2. Let $R = A(X)$ where $X \subset \mathbb{A}^n$ is an affine variety. Then we have that

$$\text{Spec} R = \{Y \subset X : Y \text{ is an irreducible subvariety}\}$$

and

$$\begin{aligned} \text{Spm}(R) &= X \\ &= \{\{x\} \subset X\}. \end{aligned}$$

Example 11.0.3. Consider $\text{Spec}(k[x])$. Then we have that

$$\text{Spec}(k[x]) = \{\langle x - a \rangle : a \in k\} \cup \{\langle 0 \rangle\}.$$

Pictorially:

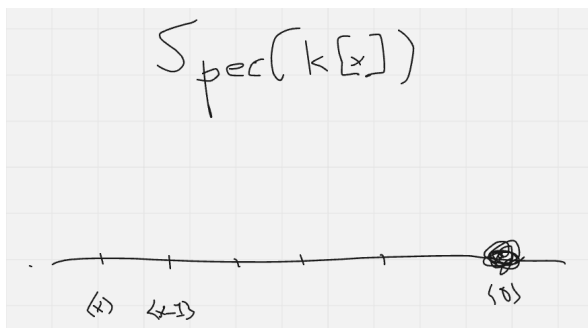


Figure 11.1: Example 10.0.28

Example 11.0.4. Consider $\text{Spec } \mathbb{Z}$. Then we have that

$$\text{Spec } \mathbb{Z} = \{\langle p \rangle : p \in \mathbb{Z} \text{ and } p \text{ prime}\} \cup \{\langle 0 \rangle\}.$$

Pictorially we have:

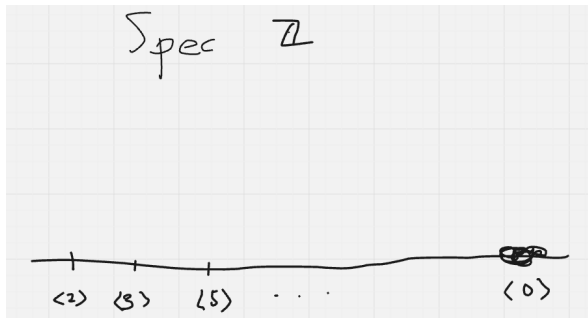


Figure 11.2: Example 10.0.29

Example 11.0.5. Consider $\text{Spec}(\mathbb{R}[x])$. Then we have that

$$\langle x^2 + 1 \rangle = \langle x - i \rangle \langle x + i \rangle$$

where $\langle x^2 + 1 \rangle = \{i, -i\}$.

More generally, we have

$$\begin{aligned} \text{spm}(\mathbb{R}[x]) &= \{z \in \mathbb{C} : \text{Im}(z) \geq 0\} \\ &= \mathbb{C}/(z \sim \bar{z}), \end{aligned}$$

where by $\text{Im}(z)$ we mean the imaginary part of $z = x + iy$. The equivalence-relation \sim identifies each such complex number z in the upper-half plane with its complex conjugate \bar{z} .

Then we have that

$$\text{Spec}(\mathbb{R}[x]) = \text{spm}(\mathbb{R}[x]) \cup \{\langle 0 \rangle\}.$$

Pictorially:

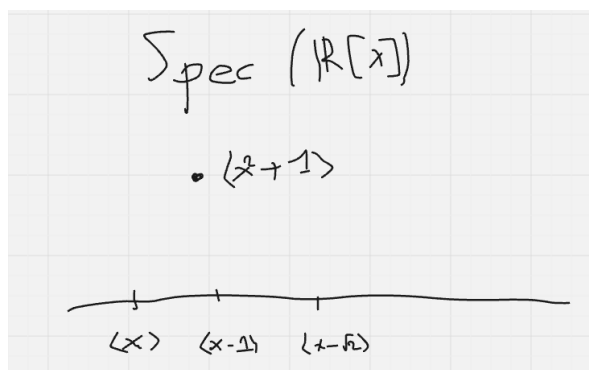
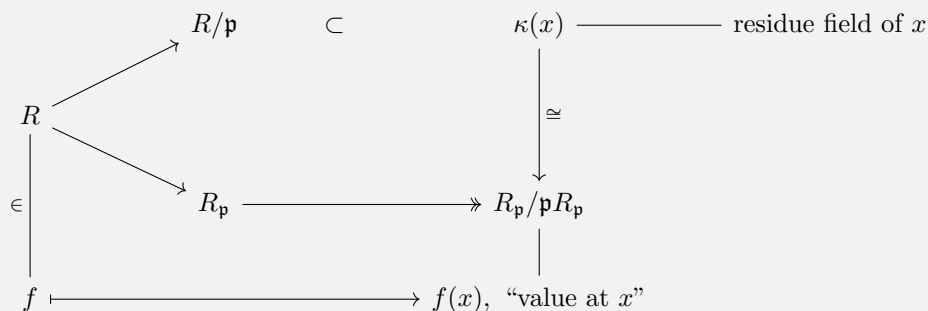


Figure 11.3: Example 10.0.31

11.0.1 Functions on $\text{Spec } R$

Functions on $\text{Spec } R$

Definition 11.0.6. Let $f \in R$ and let $x \in \text{Spec } R$ (so that x is a prime ideal in R).



Remark 11.0.7. Note that by $\mathfrak{p}R_{\mathfrak{p}}$ we mean

$$\mathfrak{p}R_{\mathfrak{p}} := \left\{ \frac{a}{s} \mid a \in \mathfrak{p}, s \in R \setminus \mathfrak{p} \right\},$$

which is the *unique* maximal ideal in the local ring $R_{\mathfrak{p}}$.

Example 11.0.8. Consider $\text{Spec } \mathbb{Z}$ with $f = 17$.

Pictorially:

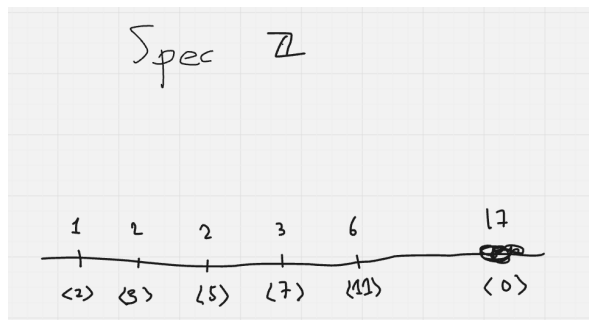


Figure 11.4: Example 10.0.32

Above, we note that

$$\begin{aligned} 17 \bmod 2 &= 1 \\ 17 \bmod 3 &= 2 \\ 17 \bmod 5 &= 2 \\ 17 \bmod 7 &= 3 \\ 17 \bmod 11 &= 6 \end{aligned}$$

We have

$$(17)((5)) = 2 \in \mathbb{Z}/5\mathbb{Z}$$

and

$$(17)((0)) = 17 \in \mathbb{Q}.$$

Example 11.0.9. Consider $\text{Spec}(k[x]/\langle x^3 \rangle) = \{\langle x \rangle\}$.

Here elements in $A := k[x]/\langle x^3 \rangle$ are things on the form $a_0 + a_1x + a_2x^2$ for $a_i \in k$ for $i = 0, 1, 2$.

We note that the *nilradical* of A , $\sqrt{\langle 0 \rangle}$, equals the intersection of all prime ideals in A by theorem. But

$$\sqrt{\langle 0 \rangle} = \{f \in A : f^n = 0\}.$$

Since $x^3 = 0$ in A we see that $x \in \sqrt{\langle 0 \rangle}$. Since every prime ideal contains the nilradical, this means that every prime \mathfrak{p} in A must contain $\langle x \rangle \subseteq A$.

We claim that $\langle x \rangle$ is in fact a maximal ideal of A , and so that $\langle x \rangle$ must be the unique prime ideal in A .

To see this, consider

$$\begin{aligned}
 A/\langle x \rangle &= (k[x]/\langle x^3 \rangle) / \langle x \rangle \\
 &= (k[x]/\langle x^3 \rangle) / \langle x \rangle / \langle x^3 \rangle \\
 &\cong k[x]/\langle x \rangle, \quad (3^{\text{rd}} \text{ isomorphism theorem}) \\
 &\cong k, \\
 \Leftrightarrow \langle x \rangle &\trianglelefteq A \text{ is a maximal ideal.}
 \end{aligned}$$

Therefore we see that $\text{Spec}(A) = \{\langle x \rangle\}$.

If $f = a_0 + a_1x + a_2x^2$ then

$$\begin{aligned}
 f(\langle x \rangle) &= f \pmod{x} \\
 &= a_0.
 \end{aligned}$$

Lastly, we have that

$$\begin{aligned}
 X &= \text{Spec}(A) \\
 &= \bullet,
 \end{aligned}$$

as already pointed out.

We now move on to define the familiar operations of $V(-)$ and $I(-)$ in the context of $\text{Spec}(-)$.

$V(I) \subset \text{Spec } R$

Definition 11.0.10. Let R be a ring and let $I \subset R$. Then we define the **zero locus** of I to be

$$\begin{aligned}
 V(I) &:= \{x \in \text{Spec } R : f(x) = 0, \forall f \in I\} \\
 &= \{x \in \text{Spec } R : x \supset I\} \subset \text{Spec } R.
 \end{aligned}$$

$I(Z)$ for $Z \subset \text{Spec } R$

Definition 11.0.11. Let $Z \subset \text{Spec } R$ for a ring R . Then we define

$$\begin{aligned}
 I(Z) &= \{f \in R : f(x) = 0, \forall x \in Z\} \\
 &= \bigcap_{x \in Z} x.
 \end{aligned}$$

Remark 11.0.12. To say more about what we mean by $f(x)$ when $x \in \text{Spec } R$ so that x is prime ideal: We have an evaluation-map $\text{ev}_x : R \rightarrow R/x \subset k(x) = \text{Frac}(R/x)$. Then $f(x) := f \pmod{x} \in R/x \subset k(x)$ (cf. 11.0.6).

We want to show that

$$V(I) = \{x \in \text{Spec } R : x \supset I\}.$$

\subseteq : Assume that $y \in \text{Spec } R$ such that $f(y) = 0$. Then this means that $f \pmod{y} = 0$ in R/y so that $f \in y$. Since this holds for all $f \in I$ this shows that $I \subset y$, so that $y \in \{x \in \text{Spec } R : x \supset I\}$.

\supseteq : Let $y \in \{x \in \text{Spec } R : x \supset I\}$. Then we see that $f \pmod{y} = 0$ since $f \in y$, for all $f \in I$ so that $f(y) = 0$ for all $f \in I$. Therefore $y \in V(I)$.

Furthermore, we want to show that

$$I(Z) = \bigcap_{p \in Z} p.$$

\subseteq : Assume that $f \in I(Z)$. Then $f(x) = 0, \forall x \in Z$. But this means precisely that $f \in x$ for all $x \in Z$, so that $f \in \bigcap_{x \in Z} x$.

\supseteq : Assume that $f \in \bigcap_{x \in Z} x$. Then f is an element of all the prime ideals in Z , so that $f(x) = 0$ for all $x \in Z$, hence $f \in I(Z)$.

11.0.2 Topology on $\text{Spec } R$

Zariski topology on $\text{Spec } R$ is the topology such that the closed sets are precisely sets of the form $V(I)$. We have

$$\begin{aligned} \emptyset &= V(\langle 1 \rangle) \\ \text{Spec } R &= V(\langle 0 \rangle) \end{aligned}$$

as well as

$$\bigcap_{\alpha} V(I_{\alpha}) = V\left(\sum_{\alpha} I_{\alpha}\right)$$

and

$$\bigcup_{i=1}^n V(I_i) = V\left(\bigcap_{i=1}^n I_i\right).$$

Generic point

Definition 11.0.13. Let X be a topological space and let $x \in X$. Then x is a **generic point** of X if $\overline{\{x\}} = X$.

New feature: Usually there exists *non-closed* points. In fact, if x is a point in $\text{Spec } R$, then

$$\begin{aligned} \overline{\{x\}} &= V(x) \\ &= \{y \in \text{Spec } R : y \supset x\}. \end{aligned}$$

Then we see that $\{x\}$ is closed, i.e. $\overline{\{x\}} = \{x\} \Leftrightarrow x$ is a maximal ideal.

Example 11.0.14. Consider $\text{Spec } \mathbb{Z}$ and let η be the point corresponding to $\langle 0 \rangle$. By definition, we then have that

$$\overline{\{\eta\}} = \text{Spec } \mathbb{Z},$$

so that “ η is the generic point”.

Example 11.0.15. We have that $\text{Spec } R = \text{Spec}\left(R/\sqrt{\langle 0 \rangle}\right)$ as topological spaces. That they are at least equal as *sets* follows from the earlier mentioned theorem that gives that the *nilradical* $\sqrt{\langle 0 \rangle}$ equals the intersection of all prime ideals of R . A corollary of this is then indeed that $\text{Spec } R = \text{Spec}(R/\sqrt{\langle 0 \rangle})$.

Example 11.0.16. Let $A = k[x, y]/\langle xy \rangle$ and consider $\text{Spec } A$.

We want to find the *non-closed* points in A . These corresponds to non-maximal prime ideals in A . By the 3rd isomorphism theorem for rings, every ideal of A is of the form $I/\langle xy \rangle$ for some ideal I that contains $\langle xy \rangle$. By the nullstellensatz we know that the maximal ideals $\mathfrak{m} \subseteq k[x, y]$ are of the form $\langle x - a, y - b \rangle$ for $a, b \in k$. If one looks at the evaluation-map $\text{ev}_{a,b} : k[x, y] \rightarrow k$ that takes $x \mapsto a$ and $y \mapsto b$ with kernel precisely $\langle x - a, y - b \rangle$ we see that for $\langle xy \rangle$ to be contained in $\ker \text{ev}_{a,b} = \langle x - a, y - b \rangle$ we need either $a = 0$ or $b = 0$. Under the correspondence to points in \mathbb{A}_k^n by the nullstellensatz, we see that such maximal ideals \mathfrak{m} corresponds to points on either the x -axis or the y -axis.

To find the *non-closed* points, we want to find prime ideals containing $\langle xy \rangle$ that are *not* maximal.

If $\mathfrak{p} \supset \langle xy \rangle$ then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Without loss of generality assume that $x \in \mathfrak{p}$. Then again under the 3rd isomorphism theorem this corresponds to the ideal

$$\mathfrak{p}/\langle x \rangle \in k[x, y]/\langle x \rangle \cong k[y].$$

Since $k[x]$ is a PID, and prime \Leftrightarrow irreducible, when k is algebraically closed we see that the prime-ideals are on the form $\langle y - b \rangle$ and $\langle 0 \rangle$.

Pulling back the ideals from $k[y]$ to $k[x, y]$ we see that the possible prime ideals are $\langle x, y - b \rangle$ and $\langle x \rangle$ (with $\langle 0 \rangle$ in $k[y]$). The first type of options just gives us a maximal ideal, so the only choice for a non-maximal prime ideal is $\langle x \rangle$. The same argument but with $\langle y \rangle \subset \mathfrak{p}$ then shows that the only non-maximal prime-ideals in A are of the form $\langle x \rangle$ and $\langle y \rangle$, corresponding to the points η_2 and η_1 in the picture below.

Remark 11.0.17. The arguments above we believe are correct, but we leave out some details (in the interest of time-management; that is, we have not looked into the argument as close as we perhaps should).

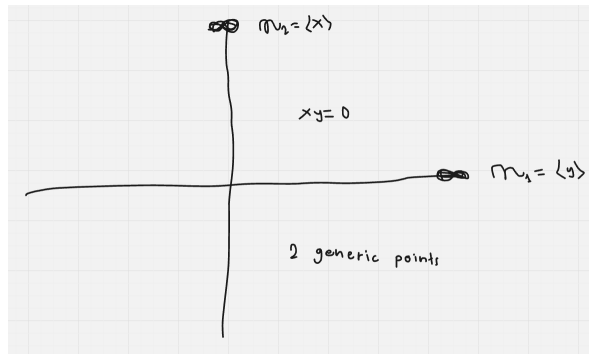


Figure 11.5: Example 10.0.38

Example 11.0.18. Consider $\text{Spec } k[x, y]$. We want to find its *non-closed* points $\mathfrak{p} \in \text{Spec } k[x, y]$, i.e. points \mathfrak{p} such that

$$V(\mathfrak{p}) \neq \mathfrak{p}.$$

This is the same as finding *prime, non-maximal* $\mathfrak{p} \in \text{Spec } k[x, y]$.

By nullstellensatz the maximal ideals are precisely those on the form $\langle x - a, y - b \rangle$ for $a, b \in k$. So we need to find prime-ideals not on the above form.

We then have that $\langle 0 \rangle$ is a prime ideal not on the above form, so that this is a non-closed point in $\text{Spec } k[x, y]$. If $\mathfrak{p} \neq \langle 0 \rangle$ then what will \mathfrak{p} look like if it is non-maximal but prime?

Lemma 11.0.19. *Let $R = k[x]$. Then the prime ideals of $R[y] = k[x, y]$ are precisely $\langle 0 \rangle$, $\langle f(y) \rangle$ for $f(y) \in R[y]$ irreducible, and maximal (hence prime) ideals $\langle p, f(y) \rangle$ for $p \in R$ prime and irreducible $f(y) \in (R/p)[y]$.*

Proof. In a PID, all non-zero prime ideals are maximal. Therefore, for $\langle 0 \rangle \neq \mathfrak{p} \subseteq R$ we have that R/\mathfrak{p} is a field. In particular, let \mathfrak{Q} be a non-zero prime ideal in $R[y]$ such that $\mathfrak{Q} \cap R = \langle \mathfrak{p} \rangle$. We then have that

$$(R/\mathfrak{p})[y] \cong R[y]/\langle \mathfrak{p} \rangle, \quad ([DF04, \text{Prop. 9.1.2}])$$

is an integral domain which is equivalent to $\langle \mathfrak{p} \rangle \subseteq R[y]$ being prime (where $\langle \mathfrak{p} \rangle$ is the set of polynomials in $R[y]$ with coefficients from \mathfrak{p}). Since $(R/\mathfrak{p})[y]$ is a PID we have that the non-zero prime ideals are precisely those generated by some irreducible polynomial $f(y)$, and that $\langle f(y) \rangle$ is again a maximal ideal. Then we see that

$$(R/\mathfrak{p})[y]/\langle f(y) \rangle$$

is a field. Furthermore, we note that by the 3rd isomorphism theorem for rings, every ideal in $(R/\mathfrak{p})[y] \cong R[y]/\langle \mathfrak{p} \rangle$ correspond to some ideal in $\mathfrak{q} \subseteq R[y]$ that *contains* $\langle \mathfrak{p} \rangle$. So in particular this means that $\langle f(y) \rangle$ corresponds to such an ideal \mathfrak{q} . The correspondence also then gives that

$$(\mathfrak{q} + \langle \mathfrak{p} \rangle)/\langle \mathfrak{p} \rangle = \langle f(y) \rangle,$$

holds in $R[y]/\langle \mathfrak{p} \rangle$ so that under the map $\varphi : R[y] \rightarrow (R/\mathfrak{p})[y]$ we have that $\varphi^{-1}(\langle f(y) \rangle) = \mathfrak{q}$.

Since φ is surjective this means that there is some $F(y) \in R[y]$ such that $\varphi(F(y)) = f(y)$. Therefore we have that

$$\begin{aligned} \varphi^{-1}(\langle f(y) \rangle) &= \langle F(y), \mathfrak{p} \rangle \\ &= \mathfrak{q}, \end{aligned}$$

where the reason that $\varphi^{-1}(\langle f(y) \rangle)$ contains \mathfrak{p} is that \mathfrak{p} gets modded out in the quotient, since it is precisely the kernel (up to isomorphism) of the projection morphism. Recall further that the preimage of a maximal ideal is maximal under a ring homomorphism. It follows that $\langle F(y), \mathfrak{p} \rangle$ is *maximal* in $R[y]$.

From the above we see that we get $\langle F(y) \rangle$ whenever we take $\mathfrak{p} = \langle 0 \rangle$. We claim that this exhaust all the prime ideals of $R[y]$ that contains a prime of R : Let $\mathfrak{Q} \subseteq R[y]$ be any prime such that $\mathfrak{Q} \cap R = \mathfrak{p}$ (we are using that R is a PID to get the right-hand side). Then $\mathfrak{Q} \supset \mathfrak{p}$ and then the argument proceeds as above. In the end, by the bijective property of the 4th isomorphism theorem note here that \mathfrak{q} must equal \mathfrak{Q} .

Remark 11.0.20. Diagrammatically, the fourth isomorphism theorem gives the correspondence below:

Correspondence theorem

Theorem 11.0.21.

$$\left\{ \begin{array}{l} J \trianglelefteq R \\ J \text{ contains } I \end{array} \right\} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \left\{ K \trianglelefteq R/I \right\}$$

$$I \subset J \longmapsto J/I$$

$$\pi^{-1}(K) \longleftarrow K$$

where $\pi : R \rightarrow R/I$ is the canonical projection homomorphism.

Consider the case where we have a prime ideal \mathfrak{p} in $R[y]$ that is not principal and does not contain a prime ideal from R . A corollary to Gauss lemma is that whenever $f, g \in R[y]$ has no common factor then they have no common factor in $K[y]$ with $K := \text{Frac}(R)$. Therefore since $\gcd(f, g) = 1$ in $K[y]$ and $K[y]$ is a Bezout-domain there are $h, \ell \in K[y]$ such that $f\ell + gh = 1$. By clearing denominators we see that $d = f\ell' + gh' \in \mathfrak{p}$ if $f, g \in \mathfrak{p}$. But d is also in R , so that $d \in \mathfrak{p} \cap R$. But then we note that $\mathfrak{p} \cap R \subset \mathfrak{p}$ and that $\mathfrak{p} \cap R$ is a prime ideal in R (it is the preimage of our prime ideal in $R[y]$ under the inclusion homomorphism) and since $d \neq 0$ it follows that \mathfrak{p} contains a non-trivial prime ideal from R , contradicting our assumption. \square

Lemma (11.0.19) then shows that all prime *non-maximal* ideals are of the form $\langle f(y) \rangle \trianglelefteq R[y]$ for $f(y) \in R[y]$ irreducible $\Leftrightarrow \langle f(x, y) \rangle \trianglelefteq k[x, y]$ for $f(x, y) \in k[x, y]$ irreducible.

