

# Graph Theory, HT2025

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# Chapter 1

## Lecture 1

### Examination:

- 2 Homeworks (18 points each).
- Paper review + presentation (mandatory, 14 points).
- Exam (50 points).

To pass requires 51 points, and 80 points for A.

### Today:

- Graphs
  - Basic definitions (graphs, subgraphs, degree, ...).
  - First examples.
- Overview
  - Motivation/highlights.

### Set of subsets of size $k$

**Definition 1.0.1.** Let  $V$  be a finite set and let  $k \in \mathbb{N}$ . Then we define

$$\binom{V}{k} := \{A \subset V : |A| = k\},$$

i.e.  $\binom{V}{k}$  is the *set of subsets of size  $k$* .

### Graph $G = (V, E)$

**Definition 1.0.2.** Let  $V$  be a finite set and let  $E \subseteq \binom{V}{2}$ . The pair  $G = (V, E)$  is called a **graph**.

Furthermore,  $V$  is called the **vertex set** or **node set** and  $E$  is called the **edge set**.

*Remark 1.0.3.* Notation:  $V(G) := V$  and  $E(G) := E$ . We may abbreviate the set  $e = \{u, v\} \in E$  as  $uv$ .

**Definition 1.0.4.** If  $uv \in E$  for a given graph  $G = (V, E)$ , we say that the nodes (or vertices)  $u$  and  $v$  in  $V$  are **adjacent** or **neighbours**.

**Example 1.0.5** (Complete graphs  $K_n$ ). For each  $n \in \mathbb{N}_{\geq 1}$  are complete graphs  $K_n = (G, V)$  with  $|V| = n$  and  $E = \binom{V}{2}$ , i.e. there is precisely one edge between each pair of the  $n$  vertices.

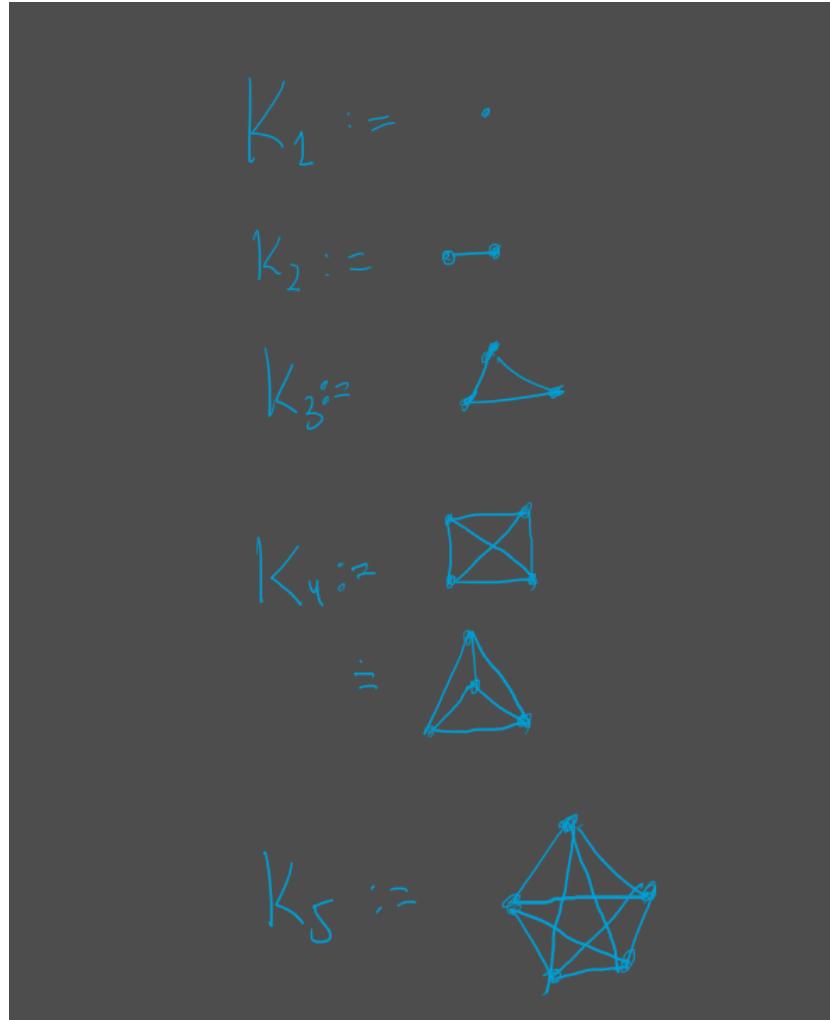


Figure 1.1:  $K_1, K_2, \dots, K_5$ .

Informal: A **graph drawing** is an illustration of a graph in the plane such that:

- Each vertex is represented by a point.
- Each edge is represented by a curve between two vertices.
- The reader can unambiguously recover  $G$  (i.e. each vertex is represented by a unique point belonging only to that vertex, and edges only pass through vertices they are meant to connect).

**Definition 1.0.6.** A drawing is called **planar** if *two edges can only intersect at their respective vertices*.

Highlight: A graph  $G$  has a planar drawing  $\Leftrightarrow$  it does not “contain”  $K_5$  or  $K_{3,3}$  (Kuratowski’s theorem)

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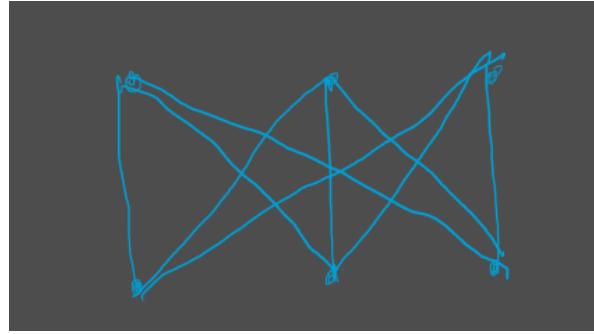


Figure 1.2:  $K_{3,3}$ , an example of a *complete bipartite graph*.

### Graph isomorphism

**Definition 1.0.7.** We say that two graphs  $G$  and  $G'$  are **isomorphic** if there exists a bijection  $\varphi : V(G) \rightarrow V(G')$  such that

$$uv \in E(G) \Leftrightarrow \varphi(u)\varphi(v) \in E(G').$$

One may write this as  $G \cong G'$ .

### Bipartite graphs

**Definition 1.0.8.** We say that a graph  $G = (V, E)$  is **bipartite** if we can *partition* as  $V = S \sqcup T$  such that for arbitrary  $e = \{u, v\} \in E$  we have that

$$\begin{aligned} |e \cap S| &= |e \cap T| \\ &= 1. \end{aligned}$$

### Complete bipartite graphs

**Definition 1.0.9.** We denote by  $K_{m,n} = (V, E)$  the *complete bipartite graph* (1.0.8) where  $|S| = m$  and  $|T| = n$  and where for every choice of  $(u, v) \in S \times T$  we have that  $uv \in E$ .

### $N(v)$ and $N(W)$ for $W \subseteq V$

**Definition 1.0.10.** Given a graph  $G = (V, E)$ , we define

$$N(v) := \{u \in V : uv \in E\}.$$

If  $W \subseteq V$  then we define

$$N(W) := \bigcup_{v \in W} N(v).$$

We also let  $\overline{N}(v) := \{v\} \cup N(v)$ .

*Remark 1.0.11.* Regarding  $\overline{N}(v)$ : Note that if we don't allow *loops* (i.e. edges  $e \in E$  on the form  $vv$  for nodes  $v$  in  $V$ ) then  $v \notin N(v)$ .

### $M \subseteq E$ Matching

**Definition 1.0.12.** Let  $G = (V, E)$  be a graph. A subset  $M$  of the edge set  $E$  is called a **matching** if  $e_1 \cap e_2 = \emptyset$  for all  $e_1 \neq e_2 \in M$ .

Highlight: Given a graph  $G = (S \sqcup T, E)$ , there exists a *perfect matching* of  $S \Leftrightarrow |N(W)| \geq |W|$  for all subsets  $W$  of  $S$  (Hall's marriage theorem; the graph-theoretic formulation is for *finite bipartite graphs*)

*Remark 1.0.13.* Note that in the case of *bipartite graphs* (1.0.8) we have that for any  $v \in S$  it holds that  $N(v) \subseteq T$ .

### 1.0.1 Paths and cycles

#### Path $P_k$

**Definition 1.0.14.** A **path** is a *non-empty* graph  $P_k = (V, E)$  of the form

$$V = \{x_0, x_1, \dots, x_k\} \text{ and } E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\} \quad (x_i \neq x_j \text{ if } i \neq j).$$

#### Cycle $C_k$

**Definition 1.0.15.** A **cycle** is a *non-empty* graph  $C_k = (V, E)$  of the form

$$V = \{x_0, x_1, \dots, x_k\} \text{ and } E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\} \quad (x_i \neq x_j \text{ if } 0 \leq i \neq j \leq k-1)$$

with the added condition that  $x_k = x_0$  (c.f. with 1.0.14).

*Remark 1.0.16.* We abbreviate  $P_k = x_0x_1x_2\dots x_k$  and  $C_k = x_0x_1x_2\dots x_{k-1}x_0$ .

*Remark 1.0.17.* In this course, all graphs are *finite* (unless otherwise stated).

**Definition 1.0.18** (Simple graph). A **simple graph** is an *undirected, unweighted* containing no multiple edges between vertices (including not containing self-edges, i.e. loops).

Observation: Simple graphs (1.0.18) encode binary relation ( $E \subseteq \binom{V}{2}$ ) that are *symmetric* and *irreflexive*.

Variants:

- *Directed graphs*  $G = (V, E)$  with an **orientation**  $o : E \rightarrow V$  such that  $o(e) \in e$ , leads to the theory of **Networks** and **Flows**.
- Multigraphs: Parallel edges and loops are allowed.
- **Weighted Graphs:**  $G = (V, E)$  together with a *weight-function*  $w : E \rightarrow \mathbb{R}_+ \rightsquigarrow$  Dijkstra's algorithm (finding the shortest path) and spanning trees of lowest weight.
- Hypergraphs
- Matroids.

### 1.0.2 Seven bridges of Königsberg

Problem: Can I take a walk in Königsberg so that I pass every bridge *exactly once*?

- Euler (1776): Not possible.

- **Highlight:** Possible if and only if at most 2 vertices (in the Königsberg context - land mass as far as we can tell) have odd degree.

**Adjacency matrix**  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$

**Definition 1.0.19.** Let  $G = (V, E)$  be a graph. Then the **adjacency matrix** of  $G$  is a matrix  $A := (a_{ij}) \in \mathbb{R}^{n \times n}$  such that

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E \\ 0, & \text{otherwise} \end{cases} .$$

*Remark 1.0.20.* Note that the adjacency matrix  $A$  may depend on the *ordering* of the vertices.

**Example 1.0.21.** The graph  $G$  below has adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} .$$

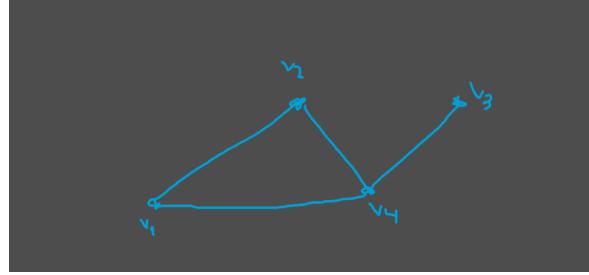


Figure 1.3: Graph  $G$ .

### Walk

**Definition 1.0.22.** A **walk** of length  $k$  in a graph  $G$  is a *non-empty alternating sequence*  $v_0 e_0 v_1 e_1 \dots e_{k-1} v_k$  of vertices  $v_i$  and edges  $e_j$  such that  $e_j = \{v_j v_{j+1}\}$ .

### Trail

**Definition 1.0.23.** A **trail** is a walk (1.0.22) with *no repeated edges*.

### Path

**Definition 1.0.24.** A **path** is a trail (1.0.23) with *no repeated vertices (and edges)*.

**Proposition 1.0.25.** Let  $G$  be a graph with  $V(G) = \{v_1, \dots, v_n\}$  and adjacency matrix  $A = (a_{ij})$ . Let  $\ell \in \mathbb{N}$  and  $A^\ell = (b_{ij})$ . Then  $b_{ij}$  is the number of walks (1.0.22) from  $v_i$  to  $v_j$  of length  $\ell$ .

*Proof.* We proceed by induction on  $\ell$ :

$\ell = 1$ : We have that  $a_{ij} = b_{ij}$  for all  $1 \leq i, j \leq n$ . Furthermore,  $b_{ij} = a_{ij} = 1 \Leftrightarrow$  there is exactly one path from  $v_i$  to  $v_j$  of length  $\ell$  (there can not be more than one edge between  $v_i$  and  $v_j$ ).

$\ell > 1$ : Let  $(c_{ij}) = A^{\ell-1}$ . A walk from  $v_i$  to  $v_j$  of length  $\ell$  consists of a walk of length one from  $v_i$  to a neighbor  $v_s$ , together with a walk of length  $\ell - 1$  from  $v_s$  to  $v_j$ . By induction,  $(c_{ij})$  counts the second part, so the number of walks of length  $\ell$  from  $v_i$  to  $v_j$  is

$$\begin{aligned} \sum_{v_s \in N(v_i)} c_{sj} &= \sum_{s=1}^n a_{is} c_{sj} \\ &= b_{ij} \quad (\text{since } A^\ell = A \cdot A^{\ell-1}). \end{aligned}$$

□

*Remark 1.0.26.* To further elaborate on the last part of the proof above: Recall that

$$N(v_i) = \{u \in V(G) : uv_i \in E(G)\} \quad (1.0.10).$$

Hence we sum  $c_{sj}$  over all  $v_s$  that are neighbors to  $v_i$ . By induction, we had that  $c_{sj}$  counts the number of walks from  $v_s$  to  $v_j$ . The index takes care of the first part, since there is exactly one choice of a walk of length one from  $v_i$  to  $v_s$ .

For the second equality, note that

$$a_{is} = \begin{cases} 1, & \text{if } v_s \text{ is a neighbor of } v_i \\ 0, & \text{otherwise.} \end{cases}$$

Hence  $a_{is}$  helps us pick out precisely those  $v_s \in N(v_i)$ , i.e.  $v_s \in N(v_i) \Leftrightarrow a_{is} = 1$ . The last equality follows directly from the definition of the matrix-product, the definition of  $A$ ,  $A^\ell = (b_{ij})$  ( $c_{ij}) = A^{\ell-1}$  together with induction on  $\ell$ .

### Subgraph

**Definition 1.0.27.** A **subgraph** of a graph  $G = (V, E)$  is a pair  $G' = (V', E')$  such that  $V' \subseteq V$  and  $E' \subseteq (E \cap \binom{V'}{2})$ .

### Induced subgraph

**Definition 1.0.28.** Given a graph  $G = (V, E)$ , we say that a subgraph  $G' = (V', E')$  (1.0.27) is an **induced subgraph** if  $E' = (E \cap \binom{V'}{2})$ .

*Remark 1.0.29.* One way to formulate the condition  $E' = (E \cap \binom{V'}{2})$  is as the condition: For all  $u, v \in V'$  we have that  $uv \in E \Leftrightarrow uv \in E'$ .

**Example 1.0.30.** Take the graph  $G = K_4$ . Then  $G_1$  below is *not induced*, while  $G_2 = K_3$  is *induced*.

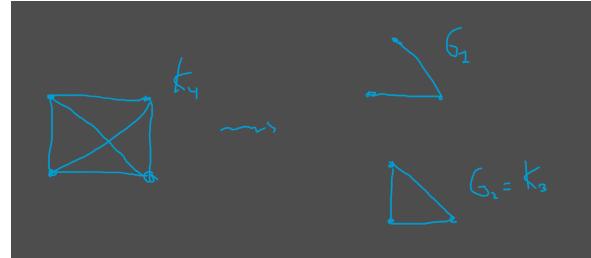


Figure 1.4: Non-induced ( $G_1$ ) and induced ( $G_2 = K_3$ ) subgraph from  $K_4$ .

# Chapter 2

## Lecture 2

Today:

- Graph class: trees.
- Characterizations.
- Spanning trees.
- Finding spanning trees (algorithms).

### Connected graph

**Definition 2.0.1.** A non-empty graph  $G$  is called **connected** if a *path* (1.0.14) exists between *any* two vertices  $u, v$  of the graph  $G$ .

### Maximal connected graph - component

**Definition 2.0.2.** A *maximal connected subgraph* of a graph  $G$  is called a **component** of  $G$ .

*Remark 2.0.3.* C.f. with *component* in a topological context, i.e. as a *maximal* connected subset of some topological space  $(T, \tau)$ .

### Bridge

**Definition 2.0.4.** An edge  $e$  in a graph  $G$  is called a **bridge** if  $G \setminus e$  has (strictly) more components than  $G$ .

**Lemma 2.0.5.** An edge  $e$  in a graph  $G$  is a bridge  $\Leftrightarrow e$  does not lie on any cycle in  $G$ .

*Proof.* Exercise:

**Lemma 2.0.6.** Adding an edge  $e$  to a graph  $H = (V, E)$  where  $e \in \binom{V}{2}$  can at most decrease the number of components by one.

*Proof.* Assume that  $e = \{u, v\}$ . We divide by cases.

Vertices  $u, v$  are in the same component of  $H$ : Two vertices are in the same component if there is a path between them. Hence adding another  $u-v$ -path does not change any connectedness property, given that one already exists in  $H$ .