I.e. we get one “generic point” for every irreducible polynomial $f(x, y) \in k[x, y]$.

Pictorially:

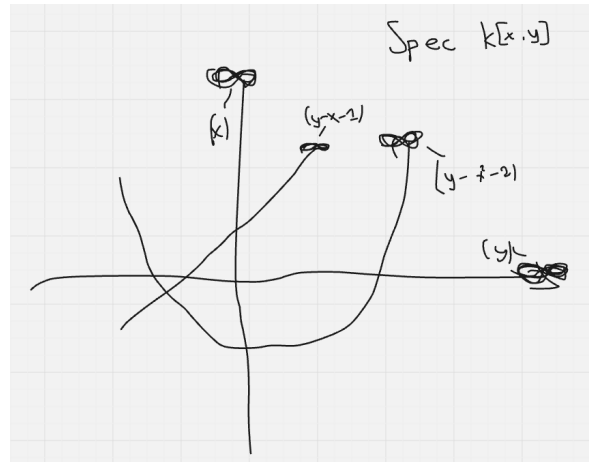


Figure 11.6: Example 10.0.39

Principal open subsets in $\text{Spec } R$

Definition 11.0.22. Let R be a ring and $f \in R$. Then we define

$$\begin{aligned} D(f) &:= \text{Spec } R \setminus V(f) \\ &= \{\mathfrak{p} \in \text{Spec } R : \mathfrak{p} \not\supset \langle f \rangle\} \\ &= \{\mathfrak{p} \in \text{Spec } R : f \notin \mathfrak{p}\}. \end{aligned}$$

as the **principal open subsets** of $\text{Spec } R$.

Remark 11.0.23. To see the equality $\{\mathfrak{p} \in \text{Spec } R : \mathfrak{p} \not\supset \langle f \rangle\} = \{\mathfrak{p} \in \text{Spec } R : f \notin \mathfrak{p}\}$:

\subseteq : If $\langle f \rangle \not\subset \mathfrak{p}$ then $f \notin \mathfrak{p}$ since otherwise $\mathfrak{p} \supset \langle f \rangle$.

\supseteq : If $f \notin \mathfrak{p}$ then it is clear that $\mathfrak{p} \not\supset \langle f \rangle$.

Exercise: $D(f) = \text{Spec}(R_f) \subset \text{Spec } R$.

Proof. By definition, we have $D(f) = \{\mathfrak{p} \in \text{Spec } R : f \notin \mathfrak{p}\}$. R_f is the localization of R at the multiplicatively closed subset $\{1, f, f^2, \dots\}$. There is a ring-homomorphism $g : R \rightarrow R_f$ defined explicitly by $h \mapsto \frac{h}{1}$. We have that the preimage of a prime ideal is a prime ideal under this ring-homomorphism. Notice that if $\mathfrak{q} \trianglelefteq R_f$ is a prime, then $g^{-1}(\mathfrak{q})$ can not contain f , since if $\frac{f}{1} \in \mathfrak{q}$ then it is not a proper ideal. Therefore we get a prime ideal $g^{-1}(\mathfrak{q})$ in R that does not contain f . This shows that under this identification $\mathfrak{q} \mapsto g^{-1}(\mathfrak{q})$, we have that $\text{Spec}(R_f) \subset D(f)$. On the other hand, if $\mathfrak{p} \in \text{Spec } R$ such that \mathfrak{p} does not contain f , then we claim that $\mathfrak{p}R_f = \left\{ \frac{r}{f^n} : r \in \mathfrak{p}, n \in \mathbb{N} \right\}$ is a prime ideal in R_f : Assume that we have $\frac{r_1}{f^{n_1}}, \frac{r_2}{f^{n_2}} \in R_f$ such that $\frac{r_1 r_2}{f^{n_1 + n_2}} \in \mathfrak{p}R_f$, then this means that there is some $r \in \mathfrak{p}$ such that $\frac{r_1 r_2}{f^{n_1 + n_2}} = \frac{r}{f^n} \Leftrightarrow$ there is some $m \in \mathbb{N}$ such that $f^m(r_1 r_2 f^n - r f^{n_1 + n_2}) = 0$ in $R \Leftrightarrow f^{m+n} r_1 r_2 = r f^{n_1 + n_2}$ but the right-hand side is in \mathfrak{p} , so the left-hand side is in \mathfrak{p} . But f^{m+n} is not in \mathfrak{p} , so $r_1 r_2$ needs to be in \mathfrak{p} . But if $r_1 r_2$ is in \mathfrak{p} then either r_1 or r_2 is in \mathfrak{p} . Either way we then see that $\frac{r_1}{f^{n_1}}$ or $\frac{r_2}{f^{n_2}}$ is in $\mathfrak{p}R_f$. Since \mathfrak{p} is an ideal it is also closed under addition (within itself) and multiplication from R_f . Therefore $\mathfrak{p}R_f$ is a prime in R_f whenever $\mathfrak{p} \in D(f)$. Therefore, under the identification $D(f) \ni \mathfrak{p} \mapsto \mathfrak{p}R_f$ we have that $D(f) \subset \text{Spec}(R_f)$. In fact, this is a part of a more general theory (see [DF04, Prop. 15.4.38.(3)]):

$$\left\{ \begin{array}{l} \text{Prime ideals } \mathfrak{p} \text{ of } R \\ \text{with } \mathfrak{p} \cap D = \emptyset \end{array} \right\} \begin{array}{c} \xrightarrow{e} \\ \xleftarrow{c} \end{array} \left\{ \text{prime ideals of } D^{-1}R \right\} \quad \text{bijection!}$$

with in our case $D = \{1, f, f^2, \dots\}$ and $e :=$ extension takes I to $g(I)$ and $c :=$ contraction takes $J \subset A/K$ to $g^{-1}(J)$. Thus, we have shown that $\text{Spec}(R_f)$ and $D(f)$ are in bijection.

We will compare their respective structure sheaves on the common distinguished open subset U of all primes not containing a given element $r \in R$.

- On $D(f)$ this is the distinguished open subset $D(f) \cap D(r) = D(fr)$. Thus we have

we have that

$$\begin{aligned} \mathcal{O}_{D(f)}(U) &:= \mathcal{O}_{\text{Spec } R}(D(fr)) && ([\text{Gat21, Remark 3.16}]) \\ &\cong R_{fr} && ([\text{Gat21, Prop. 12.19}]). \end{aligned}$$

On $\text{Spec } R_f$, we again have

$$\begin{aligned} \mathcal{O}_{\text{Spec } R_f}(U) &= \mathcal{O}_{\text{Spec } R_f}(D(r)) \\ &\cong (R_f)_r \end{aligned}$$

again by [Gat21, Prop. 12.19]. □

Claim: $R_{fr} \cong (R_f)_r$.

Proof. We will use the universal property of localization covered in [DF04, Theorem 15.4.36].

Consider the following diagrams:

$$1) \quad \begin{array}{ccc} R & \xrightarrow{\pi_{fr}} & R_{fr} \\ & \searrow \pi_f & \nearrow \exists! \psi \\ & R_f & \end{array}$$

$$2) \quad \begin{array}{ccc} R & \xrightarrow{\pi_r \circ \pi_f} & (R_f)_r \\ & \searrow \pi_{fr} & \nearrow \exists! \Phi \\ & R_{fr} & \end{array}$$

$$3) \quad \begin{array}{ccc} R_f & \xrightarrow{\psi} & R_{fr} \\ & \searrow \pi_r & \nearrow \exists! \Psi \\ & (R_f)_r & \end{array}$$

$$4) \quad \begin{array}{ccc} R & \xrightarrow{\pi_{fr}} & R_{fr} \\ & \searrow \pi_{fr} & \nearrow ! \\ & R_{fr} & \end{array}$$

$$5) \quad \begin{array}{ccc} R_f & \xrightarrow{\pi_r} & (R_f)_r \\ & \searrow \pi_r & \nearrow ! \\ & (R_f)_r & \end{array}$$

We have that

$$\begin{aligned} (\Psi \circ \Phi) \circ \pi_{fr} &= \Psi \circ (\pi_r \circ \pi_f) \\ &= \psi \circ \pi_f \\ &= \pi_{fr} \end{aligned}$$

and

$$\begin{aligned} (\Phi \circ \Psi) \circ \pi_r &= \Phi \circ (\Psi \circ \pi_r) \\ &= \Phi \circ \psi. \end{aligned}$$

We claim that $\Phi \circ \psi = \pi_r$. To see this, consider the diagram

$$\begin{array}{ccc} R & \xrightarrow{\pi_r \circ \pi_f} & (R_f)_r \\ & \searrow \pi_f & \nearrow ! \\ & R_f & \end{array}$$

and notice that

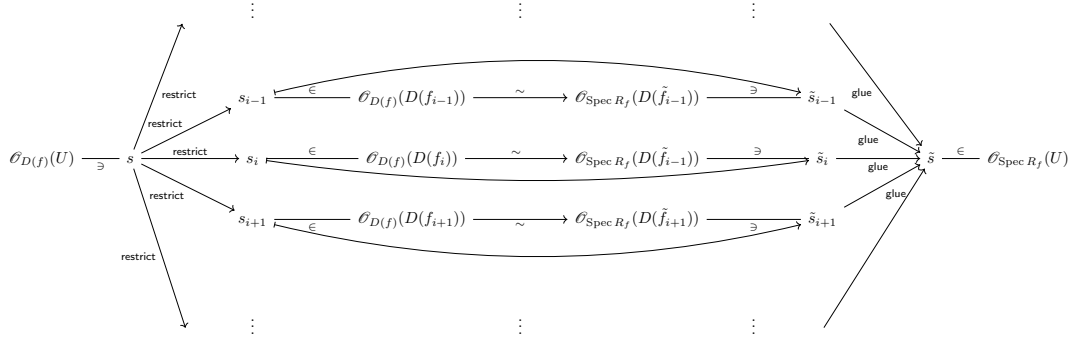
$$\begin{aligned} (\Phi \circ \psi) \circ \pi_f &= \Phi \circ (\psi \circ \pi_f) \\ &= \Phi \circ \pi_{f_r} \\ &= \pi_r \circ \pi_f. \end{aligned}$$

Hence by the universal property it follows that $\Phi \circ \psi = \pi_r$ so that

$$(\Phi \circ \Psi) \circ \pi_r = \pi_r.$$

Hence it follows that $(\Phi \circ \Psi) = \text{id}_{(R_f)_r}$ and $(\Psi \circ \Phi) = \text{id}_{R_{f_r}}$ (from diagram 4) and 5) since the identity homomorphism in both cases fulfills the universal property, so that the claim follows by *uniqueness*. ■

Therefore we see that $\mathcal{O}_{D(f)}(U) \cong \mathcal{O}_{\text{Spec } R_f}(U)$ holds for all distinguished open subsets U . But the distinguished open subsets is a basis for the topology. Therefore for a general open subset $U \subset D(f)$ (or the corresponding open subset in $\text{Spec } R_f$) it can be covered by a basis $\{D(f_i) : i \in I\}$. For any $s \in \mathcal{O}_{D(f)}(U)$, we set $s_i := s|_{D(f_i)}$. Then this gives an isomorphic copy \tilde{s}_i in $\mathcal{O}_{\text{Spec } R_f}(D(\tilde{f}_i))$. All such \tilde{s}_i can then be glued together to get a unique corresponding section \tilde{s} in $\mathcal{O}_{\text{Spec } R_f}(U)$. Diagrammatically, we have the following situation:



The same procedure also goes in the other direction. □

These principal open subsets form a basis for the topology on $\text{Spec } R$. We have

$$\text{Spec } R \setminus V(I) = \bigcup_{f \in I} D(f).$$

11.0.3 Nullstellensatz for $\text{Spec } R$

Lemma 11.0.24 ([Gat21, Prop 12.10]).

- (a) $V(I(Z)) = \overline{Z}$ for all $Z \subset \text{Spec } R$.
- (b) $I(V(J)) = \sqrt{J}$ for all ideals $J \leq R$.

Furthermore, we have the inclusion-reversing bijection

$$\{\text{closed subsets of } \text{Spec } R\} \xleftarrow{1:1} \{\text{radical ideals in } R\}$$

Proof. (a): Assume that $Z \subset \operatorname{Spec} R$. Then we have that

$$\begin{aligned} V(I(Z)) &= V\left(\bigcap_{p \in Z} p\right) \\ &= \left\{ Q \in \operatorname{Spec} R : Q \supset \bigcap_{p \in Z} p \right\}. \end{aligned}$$

We want to show that this is the *minimal* closed set containing Z . First notice that $Z \subseteq V(I(Z))$ since if $\mathfrak{p} \in Z$ then $\mathfrak{p} \supset \bigcap_{p \in Z} p$.

Now let $C \subseteq \operatorname{Spec} R$ with $Z \subseteq C$ and let C be closed. Then $C = V(J)$ for J some ideal (possibly on the form $J = \langle S \rangle$ for $S \subset R$ where S is not necessarily an ideal). Then note that $Z \subseteq C = V(J)$ means that for all $p \in Z$ we have that $p \supset J$. But this means in turn that

$$\begin{aligned} J &\subset \bigcap_{p \in Z} p = I(Z) \\ \Rightarrow V(J) &\supset V(I(Z)). \end{aligned}$$

We conclude that $V(I(Z))$ is indeed the minimal closed set containing Z , so that $V(I(Z)) = \overline{Z}$.

(b): We can show this directly:

$$\begin{aligned} I(V(J)) &= \bigcap_{p \in V(J)} p, \quad (\text{definition 11.0.11}) \\ &= \bigcap_{p \supset J} p \\ &= \sqrt{J}, \quad ([\text{DF04, Prop. 15.2.12}]). \end{aligned}$$

□

11.0.4 $\operatorname{Spec} R$ as a ringed space

Let $X = \operatorname{Spec} R$. In analogy with affine varieties, we want a sheaf of rings \mathcal{O}_X such that

1.

$$\begin{aligned} \mathcal{O}_X(D(f)) &= R_f, \quad (\forall f \in R) \\ \rightsquigarrow \mathcal{O}_X(X) &= \mathcal{O}_X(D(1)) \\ &= R_1 \\ &= R. \end{aligned}$$

2. $\mathcal{O}_{X,x} = R_{\mathfrak{p}}$, (if $x \leftrightarrow \mathfrak{p} \subset R$)

Remark 11.0.25. $\varphi \in R_{\mathfrak{p}}$ then $\varphi = \frac{f}{g}$ with $f \in R$ and $g \in R \setminus \mathfrak{p}$,

$$\begin{array}{ccc} \rightsquigarrow \frac{f}{g} \in R_g & \xrightarrow{\quad} & R_Q \quad g \notin Q \text{ prime ideal} \\ & \searrow & \uparrow \\ & & R_{\mathfrak{p}} \\ & \searrow & \\ & & \varphi \end{array}$$

$\mathcal{O}_X(U)$ for $\text{Spec } R$

Definition 11.0.26.

$$\mathcal{O}_X(U) := \left\{ (\varphi_{\mathfrak{p}} \in R_{\mathfrak{p}})_{\mathfrak{p} \in U} : \begin{array}{l} \forall x \in U \exists f, g \in R, g(x) \neq 0, \\ \text{such that } \varphi_{\mathfrak{p}} = \frac{f}{g}, \forall \mathfrak{p} \text{ in a neighborhood of } x \end{array} \right\}.$$

Proposition 11.0.27. $\mathcal{O}_X(U)$ in definition 11.0.26 is a ring.

(Easy) Exercise: $\mathcal{O}_X(U)$ in definition 11.0.26 is a sheaf.

$$U = \bigcup_{\alpha} U_{\alpha}$$

and $f_{\alpha} \in \mathcal{O}_X(U_{\alpha})$ then $f_{\alpha}|_{U_{\alpha} \cap U_{\beta}} = f_{\beta}|_{U_{\alpha} \cap U_{\beta}}$ implies that there exists $f \in \mathcal{O}_X(U)$ so that $f|_{U_{\alpha}} = f_{\alpha}$ for all α . Below, we want to show that $\mathcal{O}_{X,x} = R_{\mathfrak{p}}$, (if $x \leftrightarrow \mathfrak{p} \subset R$) holds with definition 11.0.26. We will formulate this as in [Gat21] (well, up to letting $X = \text{Spec } R$).

Lemma 11.0.28 ([Gat21, Lemma 12.18]). *Let R be a ring and let $X = \text{Spec } R$ be the associated affine scheme. Then for any point in $\mathfrak{p} \in X$ the stalk (3.0.7) $\mathcal{O}_{X,\mathfrak{p}}$ of the structure sheaf \mathcal{O}_X at \mathfrak{p} is isomorphic to the localization $R_{\mathfrak{p}}$.*

Proof. We have a well-defined ring homomorphism

$$\mathcal{O}_{X,\mathfrak{p}} \rightarrow R_{\mathfrak{p}}$$

defined by $\overline{(U, \varphi)} \mapsto \varphi|_{\mathfrak{p}}$ that maps the class of a family $\varphi = (\varphi_{\mathfrak{q}})_{\mathfrak{q} \in U} \in \mathcal{O}_X(U)$ in the stalk $\mathcal{O}_{X,\mathfrak{p}}$ to its element at $\mathfrak{q} = \mathfrak{p}$.

To see that it is well-defined:

- If $(U, \varphi) \sim (V, \psi)$ represents the same *germ* (3.0.7) in $\mathcal{O}_{X,\mathfrak{p}}$ then there is an open $W \ni \mathfrak{p}$ such that $W \subset U \cap V$ and $\varphi|_W = \psi|_W$.
- By the previous point, we then have that

$$\begin{aligned} \varphi_{\mathfrak{p}} &= (\varphi|_W)_{\mathfrak{p}} \\ &= (\psi|_W)_{\mathfrak{p}} \\ &= \psi|_{\mathfrak{p}} \end{aligned}$$

We will show that the map given above is a *bijection*:

Surjective: For any $\frac{g}{f} \in R_{\mathfrak{p}}$, the map takes $\overline{(D(f), \frac{g}{f})}$ in $\mathcal{O}_{X,\mathfrak{p}}$ to $\frac{g}{f}$.

Notice that this makes sense, since a *regular function* on $D(f) = \{\mathfrak{p} \in \text{Spec } R : f \notin \mathfrak{p}\}$ is a family $\varphi = (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in D(f)}$ such that $\varphi_{\mathfrak{p}}$ is in $R_{\mathfrak{p}}$ for all \mathfrak{p} in $D(f)$, such that the following property holds: For each $\mathfrak{p} \in D(f)$ there are elements f, g in R such that f is not in \mathfrak{p} and so that

$$\varphi_{\mathfrak{p}} = \frac{g}{f} \in R_{\mathfrak{p}}$$

for all Q in some open subset $U_{\mathfrak{p}}$ that contains \mathfrak{p} and is contained in $D(f)$. Since f is not in Q for any Q in $D(f)$, we may choose $\varphi_Q \equiv \frac{g}{f}$ for each Q in $D(f)$.

Injective: Let $\varphi \in \mathcal{O}_{\text{Spec } R}(U)$ for some open subset U of $\text{Spec } R$ that contains \mathfrak{p} , such that $\varphi_{\mathfrak{p}} = 0$. By definition, we may shrink U if necessary such that $\varphi = \frac{g}{f}$ on U for some elements f and g in R . We then see that we in particular have $\frac{g}{f} = \frac{0}{1} \in R_{\mathfrak{p}} \Leftrightarrow$ there is some $h \in R \setminus \mathfrak{p}$ such that $hg = 0$. Then if we look at the intersection of U with $D(h) = \{Q \in \text{Spec } R : h \notin Q\}$ we see that $\frac{g}{f} = 0 \in R_Q$ since the condition that $hg = 0$ holds in the ring R . Thus $\varphi = 0$ on the open neighborhood $U \cap D(h)$ of \mathfrak{p} (notice that since h is not in \mathfrak{p} we have that $\mathfrak{p} \in D(h)$). Therefore the germ of φ in $\mathcal{O}_{\text{Spec } R, \mathfrak{p}}$ is zero (since $(U \cap D(h), \varphi) \sim (U, \varphi)$). \square

Proposition 11.0.29 ([Gat21, Prop. 12.19]). *Let R be a ring and $f \in R$. Then $\mathcal{O}_{\text{Spec } R}(D(f))$ is isomorphic to the localization R_f .*

In particular, setting $f = 1$, we see that the global regular functions are $\mathcal{O}_{\text{Spec } R}(\text{Spec } R) \cong R$.

Proof. There is a well-defined ring homomorphism $\Phi : R_f \rightarrow \mathcal{O}_{\text{Spec } R}(D(f))$ defined explicitly by $\frac{g}{f^r} \mapsto \frac{g}{f^r}$ that takes $\frac{g}{f^r} \in R_f$ to the family $(\varphi_{\mathfrak{p}})_{\mathfrak{p} \in D(f)}$ with $\varphi_{\mathfrak{p}} = \frac{g}{f^r} \in R_{\mathfrak{p}}$ for all $\mathfrak{p} \in D(f)$. Notice here that if $\mathfrak{p} \in D(f)$ then f is not in \mathfrak{p} , and then f^r is not in \mathfrak{p} since \mathfrak{p} is a prime ideal so that if $f^r \in \mathfrak{p} \Rightarrow f \in \mathfrak{p}$ (the contrapositive of this statement is what we claimed). We aim to prove that this is bijective.

Injective: Let $g \in R$ and $r \in \mathbb{N}$ such that $\frac{g}{f^r} = 0$ on $D(f)$. For all $\mathfrak{p} \in D(f)$ this means that $\frac{g}{f^r} = 0 \in R_{\mathfrak{p}} \Leftrightarrow$ there is some $h \notin \mathfrak{p}$ such that $hg = 0$. If we let $J = \{h \in R : hg = 0\}$ then we see that $\mathfrak{p} \not\supset J$ (since we found a $h \in R$ such that $h \notin \mathfrak{p}$ but $hg = 0$). Consequently, since

$$V(J) = \{Q \in \text{Spec } R : Q \supset J\}$$

we have that $\mathfrak{p} \notin V(J)$. Since $\mathfrak{p} \in D(f)$ was arbitrary, we see that $D(f) \cap V(J) = \emptyset$, but $D(f) \cup V(f) = \text{Spec } R$, thus

$$\begin{aligned} V(J) \cap \text{Spec } R &= V(J) \cap (D(f) \cup V(f)) \\ &= (V(J) \cap D(f)) \cup (V(J) \cap V(f)) \\ &= V(J) \cap V(f) \\ &= V(J) \\ &\Rightarrow V(J) \subset V(f). \end{aligned}$$

In turn, the Nullstellensatz (for $\text{Spec } R$) and the inclusion-reversing property of $I(-)$ we have

$$\begin{aligned} I(V(f)) &= \sqrt{\langle f \rangle} \\ &\subset \sqrt{J} \\ &= I(V(J)). \end{aligned}$$

But $f \in \sqrt{f}$ so $f \in \sqrt{J} \Rightarrow$ there is some $k \in \mathbb{N}$ such that $f^k \in J$, which by definition of J means that $f^k g = 0$ from which it follows that $\frac{g}{f^r} = 0 \in R_f$. Therefore Φ is injective.

Surjective (cf. lemma 3.0.20): Let $\varphi \in \mathcal{O}_{\text{Spec } R}(D(f))$. Then by definition, for each $\mathfrak{p} \in D(f)$, there are $g_{\mathfrak{p}}, f_{\mathfrak{p}} \in R$ such that $\varphi = \frac{g_{\mathfrak{p}}}{f_{\mathfrak{p}}}$ on some open neighborhood $U_{\mathfrak{p}}$ of \mathfrak{p} with $U_{\mathfrak{p}} \subset D(f_{\mathfrak{p}})$.

Remark 11.0.30. Notice that $U_{\mathfrak{p}} \subset D(f_{\mathfrak{p}}) = \{Q \in \text{Spec } R : f_{\mathfrak{p}} \notin Q\}$ follows directly from [Gat21, Def. 12.16].

The distinguished open subsets form a basis for the Zariski-topology on $\text{Spec } R$ so we may assume that $U_{\mathfrak{p}} = D(h_{\mathfrak{p}})$ for some $h_{\mathfrak{p}} \in R$.

We want to show that we can assume that $f_{\mathfrak{p}} = h_{\mathfrak{p}}$ for all \mathfrak{p} for all \mathfrak{p} . Since $U_{\mathfrak{p}} \subset D(f_{\mathfrak{p}})$ and we have set $U_{\mathfrak{p}} = D(h_{\mathfrak{p}})$ we have that $D(h_{\mathfrak{p}}) \subset D(f_{\mathfrak{p}})$. Thus we have

$$\begin{aligned} V_{f_{\mathfrak{p}}} &= V(f_{\mathfrak{p}}) \cap \text{Spec } R \\ &= V(f_{\mathfrak{p}}) \cap (D(h_{\mathfrak{p}}) \cup V(h_{\mathfrak{p}})) \\ &= (V(f_{\mathfrak{p}}) \cap D(h_{\mathfrak{p}})) \cup (V(f_{\mathfrak{p}}) \cap V(h_{\mathfrak{p}})) \\ &= V(f_{\mathfrak{p}}) \cap V(h_{\mathfrak{p}}) \\ &\Rightarrow V(f_{\mathfrak{p}}) \subset V(h_{\mathfrak{p}}) \end{aligned}$$

where we in the last equality used that $D(h_{\mathfrak{p}}) \subset D(f_{\mathfrak{p}})$ implies that $D(h_{\mathfrak{p}}) \cap V(f_{\mathfrak{p}}) \subset D(f_{\mathfrak{p}}) \cap V(f_{\mathfrak{p}}) = \emptyset$ so that $D(h_{\mathfrak{p}}) \cap V(f_{\mathfrak{p}}) = \emptyset$. Applying the Nullstellensatz in the context of $\text{Spec } R$ and the inclusion-reversing nature of $I(-)$ then gives us that

$$\begin{aligned} I(V(f_{\mathfrak{p}})) &= \sqrt{\langle f_{\mathfrak{p}} \rangle} \\ &\supset \sqrt{\langle h_{\mathfrak{p}} \rangle} \\ &= I(V(h_{\mathfrak{p}})). \end{aligned}$$

Therefore, $h_{\mathfrak{p}} \in \sqrt{\langle h_{\mathfrak{p}} \rangle} \subset \sqrt{\langle f_{\mathfrak{p}} \rangle}$ which by definition means that there is some $r \in \mathbb{N}$ and some $c \in R$ such that $h_{\mathfrak{p}}^r = cf_{\mathfrak{p}}$. Thus we have that

$$\begin{aligned} \frac{cg_{\mathfrak{p}}}{h_{\mathfrak{p}}^r} &= \frac{cg_{\mathfrak{p}}}{cf_{\mathfrak{p}}} \\ &= \frac{g_{\mathfrak{p}}}{f_{\mathfrak{p}}}. \end{aligned}$$

We note that

$$\begin{aligned} D(cf_{\mathfrak{p}}) &= D(h_{\mathfrak{p}}^r) \\ &= \{Q \in \text{Spec } R : h_{\mathfrak{p}}^r \notin Q\} \\ &= \{Q \in \text{Spec } R : h_{\mathfrak{p}} \notin Q\}, \quad (\text{since } h_{\mathfrak{p}} \notin Q \Leftrightarrow h_{\mathfrak{p}}^r \notin Q \text{ by primality of } Q) \\ &= D(h_{\mathfrak{p}}) \end{aligned}$$

We then set $f_{\mathfrak{p}} := h_{\mathfrak{p}}^r$ (noting that with this replacement we have $D(f_{\mathfrak{p}}) = D(h_{\mathfrak{p}})$), and we let $g_{\mathfrak{p}} := cg_{\mathfrak{p}}$. We may then assume that $D(f)$ is covered by open subsets of the form $D(f_{\mathfrak{p}})$ and that $\varphi = \frac{g_{\mathfrak{p}}}{f_{\mathfrak{p}}}$ on $D(f_{\mathfrak{p}})$.

Next, we observe that $D(f)$ can be covered by *finitely many* such distinguished opens $D(f_{\mathfrak{p}})$. Indeed, $D(f) \subset \bigcup_{\mathfrak{p}} D(f_{\mathfrak{p}})$ is equivalent to $V(f) \supset \bigcap_{\mathfrak{p}} V(f_{\mathfrak{p}}) = V(\sum_{\mathfrak{p}} \langle f_{\mathfrak{p}} \rangle)$. Therefore

$$\begin{aligned} I(V(f)) &= \sqrt{f} \\ &\subset \sqrt{\sum_{\mathfrak{p}} \langle f_{\mathfrak{p}} \rangle} \\ &= I\left(V\left(\sum_{\mathfrak{p}} \langle f_{\mathfrak{p}} \rangle\right)\right). \end{aligned}$$

Since $f \in \sqrt{f}$ it follows that $f \in \sqrt{\sum_{\mathfrak{p}} \langle f_{\mathfrak{p}} \rangle}$ hence there is some $r \in \mathbb{N}$ such that $f^r \in \sum_{\mathfrak{p}} \langle f_{\mathfrak{p}} \rangle$. Therefore we have

$$f^r = \sum_{\mathfrak{p}} k_{\mathfrak{p}} f_{\mathfrak{p}} \tag{11.0.3}$$

with only *finitely many* $k_{\mathfrak{p}}$ in R non-zero. Hence we only have to consider finitely many $\mathfrak{p} \in \text{Spec } R$. One might ask why? To answer this, notice that $D(f) = D(f^r)$ and that if $\mathfrak{q} \in D(f^r)$ then $f^r \notin \mathfrak{q}$. If $k_{\mathfrak{p}}f_{\mathfrak{p}}$ are in \mathfrak{q} for all non-zero $k_{\mathfrak{p}}$ in the sum (11.0.3) then $f^r \in \mathfrak{q}$, which is a contradiction. Therefore there is some term $k_{\mathfrak{p}}f_{\mathfrak{p}}$ in (11.0.3) such that $k_{\mathfrak{p}}f_{\mathfrak{p}} \notin \mathfrak{q} \Rightarrow \mathfrak{q} \in D(k_{\mathfrak{p}}f_{\mathfrak{p}}) = D(k_{\mathfrak{p}}) \cap D(f_{\mathfrak{p}}) \subset D(f_{\mathfrak{p}})$. Hence

$$\begin{aligned} D(f) &= D(f^r) \\ &\subset \bigcup_{\mathfrak{p}} D(f_{\mathfrak{p}}). \end{aligned}$$

On the distinguished open subset $D(f_{\mathfrak{p}}) \cap D(f_{\mathfrak{q}})$ for some $\mathfrak{p}, \mathfrak{q}$ we have that

$$\begin{aligned} \varphi &= \frac{g_{\mathfrak{p}}}{f_{\mathfrak{p}}} \\ &= \frac{g_{\mathfrak{q}}}{f_{\mathfrak{q}}} \end{aligned}$$

By the injectivity of Φ (with $f = f_{\mathfrak{p}}f_{\mathfrak{q}}$) it follows that $\frac{g_{\mathfrak{p}}}{f_{\mathfrak{p}}} = \frac{g_{\mathfrak{q}}}{f_{\mathfrak{q}}}$ in $R_{f_{\mathfrak{p}}f_{\mathfrak{q}}} \Leftrightarrow$ there is some natural number n such that $(f_{\mathfrak{p}}f_{\mathfrak{q}})^n(g_{\mathfrak{p}}f_{\mathfrak{q}} - g_{\mathfrak{q}}f_{\mathfrak{p}}) = 0 \in R$. Since there are only finitely many \mathfrak{p} and \mathfrak{q} that we need to consider we may take $n = \max(n_{\mathfrak{p}_1}, \dots, n_{\mathfrak{p}_\ell})$ where $\mathfrak{p}_1, \dots, \mathfrak{p}_\ell$ correspond to the finitely many non-zero primes in the sum (11.0.3). We may now set $g_{\mathfrak{p}} := g_{\mathfrak{p}}f_{\mathfrak{p}}^n$ and $f_{\mathfrak{p}} := f_{\mathfrak{p}}^{n+1}$ for all \mathfrak{p} . Then we still have $\varphi = \frac{g_{\mathfrak{p}}}{f_{\mathfrak{p}}}$ on $D(f_{\mathfrak{p}})$ and

$$\begin{aligned} g_{\mathfrak{p}}f_{\mathfrak{q}} - g_{\mathfrak{q}}f_{\mathfrak{p}} &= g_{\mathfrak{p}}f_{\mathfrak{p}}^nf_{\mathfrak{q}}^{n+1} - g_{\mathfrak{q}}f_{\mathfrak{q}}^nf_{\mathfrak{p}}^{n+1} \\ &= (f_{\mathfrak{p}}f_{\mathfrak{q}})^n(g_{\mathfrak{p}}f_{\mathfrak{q}} - g_{\mathfrak{q}}f_{\mathfrak{p}}) \\ &= 0, \end{aligned}$$

for all \mathfrak{p}_i with $i = 1, \dots, \ell$.

Then write

$$f^r = \sum_{i=1}^{\ell} k_{\mathfrak{p}_i} f_{\mathfrak{p}_i}$$

and set

$$g := \sum_{i=1}^{\ell} k_{\mathfrak{p}_i} g_{\mathfrak{p}_i}.$$

Then for every $\mathfrak{q} = \mathfrak{p}_i$ we have that

$$\begin{aligned} gf_{\mathfrak{q}} &= \sum_{i=1}^{\ell} k_{\mathfrak{p}_i} g_{\mathfrak{p}_i} f_{\mathfrak{q}} \\ &= \left(\sum_{i=1}^{\ell} k_{\mathfrak{p}_i} f_{\mathfrak{p}_i} \right) g_{\mathfrak{q}} \\ &= f^r g_{\mathfrak{q}}. \end{aligned}$$

By assumption $D(f_{\mathfrak{q}}) \subset D(f)$. This means that on $D(f_{\mathfrak{q}})$ the fraction $\frac{g}{f^r}$ makes sense, but also $\frac{g_{\mathfrak{q}}}{f_{\mathfrak{q}}}$ makes sense and so since $gf_{\mathfrak{q}} = f^r g_{\mathfrak{q}}$ on R we see that

$$\frac{g}{f^r} = \frac{g_{\mathfrak{q}}}{f_{\mathfrak{q}}}$$

on $D(f_{\mathfrak{q}})$. Since the $D(f_{\mathfrak{q}})$ cover $D(f)$ we find that $\varphi = \frac{g}{f^r}$ on $D(f)$. Therefore the corresponding element $\frac{g}{f^r} \in R_f$ maps to φ under Φ , so Φ is surjective. \square

Chapter 12

Lecture 12: Schemes II

Recall: $\text{Spec } A, V(I), D(f), \mathcal{O}_{\text{Spec } A}(D(f)) \cong A_f, \mathcal{O}_{\text{Spec } A, \mathfrak{p}} = A_{\mathfrak{p}}$.

$\mathcal{O}_{\text{Spec } A}(U) = \{(\varphi_{\mathfrak{p}})_{\mathfrak{p} \in U} \text{ such that } \varphi_{\mathfrak{p}} = \frac{g}{f} \text{ “locally”}\}$.

Previously: Standing assumption that $\mathcal{O}_X(U) \subseteq \text{Fun}(U, k)$.

Now we give the following definition.

Morphism of ringed spaces (for schemes)

Definition 12.0.1. $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a **morphism of ringed spaces** if the following holds:

1. $f : X \rightarrow Y$ is continuous.
2. For all $U \subset Y$ open, we get induced ring homomorphisms $f_U^* : \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$ such that

$$f_U^*(\varphi)|_{f^{-1}(V)} = f_V^*(\varphi|_V)$$

for all $V \subset U \subset Y$ open, where this diagram below commutes:

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{f_U^*} & \mathcal{O}_X(f^{-1}(U)) \\ \downarrow \text{restr} & & \downarrow \text{restr} \\ \mathcal{O}_Y(V) & \xrightarrow{f_V^*} & \mathcal{O}_X(f^{-1}(V)) \end{array}$$

Condition 2. is now part of the definition instead of a condition.

We also have that for all $x \in X$ the maps $\{f_U^*\}$ induces ring homomorphisms $f_x^* : \mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X, x}$ between stalks.

Furthermore, recall (4.0.7). Now we give the following definition.

Locally ringed space

Definition 12.0.2. A **locally ringed space** is a ringed space (X, \mathcal{O}_X) (4.0.1) such that each stalk $\mathcal{O}_{X, \mathfrak{p}}$ for $\mathfrak{p} \in X$ is a *local ring* (i.e. at least in the case of commutative rings, has a unique maximal ideal).

Example 12.0.3. Let X be a variety, then $\mathfrak{m}_x = \{f \in \mathcal{O}_{X, x} : f(x) = 0\} \subset \mathcal{O}_{X, x}$.

Example 12.0.4. Let $X = \text{Spec } A$ be an affine scheme. Then

$$\begin{aligned} \mathfrak{m}_{\mathfrak{p}} &= \mathfrak{p}A_{\mathfrak{p}} \\ &= \{f \in A_{\mathfrak{p}} : f(x) = 0\}, \quad (f(x) \in k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \\ &\subset \mathcal{O}_{X, \mathfrak{p}} \\ &= A_{\mathfrak{p}}. \end{aligned}$$

Morphism of locally ringed spaces

Definition 12.0.5. A morphism $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of locally ringed spaces is a morphism of ringed spaces (in the sense of 12.0.1) such that for all $x \in X$ we have that $f_x^* : \mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X, x}$ is a *local ring homomorphism*, i.e. $(f_x^*)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$. Equivalently, $\text{Spec}(\mathcal{O}_{X, x}) \rightarrow \text{Spec}(\mathcal{O}_{Y, f(x)})$ induced by f_x^* maps x to $f(x)$.

Example 12.0.6. A morphism of varieties (5.0.14) is a morphism of locally ringed spaces.

Non-example: $\varphi : k[x]_x \hookrightarrow k(x)$ is not a *local ring homomorphism* since

$$\begin{aligned} \varphi^{-1}(\langle 0 \rangle) &= 0 \\ &\neq \langle x \rangle. \end{aligned}$$

To say a bit more, notice that $k(x)$ is a field, so its only maximal ideal is $\langle 0 \rangle$. φ is injective, so the preimage of $\langle 0 \rangle$ is $\langle 0 \rangle$ which is not a maximal ideal in $k[x]_x$, instead the maximal ideal of $k[x]_x$ is $\langle x \rangle$.

Proposition 12.0.7 ([Gat21, Prop. 12.27]). *For any two rings (commutative with 1) R and S there is a bijection*

$$\begin{aligned} \{\text{morphisms } \text{Spec } R \rightarrow \text{Spec } S\} &\xleftrightarrow{1-1} \{\text{ring homomorphisms } S \rightarrow R\} \\ f &\longmapsto f^*. \end{aligned}$$

In particular, this means that there is a natural bijection

$$\{\text{affine schemes}\} / \text{isomorphisms} \xleftrightarrow{1-1} \{\text{rings}\} / \text{isomorphisms}.$$

Proof. Let $f : \text{Spec } R \rightarrow \text{Spec } S$ be a morphism of affine schemes. Then this includes by definition 12.0.1 the data of ring homomorphisms $f_{\text{Spec } S}^* : \mathcal{O}_{\text{Spec } S}(\text{Spec } S) \rightarrow \mathcal{O}_{\text{Spec } R}(\text{Spec } R)$. But by 11.0.29 this is just a ring homomorphism $S \rightarrow R$.

If we instead assume $\varphi : S \rightarrow R$ is a ring homomorphism then this first of all defines a set-theoretic map $f : \text{Spec } R \rightarrow \text{Spec } S$ by $\text{Spec } R \ni \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ (inverse images of prime ideals are prime ideals). We claim that this map is continuous: Let $J \trianglelefteq S$ be an ideal. Then

$$\begin{aligned} f^{-1}(V(J)) &= \{\mathfrak{p} \in \text{Spec } R : f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}) \in V(J)\} \\ &= \{\mathfrak{p} \in \text{Spec } R : f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}) \supset J\} \\ &= \{\mathfrak{p} \in \text{Spec } R : \mathfrak{p} \supset \varphi(J)\} \\ &= V(\varphi(J)). \end{aligned}$$

In the third equality, we used that if $\varphi^{-1}(\mathfrak{p}) \supset J \Leftrightarrow \forall j \in J \varphi(j) \in \mathfrak{p} \Leftrightarrow \varphi(J) \subset \mathfrak{p}$. Thus the inverse image of a closed set under f is closed. For any prime $\mathfrak{p} \in \operatorname{Spec} R$ we get an induced map $\varphi_{\mathfrak{p}} : S_{\varphi^{-1}(\mathfrak{p})} \rightarrow R_{\mathfrak{p}}$ explicitly by $\frac{s}{t} \mapsto \frac{\varphi(s)}{\varphi(t)}$.

Remark 12.0.8. Notice that this map is well-defined since if $\frac{s}{t} = \frac{s'}{t'} \in S_{\varphi^{-1}(\mathfrak{p})}$ then there is some point $q \in S \setminus \varphi^{-1}(\mathfrak{p})$ such that $q(st' - s't) = 0 \in S$. Then $\varphi(q)(\varphi(s)\varphi(t') - \varphi(s')\varphi(t)) = 0$ in R with $\varphi(q) \notin \mathfrak{p}$ and so

$$\frac{\varphi(s)}{\varphi(t)} = \frac{\varphi(s')}{\varphi(t')}$$

in $R_{\mathfrak{p}}$ where $\varphi(t), \varphi(t') \notin \mathfrak{p}$ since $t, t' \in S \setminus \varphi^{-1}(\mathfrak{p})$.

The *sections* of the structure sheaves on $\operatorname{Spec} R$ and $\operatorname{Spec} S$ are by definition made up from elements in $R_{\mathfrak{p}}$ and $S_{\varphi^{-1}(\mathfrak{p})} = S_{f(\mathfrak{p})}$ respectively, this in turn induces componentwise ring homomorphisms $f_U^* : \mathcal{O}_{\operatorname{Spec} S}(U) \rightarrow \mathcal{O}_{\operatorname{Spec} R}(f^{-1}(U))$ for all open subsets $U \subset \operatorname{Spec} S$.

Explicitly for each s a section in $\mathcal{O}_{\operatorname{Spec} S}(U)$, we let $f_U^*(s) \in \mathcal{O}_{\operatorname{Spec} R}(f^{-1}(U))$ be defined on $\mathfrak{q} \in f^{-1}(U)$ (with $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{p}) \in U$) as

$$\begin{aligned} f_U^*(s)(\mathfrak{q}) &:= \varphi_{\mathfrak{p}}(s(f(\mathfrak{q}))) \\ &= \varphi_{\mathfrak{p}}(s(\varphi^{-1}(\mathfrak{p}))) \\ &= \varphi_{\mathfrak{p}}\left(\frac{g_{\varphi^{-1}(\mathfrak{p})}}{f_{\varphi^{-1}(\mathfrak{p})}}\right) \\ &= \frac{\varphi_{\mathfrak{p}}(g_{\varphi^{-1}(\mathfrak{p})})}{\varphi_{\mathfrak{p}}(f_{\varphi^{-1}(\mathfrak{p})})} \in R_{\mathfrak{p}}. \end{aligned}$$

We want to show that the induced maps f_U^* are compatible with restrictions. So fix $V \subset U$ and f_U^*, f_V^* (and note that $V \subset U \Rightarrow f^{-1}(V) \subset f^{-1}(U)$). Then we need to check that

$$f_V^* \circ \operatorname{restr} = \operatorname{restr} \circ f_U^*,$$

i.e. that the diagram in definition 12.0.1 commutes. To be more precise, for

- $\rho_{U,V} : \mathcal{O}_{\operatorname{Spec} S}(U) \rightarrow \mathcal{O}_{\operatorname{Spec} S}(V)$,
- $\rho'_{f^{-1}(U), f^{-1}(V)} : \mathcal{O}_{\operatorname{Spec} R}(f^{-1}(U)) \rightarrow \mathcal{O}_{\operatorname{Spec} R}(f^{-1}(V))$,

we need the following diagram to commute:

$$\begin{array}{ccc} s & \xrightarrow{\in} & \mathcal{O}_{\operatorname{Spec} S}(U) & \xrightarrow{f_U^*} & \mathcal{O}_{\operatorname{Spec} R}(f^{-1}(U)) \\ & & \downarrow \rho_{U,V} & & \downarrow \rho'_{f^{-1}(U), f^{-1}(V)} \\ & & \mathcal{O}_{\operatorname{Spec} S}(V) & \xrightarrow{f_V^*} & \mathcal{O}_{\operatorname{Spec} R}(f^{-1}(V)) \end{array}$$

We then note that $f_V^* \circ \rho_{U,V}, \rho'_{f^{-1}(U), f^{-1}(V)} \circ f_U^* \in \mathcal{O}_{\operatorname{Spec} R}(f^{-1}(V))$. Let $s \in \mathcal{O}_{\operatorname{Spec} S}(U)$. Then if $\mathfrak{q} \in f^{-1}(V)$ so that $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}) \in V$, then

$$\begin{aligned} (f_V^* \circ \rho_{U,V}(s))(\mathfrak{q}) &= (f_V^* \circ s|_V)(\mathfrak{q}) \\ &= \varphi_{\mathfrak{q}}(s|_V(f(\mathfrak{q}))) \\ &= \varphi_{\mathfrak{q}}(s(f(\mathfrak{q}))) \quad (\text{since } f(\mathfrak{q}) \in V). \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} (\rho_{f^{-1}(U), f^{-1}(V)} \circ f_U^*)(s)(\mathfrak{q}) &= (f_U^*(s))|_{f^{-1}(V)}(\mathfrak{q}) \\ &= (f_U^*(s))(\mathfrak{q}), \quad (\text{since } \mathfrak{q} \in f^{-1}(V)) \\ &= \varphi_{\mathfrak{q}}(s(f(q))). \end{aligned}$$

By comparison we see that they are equal.