Vertices  $u, v$  are in different components of  $H$ : Applying again the fact that two vertices  $u, v$  are in the same component if there is a path between them, we see that in  $H$  there is no path between  $u$  and  $v$  (given that they are in different components), but in  $H \cup e$  there is a path between  $u$  and  $v$  so they are now in the same component. For any other component in  $H$ , call it  $C$ , we have that  $C$  remains connected in  $H \cup e$  (we don't remove any paths). Any path from some vertex of  $C$  to some vertex  $x$  of  $H \cup e$  must then use  $uv$ . But for this to happen, we would need  $u$  or  $v$  to be in  $C$ , which we assumed was false. Hence no new paths from  $C$  to any vertex outside of  $C$  is created when going from  $H$  to  $H \cup e$ , and so  $C$  is still a component in  $H \cup e$ . This argument also shows that the components containing  $u$  and  $v$  do not connect with any other component in  $H \cup e$ . Thus we reduce the number of components exactly by one when going from  $H$  to  $H \cup e$ .  $\square$

$\Rightarrow$ : Assume that  $e = \{u, v\}$  lies on a cycle in  $G$ . Then there is an  $u-v$  path  $P$  between  $u$  and  $v$  in  $G \setminus e$ . This means that  $u$  and  $v$  must be in the same component  $\mathcal{K}$  of  $G \setminus e$ . By the proof of lemma 2.0.6(first case) we see that adding another  $u-v$  path does not change any components. Thus the number of components of  $G \setminus e$  is the same as the number of components of  $G$ . Hence  $e$  is not a bridge.

$\Leftarrow$ : If  $e = \{u, v\}$  and  $P$  is an  $u-v$  path in  $G \setminus e$ , then  $P \cup \{e\}$  is a cycle in  $G$ . This means that  $G \setminus e$  can not have an  $u-v$   $P$  path if  $e$  does not lie on any cycle in  $G$ . But this means that  $u$  and  $v$  must be in *different* components of  $G \setminus e$ . It follows from the proof of lemma 2.0.6(second case) that the number of components in  $G$  is exactly one smaller than the number of components in  $G \setminus e$ , so that  $e$  is a bridge (2.0.4).  $\square$

*Remark 2.0.7.* Notice that 2.0.6 says that  $\mathcal{C}(H \cup e) = \mathcal{C}(H)$  or  $\mathcal{C}(H \cup e) = \mathcal{C}(H) - 1$ . We may now exchange  $H$  for  $G \setminus e$  and  $H \cup e = G$  to get that  $\mathcal{C}(G) = \mathcal{C}(G \setminus e)$  or  $\mathcal{C}(G) = \mathcal{C}(G \setminus e) - 1$ . Hence we have that

$$\mathcal{C}(G \setminus e) - 1 \leq \mathcal{C}(G) \leq \mathcal{C}(G \setminus e).$$

### Forest

**Definition 2.0.8.** A graph  $G$  without cycles (= “acyclic graph”) is called a **forest**.

### Tree

**Definition 2.0.9.** A connected forest (2.0.1, 2.0.8) is called a **tree**.

*Remark 2.0.10.* We see from the definition above, that a forest has trees as connected components.

### Leaves

**Definition 2.0.11.** Vertices  $v$  in a tree with degree one are called **leaves**.

**Proposition 2.0.12.** Any tree  $G$  with  $|V(G)| \geq 2$  has a leaf. (2.0.11).

*Proof.* By definition of a tree, there must be precisely one path between pair of vertices  $u, v \in G$ , since if there was two different paths between  $u, v$ , one could form a cycle in  $G$ , contradicting that a tree is a forest and hence an acyclic graph. Since we are only concerned with *finite, simple graphs*, it follows all paths are of finite length. Hence there must be a maximal (with respect to length) path  $P = x_0 \dots x_{k-1}$  in  $G$ , of length  $k$ . If we assume on the contrary that the degree of each vertex in  $G$  is

at least of degree two, then in particular  $x_{k-1}$  is of degree two, so that we have an edge  $e = \{x_{k-1}, x_\ell\}$  where  $x_\ell \neq x_{k-2}$ . Then  $x_\ell$  can not be a vertex in this path, since if it was, we could form the cycle  $x_\ell P x_\ell$  in  $G$ , contradicting that  $G$  is a tree. Hence we see that  $P x_\ell$  is a *longer* path than  $P$ , contradiction! Hence there must be some vertex  $x_i$  of degree less than two. By definition a tree is connected, and so no vertex can have degree zero: There would then not exist any path between  $x_i$  and any other vertex  $x_j \neq x_i$ , and since  $|V(G)| \geq 2$ , by assumption, such a vertex  $x_j$  exists. Thus there must be a vertex of degree exactly one.  $\square$

Notation: For a graph  $G$  and a vertex subset  $W \subseteq V(G)$ , we denote by  $G \setminus W$  the *induced subgraph* (1.0.28) with vertex set  $V(G) \setminus W$ . For  $v \in V(G)$ , we also write  $G \setminus v$  for  $G \setminus \{v\}$ .

**Proposition 2.0.13.** *Let  $G$  be a graph and let  $v \in V$  with  $d(v) = 1$ . Then*

$$G \text{ is a tree} \Leftrightarrow G \setminus v \text{ is a tree.}$$

*Proof.* Exercise:

$\Rightarrow$ : Assume that  $G$  is a tree. This means that  $G$  is connected and acyclic. Removing an edge can not create a cycle, since if there was a cycle in  $G \setminus v$   $C = x_0 \dots, x_k x_0$  then this would still be a cycle in  $G$ .

We want to claim that  $G \setminus v$  is still connected. Assume that  $e = \{u, v\}$  is the unique edge from  $v$ . By assumption  $G$  is connected, so there is a path between every pair of vertices  $x, y$  in  $G$ . Since the degree of  $v$  is one, the only way  $v$  shows up in such a path is as the first or the last element in the path, i.e.  $P = vx_1 \dots x_{k-1}$  or  $P = x_0 \dots x_{k-1}v$ . Thus for any path  $P_{x,y}$  between  $x$  and  $y$ , the unique edge  $e$  out from  $v$  does not occur. Hence removing  $v$  still leaves  $G \setminus e$  connected, since we still have the path  $P_{x,y}$  in  $G \setminus e$  for any  $x, y$  where neither  $x$  nor  $y$  are equal to  $v$ .

We claim that the argument above shows that  $G \setminus v$  is connected and acyclic, hence is a tree.

$\Leftarrow$ : Assume that  $G \setminus v$  is a tree. This means that there is a path  $P_{x,y}$  between any pair of vertices  $x, y$  in  $G \setminus v$ . In particular, there is a path  $P_{u,x}$  between  $u$  and any other vertex  $x$  in  $G \setminus v$ . When we add  $v$  and  $e$ , then by going from  $v$  to  $u$  and then from  $u$  to  $x$  via  $P_{x,y}$ , we see that there is path from  $v$  to any other vertex  $x$  in  $G$  (with  $P_{u,u}$  just meaning that we stop at  $u$  after going from  $u$  from  $v$  via  $e$ ). This together with the paths  $P_{x,y}$  in  $G \setminus v \subset G$  shows us that  $G$  is connected.

To show that  $G$  is *acyclic*: There are no cycles in  $G \setminus v$  by assumption, hence the only way a cycle can show up in  $G$  is if  $v$  is a vertex in the cycle. But this would force  $d(v) \geq 2$ , contrary to our assumption. Hence  $G$  is acyclic.  $\square$

### Spanning tree

**Definition 2.0.14.** Let  $G$  be a connected (2.0.1) graph. A subgraph (1.0.27)  $T$  of  $G$  is a **spanning tree** of  $G$  if  $T$  is a tree and  $V(T) = V(G)$ , i.e. all vertices in  $G$  “show up” in  $T$ .

*Remark 2.0.15.* Related to this: Kirchhoff’s matrix tree theorem.

**Proposition 2.0.16.** *Every connected graph has a spanning tree.*

*Proof.* We prove this by induction on  $|E(G)|$ , i.e. the number of edges of  $G$ .

Case  $|E(G)| = 0, 1$ : If  $|E(G)| = 0$  then since  $G$  is connected, we must have that  $G$  has only one node, hence is acyclic and connected, so is a tree. If  $|E(G)| = 1$  then  $G$  must be  $K_2$ , which is acyclic and connected, so is a tree.

Case  $|E(G)| \geq 2$ : Assume it holds for  $|E(G)| = n$ . Consider a connected graph  $G$  such that  $|E(G)| = n + 1$ . If  $G$  is a tree then it is a spanning tree of itself. If  $G$  is not a tree, then since  $G$  is connected, it

must contain a cycle  $C = v_0 \dots v_{\ell-1} v_0$ . Let  $e = v_0 v_{\ell-1}$ . Furthermore, let  $G' = G \setminus e$ , i.e.  $V(G) = V(G')$  but  $E(G') = E(G) \setminus e$ . By 2.0.5 we see that  $e$  is not a bridge, so that by the same lemma we have that  $\mathcal{C}(G') \leq \mathcal{C}(G)$ . Since by 2.0.7 we also have  $\mathcal{C}(G) \leq \mathcal{C}(G') \Rightarrow \mathcal{C}(G) = \mathcal{C}(G')$ . Since  $G$  is connected we see that  $\mathcal{C}(G') = 1$ . Furthermore, we have that  $|E(G')| = n$ , hence it follows that by the inductive assumption,  $G'$  has a spanning tree  $T$ . Since  $V(G') = V(G) = V(T)$ ,  $T$  is also a spanning tree of  $G$ .  $\square$

**Theorem 2.0.17** (Characterization of trees). *Let  $G = (V, E)$  be a non-empty graph. Then the following are equivalent:*

- i)  $G$  is a tree.
- ii) Any two vertices in  $G$  are linked by a unique path.
- iii)  $G$  is connected and we have  $|E| = |V| - 1$ .
- iv)  $G$  is minimally connected, i.e.  $G$  is connected and  $G \setminus e$  is disconnected for every  $e \in E$ .
- v)  $G$  is maximally acyclic, i.e.  $G$  contains no cycle, but  $G \cup e = (V(G), E(G) \cup \{e\})$  has a cycle for  $e \in \binom{V(G)}{2} \setminus E$ .