By [Gat21, Lemma 12.18] we have that $\mathcal{O}_{\text{Spec } R, \mathfrak{p}} \cong R_{\mathfrak{p}}$ and so since f induced maps $\varphi_{\mathfrak{p}} : S_{\varphi^{-1}(\mathfrak{p})} \rightarrow R_{\mathfrak{p}}$ this in turn induces maps

$$f_{\mathfrak{p}}^* := \varphi_{\mathfrak{p}} : \mathcal{O}_{\text{Spec } S, f(\mathfrak{p})} \rightarrow \mathcal{O}_{\text{Spec } R, \mathfrak{p}}$$

between stalks. We need to check that, in accordance with definition 12.0.5, that $(f_{\mathfrak{p}}^*)^{-1}(\mathfrak{m}_{\mathfrak{p}}) = \mathfrak{m}_{f(\mathfrak{p})}$. But $f_{\mathfrak{p}}^* := \varphi_{\mathfrak{p}}$.

Lemma 12.0.9. *Let R be a commutative ring with 1 and let $\mathfrak{p} \trianglelefteq R$ be a prime ideal. Then $R_{\mathfrak{p}}$ has a unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}} := \{\frac{a}{b} \in R_{\mathfrak{p}} : a \in \mathfrak{p}\}$.*

Proof.

$\mathfrak{p}R_{\mathfrak{p}}$ maximal ideal: By definition, any ideal $I \supsetneq \mathfrak{p}R_{\mathfrak{p}}$ contains a unit: This follows from the fact that $\frac{a}{b}$ is invertible $\Leftrightarrow a \in R \setminus \mathfrak{p}$ and $\mathfrak{p}R_{\mathfrak{p}}$ contains precisely all elements where the numerator has an element not in $R \setminus \mathfrak{p}$. Therefore if $\mathfrak{p}R_{\mathfrak{p}}$ is strictly contained in I then there is some element $\frac{c}{d}$ with $c \in R \setminus \mathfrak{p}$ and so $\frac{c}{d}$ is invertible, but then by definition of an ideal (in particular closure under multiplication from $R_{\mathfrak{p}}$) we have that $\frac{d}{c} \cdot \frac{c}{d} = 1 \in \mathfrak{p}R_{\mathfrak{p}} \Rightarrow I = R_{\mathfrak{p}}$.

$\mathfrak{p}R_{\mathfrak{p}}$ unique maximal ideal: Any possible maximal ideal \mathfrak{m} has to be contained in the set of non-units (since a maximal ideal is *proper*). But the non-units of $R_{\mathfrak{p}}$ are precisely $\mathfrak{p}R_{\mathfrak{p}}$. Therefore $\mathfrak{m} \subseteq \mathfrak{p}R_{\mathfrak{p}}$. But then by maximality $\mathfrak{m} = \mathfrak{p}R_{\mathfrak{p}}$. \square

Elements in $\mathfrak{m}_{\mathfrak{p}} \leq R_{\mathfrak{p}}$ are on the form $\frac{a}{b}$ with $a \in \mathfrak{p}$ and $b \in R \setminus \mathfrak{p}$. Thus we have

$$\begin{aligned} \varphi_{\mathfrak{p}}^{-1}(\mathfrak{m}_{\mathfrak{p}}) &= \left\{ \frac{s}{t} \in S_{f(\mathfrak{p})} : \frac{\varphi(s)}{\varphi(t)} \in \mathfrak{m}_{\mathfrak{p}} \right\} \\ &= \left\{ \frac{s}{t} \in S_{f(\mathfrak{p})} : \varphi(s) \in \mathfrak{p} \right\} \\ &= \left\{ \frac{s}{t} \in S_{f(\mathfrak{p})} : s \in \varphi^{-1}(\mathfrak{p}) \right\} \\ &= \left\{ \frac{s}{t} \in S_{f(\mathfrak{p})} : s \in f(\mathfrak{p}) \right\}, \end{aligned}$$

but this is precisely $\mathfrak{m}_{f(\mathfrak{p})}$. \square

Corollary 12.0.10. *There is an anti-equivalence of categories between \mathbf{Ring}^{op} and $\mathbf{AffSch} \subset \mathbf{LRS}$. The category \mathbf{Ring}^{op} has as objects rings and as morphisms ring homomorphisms, while \mathbf{AffSch} has as objects (affine) schemes and morphisms morphisms of locally ringed spaces.*

12.0.1 Closed Subschemes (of affine schemes)

Warning: If $X = \text{Spec } A$, $Z \subset X$ closed subset then there are many different structures as closed subscheme. Have to choose one.

Affine subscheme

Definition 12.0.11. An **affine subscheme** of $\text{Spec } A$ is:

- An affine scheme $\text{Spec } B$, and
- a morphism $i : \text{Spec } B \rightarrow \text{Spec } A$,

such that the ring homomorphism $\varphi := i^* : A \rightarrow B$ induced by proposition 12.0.7 is *surjective*.

Remark 12.0.12. In class this was called “closed subscheme” but not to confuse this with a later definition, and following [Gat21] we will call it an affine subscheme.

Remark 12.0.13. If $\varphi : A \rightarrow B$ is surjective then we claim that i is injective. We have that i is then defined by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. If $\varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}')$, then we have that

$$\begin{aligned} b = \varphi(a) \in \mathfrak{p} &\Leftrightarrow a \in \varphi^{-1}(\mathfrak{p}) \\ &\Leftrightarrow a \in \varphi^{-1}(\mathfrak{p}') \\ &\Leftrightarrow b = \varphi(a) \in \mathfrak{p}' \\ &\Rightarrow \mathfrak{p} = \mathfrak{p}', \end{aligned}$$

so that i is indeed injective.

Remark 12.0.14. Notice that if $\varphi : A \rightarrow B$ is surjective then by the first isomorphism theorem we have that $A/\ker \varphi \cong B$. Then by proposition 12.0.7 we have that the ring isomorphism $\psi : A/\ker \varphi \cong B$ corresponds to some morphism of schemes $\text{Spec } B \rightarrow \text{Spec}(A/\ker \varphi)$. If we assume without proof for now that 12.0.7 just recounts the categorical fact given in 12.0.10 in more explicit terms, and noting that the anti-equivalence of 12.0.10 can be stated functorially, then it is clear that the image of ψ is an isomorphism of locally ringed spaces $\text{Spec}(A/\ker \varphi)$ and $\text{Spec } B$ (since functors take isomorphisms to isomorphisms). We then have that

$$\text{Spec } A/I \xrightarrow{\sim} V(I) \subseteq \text{Spec } A$$

and this isomorphism comes from the *correspondence isomorphism theorem for rings* (since $V(I)$ consists precisely of the prime ideals in A that contain I):

$$\{\mathfrak{q} \subset A/I \text{ prime}\} \longleftrightarrow \{I \subset \mathfrak{p} \subset A \text{ prime}\}, \quad \mathfrak{q} \longmapsto \pi^{-1}(\mathfrak{q})$$

Thus, for a given closed set $Z \subseteq \text{Spec } A$, we choose

$$I \subset A \text{ such that } V(I) = Z.$$

There is a maximal such set, namely the radical

$$I(Z) = \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}.$$

since by the Nullstellensatz we have $V(I(Z)) = Z$. We show maximality (with respect to inclusion) of $I(Z)$: If J is a subset of A such that $V(J) = Z$ then by definition of $V(J) = \{Q \in \text{Spec } A : Q \supset J\}$ then note that Z is a set of prime ideals, and so that if $V(J) = Z$ then by definition, $J \subset \mathfrak{p}$ for each $\mathfrak{p} \in Z$, and so $J \subseteq \bigcap_{\mathfrak{p} \in Z} \mathfrak{p} = I(Z)$.

We have that

$$\left\{ \begin{array}{c} \text{closed subschemes} \\ \text{of } \text{Spec } A \end{array} \right\} \xleftrightarrow{\text{inclusion-reversing}} \left\{ \begin{array}{c} \text{ideals of} \\ A \end{array} \right\}.$$

The reason for the inclusion-reversing nature of this correspondence comes from the fact that $i : \text{Spec } A/I \hookrightarrow \text{Spec } A/J \Leftrightarrow I \supset J$. To see this notice that if $\text{Spec } A/I \subset \text{Spec } A/J$ then by definition

there is a surjective ring-homomorphism $\varphi : A/J \rightarrow A/I$. But this ring-homomorphism is well-defined precisely when $I \supset J$, since if not, then there is some $j \in J$ such that $j \notin I$. Then we see that $\varphi(0 + J) = \varphi(j + J) = j + I \neq 0 + I$ so that φ does not take zero to zero, hence is not a ring-homomorphism.

On the other hand, if $I \supset J$ then by the inclusion-reversing property of $V(-)$ we see that $\text{Spec } A/I \cong V(I) \subset V(J) \cong \text{Spec } A/J$.

12.0.2 Intersections and unions of closed (sub-)schemes

Lemma 12.0.15.

- $\text{Spec } R/J_1 \cap \text{Spec } R/J_2 := \text{Spec } R/(J_1 + J_2)$.
- $\text{Spec } R/J_1 \cup \text{Spec } R/J_2 := \text{Spec } R/(J_1 \cap J_2)$.

Proof. Consider that $V(J_i) \cong \text{Spec } R/J_i$. Then notice that

- $V(J_1) \cap V(J_2) = V(J_1 + J_2)$.
- $V(J_1) \cup V(J_2) = V(J_1 \cap J_2)$.

□

Remark 12.0.16. Note that the proof only shows isomorphisms. We seem to want to identify them *up to isomorphism* (cf. 12.0.7).

Example 12.0.17. Let $R = k[x, y]$ and consider $\text{Spec } R$. Let

$$\begin{cases} C &:= \text{Spec } R/\langle y^2 - x \rangle \\ D &:= \text{Spec } R/\langle y - 1 \rangle \\ D' &:= \text{Spec } R/\langle y \rangle. \end{cases}$$

Then by lemma 12.0.15 we have

$$\begin{aligned} C \cap D &= \text{Spec } R/\langle y - x^2, y - 1 \rangle \\ &= \text{Spec } R/\langle y - 1, (x + 1)(x - 1) \rangle \\ &= \text{Spec}(k \times k). \end{aligned}$$

The second equality follows from the claim that $\langle y - x^2, y - 1 \rangle = \langle y - 1, x^2 - 1 \rangle$ are equal. We now prove this.

Proof.

\subseteq : We have $(y - 1) - (x^2 - 1) = y - x^2$ and so $r_1(y - x^2) + r_2(y - 1)$ are in $\langle y - 1, x^2 - 1 \rangle$ for all $r_1, r_2 \in R$ from which the inclusion follows.

\supseteq : We have that $-(y - x^2) + (y - 1) = x^2 - 1$ so that $r_1(y - 1) + r_2(x^2 - 1)$ are in $\langle y - x^2, y - 1 \rangle$ for all $r_1, r_2 \in R$. □

Furthermore, we claim that $k[x, y]/\langle y - 1, x^2 - 1 \rangle \cong k \times k$. We prove this.

Proof. Define

$$\varphi : k[x, y] \longrightarrow k[x]/(x^2 - 1) \quad \text{by} \quad \varphi(x) = \bar{x}, \quad \varphi(y) = 1,$$

where \bar{x} denotes the class of x in $k[x]/(x^2 - 1)$.

Any polynomial $f \in k[x, y]$ can be written uniquely as

$$f(x, y) = \sum_{i=0}^n a_i(y) x^i, \quad a_i(y) \in k[y].$$

We then set

$$\varphi(f(x, y)) = \sum_{i=0}^n a_i(1) \bar{x}^i.$$

We check the ring-homomorphism properties directly:

$$\begin{aligned} \varphi(f + g) &= \sum_i (a_i(1) + b_i(1)) \bar{x}^i \\ &= \sum_i a_i(1) \bar{x}^i + \sum_i b_i(1) \bar{x}^i \\ &= \varphi(f) + \varphi(g), \\ \varphi(fg) &= \sum_k \left(\sum_{i+j=k} a_i(1) b_j(1) \right) \bar{x}^k \\ &= \varphi(f) \varphi(g), \\ \varphi(1) &= 1, \quad \varphi(c) = c \quad (\forall c \in k). \end{aligned}$$

Finally, $\varphi(y - 1) = 0$ and $\varphi(x^2 - 1) = \bar{x}^2 - 1 = 0$, so $\langle y - 1, x^2 - 1 \rangle \subseteq \ker \varphi$. On the other hand let $f = \sum_{i=0}^n a_i(y) x^i$ so that $f \in \ker \varphi$. This means that

$$\begin{aligned} \varphi(f) &= \bar{f} \\ &= \sum_{i=0}^n a_i(1) \bar{x}^i \\ &= 0 \end{aligned}$$

which means that $f(x, 1) \in \langle x^2 - 1 \rangle$. Therefore there is some $h(x) \in k[x]$ such that $f(x, 1) = h(x)(x^2 - 1)$. We have that

$$\begin{aligned} f(x, y) - f(x, 1) &= \left(\sum_{i=0}^n a_i(y) x^i \right) - \left(\sum_{i=0}^n a_i(1) x^i \right) \\ &= \sum_{i=0}^n (a_i(y) - a_i(1)) x^i \\ &= \sum_{i=0}^n (h_i(y)(y - 1)) x^i, \quad (\text{since } a_i(y) - a_i(1) \text{ has a zero at } y = 1) \\ \Rightarrow f(x, y) &= (y - 1) \sum_{i=0}^n (h_i(y)) x^i + f(x, 1) \\ &= (y - 1) \sum_{i=0}^n (h_i(y)) x + h(x)(x^2 - 1) \in \langle y - 1, x^2 - 1 \rangle. \end{aligned}$$

We then have

$$\begin{aligned} k[x, y]/\langle y - 1, x^2 - 1 \rangle &\cong k[x]/\langle x^2 - 1 \rangle \\ &\cong k[x]/\langle x + 1 \rangle \times k[x]/\langle x - 1 \rangle, \quad (\text{by Chinese Remainder Theorem}) \\ &\cong k \times k \end{aligned}$$

where in the second isomorphism we used that $x + 1, x - 1$ are *coprime* since they generate R . The last isomorphism follows by evaluation at $x = 1$ in the first factor and $x = -1$ in the second factor. \square

$$\begin{aligned} C \cap D' &= \text{Spec } R/\langle y - x^2, y \rangle \\ &= \text{Spec } R/\langle y, x^2 \rangle \\ &= \text{Spec } k[x]/\langle x^2 \rangle \\ &= k \oplus kx, \end{aligned}$$

where the second equality follows from the claim that $\langle y - x^2, y \rangle = \langle y, x^2 \rangle$. We show this claim.

Proof.

\subseteq : We see that $y - x^2$ and y are in $\langle y, x^2 \rangle$ so $r_1(y - x^2) + r_2(y) \in \langle y, x^2 \rangle$ for arbitrary $r_1, r_2 \in R$, so the statement follows.

\supseteq : Notice that $y - x^2 - y = -x^2$ is in $\langle y - x^2, y \rangle$ and so $(-1)(-x^2) = x^2$ is in $\langle y - x^2, y \rangle$. Therefore $r_1(y) + r_2(x^2)$ is in $\langle y - x^2, y \rangle$ for arbitrary $r_1, r_2 \in R$. \square

12.0.3 Distinguished subschemes

If (X, \mathcal{O}_X) is a locally ringed space and $U \subset X$ is open $\rightsquigarrow (U, \mathcal{O}_X|_U)$ is a locally ringed space.

Proposition 12.0.18 ([Gat21, Prop. 12.29]). *Let R be a commutative ring with unity, and let $f \in R$. Then the distinguished open subset $D(f) \subset \text{Spec } R$ is isomorphic to the affine scheme $\text{Spec } R_f$.*

Proof. \square

Scheme

Definition 12.0.19. A locally ringed space (X, \mathcal{O}_X) is a **scheme** if there exists an open cover $X = \bigcup_{\alpha} U_{\alpha}$ such that $(U_{\alpha}, \mathcal{O}_X|_{U_{\alpha}})$ is an affine scheme for all α (“atlas”).

Closed subscheme

Definition 12.0.20. A **closed subscheme** of a scheme X is a scheme Y together with a morphism $i : Y \rightarrow X$ such that X has an affine open cover $\{U_k : k \in I\}$ for which each restriction $i|_{i^{-1}(U_k)} : i^{-1}(U_k) \rightarrow U_k$ is an affine subscheme in the sense of definition 12.0.11.

Open subscheme

Definition 12.0.21. Given $U \subset X$ open we have that $(U, \mathcal{O}_X|_U)$ is a scheme, called an **open subscheme**.

Idea: If $X = \bigcup_{\alpha} U_{\alpha}$ affine open covering $\rightsquigarrow U = \bigcup (U \cap U_{\alpha})$ open covering, but $U \cap U_{\alpha}$ need not be affine, $U \cap U_{\alpha} \subset U_{\alpha}$ but $U \cap U_{\alpha} = \bigcup_{\beta} \underbrace{D(f_{\alpha\beta})}_{\text{affine}}$ so that $U = \bigcup_{\alpha, \beta} D(f_{\alpha\beta})$.

12.0.4 Properties of schemes

A scheme over Y

Definition 12.0.22. Let Y be a scheme. A **scheme over Y** is a scheme X together with a morphism $f : X \rightarrow Y$. A morphism of schemes $f_1 : X_1 \rightarrow Y$ and $f_2 : X_2 \rightarrow Y$ over Y is a morphism $f : X_1 \rightarrow X_2$ with $f_1 = f_2 \circ f$, i.e. such that the following diagram commutes

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ & \searrow f_1 & \swarrow f_2 \\ & Y & \end{array} .$$

Remark 12.0.23. A scheme over an affine scheme $Y = \text{Spec } C$ is also called a **scheme over S** . If $X = \text{Spec } R$ is affine as well, with X a scheme over Y , then the morphism $f : X \rightarrow Y$ corresponds to a ring homomorphism $f^* : S \rightarrow R$ by proposition 12.0.7. Then this gives us an S -module structure on R by $s \cdot r := f^*(s) \cdot r$. Since R is commutative we have that the image of S under f is contained in the center $Z(R)$ of R , so that R is an S -algebra.

Of finite type over Y

A scheme $f : X \rightarrow Y$ over Y is said to be **of finite type over Y** if there is an open cover of Y by affine schemes $U_i = \text{Spec } S_i$ such that $f^{-1}(U_i)$ has a *finite* open cover by affine schemes $\text{Spec } R_{i,j}$ where each $R_{i,j}$ is a finitely generated S_i -algebra.

Example 12.0.24. If $A \rightarrow B$ f.g. then $\text{Spec } B \rightarrow \text{Spec } A$ is of finite type.

Reduced scheme

Definition 12.0.25. A scheme X is **reduced** if there exists an affine open cover $\bigcup_{\alpha} U_{\alpha} = X$ of X such that $\mathcal{O}_X(U_{\alpha})$ is reduced, i.e. has no nilpotent elements for all α , or equivalently that for all $x \in X$ we have that $\mathcal{O}_{X,x}$ is reduced (i.e. has no nilpotent elements).

Remark 12.0.26. One can change the above definition to wanting any open $U \subset X$ to be such that $\mathcal{O}_X(U)$ is reduced.

If X is a (pre-)variety over an algebraically closed field k then $X = \bigcup_{i=1}^n U_i$ is a union of affine varieties $\rightsquigarrow X_{\text{Sch}} := \bigcup_{i=1}^n \text{Spec}(A(U_i))$ where $A(U_i)$ is a reduced, finitely generated k -algebra (cf. [Gat21, Prop. 12.39]). So X_{Sch} is a reduced scheme of finite type over $\text{Spec } k = \{(0)\}$.

We have

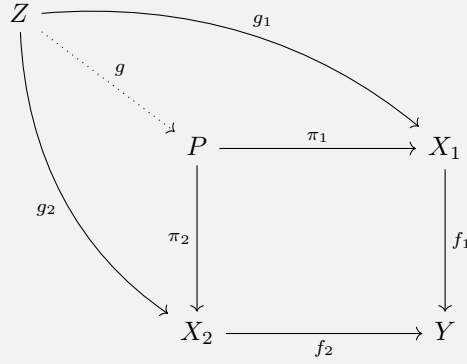
$$\left\{ \begin{array}{c} \text{pre-varieties} \\ \text{over } k \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{reduced } k\text{-schemes} \\ \text{of finite type over } k \end{array} \right\}$$

Fiber product

Definition 12.0.27. Let $f_1 : X_1 \rightarrow Y$ and $f_2 : X_2 \rightarrow Y$ be schemes over Y . A **fiber product** of X_1 and X_2 is a scheme P together with morphisms $\pi_1 : P \rightarrow X_1$ and $\pi_2 : P \rightarrow X_2$ such that the square in the diagram below commutes and such that the following universal property holds: For any two morphisms $g_1 : Z \rightarrow X_1$ and $g_2 : Z \rightarrow X_2$ from another scheme Z that commute with f_1 and f_2 , i.e. such that

$$f_1 \circ g_1 = f_2 \circ g_2,$$

there is a unique morphism $g : Z \rightarrow P$ that makes the complete diagram below commutative.



Example 12.0.28.

$$\begin{array}{ccccc} \mathrm{Spec} B \otimes_A C & \cong & \mathrm{Spec} B \times_{\mathrm{Spec} A} \mathrm{Spec} C & \longrightarrow & \mathrm{Spec} C \\ & & \downarrow & & \downarrow \\ & & \mathrm{Spec} B & \longrightarrow & \mathrm{Spec} A \end{array}$$

Example 12.0.29. Given $X \rightarrow S \rightsquigarrow X \xrightarrow{\Delta_X} X \times_S X$ with $\Delta_X = (\mathrm{id}_X, \mathrm{id}_X)$ with setting $g_1 = g_2 = \mathrm{id}_X$ in 12.0.27.

Example 12.0.30. Given closed subschemes $Z_1 \hookrightarrow X, Z_2 \hookrightarrow X \rightsquigarrow Z_1 \cap Z_2 = Z_1 \times_X Z_2$. The same holds for open subschemes.

Separated scheme

Definition 12.0.31. A scheme X is **separated** if $X \xrightarrow{\Delta_X} X \times_S X$ is a closed subscheme.

Remark 12.0.32. Δ_X is always injective. We have that X is separated $\Leftrightarrow \Delta_X(X)$ is a closed subscheme.

Proposition 12.0.33 ([Gat21, Prop. 12.39]). *For an algebraically closed field k , there is a bijection (equivalence)*

$$\{\text{varieties over } k\} \xleftrightarrow{1-1} \{\text{separated, reduced schemes of finite type over } k\},$$

sending

$$X \longmapsto X_{\text{sch}},$$

and morphisms of varieties correspond exactly to morphisms of the associated schemes over k .

Remark 12.0.34 ([Gat21, cf. Remark 12.40]). In light of the above proposition, we should perhaps (not as far as I can tell covered in lecture) then say that from now on, we will identify varieties X with their associated scheme X_{Sch} . Therefore:

- From now on, a variety will always be a separated, reduced scheme of finite type over an algebraically closed field k .
- Morphisms of varieties are always morphisms over k .

Chapter 13

Lecture 13

13.0.1 Motivation

Let R be a ring and let M be an R -module. This gives:

- (1) *Localization* at an element f ($f \in R$)

$$\rightsquigarrow M_f \cong M \otimes_R R_f \quad (R_f\text{-module}).$$

- (2)

$$M_{\mathfrak{p}} \cong M \otimes_R R_{\mathfrak{p}} \quad (R_{\mathfrak{p}}\text{-module}).$$

- (3)

$$\begin{aligned} M|_{\mathfrak{p}} &:= M \otimes_R k(\mathfrak{p}) \\ &= M_{\mathfrak{p}} / \mathfrak{p}M_{\mathfrak{p}} \quad (k(\mathfrak{p})\text{-vector space}). \end{aligned}$$

- (4) $M \oplus N, M \otimes N$ etc ...

Today: Let X be a *scheme* (locally ringed space; or more generally a topological space), let \mathcal{F} be a sheaf of \mathcal{O}_X -modules.

- (1) $\mathcal{F}(U)$ will be an $\mathcal{O}_X(U)$ -module.
- (2) $\mathcal{F}_{\mathfrak{p}}$ will be an $\mathcal{O}_{X,\mathfrak{p}}$ -module ($\mathcal{F}_{\mathfrak{p}}$ being the stalk of \mathcal{F} at $\mathfrak{p} \in X$).
- (3) $\mathcal{F}|_{\mathfrak{p}}$ will be a $k(\mathfrak{p})$ -vector space (fiber) .

Example 13.0.1. Let $\mathcal{F} = T_X$ (where T_X is to be defined).

$$\begin{aligned} \stackrel{(3)}{\rightsquigarrow} \mathcal{F}|_{\mathfrak{p}} &= T_X(\mathfrak{p}) \\ &\cong (\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2)^{\vee}, \end{aligned}$$

where $\mathfrak{m}_{\mathfrak{p}}$ is the maximal ideal in the local ring $R_{\mathfrak{p}}$.

$s \in T_X(U) \approx s_x \in T_x X$ tangent vector at x for all $x \in U$ but “that varies continuously” (not ok with nilpotents).

We give a schematic picture below.

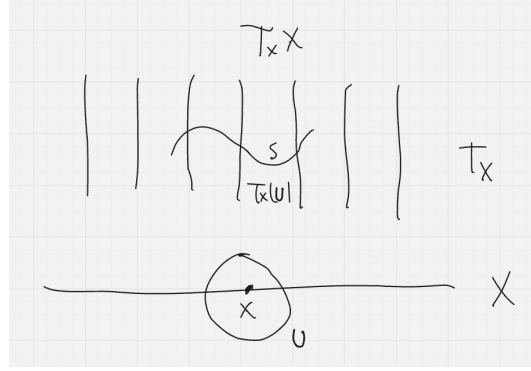


Figure 13.1: Schematic picture

Remark 13.0.2. Below, we formulate the definition of a presheaf of \mathcal{O}_X -modules slightly differently than in class, following [Gat21, Def. 13.2].

13.0.2 Sheaves of modules

Presheaf of \mathcal{O}_X -modules

Definition 13.0.3. Let X be a scheme (12.0.19). A **presheaf of \mathcal{O}_X -modules** is a (pre-)sheaf \mathcal{F} in the sense of (3.0.1) such that the following holds:

1. $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -module for all open $U \subset X$.
2. For all $U \subset V \subset X$ open, there are restriction maps $\rho_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ that are \mathcal{O}_X -module homomorphisms in the sense that

•

$$\begin{aligned} \rho_{V,U}(\varphi + \psi) &= (\varphi + \psi)|_U \\ &= \varphi|_U + \psi|_U \\ &= \rho_{V,U}(\varphi) + \rho_{V,U}(\psi), \end{aligned}$$

•

$$\begin{aligned} \rho_{V,U}(\lambda \cdot \varphi) &= (\lambda \cdot \varphi)|_U \\ &= \lambda|_U \cdot \varphi|_U \\ &= \lambda|_U \cdot \rho_{V,U}(\varphi), \end{aligned}$$

for $\varphi, \psi \in \mathcal{F}(V)$ and $\lambda \in \mathcal{O}_X(V)$.

One may formulate this second requirement as needing the following diagram to commute:

$$\begin{array}{ccccc} (\lambda, s) & \xrightarrow{\in} & \mathcal{O}_X(V) \times \mathcal{F}(V) & \xrightarrow{\text{scalar mult.}} & \mathcal{F}(V) \\ & & \downarrow (|_U, |_U) & & \downarrow |_U \\ & & \mathcal{O}_X(U) \times \mathcal{F}(U) & \xrightarrow{\text{scalar multi.}} & \mathcal{F}(U) \end{array}$$

Remark 13.0.4. Let \mathcal{F} be a sheaf on a scheme X . Then the \mathcal{O}_X -module structure makes each stalk $\mathcal{F}_{\mathfrak{p}}$ for points $\mathfrak{p} \in X$ into an $\mathcal{O}_{X,\mathfrak{p}}$ -module, with **fiber**

$$\begin{aligned}\mathcal{F}|_{\mathfrak{p}} &:= \mathcal{F}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}\mathcal{F}_{\mathfrak{p}} \\ &= \mathcal{F}_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} k(\mathfrak{p}),\end{aligned}$$

which is a $k(\mathfrak{p})$ -vector space (it has a natural $k(\mathfrak{p})$ -module structure, but $k(\mathfrak{p})$ is a field, hence this is indeed a $k(\mathfrak{p})$ -vector space).

To see this more generally first, before going back to the above case: Consider the short exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ where $I \trianglelefteq R$ is an ideal of a commutative ring R with unity. Upon tensoring with $-\otimes_R M$ we obtain the exact sequence $I \otimes_R M \rightarrow M \rightarrow (R/I) \otimes_R M$ but the image of the leftmost map is IM and so by exactness and the first isomorphism theorem we have that $M/IM \cong (R/I) \otimes_R M$. Going back to our specific case: If we let $M = \mathcal{F}_{\mathfrak{p}}$ and $R = \mathcal{O}_{X,\mathfrak{p}}$ then we see that

$$\begin{aligned}\mathcal{F}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}\mathcal{F}_{\mathfrak{p}} &\cong (\mathcal{O}_{X,\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}) \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \mathcal{F}_{\mathfrak{p}} \\ &\simeq \mathcal{F}_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} (\mathcal{O}_{X,\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}) \\ &= \mathcal{F}_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} (\mathcal{O}_{X,\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{X,\mathfrak{p}}) \\ &\cong \mathcal{F}_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} k(\mathfrak{p}).\end{aligned}$$

Definition 13.0.5. \mathcal{F} is a **sheaf of \mathcal{O}_X -modules** if the usual *gluing* conditions hold.

Example 13.0.6.

Sheaf \mathcal{F}	$\mathcal{F}(U)$	$\mathcal{F}_{\mathfrak{p}}$
\mathcal{O}_X	$\mathcal{O}_X(U)$	$\mathcal{O}_{X,\mathfrak{p}}$
$\mathcal{O}_X^{\oplus d}$	$\mathcal{O}_X^{\oplus d}(U)$	$\mathcal{O}_{X,\mathfrak{p}}^{\oplus d}$
$\mathcal{F} \oplus \mathcal{G}$	$\mathcal{F}(U) \oplus \mathcal{G}(U)$	$\mathcal{F}_{\mathfrak{p}} \oplus \mathcal{G}_{\mathfrak{p}}$
$\bigoplus_{\alpha} \mathcal{F}_{\alpha}$	Somewhat complicated	$\bigoplus_{\alpha} (\mathcal{F}_{\alpha})_{\mathfrak{p}}$
$\prod_{\alpha} \mathcal{F}_{\alpha}$	$\prod_{\alpha} \mathcal{F}_{\alpha}(U)$	Complicated
$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$	Somewhat complicated	$\mathcal{F}_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \mathcal{G}_{\mathfrak{p}}$
$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$	$\text{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$	Sometimes complicated

Direct image/Push-forward of sheaves

Definition 13.0.7. Let $f : X \rightarrow Y$ be a morphism of schemes (12.0.1) and let \mathcal{F} be a sheaf on X . For all open subsets $U \subset Y$ we define

$$(f_*\mathcal{F})(U) := \mathcal{F}(f^{-1}(U)). \quad (13.0.1)$$

Remark 13.0.8. One checks that the above construction is a sheaf on Y ; in fact it is a sheaf of \mathcal{O}_Y -modules by setting $\lambda \cdot \varphi := f_U^* \lambda \cdot \varphi$ for $\lambda \in \mathcal{O}_Y(U)$ and $\varphi \in \mathcal{F}(f^{-1}(U))$. We just recall that $f_U^* : \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$ is in the data given for a locally ringed space, in our case scheme, X .

Exc: Restriction maps $f_*\mathcal{F}$ is a sheaf.

Example 13.0.9. Let X be a scheme or topological space, and let $\mathfrak{p} \in X$ be a closed point. Then $i : \text{Spec}(k(\mathfrak{p})) \rightarrow X$ closed subscheme \rightsquigarrow a sheaf of modules on $\mathfrak{p} \leftrightarrow \mathcal{F}(\mathfrak{p}) := V$ a $k(\mathfrak{p})$ -vector space ($\mathcal{F}(\emptyset) = 0$).

A sheaf of modules generalizes vector spaces. $i_*\mathcal{F}$ a sheaf of \mathcal{O}_X -modules, with (for $U \subset X$ open)

$$\begin{aligned} (i_*\mathcal{F})(U) &= \mathcal{F}(i^{-1}(U)) \\ &= \begin{cases} V, & \text{if } x \in U \text{ as an } \mathcal{O}_X(U)\text{-module} \\ 0, & \text{if } x \notin U \text{ as an } \mathcal{O}_X(U)\text{-module} \end{cases}, \end{aligned}$$

$$(i_*\mathcal{F})_y = \begin{cases} V, & \text{if } y = x \text{ as an } \mathcal{O}_{X,x}\text{-module} \\ 0, & \text{if } y \neq x \text{ as an } \mathcal{O}_{X,x}\text{-module} \end{cases},$$

and

$$(i_*\mathcal{F})|_y = \begin{cases} V, & \text{as } k(x)\text{-vector space, if } y = x \\ 0, & \text{as } k(x)\text{-vector space, if } y \neq x \end{cases}.$$

We have the supporting picture:

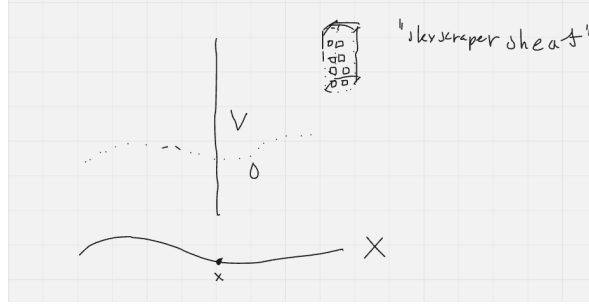


Figure 13.2: Skyscraper sheaf

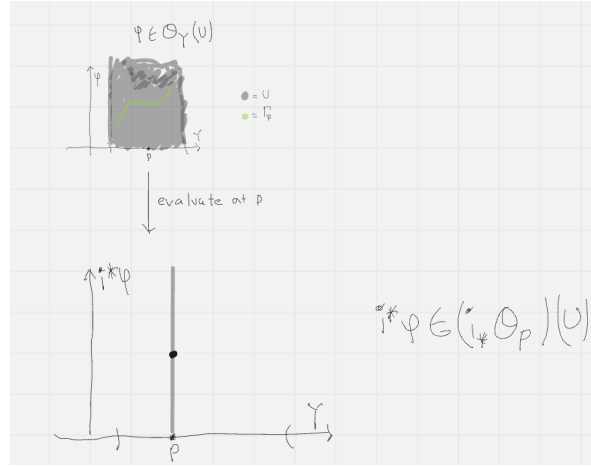
Below, we give a (perhaps) simplified version of the former example, taken from [Gat21, Example 3.10.(b)], not given in class.

Example 13.0.10. Let P be a point on a variety Y (with base field k), and let $i : P \hookrightarrow Y$ be the inclusion. Then we have that $\mathcal{O}_P(P) = k$ and $\mathcal{O}_P(\emptyset) = \{0\}$.

Hence the sheaf $i_*\mathcal{O}_P$ is given by

$$\begin{aligned} (i_*\mathcal{O}_P)(U) &= \mathcal{O}_P(i^{-1}(U)) \\ &= \begin{cases} k, & \text{if } P \in U, \\ 0, & \text{if } P \notin U \end{cases} \quad \text{for all open subsets } U \subset Y. \end{aligned}$$

The morphism $\mathcal{O}_Y \rightarrow i_*\mathcal{O}_P$ given explicitly by $\varphi \mapsto i^*\varphi$ is by evaluating a regular function φ on $U \subset Y$ at the point P (if it lies in U), as shown below:

Figure 13.3: Evaluation at P

Sections of $i_*\mathcal{O}_P$ can thus be interpreted as “functions on Y that only have a value at P ”. The sheaf $i_*\mathcal{O}_P$ is therefore usually denoted k_P (“The field k concentrated at the point P ”) and called the **skyscraper sheaf on Y at P** (because of the shape of the shaded region above in which the graph of the function lies).

13.0.3 Twisting sheaf (on \mathbb{P}^n)

- Projective varieties/schemes. We use the *twisting sheaf* to get more sections.

Twisting sheaves on \mathbb{P}^n

Definition 13.0.11. Let $n \in \mathbb{N}_{>0}$ and $d \in \mathbb{Z}$. For a non-empty open subset $U \subset \mathbb{P}^n$ we define

$$(\mathcal{O}_{\mathbb{P}^n}(d))(U) := \left\{ \frac{g}{f} : f \in k[x_0, \dots, x_n]_e \text{ and } g \in k[x_0, \dots, x_n]_{e+d} \text{ for some } e \in \mathbb{Z}, \right. \\ \left. \text{such that } f(P) \neq 0, \text{ for all } P \in U \right\},$$

as a subset of the quotient field (i.e. $\text{Frac}(k[x_0, \dots, x_n])$) of $k[x_0, \dots, x_n]$, and

$$(\mathcal{O}_{\mathbb{P}^n}(d))(\emptyset) := \{0\}.$$

The sheaves $\mathcal{O}_{\mathbb{P}^n}(d)$ are called **twisting sheaves on \mathbb{P}^n** .

Remark 13.0.12. We see that

$$\begin{aligned} (\mathcal{O}_{\mathbb{P}^n}(d))(\mathbb{P}^n) &= \left\{ \frac{g}{f} : f(x) \neq 0, \forall x \in \mathbb{P}^n \right\} \\ &= k[x_0, \dots, x_n]_d \end{aligned}$$

where the last equality follows from the fact that any non-constant homogeneous (of degree e) polynomial $f \in k[x_0, \dots, x_n]_e$ must have some zero. But this forces $e = 0$ and so $f \in k^* = k \setminus \{0\}$.

In particular, notice that $\mathcal{O}_{\mathbb{P}^n}(0)$ is the conventional sheaf of rings on \mathbb{P}^n .

Remark 13.0.13. For any $\ell \in \mathbb{Z}$ there is a multiplication map

$$(\mathcal{O}_{\mathbb{P}^n}(d))(U) \times (\mathcal{O}_{\mathbb{P}^n}(\ell))(U) \rightarrow (\mathcal{O}_{\mathbb{P}^n}(d+\ell))(U)$$

defined explicitly as $(\varphi, \psi) \mapsto \varphi\psi$, i.e. $\frac{g}{f} \cdot \frac{g'}{f'} = \frac{gg'}{ff'}$ where

$$gg' \in k[x_0, \dots, x_n]_{d+\ell}.$$

To see this note that $g \in k[x_0, \dots, x_n]_{d+e}$ for some $e \in \mathbb{Z}$ and $g' \in k[x_0, \dots, x_n]_{\ell+e'}$ for some $e' \in \mathbb{Z}$ so that $gg' \in k[x_0, \dots, x_n]_{(d+\ell)+(e+e')} = k[x_0, \dots, x_n]_{(d+\ell)+e}$ for some $e := (e + e') \in \mathbb{Z}$, where also since $f, f' \neq 0$ on U it holds that $ff' \neq 0$ on U and $ff' \in k[x_0, \dots, x_n]_e$. But since by commutativity, shows that we can give $\mathcal{O}_{\mathbb{P}^n}(d)$ a $\mathcal{O}_{\mathbb{P}^n}(0)$ -module structure, i.e. $\mathcal{O}_{\mathbb{P}^n}(d)$ is a sheaf on \mathbb{P}^n .

Remark 13.0.14. Below, we expound (we think) on an example given in class, using [Gat21, Example 13.5]

Example 13.0.15. Let $n \in \mathbb{N}_{>0}$ and let $d \in \mathbb{Z}$.

- (a) Let $\varphi = \frac{g}{f}$ be a global section in $(\mathcal{O}_{\mathbb{P}^n}(d))(\mathbb{P}^n)$. Then it follows that $V(f) = \emptyset$. Since any non-constant polynomial has a zero, it follows that f is constant (i.e. this forces $e = 0$ in 13.0.11), so that $f \in k^\times$. Therefore $\frac{g}{f} \in k[x_0, \dots, x_n]_d$. Hence we can exhibit an isomorphism between $k[x_0, \dots, x_n]_d$ and $(\mathcal{O}_{\mathbb{P}^n}(d))(\mathbb{P}^n)$. Then, by convention (we believe) it follows that $(\mathcal{O}_{\mathbb{P}^n}(d))(\mathbb{P}^n) = \{0\}$.
- (b) Let $U_0 := \{(x_0 : x_1) \in \mathbb{P}^1 : x_0 \neq 0\} \subset \mathbb{P}^1$. Then with $f = x_0 \in k[x_0, \dots, x_n]_1$ and $g = 1 \in k[x_0, \dots, x_n]_{-1+1=0}$ we see that $\frac{1}{x_0} \in (\mathcal{O}_{\mathbb{P}^n}(-1))(U_0)$.
- (c) Let $f \in k[x_0, \dots, x_n]_e$. We then claim that we have a *morphism of sheaves* (to be defined; see 13.0.16)

$$\mathcal{O}_{\mathbb{P}^n}(d) \rightarrow \mathcal{O}_{\mathbb{P}^n}(d+e), \quad \varphi \mapsto f\varphi.$$

- (d) For $d \neq 0$, the sheaf $\mathcal{O}_{\mathbb{P}^n}(d)$ and the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$ are not isomorphic on \mathbb{P}^n since their spaces of global sections are $(\mathcal{O}_{\mathbb{P}^n}(d))(\mathbb{P}^n) = k[x_0, \dots, x_n]_d$ and $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$ (the latter since the denominator is forced to be a constant, and then the numerator is forced to be of the same degree as the constant, hence constant). However, we claim that they become isomorphic *locally*. Let $U_i = \{(x_0 : \dots : x_n) \in \mathbb{P}^n : x_i \neq 0\}$ for $i = 0, \dots, n$. Then let

$$f : \mathcal{O}_{\mathbb{P}^n}|_{U_i} \rightarrow \mathcal{O}_{\mathbb{P}^n}(d), \quad \varphi \mapsto x_i^d \varphi \quad \text{with inverse} \quad f^{-1} : \mathcal{O}_{\mathbb{P}^n}(d)|_{U_i} \rightarrow \mathcal{O}_{\mathbb{P}^n}|_{U_i}, \quad \varphi \mapsto \frac{\varphi}{x_i^d}.$$

13.0.4 Morphism of sheaves

Morphism of sheaves $f : \mathcal{F} \rightarrow \mathcal{G}$

Definition 13.0.16. A **morphism** $f : \mathcal{F} \rightarrow \mathcal{G}$ of (pre-)sheaves \mathcal{F}, \mathcal{G} of modules on X is given by the data of $\mathcal{O}_X(U)$ -module homomorphisms $f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for all open subsets $U \subset X$ that are compatible with restrictions, i.e. such that $f_V(\varphi)|_U = f_U(\varphi|_U)$ for all open subsets $U \subset V$ and $\varphi \in \mathcal{F}(V)$.

Remark 13.0.17. We gave an example of a morphism of sheaves in example 13.0.15.(c).