*Remark 2.0.18.* I have added at least one more direction in the proof below (as compared to the lecture notes). However we do not claim that the added direction/directions are necessary.

*Proof.* i)  $\Rightarrow$  ii): If there are two different paths  $P_1, P_2$  between two vertices  $x$  and  $y$  in  $G$ , then we may form a cycle in  $G$ , contradicting that  $G$  is a tree.

i)  $\Rightarrow$  iii): Since  $G$  is a tree, it is connected. We use induction on  $n = |V|$  to prove that  $|E| = |V| - 1$ . If  $n = 1$ , then  $|E| = 0$ . Suppose  $|V(G)| = n \geq 2$ . Then by proposition 2.0.12 it has a leaf  $v$ , and  $G \setminus v$  is a tree by proposition 2.0.13. By the inductive assumption we then have that

$$\begin{aligned} |E(G \setminus v)| &= |V(G \setminus v)| - 1 \\ &= |V(G)| - 2. \end{aligned}$$

But since  $d(v) = 1$ , we have that  $|E(G \setminus v)| = |E(G)| - 1$ . Together, this gives us that

$$\begin{aligned} |E(G)| - 1 &= |V(G)| - 2 \\ \Leftrightarrow |E(G)| &= |V(G)| - 1, \end{aligned}$$

which is what we wanted to show.

iii)  $\Rightarrow$  ii): We proceed by induction on  $|V(G)| = n$ . For  $n = 1$  the statement is vacuously true, since there is only one vertex. For  $n > 1$ , we claim that there exists a vertex  $v'$  of degree one in  $G$ , since otherwise (using that since  $G$  is connected each vertex must have degree at least one) we have (using [Die17, p. 5]) that

$$\begin{aligned} 2|E(G)| &= 2(|V(G)| - 1) \\ &= 2n - 2 \\ &= \sum_{v \in V(G)} d(v) \\ &\geq 2|V(G)| \\ &= 2n, \end{aligned}$$

contradiction!

By induction, every pair of vertices in  $G \setminus v'$  are linked by a unique path. Hence the same must hold for  $G = (G \setminus v') \cup v'$ : Note that there is only one edge  $e = v', u$  going “out” from  $v'$ . We know that there is a unique path  $P_y = v_0 = u \dots v_{k-1} = y$  for every other vertex  $y \neq v'$  in  $G$ . First we then see that  $e + P_y$  with  $P_u = 0$  gives us a path from  $v'$  to any other vertex  $y$  in  $G$ . Since any path  $P$  starting at  $v'$  must go through  $e$ , if there was two different paths  $P_1, P_2$  from  $v'$  to  $y \neq u$  then  $uP_1, uP_2$  would be two distinct paths from  $u$  to  $y$ , a contradiction! Since  $d(v') = 1$ , there is also a unique path from  $v'$  to  $u$ .

*ii)  $\Rightarrow$  iv):* Since every pair of vertices is linked by a unique path,  $G$  is connected. Let  $e = uv \in E(G)$ . Then by assumption,  $e$  is the unique path connecting  $u$  and  $v$ , so  $G \setminus e$  is disconnected.

*iv)  $\Rightarrow$  v):* Since  $G \setminus e$  is disconnected for every  $e \in E(G)$ , we know that the number of components of  $G \setminus e$  must be greater than one for every  $e \in E(G)$ . But  $G$  is connected by assumption, i.e.  $\mathcal{C}(G) = 1$ , and  $\mathcal{C}(G \setminus e) > 1$  for every  $e \in E(G)$ . Hence every edge  $e$  is a bridge  $\Rightarrow e$  does not lie on any cycle in  $G$ . Since  $e$  was arbitrary, there can be no cycle (since any cycle would include more than one edge  $e$ , so surely one). Hence  $G$  is acyclic. Let  $uv \in \binom{V(G)}{2} \setminus E(G)$ . Since  $G$  is connected, there is a path  $P$  linking  $u$  and  $v$  in  $G$ . Therefore,  $G \cup e$  contains the cycle that goes from  $u$  to  $v$  by  $P$ , and then from  $v$  to  $u$  by  $e$  (note that  $P$  and  $e$  can not coincide since  $e$  was by assumption not an edge in  $G$ , and  $P$  must consist of edges in  $G$ ).

*v)  $\Rightarrow$  i):* Since  $G$  is acyclic,  $G$  is a forest (recall definition 2.0.8). It remains to show that  $G$  is connected. Assume on the contrary that  $u, v$  are a pair of vertices in  $G$  that belongs to different components. Then  $e = uv$  is a bridge in  $G \cup e$ . By lemma 2.0.5  $e$  does not lie on any cycle in  $G \cup e$ . Since  $G$  was acyclic, and we added an edge  $e$  that does not lie on any cycle in  $G \cup e$ , it follows that  $G \cup e$  is still acyclic: If there was a cycle  $C = v_0 \dots v_{k-1}v_0$  in  $G$ , then since  $e$  does not show up in this cycle, the cycle must consist entirely of edges  $v_i v_{i+1} \in E(G)$ . But then  $C$  is still a cycle in  $G$ , contradicting that  $G$  was acyclic. But this contradicts that  $G$  was maximally acyclic, hence  $u, v$  must be in the same component for arbitrary  $u, v \in V(G) \Rightarrow G$  is connected  $\Rightarrow G$  is a tree.  $\square$

### Spanning forest

**Definition 2.0.19.** A **spanning forest** of a graph  $G$  is a *subgraph* (recall definition 1.0.27)  $W$  with the *same number of components* as  $G$  and such that the components of  $W$  are *spanning trees* of the *components of  $G$* .

*Remark 2.0.20.* Any graph has a spanning forest.

**Corollary 2.0.21.** A graph  $G$  with  $t$  components is a forest if and only if  $|E(G)| = |V(G)| - t$ .

#### 2.0.1 Finding spanning trees

Algorithm 1: Let  $G = (V, E)$  be a graph with  $E = \{e_1, \dots, e_q\}$ . We inductively construct edge-sets  $E_0 \subseteq E_1 \subseteq \dots \subseteq E$  as follows:

1. Let  $E_0 := \emptyset$ .
2. If  $E_{i-1}$  (for  $i \geq 1$ ) is already constructed, then

$$E_i := \begin{cases} E_{i-1} \cup \{e_i\}, & \text{if } (V, E_{i-1} \cup \{e_i\}) \text{ has no cycle,} \\ E_{i-1}, & \text{otherwise} \end{cases}$$

The algorithm stops if  $|E_i| = |V| - 1$  or  $i = q$ . In this case, it returns  $T = (V, E_i)$ .

**Example 2.0.22.**

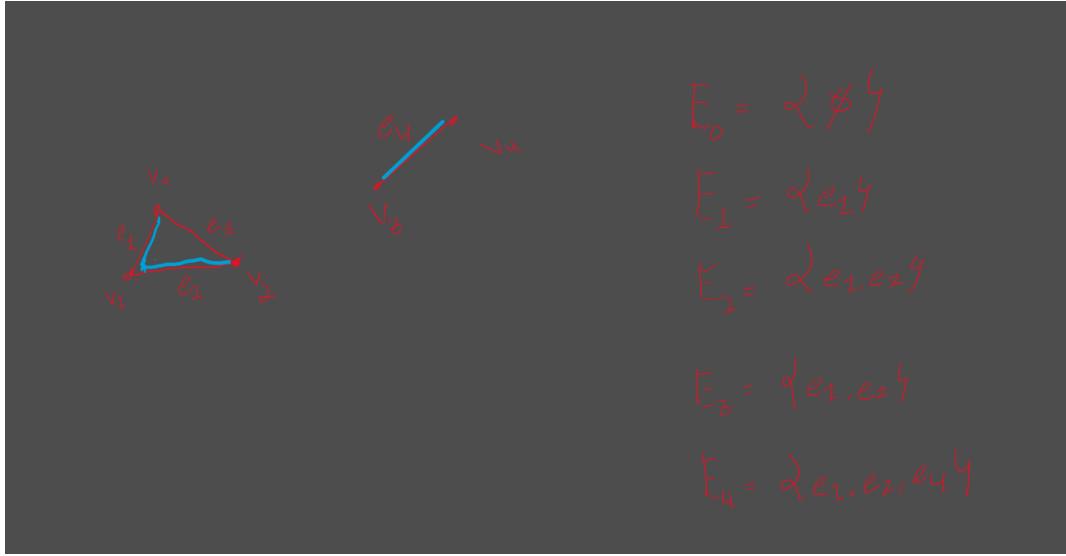


Figure 2.1: Example of algorithm 1

**Proposition 2.0.23.** Let  $|V| = n$ . With the notation in algorithm 1:

- i) If the algorithm ends with a graph  $T$  with  $n - 1$  edges, then  $G$  is connected and  $T$  is a spanning tree.
- ii) If  $T$  has  $k < n - 1$  edges then  $G$  is not connected and has  $n - k$  components.