Kernel sheaf

Definition 13.0.18. Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on a scheme X . For any open subset $U \subset X$, we set

$$\ker(f)(U) := \ker(f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)).$$

We show below that $\ker(f)$ is in fact a sheaf, and we call $\ker(f)$ the **kernel sheaf** of f .

We will prove that $\ker(f)$ for $f : \mathcal{F} \rightarrow \mathcal{G}$ a morphism of sheaves on a scheme X is in fact a sheaf:

Proof.

Presheaf:

- Notice that by definition 13.0.16 we have that f_U is an $\mathcal{O}_X(U)$ -module homomorphism, so that by the 1st isomorphism theorem for rings, we have that for an open subset $U \subset X$ of a scheme X , $\ker(f)(U)$ is a $\mathcal{O}_X(U)$ -submodule of $\mathcal{F}(U)$.
- Let $V \subset U \subset X$ be open sets. Then from the structure on \mathcal{F} as a sheaf we get a restriction-map $\rho_{U,V}^{\mathcal{F}} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$. We claim that this map gives us a restriction map $\rho' : \ker(f)(U) \rightarrow \ker(f)(V)$ induced by the restriction map $\rho_{U,V}$. Recall that from definition 13.0.16 we have that for $V \subset U \subset X$ open, it holds that $f_U(\varphi)|_V = f_V(\varphi|_V)$ for $\varphi \in \mathcal{F}(U)$. Let $\varphi \in \ker(f)(U)$, i.e. such that $f_U(\varphi) = 0$, then

$$\begin{aligned} f_V(\varphi|_V) &= f_U(\varphi)|_V \\ &= 0|_V \\ &= \rho_{U,V}^{\mathcal{G}}(0) \\ &= 0, \quad (\text{since } \rho_{U,V}^{\mathcal{G}} \text{ is a homomorphism}) \end{aligned}$$

so that $\varphi|_V$ is in $\ker(f)(V)$. This first shows that ρ' indeed does what we claimed. It remains to show that ρ' is a homomorphism, but this follows directly from the fact that $\rho' = \rho_{U,V}|_{\ker(f)(U)}$ is the restriction of a homomorphism (to be fully rigorous we should perhaps also *corestrict* the codomain to $\ker(f)(V)$).

- We have that

$$\begin{aligned} \ker(f)(\emptyset) &= \ker(f_{\emptyset} : \mathcal{F}(\emptyset) \rightarrow \mathcal{G}(\emptyset)) \\ &= \ker(f_{\emptyset} : 0 \rightarrow 0) \\ &= 0, \end{aligned}$$

where we used that $\mathcal{F}(\emptyset), \mathcal{G}(\emptyset) = 0$ and that f_{\emptyset} takes 0 to 0.

- We have

$$\begin{aligned} \rho'_{U,U} &:= \rho_{U,U}|_{\ker(f)(U)} \\ &= \text{id}_U|_{\ker(f)(U)} \\ &= \text{id}_{\ker(f)(U)}, \end{aligned}$$

for arbitrary open $U \subset X$.

- Lastly, we want to show that for $V \subset U \subset W$ open, we have that $\rho'_{U,V} \circ \rho'_{W,U} = \rho'_{W,V}$. But

$$\begin{aligned} \rho'_{U,V} \circ \rho'_{W,U} &= \rho_{U,V}|_{\ker(f)(U)} \circ \rho_{W,U}|_{\ker(f)(W)} \\ &= \rho_{W,V}|_{\ker(f)(W)} \\ &= \rho'_{W,V}, \end{aligned}$$

Sheaf: Let $\{U_i : i \in I\}$ be an open cover of an open subset $U \subset X$, and assume that we are given sections $\varphi_i \in \ker(f)(U_i) \subset \mathcal{F}(U_i)$ that agree on overlaps, i.e. for all $i, j \in I$ we have that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. Then since \mathcal{F} is a sheaf, they glue to a unique section $\varphi \in \mathcal{F}(U)$. It

remains to show that $\varphi \in \ker(f)(U)$, i.e. that $f_U(\varphi) = 0 \in \mathcal{G}(U)$. But consider that $\mathcal{G}(U)$ is a sheaf as well. We see that

$$\begin{aligned} 0|_{U_i} &= f_U(\varphi)|_{U_i} \\ &= f_{U_i}(\varphi|_{U_i}), \quad (\text{by definition 13.0.16}) \\ &= f_{U_i}(\varphi_i) \end{aligned}$$

and so if $s := f_U(\varphi_i) \in \mathcal{G}(U)$ is such that $s|_{U_i} := f_{U_i}(\varphi_i)$ is zero on each U_i and it is clear that the $s|_{U_i}, s|_{U_j}$ for $i, j \in I$ agree on overlaps (they are just zero). By *uniqueness* (since $0|_{U_i} = s|_{U_i} = 0$ for each $i \in I$) it follows that $f_U(\varphi) = 0$ and so $\varphi \in \ker(f)(U)$. □

13.0.5 Image (pre-)sheaf, cokernel (pre-)sheaf, tensor (pre-)sheaf, etc.

Image presheaf $\text{Im}'(f)$

Definition 13.0.19. For a morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves on a scheme X and any open subset $U \subset X$ we set

$$(\text{Im}'(f))(U) := \text{Im}(f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)).$$

We call this construction the **image presheaf** of f .

Remark 13.0.20. Notice that the image of an \mathcal{O}_X -module is a *submodule* of the codomain (1st isomorphism theorem for modules). Furthermore, let $V \subset U$ be open for some open subset U of a scheme X . Then we want to show that there is a restriction map $\rho_{U,V}^{\text{Im}'(f)} : \text{Im}'(f)(U) \rightarrow \text{Im}'(f)(V)$. If $s \in \text{Im}'(f)(U)$ then $s = f_U(t)$ for some $t \in \mathcal{F}(U)$. By the sheaf structure on \mathcal{G} we then have that there is a given $\rho_{U,V}^{\mathcal{G}} : \mathcal{G}(U) \rightarrow \mathcal{G}(V)$. We then see that

$$\begin{aligned} \rho_{U,V}^{\text{Im}'(f)}(s) &= \rho_{U,V}^{\mathcal{G}}(f_U(t)) \\ &= f_U(t)|_V \\ &= f_V(t|_V) \in \text{Im}'(f)(V). \end{aligned}$$

Cokernel presheaf $\text{coker}'(f)$

Definition 13.0.21. For a morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves on a scheme X and open subset $U \subset X$ we set

$$\text{coker}'(f)(U) := \text{coker}(f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)).$$

We call this the **cokernel presheaf** of f .

Tensor presheaf $\mathcal{F} \otimes' \mathcal{G}$

Definition 13.0.22. Let \mathcal{F} and \mathcal{G} be sheaves of \mathcal{O}_X -modules on a scheme X and let $U \subset X$ be an open subset. Then we set

$$(\mathcal{F} \otimes' \mathcal{G})(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

We call this the **tensor presheaf** of f .

Direct product presheaf $(\bigoplus'_\alpha \mathcal{F}_\alpha)$

Definition 13.0.23. Let \mathcal{F}_α be a presheaf for all α in some index set. Then we define

$$\left(\bigoplus'_\alpha \mathcal{F}_\alpha\right)(U) := \bigoplus_\alpha \mathcal{F}_\alpha(U)$$

The above constructions (except from the kernel sheaf) are in general *not sheaves*. This leads to our next topic: *sheafification*.

13.0.6 Sheafification

If \mathcal{F} is a presheaf, then the sheafification produces something that is sometimes denoted as $\mathcal{F}^+ = a\mathcal{F}, \mathcal{F}^\sharp, \mathcal{F}^{\text{sh}}, L\mathcal{F}, \widehat{\mathcal{F}}, \dots$. David used \mathcal{F}^+ in class, but we will instead follow the notation in [Gat21, §13] with \mathcal{F} for the *sheafification* of a presheaf \mathcal{F}' .

Sheafification

Definition 13.0.24. Let \mathcal{F}' be a presheaf on a scheme X . For an open subset $U \subset X$, we set

$$\mathcal{F}(U) := \left\{ \begin{array}{l} \varphi = (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in U} : \varphi_{\mathfrak{p}} \in \mathcal{F}'_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in U, \text{ and} \\ \text{for all } \mathfrak{p} \text{ there is an open neighborhood } U_{\mathfrak{p}} \text{ of } \mathfrak{p} \text{ contained in } U, \\ \text{and a section } s \in \mathcal{F}'(U_{\mathfrak{p}}) \text{ with } \varphi_{\mathfrak{q}} = s_{\mathfrak{q}} \text{ for all } \mathfrak{q} \in U_{\mathfrak{p}}. \end{array} \right\},$$

where $s_{\mathfrak{q}} \in \mathcal{F}'_{\mathfrak{q}}$ denotes the *germ* of s in \mathfrak{q} (one may write the last condition as $\varphi = s$ on $U_{\mathfrak{q}}$). This is a local construction, hence we claim that this is in fact a sheaf (exercise), and we call \mathcal{F} the **sheafification** of \mathcal{F}' , or **sheaf associated** to \mathcal{F}' .

Exercise: The sheafification \mathcal{F} of a presheaf \mathcal{F}' on a scheme X is in fact a sheaf.

Proof.

□

Remark 13.0.25 ([Gat21, Remark 13.16]). Any presheaf \mathcal{F}' on a scheme X admits a natural morphism to its sheafification \mathcal{F} defined by

$$\theta : \mathcal{F}'(U) \rightarrow \mathcal{F}(U), \quad s \mapsto (s_{\mathfrak{p}})_{\mathfrak{p} \in U}$$

for all open subsets $U \subset X$.

Lemma 13.0.26. [Gat21, Prop. 13.17] Let \mathcal{F}' be a presheaf on a scheme X , and let \mathcal{F} be its sheafification, together with natural morphism $\theta : \mathcal{F}' \rightarrow \mathcal{F}$ (from 13.0.25). Then

- (a) For all $\mathfrak{p} \in X$ the morphism θ induces an isomorphism $\theta_{\mathfrak{p}} : \mathcal{F}'_{\mathfrak{p}} \cong \mathcal{F}_{\mathfrak{p}}$.
- (b) If \mathcal{F}' is already a sheaf, then $\theta : \mathcal{F}' \rightarrow \mathcal{F}$ is an isomorphism.

Proof.

(a): $\theta : \mathcal{F}' \rightarrow \mathcal{F}$ induces an $\mathcal{O}_{X,\mathfrak{p}}$ -module homomorphism of stalks $\mathcal{F}'_{\mathfrak{p}} \rightarrow \mathcal{F}_{\mathfrak{p}}$: Let $(U, s) \sim (V, t)$ in $\mathcal{F}_{\mathfrak{p}}$. Then there is some open subset $W \ni \mathfrak{p}$ of X such that $W \subset U \cap V$ and $s|_W = t|_W$. Then we have that

$$\begin{aligned}
\theta_U(s)|_W &= \theta_W(s|_W) \\
&= \theta_W(t|_W) \\
&= \theta_V(t)|_W,
\end{aligned}$$

so that $[\theta_U(s)] = [\theta_V(t)]$ in $\mathcal{F}_{\mathfrak{p}}$ (notice how we used that θ_W is well-defined in the 2nd equality); that is $\theta_{\mathfrak{p}}$ is defined by $\overline{(U, \varphi)} \mapsto \overline{(U, \theta_U(\varphi))}$.

Conversely, there is a natural $\mathcal{O}_{X, \mathfrak{p}}$ -module homomorphism $\mu_{\mathfrak{p}} : \mathcal{F}_{\mathfrak{p}} \rightarrow \mathcal{F}'_{\mathfrak{p}}$ defined by $\overline{(U, \varphi)} \mapsto \varphi_{\mathfrak{p}}$ that takes $\varphi = (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in U} \in \mathcal{F}(U)$ to its element at \mathfrak{p} . These two maps are inverses to each other: Let $\overline{(U, \varphi)} \in \mathcal{F}'_{\mathfrak{p}}$. Then by definition $\varphi = (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in U}$. Thus

$$\begin{aligned}
(\mu_{\mathfrak{p}} \circ \theta_{\mathfrak{p}})(\varphi_{\mathfrak{p}}) &= \mu_{\mathfrak{p}}(\overline{(U, \theta_U(\varphi))}) \\
&= \mu_{\mathfrak{p}}(\overline{(U, (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in U})}) \\
&= \varphi_{\mathfrak{p}}.
\end{aligned}$$

(b):

Lemma 13.0.27 ([Gat21, Exercise 13.8]). *A morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves on a scheme X is an isomorphism \Leftrightarrow the induced map $f_{\mathfrak{p}} : \mathcal{F}_{\mathfrak{p}} \rightarrow \mathcal{G}_{\mathfrak{p}}$ is an isomorphism for all points \mathfrak{p} in X .*

If \mathcal{F}' is already a sheaf then θ is a morphism of sheaves (we claim). By (a) it induces an isomorphism $\mathcal{F}'_{\mathfrak{p}} \cong \mathcal{F}_{\mathfrak{p}}$ for all $\mathfrak{p} \in X$ and so by lemma 13.0.27 it is an isomorphism. \square

Proposition 13.0.28 (Universal property of sheafification [Gat21, Exercise 13.18]). *Let \mathcal{F}' be a presheaf on a scheme X and denote as in the remark above by $\theta : \mathcal{F}' \rightarrow \mathcal{F}$ the natural morphism to its sheafification \mathcal{F} . Then we want to claim that for any morphism $f' : \mathcal{F}' \rightarrow \mathcal{G}$ with \mathcal{G} a sheaf, this morphism factors uniquely through θ , i.e. that there is a unique morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ with $f' = f \circ \theta$, or diagrammatically*

$$\begin{array}{ccc}
\mathcal{F}' & \xrightarrow{f'} & \mathcal{G} \\
& \searrow \theta & \nearrow \exists! f \\
& \mathcal{F} &
\end{array}$$

With 13.0.24 in place, we give the following definitions (for clarity, now with \mathcal{F}^+ for sheafification).

Some sheafifications

Definition 13.0.29.

$$\begin{aligned}
\text{Im}(f) &:= (\text{Im}'(f))^+, \\
\text{coker}(f) &:= (\text{coker}'(f))^+, \\
\mathcal{F} \otimes \mathcal{G} &:= (\mathcal{F} \otimes' \mathcal{G})^+, \\
\bigoplus_{\alpha} \mathcal{F}_{\alpha} &:= \left(\bigoplus_{\alpha} \mathcal{F}_{\alpha} \right)^+, \\
\mathcal{F} / \mathcal{G} &:= (\mathcal{F} /' \mathcal{G})^+.
\end{aligned}$$

Injective and surjective morphism of sheaves

Definition 13.0.30. Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on a scheme X . Then

- we say that f is **injective** if $\ker(f) = \{0\}$,
- we say that f is **surjective** if $\mathrm{Im}(f) \cong \mathcal{G}$ (more precisely, the induced map from 13.0.28 is an isomorphism).

Remark 13.0.31. Notice that the induced map (with respect to surjectivity) above comes from the following diagram

$$\begin{array}{ccc} \mathrm{Im}'(f) & \xrightarrow{i'} & \mathcal{G} \\ & \searrow \theta & \nearrow \exists! i \\ & \mathrm{Im}(f) & \end{array} .$$

That is we, need $i : \mathrm{Im}(f) \rightarrow \mathcal{G}$ to be an isomorphism.

Chapter 14

Lecture 14

Remark 14.0.1. First, a notational difference from the lecture notes: We will use \varinjlim for colim.

14.0.1 Pullback

Back to modules: If $\varphi : R \rightarrow R'$ is a ring-homomorphism, and M is an R' module, we may form $M \otimes_R R'$, called **extension of scalars**. On the other hand, if we have an R' -module M' , we may treat it as an R -module M'_φ by $r \cdot m' := \varphi(r) \cdot m'$ for $r \in R$ and $m' \in M'$, called **extension of scalars**.

Let $X' = \text{Spec } R'$ and let $X = \text{Spec } R$ and assume that we have a morphism of affine schemes $f : X' \rightarrow X \rightsquigarrow$ ring homomorphism $f^\sharp : R \rightarrow R'$. Then $\widetilde{M}' \rightsquigarrow f_* \widetilde{M}' = \widetilde{M}'_\varphi$ is the **pushforward** and $\widetilde{M} \rightsquigarrow f^* \widetilde{M} := \widetilde{M \otimes_R R'}$, called the **pullback**.

We have the following:

$$\begin{array}{ccc}
 \mathbf{Mod}_R & \begin{array}{c} \xrightarrow{- \otimes_R R'} \\ \xleftarrow{(-)_\varphi} \end{array} & \mathbf{Mod}_{R'} \\
 \downarrow \widetilde{(-)} & & \downarrow \widetilde{(-)} \\
 \text{QCoh}(X) & \begin{array}{c} \xrightarrow{f^*} \\ \xleftarrow{f_*} \end{array} & \text{QCoh}(X') \\
 \downarrow & & \downarrow \\
 \mathbf{Mod}_{\mathcal{O}_X} & \begin{array}{c} \xrightarrow{f^*} \\ \xleftarrow{f_*} \end{array} & \mathbf{Mod}_{\mathcal{O}_{X'}}
 \end{array} \tag{14.0.1}$$

where $f_* : \text{QCoh}(X) \rightarrow \text{QCoh}(X')$ usually *preserves* quasi-coherence.

14.0.2 Definition of pullback of sheaves of abelian groups

Inverse image functor f^{-1}

Definition 14.0.2. Let $f : X \rightarrow Y$ be a morphism of schemes (or continuous map of topological spaces) and let \mathcal{F} be a sheaf of abelian groups on Y (i.e. $\mathcal{F}(V)$ is an abelian group for open $V \subset Y$). Then $f^{-1}\mathcal{F}$ is the sheaf (obtained by *sheafification*) associated to the presheaf

$$\begin{array}{c} U \longmapsto \text{“}\mathcal{F}(f(U))\text{”} \xlongequal{\quad} \text{“}\varinjlim_{\substack{V \supseteq f(U) \\ V \text{ open}}} \mathcal{F}(V)\text{”} \xlongequal{\quad} \left\{ (V, \varphi) : Y \supseteq V \supseteq f(U) \atop \varphi \in \mathcal{F}(V) \right\} / \sim \\ \downarrow \qquad \qquad \qquad \uparrow \\ X \qquad \qquad \text{Not necessarily open} \end{array},$$

where $(V, \varphi) \sim (V', \varphi')$ if there is some open W contained in $V \cap V'$ and containing $f(U)$ such that $\varphi|_W = \varphi'|_W$.

Remark 14.0.3. $\mathcal{F}_y = \varinjlim_{\substack{V \ni y \\ V \text{ open}}} \mathcal{F}(V)$. If $i : \{y\} \rightarrow Y$ is the inclusion then $i^{-1}\mathcal{F}(\{y\}) = \mathcal{F}_y$.

To say a bit more about this last point, $f : \{y\} \rightarrow Y$ is a continuous map ($\{y\}$ carries the discrete topology). Then notice that $f(y) = *$ is a one point space and so we in fact see that by definition we have that the rightmost definition in the “diagram” above coincides with the stalk \mathcal{F}_y . The claim is that sheafification does not change stalks, so we get that $f^{-1}\mathcal{F}(y) \cong \mathcal{F}_y$.

We may then consider $\{x\} \hookrightarrow X \xrightarrow{f} Y$. It then follows that since $i^{-1}\mathcal{G}(\{x\}) = \mathcal{G}_x$ essentially directly from the definition of a stalk and i^{-1} we have that on applying this to the sheaf $\mathcal{G} := f^{-1}\mathcal{F}$, we get that

$$\begin{aligned} (f^{-1}\mathcal{F})_x &= (i^{-1}f^{-1}\mathcal{F})(\{x\}) \\ &= (f \circ i)^{-1}\mathcal{F}(\{x\}) \\ &= \left(\varinjlim_{\substack{V \ni f(x) \\ V \text{ open}}} \mathcal{F}(V) \right)^+ \\ &\cong \mathcal{F}_{f(x)} \end{aligned}$$

where we in the second equality used the facts:

- $g_* \circ f_* = (g \circ f)_*$.
- $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$.

In the last equality we used that sheafifications do not change stalks together with definition 14.0.2.

Problem: If \mathcal{F} is an \mathcal{O}_Y -module, $f : X \rightarrow Y$ say morphism of schemes, then $f^{-1}\mathcal{F}$ is not an \mathcal{O}_X -module (it is however an $f^{-1}\mathcal{O}_Y$ -module): E.g. $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y, f(x)}$ is not an $\mathcal{O}_{X, x}$ -module: Take the affine case with $\varphi^\# : A \rightarrow B \hookrightarrow \varphi : \text{Spec } B \rightarrow \text{Spec } A$. Then the induced map on stalks from $\varphi^\#$ is

$$\varphi^\# : A_{\varphi^{-1}(\mathfrak{p})} \rightarrow B_{\mathfrak{p}}$$

i.e. with $A_{\varphi^{-1}(\mathfrak{p})} = (f^{-1}\mathcal{O}_Y)_x$ and $\mathcal{O}_{X, x} = B_{\mathfrak{p}}$ (implicitly we might say that we are using the adjunction $f^{-1} \dashv f_*$, i.e. the that the inverse image functor is left-adjoint to the pushforward-functor). But

$A_{\varphi^{-1}(\mathfrak{p})}$ is not forced to have an $B_{\mathfrak{p}}$ -module structure from this (the claim in class seem stronger but we can't see why it could not have in certain cases, e.g. when f is the identity morphism; we believe the point is to elucidate why it need not hold generally).

14.0.3 Definition of a pullback for \mathcal{O}_X -modules

Observations:

- $f^{-1}\mathcal{O}_Y$ is a sheaf of rings.
- $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ is a ring homomorphism (note $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X \rightsquigarrow \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ by $f^{-1} \dashv f_*$) ["pullback of functions"].
- If \mathcal{F} is an \mathcal{O}_Y -module then $f^{-1}\mathcal{F}$ is an $f^{-1}\mathcal{O}_Y$ -module.

Pullback of sheaves; f^*

Definition 14.0.4. $f^*\mathcal{F} := (f^{-1}\mathcal{F}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, (an \mathcal{O}_X -module).

Remark 14.0.5. $(f^*\mathcal{F})_x := \mathcal{F}_x \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$.

We may compare this with $\varphi : A \rightarrow B$ a ring homomorphism (so that both A and B are A -modules) and M an A -module, then $(M \otimes_A B)_{\mathfrak{p}} = M_{\varphi^{-1}(\mathfrak{p})} \otimes_{A_{\varphi^{-1}(\mathfrak{p})}} B_{\mathfrak{p}}$ (use what we have showed earlier together with [DF04, Exercise 15.4.16]).

14.0.4 Properties of the pullback

Remark 14.0.6. David mentions that a lot the equations given here are really canonical isomorphisms.

Fact: $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $f^* \circ g^* = (g \circ f)^*$.

Fact (14.0.4): If $R \xrightarrow{\varphi^\#} R'$ is a ring homomorphism then $\varphi^*\widetilde{M'} = \widetilde{M' \otimes_R R'}$, i.e. top diagram in 14.0.1 commutes.

Remark 14.0.7. In the lecture notes it called our $\varphi^\#$, φ . But we think it is more consistent with the notation to let the ring-homomorphism be denoted by $\varphi^\#$.

Proof sketch: First construct a morphism of sheaves, then show it is an isomorphism on stalks.

Proposition 14.0.8. If \mathcal{F} is a quasi-coherent \mathcal{O}_Y -module then $f^*\mathcal{F}$ is a quasi-coherent \mathcal{O}_X -module.

Proof. Let $Y = \bigcup_i U_i$ be an affine open covering such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ where $U_i = \text{Spec } R_i$ and M_i an R_i -module. Then $f^{-1}(U_i)$ is not necessarily affine but we may cover as $f^{-1}(U_i) = \bigcup_j V_{ij}$ with affine $V_{ij} = \text{Spec } B_{ij}$ (i.e. take $\bigcup_j f^{-1}(U_i) \cap S_j$ where $X = \bigcup_j S_j$ is the affine open cover of X).

$$\begin{array}{ccc}
 \text{Spec } R_{ij} & & \text{Spec } R_i \\
 \simeq \downarrow & & \simeq \downarrow \\
 V_{ij} & \xrightarrow{f_{ij}} & U_i \\
 \downarrow j & & \downarrow i \\
 X & \xrightarrow{f} & Y
 \end{array}$$

Then $(f^*\mathcal{F})|_{V_{ij}} = f_{ij}^*(\mathcal{F}|_{U_i}) = \widetilde{M_i \otimes_{R_i} B_{ij}}$.

Conclusion: $f^*\mathcal{F}|_{V_{ij}}$ is the $\widetilde{(-)}$ of a module, i.e. quasi-coherent, so $f^*\mathcal{F}$ is quasi-coherent. \square

Remark 14.0.9. To say a bit more about the next to last step in the proof: If $i : U_i \hookrightarrow Y$ then $i^{-1}\mathcal{F} = i^*\mathcal{F} = \mathcal{F}|_{U_i}$. Similarly let $j : V_{ij} \rightarrow X$ be the inclusion so that $j^*f^*\mathcal{F} = (f^*\mathcal{F})|_{V_{ij}}$.

Remark 14.0.10. The reason (we claim) being that $i^*\mathcal{F} = i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{O}_Y} \mathcal{O}_{U_i} = \mathcal{F}|_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{O}_{U_i} \cong \mathcal{F}|_{U_i}$.

We might want to argue for why $i^{-1}\mathcal{O}_Y = \mathcal{O}_{U_i}$ whenever $i : U_i \hookrightarrow X$.

Notice that for $V \subset U_i$, we have that $i^{-1}\mathcal{O}_Y(V) = \varinjlim_{\substack{W \supset V \\ W \text{ open in } U_i}} \mathcal{O}_Y(W)$. But $W = V$ is in these limit,

and is in fact the *initial object* with respect to this diagram (follows by considering restriction maps of this colimit) hence $i^{-1}\mathcal{O}_Y(V) = \mathcal{O}_Y(V)$. Since $V \subset U$ was arbitrary, we claim that in fact we have that $i^{-1}\mathcal{O}_Y = \mathcal{O}_{U_i}$ (this argument we see as applying more broadly for any inclusion $W \hookrightarrow X$ of some open subset W of X). In the sequence of equalities below, we will apply this reasoning in the first step to see that $j^*(f^*\mathcal{F}) = f^*\mathcal{F}|_{V_{ij}}$ since one then see by essentially the same reasoning that $j^{-1}f^*\mathcal{F} \cong (f^*\mathcal{F})_{V_{ij}}$

Then $i \circ f_{ij} = f \circ j$ we claim (the reason is just that we define f_{ij} as the corestriction of $f|_{V_{ij}}$ to U_i [which is the same as $f|_{V_{ij}}$ up to codomain; note that $f(V_{ij}) \subset U_i$ since $V_{ij} \subset f^{-1}(U_i)$ so that $f(V_{ij}) \subset f f^{-1}(U_i) \subset U_i$]).

We then have that

$$\begin{aligned} f^*\mathcal{F}|_{V_{ij}} &= j^*f^*\mathcal{F} \\ &= (f \circ j)^*\mathcal{F} \\ &= (i \circ f_{ij})^*\mathcal{F} \\ &= f_{ij}^* \circ i^*\mathcal{F} \\ &= f_{ij}^*(\mathcal{F}|_{U_i}). \end{aligned}$$

Proposition 14.0.11. *If \mathcal{F}, \mathcal{G} are (quasi-coherent) \mathcal{O}_Y -modules then $f^*(\mathcal{F} \oplus \mathcal{G}) = f^*\mathcal{F} \oplus f^*\mathcal{G}$ and $f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$.*

Proof. Pick $Y = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ and $\mathcal{G}|_{U_i} \cong \widetilde{N_i}$. Pick $f^{-1}(U) = \bigcup_j V_{ij}$ affine covering.

Then

$$\begin{aligned} f^*(\mathcal{F} \oplus \mathcal{G})|_{V_{ij}} &= f_{ij}^*(\mathcal{F} \oplus \mathcal{G})|_{U_i} \\ &= f_{ij}^*(\widetilde{M_i \oplus N_i}) \\ &= f_{ij}^*(\widetilde{M_i \oplus N_i}) \\ &= \widetilde{(M_i \oplus N_i) \otimes_{R_i} B_{ij}} \\ &= \widetilde{(M_i \otimes_{R_i} B_{ij}) \oplus (N_i \otimes_{R_i} B_{ij})} \\ &= \widetilde{M_i \otimes_{R_i} B_{ij}} \oplus \widetilde{N_i \otimes_{R_i} B_{ij}} \\ &= f_{ij}^*\widetilde{M_i} \oplus f_{ij}^*\widetilde{N_i}, \quad (\text{by fact 14.0.4}) \\ &= f^*\mathcal{F}|_{V_{ij}} \oplus f_{ij}^*\mathcal{F}|_{V_{ij}} \\ &= (f^*\mathcal{F} \oplus f^*\mathcal{G})|_{V_{ij}}. \end{aligned}$$

Notice that we in the next to last equality used that $\widetilde{M}_i = \mathcal{F}|_{U_i}$ so that

$$f_{ij}^* \widetilde{M}_i = f_{ij}^* \mathcal{F}|_{U_i} = i^* f_{ij}^* \mathcal{F} = f^* \mathcal{F}|_{V_{ij}}.$$

David says that the proof is similar for \otimes . □

14.0.5 Locally free sheaves (of finite rank)

Locally free

Definition 14.0.12. An \mathcal{O}_X -module of sheaves \mathcal{F} is **locally free** (of finite rank) if there exists an open covering $X = \bigcup_{i \in I} U_i$ of X such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r_i}$ for all $i \in I$. \mathcal{F} is then locally free of **finite rank** r if it holds that $r_i = r$ for all $i \in I$.

Remark 14.0.13. We may assume that U_i is affine in the above definition by further restrictions since $(\mathcal{O}_{U_i}^{\oplus r_i})|_V \cong \mathcal{O}_V^{\oplus r_i} (\cong B^{\oplus r_i} \text{ if } V = \text{Spec } B)$. *Addendum:* To be more precise here, the isomorphism $\mathcal{O}_V^{\oplus r_i} \cong B^{\oplus r_i}$ is with respect to *global sections*. It would perhaps be more accurate to say that $\mathcal{O}_V^{\oplus r_i} \cong \widetilde{B^{\oplus r_i}}$.

Remark 14.0.14. It is not true that if \mathcal{F} is *locally free*, then $\forall X = \bigcup_i U_i$ with U_i affine, we have that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r_i}$.

Furthermore, \mathcal{F} *locally free* implies \mathcal{F} *quasi-coherent*.

Remark 14.0.15. Another (notational) way to write $\mathcal{O}_X^{\oplus r_i}$ is as $\bigoplus_{i=1}^{r_i} \mathcal{O}_X$.

Example 14.0.16. $\mathcal{F} = \mathcal{O}_X$ is locally free of rank 1, “trivially”, since take cover $X = X$ then $\mathcal{O}_X = \mathcal{O}_X^{\oplus 1}$.

Example 14.0.17. $\mathcal{F} = \mathcal{O}_X^{\oplus n}$ is locally free of rank n .

Example 14.0.18. $\mathcal{O}_{\mathbb{P}^n}(d)$ is locally free of rank 1 because on the affine cover $\mathbb{P}^n = \bigcup_{i=1}^n D_+(x_i)$ ($D_+(x_i) := U_i$) we have that $\mathcal{O}_{\mathbb{P}^n}(d)|_{D_+(x_i)} \cong \mathcal{O}_{\mathbb{P}^n}|_{D_+(x_i)} := \mathcal{O}_{D_+(x_i)}$.

Proposition 14.0.19. If \mathcal{F}, \mathcal{G} are locally free of rank r, s respectively, then:

- $\mathcal{F} \oplus \mathcal{G}$ locally free of rank $r + s$.
- $\mathcal{F} \otimes \mathcal{G}$ locally free of rank rs .
- $f^* \mathcal{F}$ locally free of rank r .
- Dual $\mathcal{F}^\vee := \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ locally free of rank r .

Proof. Reduce to affine, free situation, eg. $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r_i}$. Then

$$\begin{aligned} \mathcal{F}^\vee|_{U_i} &\cong \text{Hom}_{\mathcal{O}_{U_i}}(\mathcal{F}|_{U_i}, \mathcal{O}_X|_{U_i}) \\ &\cong \text{Hom}_{\mathcal{O}_{U_i}}(\mathcal{O}_{U_i}^{\oplus r_i}, \mathcal{O}_{U_i}) \\ &\cong \text{Hom}_{R_i}(R_i^{\oplus r_i}, R_i) \\ &\cong R_i^{\oplus r_i} \end{aligned}$$

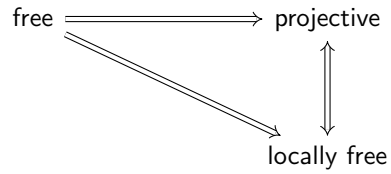
where the next-to-last isomorphism *depends on a choice of basis*. □

Example 14.0.20. $\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_{\mathbb{P}^n}(e) \cong \mathcal{O}_{\mathbb{P}^n}(d + e)$.

Example 14.0.21. $\mathcal{O}_{\mathbb{P}^n}(a)^\vee = \mathcal{O}_{\mathbb{P}^n}(-a)$.

Fact: If R is a *Dedekind domain* then *every* ideal is locally free of rank ≤ 1 (every *torsion-free* sheaf is *locally free*).

Fact: If M is an R -module then M is finitely generated and projective $\Leftrightarrow \widetilde{M}$ is locally free.



Not mentioned in Gathmann:

Example 14.0.22. If $X \xhookrightarrow{\varphi} \mathbb{P}^n$ is a projective variety with chosen embedding φ into \mathbb{P}^n . Then $\mathcal{O}_X(d) = \mathcal{O}_{\mathbb{P}^n}(d)|_X$ is locally free of rank 1.

We have $\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(e) \cong \mathcal{O}_X(d+e)$.

Warning!: $\mathcal{O}_X(d)$ depends on chosen embedding $X \hookrightarrow \mathbb{P}^n$.

Chapter 15

Lecture 15

Today: Quasi-coherent sheaves.

- \tilde{M} , $\underbrace{f_*, f^*}_{\text{on coarse-grained sheaves}}$.

15.0.1 Surjectivity (key thing about sheaves as compared to e.g. presheaves)

Surjective morphism of sheaves

Definition 15.0.1. Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then we say that f is **surjective** if the following *equivalent* conditions hold:

- (a) $\text{Im}(f) = \mathcal{G}$.
- (b) $\text{coker}(f) = 0$.
- (c) $f_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is surjective for all $x \in X$ (local condition).
- (d) For all $U \subset X$ open, and $\varphi \in \mathcal{G}(U)$ there exists an open subset $V \subset U$ and $\psi \in \mathcal{F}(V)$ such that $f(\psi) = \varphi|_V$ (“ f is locally surjective”).

Injective morphism of sheaves

Definition 15.0.2. Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then we say that f is **injective** if the following equivalent conditions hold:

- (a) $\ker(f) = 0$.
- (b) $f_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is injective for all $x \in X$.
- (c) $f_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is injective for all open $U \subset X$.

Example 15.0.3. Consider $\text{Spec}(k[x, y]/\langle y \rangle) = \mathbb{A}^1 \xrightarrow{i} \mathbb{A}^2 = \text{Spec } k[x, y]$. This is defined as $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ where $\varphi : k[x, y] \rightarrow k[x, y]/\langle y \rangle$.

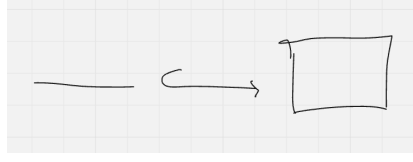


Figure 15.1:

This leads to

$$\begin{aligned}
 \mathcal{O}_{\mathbb{A}^2} &\xrightarrow{\#} i_* \mathcal{O}_{\mathbb{A}^1} \\
 \rightsquigarrow i^\# : \underbrace{\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)}_{=k[x,y]} &\twoheadrightarrow (i_* \mathcal{O}_{\mathbb{A}^1})(\mathbb{A}^2) = \mathcal{O}_{\mathbb{A}^1}(i^{-1}(\mathbb{A}^2)) \\
 &= \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1) \\
 &= k[x] \\
 &\cong k[x, y]/\langle y \rangle.
 \end{aligned}$$

Note that induced map $i^\#$ is surjective because we can view $i : \text{Spec}(k[x, y]/\langle y \rangle) = \mathbb{A}^1 \rightarrow \text{Spec}(k[x, y]) = \mathbb{A}^2$ as induced by the canonical quotient map $\pi : k[x, y] \twoheadrightarrow k[x, y]/\langle y \rangle$. By the (anti-)equivalence between **AffSch** and **Ring** we claim that we then get back $i^\# = \pi$.

Example 15.0.4. Notice that

- $\text{Spec } k[x, y] = \mathbb{A}^2$.
- We have a ring-homomorphism $\varphi : k[x, y] \twoheadrightarrow k[x, y]/\langle y \rangle$, and our morphism of sheaves $\mathcal{O}_{\mathbb{A}^2} \rightarrow \mathcal{O}_{\mathbb{A}^1}$. Let $D(x) \subset \mathbb{A}^2$, which is open. Then

$$\begin{aligned}
 i^{-1}(D(x)) &= \{\mathfrak{p} \in \mathbb{A}^1 : i(\mathfrak{p}) \in D(x)\} \\
 &= \{\mathfrak{p} \in \text{Spec}(k[x, y]/\langle y \rangle) : i(\mathfrak{p}) \in D(x)\} \\
 &= \{\mathfrak{p} \in \text{Spec}(k[x, y]/\langle y \rangle) : \varphi^{-1}(\mathfrak{p}) \in D(x)\} \\
 &= \{\mathfrak{p} \in \text{Spec } k[x] : \varphi^{-1}(\mathfrak{p}) \in D(x)\} \\
 &= \{\mathfrak{p} \in \text{Spec } k[x] : \varphi^{-1}(\mathfrak{p}) \not\ni x\}
 \end{aligned}$$

Here the map φ sends $x \mapsto x$ and $y \mapsto 0$. This means that $\varphi^{-1}(\mathfrak{p})$ are all polynomials $f(x, y) \in k[x, y]$ such that $f(x, 0) \in \mathfrak{p}$. If we want $\varphi^{-1}(\mathfrak{p})$ to not include x . Recall that $D(x)$ is equal to $\text{Spec } k[x, y] \setminus V(x) = \{\mathfrak{q} \in \text{Spec } k[x, y] : x \notin \mathfrak{q}\}$. We see that $x \in \varphi^{-1}(\mathfrak{p})$ if and only if $\varphi(x) = x \in \mathfrak{p}$. This is the same as saying that $\varphi^{-1}(\mathfrak{p}) \not\ni x \Leftrightarrow x \notin \mathfrak{p}$. Hence we have that

$$\begin{aligned}
 i^{-1}(D(x)) &= \{\mathfrak{p} \in \text{Spec } k[x] : x \notin \mathfrak{p}\} \\
 &= \text{Spec } k[x] \setminus V(x) \\
 &= D(x) \subset \text{Spec } k[x].
 \end{aligned}$$

We then see that

$$\begin{aligned}
 \mathcal{O}_{\mathbb{A}^2}(D(x)) &= \mathcal{O}_{\text{Spec } k[x, y]}(D(x)) \\
 &= (k[x, y])_x \\
 &= k \left[x, y, \frac{1}{x} \right]
 \end{aligned}$$

and

$$\begin{aligned}
 i_* \mathcal{O}_{\mathbb{A}^1}(D(x)) &= i_* \mathcal{O}_{\text{Spec } k[x]}(D(x)) \\
 &= \mathcal{O}_{\text{Spec } k[x]}(D(x)) \\
 &= (k[x])_x \\
 &= k \left[x, \frac{1}{x} \right] \\
 &= k \left[x, y, \frac{1}{x} \right] / \langle y \rangle.
 \end{aligned}$$

Hence we get an induced map

$$\mathcal{O}_{\mathbb{A}^2}(D(x)) = k \left[x, y, \frac{1}{x} \right] \rightarrow k \left[x, y, \frac{1}{x} \right] / \langle y \rangle = i_* \mathcal{O}_{\mathbb{A}^1}(D(x)).$$

On the other hand, we have that $\mathcal{O}_{\mathbb{A}^2}(D(y)) \cong k \left[x, y, \frac{1}{y} \right]$ and $i_* \mathcal{O}_{\mathbb{A}^1}(D(y))$. We calculate $i^{-1}(D(y))$; we have that

$$\begin{aligned}
 i^{-1}(D(y)) &= \{ \mathfrak{p} \in \text{Spec } k[x] : i(\mathfrak{p}) \in D(y) \} \\
 &= \{ \mathfrak{p} \in \text{Spec } k[x] : \varphi^{-1}(\mathfrak{p}) \in D(y) \}.
 \end{aligned}$$

But notice that since $\varphi(y) = 0$ we have that $\varphi^{-1}(\mathfrak{p}) \ni y$ for every $\mathfrak{p} \in \text{Spec } k[x]$. But

$$D(y) = \{ \mathfrak{q} \in \text{Spec } k[x, y] : y \notin \mathfrak{q} \}.$$

But this means that there are no $\mathfrak{p} \in \text{Spec } k[x]$ such that $\varphi^{-1}(\mathfrak{p}) \in D(y)$. Therefore $i^{-1}(D(y)) = \emptyset$ and so we have

$$\mathcal{O}_{\mathbb{A}^2}(D(y)) = k \left[x, y, \frac{1}{y} \right] \rightarrow 0.$$

- We have $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) = k[x, y]$, whereas for $i_* \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^2 \setminus \{0\})$ we first note that

$$\begin{aligned}
 \mathbb{A}^2 \setminus \{(0, 0)\} &= \mathbb{A}^2 \setminus (V(x) \cap V(y)) \\
 &= (\mathbb{A}^2 \setminus V(x)) \cup (\mathbb{A}^2 \setminus V(y)) \\
 &= D(x) \cup D(y).
 \end{aligned}$$

Thus

$$\begin{aligned}
 i^{-1}(\mathbb{A}^2 \setminus 0) &= i^{-1}(D(x)) \cup i^{-1}(D(y)) \\
 &= D(x).
 \end{aligned}$$

Hence we have an induced map

$$\begin{aligned}
 \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus 0) &= k[x, y] \rightarrow \mathcal{O}_{\mathbb{A}^1}(D(x)) \\
 &= \mathcal{O}_{\text{Spec } k[x]}(D(x)) \\
 &= k[x]_x \\
 &= k \left[x, \frac{1}{x} \right].
 \end{aligned}$$

But this map is explicitly defined as $\varphi(f(x, y)) = f(x, 0)$. Therefore its image is $k[x] \subset k \left[x, \frac{1}{x} \right]$, and so is not surjective.

Fact: $\mathcal{O}_{\mathbb{A}^2} \rightarrow i_* \mathcal{O}_{\mathbb{A}^1}$ is *locally surjective*. We will see this later today.

Remark 15.0.5. Generally, the induced map i^\sharp is defined on arbitrary $U \subset \mathbb{A}^2$ open as sending a section $s \in \mathcal{O}_{\mathbb{A}^2}(U)$ to $(\mathfrak{p} \mapsto s(i(\mathfrak{p})) = s(\varphi^{-1}(\mathfrak{p})))$, or $s \mapsto s \circ i$.

$$\begin{array}{ccc}
 \mathcal{O}_{\mathbb{P}^n}(d) \otimes' \mathcal{O}_{\mathbb{P}^n}(e) & \xrightarrow{\text{not surjective morphism of presheaves}} & \mathcal{O}_{\mathbb{P}^n}(d+e) \\
 \searrow \text{sheafify} & & \nearrow \cong \\
 & \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_{\mathbb{P}^n}(e) &
 \end{array}$$

Proof.

$$U_i = D_+(x_i)$$

$$\begin{array}{ccc}
 \varphi \otimes \psi & \xrightarrow{\quad} & \varphi\psi \\
 \\
 \mathcal{O}_{\mathbb{P}^n}(d)(U_i) \otimes_{\mathcal{O}_{\mathbb{P}^n}(U_i)} \mathcal{O}_{\mathbb{P}^n}(e)(U_i) & \xrightarrow{\text{mult.}} & \mathcal{O}_{\mathbb{P}^n}(d+e)(U_i) \\
 \downarrow \cong x_i^{-d} \otimes x_i^{-e} & & \downarrow \cong x_i^{-d-e} \\
 \mathcal{O}_{\mathbb{P}^n}(U_i) \otimes_{\mathcal{O}_{\mathbb{P}^n}(U_i)} \mathcal{O}_{\mathbb{P}^n}(U_i) & \xrightarrow[\cong]{\text{mult.}} & \mathcal{O}_{\mathbb{P}^n}(U_i)
 \end{array}$$

Isomorphisms on stalks \Rightarrow becomes isomorphism of sheaves (cf. [Gat21, Exc. 13.8]). □

15.0.2 Quasi-coherent sheaves (affine case)

Let $X = \text{Spec } R$ be an affine scheme. We have

$$\begin{array}{ccc}
 \mathbf{Mod}_R & \xrightleftharpoons[\text{global sections}]{\widetilde{(-)}} & \mathbf{Mod}_{\mathcal{O}_X} \\
 U & \xrightarrow{\quad} & \widetilde{U} \\
 \Gamma(\mathcal{F}) & \xleftarrow{\quad} & \mathcal{F}
 \end{array}$$

This is *not* an equivalence of categories. We want to rectify this, to get an equivalence of categories.