*Proof.* By construction,  $T$  does not contain cycles and is thus a forest (i.e. an acyclic graph). Let  $k = |E(T)|$  and let  $\ell$  be the number of components of  $T$ . Then, by corollary 2.0.21 we have that

$$\begin{aligned}
 |E(T)| &= k \\
 &= |V(T)| - \ell \\
 &= n - \ell \\
 \Rightarrow n - \ell &= k \\
 \Rightarrow n - k &= \ell
 \end{aligned}$$

where we used that  $T = (V, E_i)$  so that  $|V(G)| = |V(T)|$ , and  $|V(G)| = n$  with our notation. Hence the number of components  $\ell$  of  $T$  equal  $n - k$ . In particular, if

$$\begin{aligned}
 k &= |E(T)| \\
 &= n - 1 \\
 \Rightarrow \ell &= n - k \\
 &= n - (n - 1) \\
 &= 1,
 \end{aligned}$$

so that the number of components of  $T$ ,  $\ell$ , equals one, hence  $T$  is connected. Since  $T$  is a subgraph of  $G$  that includes all vertices of  $G$ , it follows that there is a path between any pair of vertices in  $G$ , so that  $G$  is connected. Since  $T$  is constructed to contain no cycles, we see that  $T$  is a tree. We claim this shows (i).

If  $k < n - 1$ , then since removing edges from  $G$  can only result in more components (never less), we know that  $T$  must have *at least* as many components as  $G$ . Suppose that  $T$  has more components

than  $G$ . Then there exists vertices  $u, v$  in one component in  $G$  that lie in different components in  $T$ . Let  $H$  be the component of  $T$  in which  $u$  is in, let  $P = u_0 \dots u_t$  be a path with  $u_0 = u$  and  $u_t = v$ , and let  $i$  be the maximal index in the path  $P$  such that  $v_i \in H$ . Since  $u, v$  are by assumption in different components in  $T$  it follows that  $i < t$  and that  $v_{i+1} \notin H$ . If the edge  $e = v_i v_{i+1}$  was in  $E(T)$ , then  $v_i$  and  $v_{i+1}$  would be in the same component, contradiction! Hence  $e \notin E(T)$ . Since  $|E(T)| = k < n - 1 = |V| - 1$  edges, the algorithm must have covered all edges of  $G$ , so in particular must have covered the edge  $e$  and decided that it would create a cycle with the other edges in  $T$  (which is why  $e \notin E(T)$ ). This means that  $e$  lies on a cycle in  $T + e := (T, E(T) \cup \{e\})$ . By 2.0.6 this means that  $e$  is not a bridge in  $T + e$ , hence  $T + e$  must have the same number of components as  $(V(T), E(T))$ . But  $e$  links two vertices  $u, v$  that lie in different components in  $T$ , so that  $T + e$  must have less components than  $T$ , contradiction! Thus  $T$  must have the same number of components as  $G$ . But  $T$  has  $\ell = n - k$  components, hence so does  $G$ . Since  $k < n - 1$  we see that  $\ell = n - k > n - (n - 1) = 1$  so that  $G$  has more than one component, hence is not connected, which is what we wanted to show.

□

*Remark 2.0.24.* Note that since algorithm 1 stops if  $|E_i| = |V(G)| - 1$  or  $i = q$ , then whenever  $|V(G)| = n$  we have that  $|E_i| = \min\{|V(G)| - 1, q\}$  so that  $|E_i| \leq n - 1$ , which was used to explore all possibilites in the proof above.

# Chapter 3

## Lecture 3

Today:

- Minimal spanning trees.
- Shortest paths.
- Counting spanning trees.

### Weighted graph and minimal spanning tree

**Definition 3.0.1.** A graph  $G = (V, E)$  together with a map  $w : E \rightarrow \mathbb{R}$  is called a **weighted graph**. A **minimal spanning tree** (forest) of  $G$  is a spanning tree (spanning forest)  $T \subseteq G$  such that the value

$$\sum_{e \in E(T)} w(e)$$

is minimal.

Finding a mimal spanning tree/forest: Algorithm 2 (Kruskal's algorithm). Let  $G = (V, E)$  be a weighted graph (definition 3.0.1) with weight-function  $w : E \rightarrow \mathbb{R}$ . Let  $E = \{e_1, \dots, e_r\}$  such that

$$w(e_1) \leq w(e_2) \leq \dots \leq w(e_r).$$

Then apply algorithm 1 (2.0.1).

**Theorem 3.0.2** (Kruskal). *With the notation in algorithm 1 and algorithm 2,  $T$  is a minimal spanning forest (definition 3.0.1) for  $G$ .*

**Lemma 3.0.3.** *Let  $(T, E)$  and  $(T, E')$  be forests with  $|E| < |E'|$ . Then there exists and edge  $e \in E' \setminus E$  such that  $T \cup \{e\} = (V, E \cup \{e\})$  is a forest.*

*Proof.* Exercise:

**Lemma 3.0.4.** *For a forest  $G = (V, E)$  it holds that  $|V| = E - c(G)$  where  $c(G)$  is the number of components in  $G$ .*

*Proof.* Recall that for a tree  $T$  we have that  $|E| = |V| - 1$  (theorem 2.0.17). Since  $G$  is a forest, it is

composed of a set of disjoint trees  $T_1 = (V_1, E_1), \dots, T_n = (V_n, E_n)$ . Thus it holds that

$$\begin{aligned} |E| &= \sum_{i=1}^n |E_i| \\ &= \sum_i^n (|V_i| - 1) \\ &= |V| - c(G) \end{aligned}$$

since  $\sum_{i=1}^n |V_i| = |V|$  and since the the trees  $T_1, \dots, T_n$  are exactly the components of  $G$ .  $\square$

If  $|E| < |E'|$  (and since they share the same vertex-set) then by lemma 3.0.4 we have that

$$\begin{aligned} |E| &= |V| - c(T) \\ &< |V| - c(T') \\ &= |E'|. \end{aligned}$$

Thus, we have that

$$\begin{aligned} |V| - c(T) &< |V| - c(T') \\ \Leftrightarrow -c(T) &< -c(T') \\ \Leftrightarrow c(T') &< c(T). \end{aligned}$$

Hence  $T$  has more components than  $T'$ . Assume that there is no edge  $e'$  in  $T'$  with endpoints in different  $T$ -components  $C_i, C_j$  (with  $i \neq j$ ). Let  $D_1, \dots, D_m$  be the components of  $T'$ . Pick a vertex  $v_i \in V$ . Then  $v_i$  belongs to some component  $C_i$  of  $T$ , and some component  $D_j$  of  $T'$ . Assume that there was some  $v_j \in D_j$  not in  $C_i$ . Then there is a path in  $T'$  from  $v_i$  to  $v_j$ , by connectedness of  $D_j$ . Since the components  $C_1, \dots, C_n$  partition  $T$  (and share vertex set with  $T'$ ), there must be some  $C_j$  that contains  $v_j$  as a node. Let  $w = w_1 = v_i, \dots, w_k = v_j$  be the path (in  $T'$ ) from  $v_i$  to  $v_j$ . Since  $C_i, C_j$  are components in different components, there must be a first edge  $e' = \{w_i, w_{i+1}\}$  where  $w_i \in C_i$  and  $w_{i+1} \in C_j$ . But this is a contradiction to our assumption. Hence any  $v_j \in D_j$  must be in  $C_i$ , so that  $D_j \subset C_i$ . If we pick one  $v_i$  from each component  $C_i$ , we get  $n$  different nodes  $\{v_1, \dots, v_n\}$  and associated components of  $T'$   $D_1, \dots, D_n$  that contains  $v_1, \dots, v_n$ , where say (after possible reordering)  $v_i \in D_i \subset C_i$  for  $i = 1, \dots, n$ . If  $D_i \cap D_j \neq \emptyset$  then  $C_i \cap C_j \neq \emptyset$ , which is a contradiction to the  $C_i$  partitioning the vertex set of  $V(T) = V(T')$ . Thus the  $n$  components  $D_1, \dots, D_n$  must all be distinct. It follows that  $c(T') \geq c(T)$ , contrary to what we showed earlier, contradiction! Therefore, there must be an edge  $e' \in T'$  that links components of  $T$ . It is clear that this edge  $e'$  can not be an edge in  $E$  (since the components are disconnected). Therefore  $e' \in E' \setminus E$ . Notice that  $T \cup \{e'\}$  is such that  $T$  has more components than  $T \cup \{e'\}$  (since  $e'$  collapses two components  $C_a, C_b$  to a single new component  $C_{ab}$  and leave the other components unchanged). By lemma 2.0.5 it follows that  $e'$  does not lie on any cycle in  $T \cup \{e'\}$ . Assume there was a cycle  $C = c_1 \dots c_k$  in  $T \cup \{e'\}$ . Since we showed that  $e'$  is not on any cycle, we must have that  $c_i \neq e'$  for  $i = 1, \dots, k$ . But then  $C$  is already a cycle in  $T$ , contradicting that  $T$  is a forest. Hence  $T \cup \{e'\}$  must be acyclic, i.e. a forest.  $\square$

We prove theorem 3.0.2.

*Proof.* By 2.0.23  $T$  is a spanning tree/forest. It remains to show minimality. Let  $T$  have edges  $\{e_{i_1}, \dots, e_{i_r}\}$  such that  $w(e_{i_1}) \leq w(e_{i_2}) \leq \dots \leq w(e_{i_r})$  (the images are in  $\mathbb{R}$  which is totally ordered with respect to  $\leq$ ). Suppose there is a spanning forest  $T'$  with edges  $f_{i_1}, \dots, f_{i_r}$  (the number of edges in a spanning forest are fixed by the graph) such that  $w(f_{i_1}) \leq w(f_{i_2}) \leq \dots \leq w(f_{i_r})$  and

$$\sum_{f \in E(T')} w(f) < \sum_{e \in e(T)} w(e).$$

Let  $1 \leq k \leq r$  be minimal such that  $w(f_{i_k}) < w(e_{i_k})$  (the set of such edges are non-empty by the above inequality). Consider  $T_{k-1} = (V(G), \{e_{i_1}, \dots, e_{i_{k-1}}\})$  and  $F_k = (V(G), \{f_{i_1}, \dots, f_{i_k}\})$ . By lemma 3.0.3 we have that there exists  $1 \leq j \leq k$  such that

$$\{e_{i_1}, \dots, e_{i_{k-1}}, f_{i_j}\}$$

is a forest and by the lemma,  $f_{i_j}$  is not in  $E(T_{k-1})$ . In particular,

$$w(f_{i_j}) \leq w(f_{i_k}) < w(e_{i_k})$$

But this means that in the sorting of the weights of the edges of  $G$ , initially in the Kruskal-algorithm, one would have that  $f_{i_j}$  was sorted before  $e_{i_k}$  since its weight is less than  $e_{i_k}$ . Since  $f_{i_j}$  is by construction not already in  $T_{k-1}$ , we must have that  $w(f_{i_j}) \geq w(e_{i_s})$  for  $1 \leq s \leq k-1$ . But in  $k^{\text{th}}$  step of Kruskal algorithms, it would then either add  $f_{i_j}$  or some other edge  $t$  with  $w(t) \leq w(f_{i_j})$  and  $t \neq e_{i_k}$  and  $t$  not in  $E(T_{k-1})$  such that  $t$  does not create a cycle when added. If no such  $t$  exists, it would add  $f_{i_j}$ . Hence, this contradicts the procedure of the algorithm.

Perhaps to state the last step more formally: If  $\mathcal{A}_k = \{e \notin E(T_{k-1}) : T_{k-1} \cup \{e\} \text{ does not create a cycle}\}$  then  $e_{i_k} = \arg_{e \in \mathcal{A}_k} \min w(e)$ . So  $w(e_k) \leq w(f_{i_j})$  since  $f_{i_j} \in \mathcal{A}_k$ , contradiction!  $\square$

*Remark 3.0.5.* If  $w$  is injective then the minimal spanning tree is unique. This is clear in the sense that the sorting is then unique (i.e.  $\leq$  never shows up when comparing edges, but only  $<$ ; we get a monotone sequence of weights so the sorting is unique determined by the weights).

### Weighted path $\ell_w(P)$

**Definition 3.0.6.** Let  $G = (V, E)$  be a weighted connected graph with weight-function  $w : E \rightarrow \mathbb{R}_{\geq 0}$ . The **weighted length** of a path  $P$  in  $G$  is

$$\ell_w(P) := \sum_{e \in P} w(e).$$

## 3.1 Finding shortest path

Algorithm 3 (Dijkstra's algorithm): Given  $G = (V, E)$  a connected weighted graph with  $w : E \rightarrow \mathbb{R}_{\geq 0}$  a weight-function and  $v \in V$ , we construct sets  $V_0 \subseteq V_1 \subseteq \dots \subseteq V$ , and  $\emptyset \subseteq E_0 \subseteq E_1 \subseteq \dots \subseteq E$ , and sequence of vertices  $v_0, v_1, \dots$  and  $\ell(v_0), \ell(v_1), \dots$  as follows:

- (1) Set  $v_0 = v$ ,  $\ell(v_0) = 0$ ,  $V_0 = \{v_0\}$  and  $E_0 = \emptyset$ .
- (2) If the sequence  $v_0, v_1, \dots, v_i$  with  $V_i = \{v_0, v_1, \dots, v_i\}$ ,  $E_i = \{e_1, e_2, \dots, e_i\}$  and  $\ell(v_0), \ell(v_1), \dots, \ell(v_i)$  are constructed, then
  - If  $i < |V| - 1$  then choose an edge  $e = vw$  where  $v \in V_i$  and  $w \in V \setminus V_i$  such that  $\ell(v) + w(e)$  is minimal, and set

$$\begin{aligned} v_{i+1} &= w \\ \ell(v_{i+1}) &= \ell(v) + w(e) \\ V_{i+1} &= V_i \cup \{v_{i+1}\} \\ E_{i+1} &= E_i \cup \{e\} \end{aligned}$$

- If  $i = |V| - 1$  then return  $T = (V_i, E_i)$ .

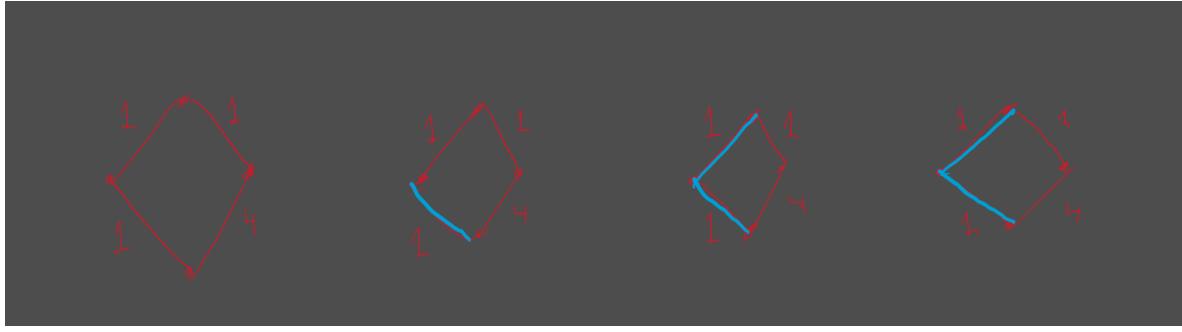


Figure 3.1: Example of Dijkstra's algorithm

**Example 3.1.1.** Above, we provide a (schematic) example of how Dijkstras algorithm would work in practice, where we have a weighted connect graph  $G = (V, E)$  with  $|V| = 4$ , hence the path should be of length  $|V| - 1 = 3$ .

*Remark 3.1.2.* Note below that we define the distance (with respect to a weight-function  $w$ ) in  $G$  from some initial node  $v_0$  to some other node  $u$  as

$$d(u) = \min_{P: v_0 \sim u} \ell_w(P)$$

where  $\ell_w(P)$  is defined the same as in definition 3.0.6.

**Theorem 3.1.3** (Dijkstra's). *With the notation in algorithm 3,  $T$  is a spanning tree of  $G$  such that for all  $v \in V(G)$  the unique path between  $u$  and  $v$  in  $T$  is a path of minimal weighted length  $\ell(v)$  in  $G$ .*

*Proof.* By construction all graphs  $T_i = (V_i, E_i)$  are connected graphs on  $V_i$  and  $|E_i| = |V_i| - 1$ . Hence by 2.0.17 we have that  $T_i$  is a tree for all  $i$ , and the last tree is a spanning tree of  $G$  (since we get  $V_{|V|-1} = \{v_0, v_1, \dots, v_{|V|-1}\}$  with cardinality  $|V|$  and all nodes are different).

We will show, by induction on  $i$ , that  $T_i$  has the claimed minimality property for all  $v \in V_i$ . For  $i = 0$  this is true since there is only the element  $v_0 \in V_i$  with  $\ell(v_0) = 0$  and  $d(v_0) = 0$  (say by convention).

Assume  $T_{i-1}$  satisfies the assumption, and notice that  $T_{i-1} \subset T_i$ . Thus, it suffices to show the minimality of the length of the unique (since it is a tree; otherwise has a cycle) path from  $v$  to  $v_i$ . Let  $Q = v_0, \dots, v_\ell$  with  $v_\ell = v_i$  and  $v_0 = v$  be a path of minimal length in  $G$ . Let  $0 \leq j \leq \ell-1$  be maximal such that  $v_j \in V_{i-1}$  but  $v_{j+1} \notin V_{i-1}$  and let  $Q_0 = v_0, \dots, v_j$  and  $Q_1 = v_{j+1}, \dots, v_\ell$  be subpaths of  $Q$ .

Let  $d(v_j)$  denote the shortest path in  $G$  from  $v$  to  $v_j$ , for any other node  $v_j$ . Notice that if  $\ell_w(Q_0) > d(v_j)$  then take the path  $P^*$  of distance  $d(v_j)$  from  $v$  to  $v_j$ , composed with  $Q_1$ . This gives a shorter path in  $G$  from  $v$  to  $v_i$ , contradicting the minimality of  $Q$ . Thus  $\ell_w(Q_0) \leq d(v_j)$ . Since  $d(v_j)$  is minimal by assumption in fact we have that  $\ell_w(Q_0) = d(v_j)$ . But since  $v_j \in V_{i-1}$  by the inductive hypothesis we have that  $d(v_j) = \ell(v_j)$ .