Note: $\Gamma(\mathcal{F}) := \Gamma(X, \mathcal{F})$ is not in [Gat21], but is the same as $\mathcal{F}(X)$, i.e. the *global sections* of the \mathcal{O}_X -module of sheaves \mathcal{F} .

Sheaf associated to a module; \widetilde{M}

Definition 15.0.6. Let $X = \operatorname{Spec} R$ be an affine scheme, and let M be an R -module. For an open subset $U \subset X$ we set

$$\widetilde{M}(U) := \left\{ \begin{array}{l} \varphi = (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in U} : \varphi_{\mathfrak{p}} \in M_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in U \\ \text{and for all } \mathfrak{p} \in U \text{ there exists } U_{\mathfrak{p}} \subset U \text{ open} \\ \text{with } \mathfrak{p} \in U \text{ and } g \in M, f \in R \text{ with } \varphi_Q = \frac{g}{f}, \forall Q \in U_{\mathfrak{p}}. \end{array} \right\}.$$

Perhaps a shorter way to write this is as

$$\widetilde{M}(U) := \left\{ \varphi = (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in U} : \varphi_{\mathfrak{p}} \in M_{\mathfrak{p}} \text{ and } \varphi_Q = \frac{g}{f} \text{ with } g \in M \text{ and } f \in R, \text{ locally} \right\}.$$

Notice that since $\varphi_Q \in M_Q$ we have that $f \notin Q$. Furthermore, to see that \widetilde{M} is a sheaf:

- Restriction maps for $V \subset U$ are given by $\varphi = (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in U} \mapsto (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in V}$ from which the identity axiom and the composition-axiom for presheafs follows directly.
- $M(\emptyset)$ are elements $\varphi = (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in \emptyset}$. We may think of this as $U = \emptyset \rightarrow \bigcup_{\mathfrak{p} \in \emptyset} M_{\mathfrak{p}} = \emptyset$. There is only one such function, and $\mathcal{O}_X(\emptyset) = 0$.

Fact: \widetilde{M} is a sheaf (easy; locally defined).

- $\widetilde{M}_x = M_{\mathfrak{p}}$ (fairly easy: $(U, \varphi) \mapsto \varphi_x$).
- $\widetilde{M}(D(f)) \cong M_f$ (more difficult); setting $f = 1$ we recover the module M (“it *remembers* the module”).

There are natural maps:

$$\begin{array}{ccc} M & \xrightarrow{\cong} & \widetilde{M}(X), \quad \forall M \in \mathbf{Mod}_R \\ \widetilde{\mathcal{F}}(X) & \longrightarrow & \mathcal{F}, \quad \forall \mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}. \\ & \underbrace{\hspace{1cm}} & \\ & \text{not iso} & \\ & \text{in general} & \end{array}$$

Quasi-coherent sheaves (affine case)

Definition 15.0.7. \mathcal{O}_X -modules \mathcal{F} on the form \widetilde{M} are called **quasi-coherent sheaves**.

Lemma 15.0.8 ([Gat21, Lemma 14.7]).

(a)

$$\{ {}_{R\text{-module}} M \rightarrow N \} \xleftarrow{1-1} \{ {}_{\mathcal{O}_X\text{-module}} \widetilde{M} \rightarrow \widetilde{N} \}$$

Remark 15.0.9. The functor $\mathbf{Mod}_R \rightarrow \mathbf{Mod}_{\mathcal{O}_X}$ ($M \mapsto \widetilde{M}$) is *fully faithful*: Thus

$$\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$$

for all $M, N \in \mathbf{Mod}_R$.

(b)

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \text{ exact seq.} \Leftrightarrow 0 \rightarrow \widetilde{L} \rightarrow \widetilde{M} \rightarrow \widetilde{N} \rightarrow 0 \text{ exact seq.}$$

- (c) $\widetilde{M \oplus N} = \widetilde{M} \oplus \widetilde{N}$.
- (d) $\widetilde{M \otimes_R N} = \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.
- (e) $\widetilde{(\bigoplus_{i \in I} M_i)} \cong \bigoplus_{i \in I} \widetilde{M_i}$ (but not for $\prod_{i \in I} M_i$).

Proof.

(a): (Exists natural maps in both directions that are inverses of each other). In one direction given $\widetilde{M} \rightarrow \widetilde{N}$ then $\widetilde{M}(X) \rightarrow \widetilde{N}(X)$ gives $M \rightarrow N$ (“ $\widetilde{(-)}$ ” remembers M or N). On the other hand, given $M \rightarrow N \rightsquigarrow \phi_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ for every $\mathfrak{p} \in X = \text{Spec } R$ (use that localization is a functor).

(c)-(d): Direct sums and direct products commutes with localization, i.e. $S^{-1}(M \oplus N) \cong S^{-1}M \oplus S^{-1}N$ and $S^{-1}(M \otimes_R N) \cong S^{-1}M \otimes_R S^{-1}N$. Then let $S = R - \mathfrak{p}$. We then want to claim that this gives rise to a morphism $\widetilde{M \oplus N} \rightarrow \widetilde{M} \oplus \widetilde{N}$ which we want claim one can check on the stalks is an isomorphism by [Gat21, Exercise 13.8]. \square

15.0.3 Quasi-coherent sheaves (general case)

Quasi-coherent sheaves (general case)

Definition 15.0.10. An \mathcal{O}_X -module \mathcal{F} is a **quasi-coherent sheaf** if there exists an open *affine* cover $\{U_i : i \in I\}$ such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ where $U_i \cong \text{Spec } R_i$ and M_i is an R_i -module.

Remark 15.0.11. [Gat21, Remark 14.5] mentions that it can be shown that in fact for a quasi-coherent sheaf \mathcal{F} it is the case that for *any* affine open $U \cong \text{Spec } R$ one has that $\mathcal{F}|_U \cong \widetilde{M}$ where M is an R -module. We think this is what David means when he says that this holds for all $X = \bigcup_i U_i$ open affine covering (and not just the one we know exists given \mathcal{F} quasi-coherent). Compare the above definition to the affine case and see how they agree, if you like.

Noetherian Scheme

Definition 15.0.12. We say that a scheme X is a **Noetherian scheme** if it can be covered by *finitely* many affine open schemes $\text{Spec } R_i$ such that each R_i is a noetherian ring (i.e. satisfies the ascending chain condition).

Coherent sheaf

Definition 15.0.13. Let X be a noetherian scheme. Then we say that an \mathcal{O}_X -module \mathcal{F} is **coherent** if it is quasi-coherent and there is an open cover $X = \bigcup_i U_i$ such that $\mathcal{O}_X|_{U_i} \cong \widetilde{M_i}$ and M_i is a *finitely generated* R_i -module.

Example 15.0.14. The structure sheaf \mathcal{O}_X is quasi-coherent since for each affine open $U_i \cong \text{Spec } R_i$ we have that

$$\begin{aligned} \mathcal{O}_X|_{U_i} &= \mathcal{O}_{U_i} \\ &= \mathcal{O}_{\text{Spec } R_i} \\ &= \widetilde{R_i}, \end{aligned}$$

where (of course) R_i is an R_i -module.

Example 15.0.15. $\mathcal{O}_{\mathbb{P}^n}(d)$ is a quasi-coherent sheaf: Locally on $D_+(x_i)(=U_i)$ we have that

$$\begin{aligned}\mathcal{O}_{\mathbb{P}^n}(d)|_{D_+(x_i)} &\cong \mathcal{O}_{D_+(x_i)} \\ &= \mathcal{O}_{U_i}.\end{aligned}$$

but $\mathcal{O}_{U_i} \cong k[x_1, \dots, x_n]$ since $U_i \cong \mathbb{A}^n = \operatorname{Spec} k[x_1, \dots, x_n]$.

Remark 15.0.16. \mathcal{O}_X and $\mathcal{O}_{\mathbb{P}^n}(d)$ for $d \in \mathbb{Z}$ are examples of what are called “**line bundles**”.

Corollary 15.0.17 (Lemma 15.0.8 \Rightarrow). $\mathcal{F} \xrightarrow{f} \mathcal{G}$ quasi-coherent then $\ker f, \operatorname{Im} f, \operatorname{coker} f, \mathcal{F} \oplus \mathcal{G}, \mathcal{F} \otimes \mathcal{G}$ quasi-coherent.

15.0.4 Push-forward

Let say $f : X \rightarrow Y$ be a morphism of schemes and let \mathcal{F} be an \mathcal{O}_X -module. We have seen that there is a sheaf $f_*\mathcal{F}$ of \mathcal{O}_Y -modules ($U \subset Y$ open $\rightsquigarrow (f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$), called the “direct image sheaf” or “pushforward”.

Fact: If \mathcal{F} is quasi-coherent and f satisfies very mild assumptions (quasi-compact + quasi-separated) then $f_*\mathcal{F}$ is quasi-coherent (we only do closed immersions).

Remark 15.0.18. The above two definitions was (as far as we can recall) not covered in class.

Closed immersion

Definition 15.0.19. A **closed immersion** is a morphism of schemes $f : X \rightarrow Y$ such that the underlying continuous map $f : \operatorname{sp}(X) \rightarrow \operatorname{sp}(Y)$ induces a homeomorphism onto some *closed* subset of $\operatorname{sp}(Y)$, and such that the induced map $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ of sheaves on Y is surjective (\Leftrightarrow surjective on stalks $\mathcal{O}_{Y,f(p)} \rightarrow \mathcal{O}_{X,p}$).

Closed subscheme

Definition 15.0.20. A **closed subscheme** of a scheme Y is an *equivalence class* of closed immersions (15.0.19) where we say that $f : X \rightarrow Y$ and $f' : X' \rightarrow Y$ are equivalent if there is an isomorphism $i : X' \rightarrow X$ such that $f' = f \circ i$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{\quad i \quad} & X \\ & \searrow f' & \swarrow f \\ & Y & \end{array}$$

Lemma 15.0.21 ([Gat21, Lemma 14.8]). Let $i : Z \rightarrow X$ be a closed immersion (15.0.19).

1. If \mathcal{F} is a quasi-coherent \mathcal{O}_Z -module then $i_*\mathcal{F}$ is a quasi-coherent \mathcal{O}_X -module.
2. $\mathcal{O}_X \rightarrow i_*\mathcal{O}_Z$ is surjective

$$\rightsquigarrow 0 \rightarrow I_Z \rightarrow \mathcal{O}_X \rightarrow i_*\mathcal{O}_Z \rightarrow 0 \quad \text{exact.}$$

3.

Remark 15.0.22. I_Z the ideal sheaf.

Proof. (a): We reduce to the affine case with $X = \operatorname{Spec} R$. Then by definition we have that $Z = \operatorname{Spec} R/\overline{I} \rightarrow \operatorname{Spec} R$. This in turn induces a map between global sections $R \twoheadrightarrow R/I$. By quasi-coherence we have that $\mathcal{F} = \widetilde{M}$ for some R/I -module M . We then have that $i_*\mathcal{F}$ have the same sections as \mathcal{F} so that $i_*\mathcal{F} = \widetilde{M}$ but where $M := M_\varphi$ is viewed as an R -module by restriction of scalars through $R \xrightarrow{\varphi} R/I$.

The general case reduces to the above by doing the same on each affine open in a cover of X

(b): In the affine case this corresponds to

$$\begin{aligned} 0 \rightarrow I \rightarrow R \twoheadrightarrow R/I \rightarrow 0 \text{ exact} \\ \xRightarrow{[\text{Gat21, Lemma 14.7.(b)}]} 0 \rightarrow \widetilde{I} \rightarrow \widetilde{R} \twoheadrightarrow \widetilde{R/I} \rightarrow 0 \text{ exact} \end{aligned}$$

but the second sequence is precisely $0 \rightarrow I_Z \rightarrow \mathcal{O}_X \rightarrow i_*\mathcal{O}_Z \rightarrow 0$. \square

Example 15.0.23 (cf. example 15.0.3).

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_{\mathbb{A}^1/\mathbb{A}^2} & \longrightarrow & \mathcal{O}_{\mathbb{A}^2} & \longrightarrow & i_*\mathcal{O}_{\mathbb{A}^1} \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & \widetilde{\langle y \rangle} & \longrightarrow & \widetilde{k[x, y]} & \longrightarrow & \widetilde{k[x, y]/\langle y \rangle} \longrightarrow 0 \end{array}$$

Before the next example we will revisit some things not covered in class.

15.0.5 Interlude: Skyscraper sheaf

When we consider an inclusion $i : P \hookrightarrow \mathbb{P}^1$ for some point, say $P := (1 : a) \in \mathbb{P}^1$, we are really considering $P := \operatorname{Spec}(\kappa(P))$. Here $\kappa(P)$ is the field $\mathcal{O}_{\mathbb{P}^1, P}/\mathfrak{m}_P$ where \mathfrak{m}_P is the maximal ideal of the *local ring* $\mathcal{O}_{\mathbb{P}^1, P}$ (stalk). Notice that we have a surjection of rings $\mathcal{O}_{\mathbb{P}^1, P} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1, P}/\mathfrak{m}_P = \kappa(P)$. This induces a morphism of schemes $\operatorname{Spec}(\kappa(P)) \rightarrow \operatorname{Spec}(\mathcal{O}_{\mathbb{P}^1, P})$. Note that we may consider that P is in either $D_+(x_0)$ or $D_+(x_1)$. Thus $\mathcal{O}_{\mathbb{P}^1, P} = \mathcal{O}_{D_+(x_i), P} \cong k[t]_{(t-P)}$.

15.0.6 Back to lecture: epilogue

Example 15.0.24.

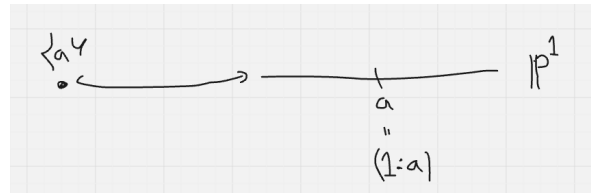


Figure 15.2: Example 15.0.24

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{x_1 - ax_0} \mathcal{O}_{\mathbb{P}^1}(0) \rightarrow i_*\mathcal{O}_{\{a\}} \rightarrow 0.$$

We have that $x_1 - ax_0 \in \mathcal{O}_{\mathbb{P}^1}(1) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = k[x_0, x_1]_1 = k[x_0, x_1]$ (also sometimes denoted as $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$). Also $I_{\{a\}/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$

Chapter 16

Lecture 16

Recap:

- R ring $\rightsquigarrow X = \text{Spec } R$.
 - $\mathfrak{p} \trianglelefteq R$ prime \rightsquigarrow points $x \in X$.
 - M R -module $\rightsquigarrow \mathcal{F} = \widetilde{M}$ quasi-coherent sheaf.
- We have that $M_f = \mathcal{F}(D(f))$.
- Stalk: $M_{\mathfrak{p}} = \mathcal{F}_x$, which is an $R_{\mathfrak{p}} = \mathcal{O}_{X,x}$ -module.
- We have

$$\begin{aligned}
 M \otimes_R \underbrace{k(\mathfrak{p})}_{= R_{\mathfrak{p}} / \mathfrak{p} R_{\mathfrak{p}}} &= M_{\mathfrak{p}} / \mathfrak{p} M_{\mathfrak{p}} \\
 &= \underbrace{\mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x}_{= \mathcal{F}(x)} \\
 &= \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x),
 \end{aligned}$$

which is a vector space over $k(x)$!

Let $i : \{x\} \hookrightarrow X$ be the inclusion. We may think of $\{x\}$ as $\text{Spec } k(x)$. Then $i^* \mathcal{F} = \mathcal{F}(x)$ is the **fiber** at x , and $i^{-1} \mathcal{F} = \mathcal{F}_x$ is the **stalk** at x .

Today: Ω_X quasi-coherent sheaf. - $\Omega_X(x) = (T_x X)^\vee$.

Example 16.0.1. “The duals assemble into a quasi-coherent sheaf”.

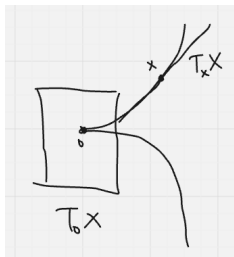


Figure 16.1:

16.0.1 Derivations

Example 16.0.2. Let $R := k[x_1, \dots, x_n]$. Can define a map $\frac{\partial}{\partial x_i} : R \rightarrow R$ that satisfies:

- (1) $D(f + g) = D(f) + D(g)$ for all f, g in R .
- (2) $D(fg) = fD(g) + gD(f)$ for all f, g in R .
- (3) $D(f) = 0$ if $f \in k$.

k-derivation

Definition 16.0.3. Let R be a commutative ring which is a k -algebra. Then a ***k*-derivation** is map $D : R \rightarrow M$, where M is an R -module such that (1)-(3) in example 16.0.2 holds.

Remark 16.0.4. We may note that (1)-(3) implies that D in 16.0.3 is k -linear as follows: Let $f \in k$ and let $g \in R$. Then

$$\begin{aligned} D(fg) &= g \underbrace{D(f)}_{=0 \text{ by (3)}} + fD(g) \\ &= fD(g). \end{aligned}$$

Kähler differentials $\Omega_{R/k}$

Definition 16.0.5. Let R be a k -algebra (and commutative ring with unit). We then define Ω_R or $\Omega_{R/k}$ (if we want to underscore the base field k) is the *free* R -module generated by all formal symbols df for $f \in R$, *modulo* the relations:

1. $d(f + g) = df + dg$ for all $f, g \in R$;
2. $d(fg) = gdf + f dg$ for all $f, g \in R$;
3. $df = 0$ for all $f \in k$.

That is, $\Omega_{R/k} = R\langle df \mid f \in R \rangle / \dots$ where \dots denotes the above relations, and where $R\langle df \mid f \in R \rangle$ is a free R -module.

Then the map

$$d : R \rightarrow \Omega_{R/k}, \quad f \mapsto df$$

is a derivation (16.0.3) by construction. The elements of $\Omega_{R/k}$ are called the **(Kähler) differentials** of R over k .

Remark 16.0.6. By (slight) abuse of notation, we will use df also for the image $\overline{df} \in \Omega_{R/k}$ (as compared to $df \in R\langle df \mid f \in R \rangle$).

Compare: For commutative ring R , there is a similar way we construct the tensor product $M \otimes_R N$ by $\overline{M \times N} \rightarrow M \otimes_R N$, $(m, n) \mapsto m \otimes n$ by modding out the relations of a bilinear map (so $M \otimes_R N := F(M \times N) / \dots$).

Proposition 16.0.7 (Universal property). *For all $D : R \rightarrow M$ that are k -derivations, there is a unique R -linear map φ such that $D = \varphi \circ d$. That is, the following diagram commutes*

$$\begin{array}{ccc} R & \xrightarrow{D} & M \\ & \searrow d & \nearrow \exists! \varphi \\ & \Omega_{R/k} & \end{array} .$$

Thus d is the “universal derivation”, and since d takes f in R to df , we see that $\varphi(df) = Df$ for all f in R .

Localization: Let S be a multiplicatively closed subset in a k -algebra R . Then for any $g \in R$ and $f \in S$ we have in $\Omega_{S^{-1}R}$ that

$$\begin{aligned} 0 &= d(1) \quad (\text{by (3) and well-definedness}) \\ &= d\left(\frac{1}{f} \cdot f\right) \\ &= f d\left(\frac{1}{f}\right) + \frac{1}{f} df \\ \Rightarrow d\left(\frac{1}{f}\right) &= -\frac{1}{f^2} df. \end{aligned}$$

Therefore we have that for any g in R and f in S , it holds that

$$\begin{aligned} d\left(\frac{g}{f}\right) &= \frac{1}{f} dg + g d\left(\frac{1}{f}\right) \\ &= \frac{1}{f} dg - \frac{g}{f^2} df. \end{aligned}$$

That is, the (Kähler) differentials for a localized k -algebra satisfy not only the “usual” differentiation rules for sums, products and constants, but also for quotients.

The above computation also shows that the differential $d\left(\frac{g}{f}\right)$ of the quotient $\frac{g}{f} \in S^{-1}R$ can be expressed as an $S^{-1}R$ -linear combination of the differentials df, dg of f, g in R . In this way, one gets an isomorphism $S^{-1}\Omega_{R/k} \cong \Omega_{S^{-1}R/k}$ of $S^{-1}R$ -modules $S^{-1}\Omega_{R/k}$ and $\Omega_{S^{-1}R/k}$, defined by

$$d\left(\frac{g}{f}\right) \mapsto \frac{1}{f} dg - \frac{g}{f^2} df.$$

We check that this map is well-defined and bijective.

Well-defined:

In $\Omega_{S^{-1}R}$ we have that $d\left(\frac{g}{f}\right) = \frac{1}{f} dg - \frac{g}{f^2} df$ for $g \in R$ and $f \in S$ and so $d\left(\frac{g}{f}\right) = d\left(\frac{g'}{f'}\right)$ immediately implies that

$$\frac{1}{f} dg - \frac{g}{f^2} df = \frac{1}{f'} dg' - \frac{g'}{f'^2} df'$$

in $\Omega_{S^{-1}R/k}$, for $g, g' \in R$ and $f, f' \in S$.

Injective:

Surjective: Elements in $S^{-1}\Omega_{R/k}$ are on the form $\frac{g}{s}$ with $g \in \Omega_{R/k}$ and $s \in S$. But if $g \in \Omega_{R/k}$ then g is on the form dg' for some $g' \in R$. So we can rewrite this as $\frac{dg'}{s}$.

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