Since by assumption  $v_i$  is the chosen vertex in step  $i$ , and we presume that  $v_j \in V_{i-1}$  and  $v_{j+1} \notin V_{i-1}$ , it follows (by definition) that  $\ell(v_i) \leq \ell(v_j) + w(v_j v_{j+1})$ .

By the inductive assumption, we have that

$$\begin{aligned} \ell(v_i) &= \min_{u \in V_{i-1}} (\ell(u) + w(u v_i)) \\ &= \min_{u \in V_{i-1}} (d(u) + w(u v_i)). \end{aligned}$$

Assume that  $u$  is chosen as  $p$ , i.e. we have that  $\ell(v_i) = d(p) + w(pv_i)$ . By definition,  $d(p)$  is the distance of shortest path in  $G$  from  $v$  to  $p$ . Then notice that the unique path from  $v_0$  to  $p$  in  $T_{i-1}$  (which by induction have distance  $d(p)$ ), together with the edge  $pv_i$ , is a path in  $T_i$  from  $v_0$  to  $v_i$ . But  $T_i$  is a tree, so this must be the same path as  $P$ . Hence this path  $P$  must have distance

$$\begin{aligned}\ell_w(P) &= d(p) + w(pv_i) \\ &= \ell(v_i).\end{aligned}$$

Furthermore, notice that  $\ell_w(Q_1) \geq 0$ .

If we put this all together we see that:

$$\begin{aligned}\ell_w(Q) &= \ell_w(Q_0) + w(v_j v_{j+1}) + \ell_w(Q_1) \\ &= \ell(v_j) + w(v_j v_{j+1}) + \ell_w(Q_1) \\ &\geq \ell(v_i) + \underbrace{\ell_w(Q_1)}_{\geq 0} \\ &\geq \ell(v_i).\end{aligned}$$

Since by assumption we already have  $\ell_w(Q) \leq \ell_w(P) \Rightarrow \ell_w(P) = \ell_w(Q)$ , which is what we wanted to show.  $\square$

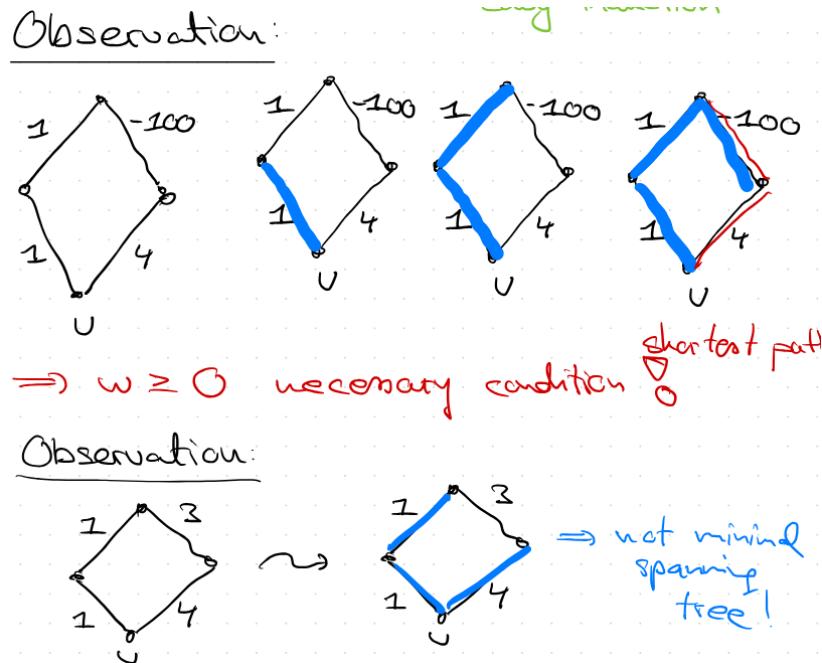


Figure 3.2: Example 3.1.4.1 (upper) and example 3.1.4.2(lower)

### Example 3.1.4.

The notation is the source as  $U$ , left one  $L$ , right one  $R$  and upper one  $T$ . Example 3.1.4.1: Note that does not take the shortest path from  $U$  to  $T$ . We get a path of length 2, but there exists a path of length  $4 + (-100) = -96$ !

### Example 3.1.4.2:

1. Initialize with  $U$ , set  $\ell(U) = 0$ ,  $V_0 = \{U\}$  and  $E_0 = \emptyset$ .
2. Add  $L$ , set  $\ell(L) = \ell(U) + 1 = 1$ ,  $V_1 = \{U, L\}$  and  $E_1 = \{UL\}$ .
3. Add  $T$ , set  $\ell(T) = \ell(L) + 1 = 2$ ,  $V_2 = \{U, L, T\}$  and  $E_2 = \{UL, LT\}$ .
4. In this step, there are only two choices, but  $\ell(T) + w(TR) = 2 + 3 = 5$  and  $\ell(U) + w(UR) = 4$ , so Dijkstra's algorithm chooses the edge  $UR$ . But this is not a *minimal spanning tree* (definition 3.0.1) since  $1 + 1 + 4 = 6 > 1 + 1 + 3 = 5$ .

### Laplacian matrix

**Definition 3.1.5.** The **Laplacian matrix** of a graph  $G = (V, E)$  with  $V = \{v_1, \dots, v_n\}$  is an  $n \times n$ -matrix  $L(G) = (a_{ij})$  defined by

$$a_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j \\ -1, & \text{if } i \neq j \text{ and } v_j \text{ is adjacent to } v_i, \\ 0, & \text{otherwise} \end{cases} .$$

**Example 3.1.6.** Consider the graph in the picture.

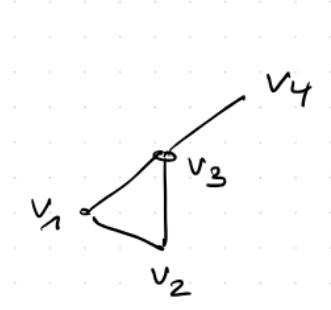


Figure 3.3: Enter Caption

Its Laplacian matrix is

$$L(G) = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} .$$

One may compute the determinant of the  $(3, 3)$  and  $(2, 3)$  minor. The former has determinant 3 and the latter has determinant  $-3$ . Notice that the former is a **principal minor**, i.e. on the form  $(n, n)$ ,

*Remark 3.1.7.* For an  $n \times n$ -matrix  $M$  let  $M_{\ell, m}$  be the matrix obtained by deleting the  $\ell^{\text{th}}$  row and  $m^{\text{th}}$  column.

**Theorem 3.1.8** (Kirchhoff's theorem). *Let  $G$  be a connected graph. Then the number of spanning trees  $\tau(G)$  is given by*

$$\tau(G) = (-1)^{\ell+m} \det(L(G)_{\ell, m})$$

for all  $1 \leq \ell, m \leq n$ .

*Remark 3.1.9.* Kirschoff's theorem is equivalent to

$$\text{adj}(L(G)) = \tau(G) \cdot \mathbf{1}_n$$

where  $\mathbf{1}_n \in \mathbb{R}^{n \times n}$  denotes the all-ones matrix. The claim is the same as saying that if  $C$  is the cofactor-matrix associated with  $L(G)$ , then  $C = \tau(G) \cdot \mathbf{1}_n$ . If we take the transpose of both sides, then since  $\tau(G)$  is a scalar and  $\mathbf{1}_n^T = \mathbf{1}_n$ , nothing happens to the right-side of the equation, while  $C^T = \text{adj}(L(G))$ .

**Lemma 3.1.10.**

$$\text{adj}(L(G)) = \alpha \cdot \mathbf{1}_n$$

for some  $\alpha \in \mathbb{R}$ .

*Remark 3.1.11.* Below, we only give a proof for connected graphs  $G$ .

*Proof.* Exercise: For an  $n \times n$ -matrix  $L(G)$  we have that  $L(G)\text{adj}(L(G)) = \det(L(G))I_n$ . We note that by construction, the sum of each row of the Laplacian matrix  $L(G)$  is zero. Therefore  $L(G)\mathbf{1} = 0$  where  $\mathbf{1}$  is the  $n \times 1$ -matrix with all ones. Therefore  $L(G)x = 0$  has a non-trivial solution which by linear algebra theory is equivalent to  $\det(L(G)) = 0$ .

Therefore, we see that  $L(G) \cdot \text{adj}(L(G)) = 0$ . Hence each column of  $\text{adj}(L(G))$  is in the kernel of  $L(G)$  (if we think of  $L(G)$  as a linear map).

**Lemma 3.1.12.**  $L(G) = D(G) - A(G)$  where  $D(G)$  is the diagonal matrix with  $\deg(v_i)$  on matrix-element  $(i, i)$ , and  $A(G)$  is the adjacency matrix of  $G$  (1.0.19).

*Proof.* By construction of  $L(G)$ . □

Since  $x^T M x = \sum_{i,j} x_i M_{ij} x_j$  for any matrix  $M$  of dimension  $n \times n$ , we have that

$$\begin{aligned} x^T L(G) x &= x^T (D - A)x \\ &= x^T Dx - x^T Ax \\ &= \sum_{u,v} x_u D_{uv} x_v - \sum_{u,v} x_u A_{uv} x_v \\ &= \sum_u x_u D_{u,u} x_u - 2 \sum_{\{u,v\} \in E(G)} x_u x_v \\ &= \sum_{u \in V(G)} \deg(u) x_u^2 - 2 \sum_{\{u,v\} \in E(G)} x_u x_v \\ &= \sum_{\{u,v\} \in E(G)} (x_u^2 + x_v^2) - 2 \sum_{\{u,v\} \in E(G)} x_u x_v \\ &= \sum_{\{u,v\} \in E(G)} (x_u - x_v)^2. \end{aligned}$$

*Remark 3.1.13.* Here we in the next to last equality used that

$$\begin{aligned} \sum_u \deg(u)x_u^2 &= \sum_u x_u^2 \deg(u) \\ &= \sum_u x_u^2 \left( \sum_v A_{uv} \right) \\ &= \sum_{u,v} A_{uv} x_u^2 \\ &= \frac{1}{2} \sum_{u,v} A_{uv} (x_u^2 + x_v^2) \\ &= \sum_{u,v} x_u^2 + x_v^2 \end{aligned}$$

where  $u, v$  in the lower-index is to be understood as  $\{u, v\} \in E(G)$ . To see this, use that  $A_{uv}$  is symmetric and count each edge two times.

Thus, we have that  $x^T L(G)x = \sum_{\{u,v\} \in E(G)} (x_u - x_v)^2$ . Since each term in the sum is greater than or equal to 0, it follows that  $x^T L(G)x = 0 \Leftrightarrow x_u = x_v$  for all  $u, v$  such that  $\{u, v\} \in E(G)$ . In particular, if  $G$  is connected then  $x^T L(G)x \Leftrightarrow x_u = x_v$  for all  $1 \leq u, v \leq n$  which is the same as saying that  $x \in \text{span}\{\mathbf{1}\}$ .

Since we showed that each column of  $\text{adj}(L(G))$  is in the kernel of  $L(G)$ , it follows that each column  $\text{adj}(L(G))_i \in \text{span}\{\mathbf{1}\} \Rightarrow \text{adj}(L(G))_i = \alpha_i \mathbf{1}$ .

Hence

$$\text{adj}(L(G)) = \begin{pmatrix} \alpha_1 & \dots & \alpha_n \\ \vdots & \ddots & \vdots \\ \alpha_1 & \dots & \alpha_n \end{pmatrix}$$

From the identity

$$\begin{aligned} \text{adj}(L(G))L(G) &= \det(L(G))I_n \\ &= 0, \end{aligned}$$

it also holds that the rows of  $\text{adj}(L(G))$  are in the kernel of  $L(G)$ . Again, using that  $x^T L(G)x = \sum_{u,v} (x_u - x_v)^2$  for connected  $G$ , we see that the rows of  $\text{adj}(L(G))_j = \beta_j \mathbf{1}$ . Thus we have that

$$\text{adj}(L(G)) = \begin{pmatrix} \beta_1 & \dots & \beta_1 \\ \vdots & \ddots & \vdots \\ \beta_n & \dots & \beta_n \end{pmatrix}.$$

Hence, we have that

$$\begin{aligned} \text{adj}(L(G)) &= \begin{pmatrix} \alpha_1 & \dots & \alpha_n \\ \vdots & \ddots & \vdots \\ \alpha_1 & \dots & \alpha_n \end{pmatrix} \\ &= \begin{pmatrix} \beta_1 & \dots & \beta_1 \\ \vdots & \ddots & \vdots \\ \beta_n & \dots & \beta_n \end{pmatrix} \end{aligned}$$

Therefore,  $\alpha_1 = \dots = \alpha_n = \beta_1 := \alpha$ . So indeed, we have that  $\text{adj}(L(G)) = \alpha \mathbf{1}$ .

□

(Oriented) vertex-edge incidence matrix  $N(G)$

**Definition 3.1.14.** The **oriented vertex-edge incidence matrix** of a graph  $G$  with vertices  $v_1, \dots, v_n$  and edges  $e_1, \dots, e_m$  is the  $n \times m$ -matrix  $N = N(G)$  with elements

$$N_{ij} = \begin{cases} 1, & \text{if } e_j = v_i v_k \text{ with } i < k \\ -1, & \text{if } e_j = v_i v_k \text{ with } i > k \\ 0, & \text{otherwise} \end{cases}$$

**Lemma 3.1.15.**

$$L(G) = N(G)N(G)^T$$

*Proof.* Exercise:

□

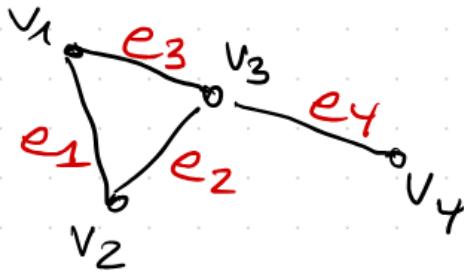


Figure 3.4: Enter Caption

**Example 3.1.16.** The graph  $G$  in figure 3.4 have the following oriented vertex-edge incidence matrix  $N(G)$ :

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (3.1.1)$$

Recall: The Cauchy-Binet formula: For  $n \geq m$ ,  $A \in \mathbb{R}^{m \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  we have that

$$\det(AB) = \sum_{S \in \binom{[n]}{m}} \det(A_{[m], S}) \det(B_{S, [m]})$$

where  $A_{I,J}$  denotes the submatrix of  $A$  with rows respectively columns indexed by in  $I$ , respectively  $J$ .

*Proof.*

□

**Corollary 3.1.17** (Cayley's formula). *The complete graph  $K_n$  has  $n^{n-2}$  spanning trees.*

*Proof.* Exercise.

□

# Chapter 4

## Lecture 4: Connectivity

Recall: A graph  $G$  is connected if any pair of vertices  $u, v$  in  $G$  are linked by some path  $P$ .

Today: “Higher connectivity”: Graphs that remain connected even when vertices (or edges) are removed. Applications: Failure tolerant networks (telecome trains etc).

**Definition 4.0.1** ( $k$ -connected). For  $k \geq 0$ , a graph  $G = (V, E)$  is  **$k$ -connected** if:

1. The set of nodes in  $G$  are (strictly) greater than  $k$ .
2.  $G \setminus S$  is connected for all subsets  $S \subseteq V$  such that  $|S|$  has cardinality less than  $k$ .

**Definition 4.0.2** (vertex connectivity  $\kappa(G)$ ). The largest  $k$  such that a graph  $G$  is  $k$ -connected is the **vertex connectivity**  $\kappa(G)$  of  $G$ .

- Any non-empty graph  $G$  is 0-connected.
- 1-connected non-trivial ( $\neq K_1$ ) graphs are exactly the connected graphs.
- The complete graph  $K_n$  on  $n$  vertices is  $n - 1$ -connected.
- If  $G$  is a tree, and  $|V(G)| \geq 2$  then  $\kappa(G) = 1$ .

Question: How do 2-connected graphs look like?

**Definition 4.0.3** (Independent paths). Two or more paths in a graph  $G$  between the same points  $u, v$  are **independent** if they do not have an *inner vertex* in common.

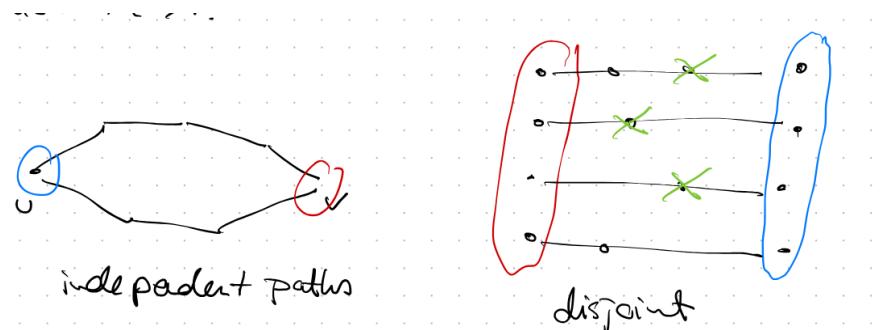


Figure 4.1: Illustration of independent/disjoint paths

**Definition 4.0.4** ( $k$ -connected: Alternative definition). A graph  $G$  is  $k$ -connected if any two pair of vertices can be joined by  $k$  independent paths.

**Theorem 4.0.5** (Menger, 1927). *These two definitions of  $k$ -connected are equivalent.*

## 4.1 Paths between sets

**Definition 4.1.1** ( $A$ - $B$  path). Given sets  $A, B \subseteq V$ , a path  $P = x_0 \dots x_k$  is an  $A$ - $B$  path if  $V(P) \cap A = \{x_0\}$  and  $V(P) \cap B = \{x_k\}$ .

**Definition 4.1.2** (Separator). If  $A, B \subseteq V$  and  $X \subset V \cup E$  are such that every  $A$ - $B$  path in  $G$  contains a vertex or an edge of  $X$ , we say that  $X$  separates  $A, B$  in  $G$ , or that  $X$  is an **separator** of  $A, B$ .

We also say that  $X$  separates  $G$  if  $G - X$  is disconnected, i.e. if  $X$  separates two vertices that are not in  $X$ .

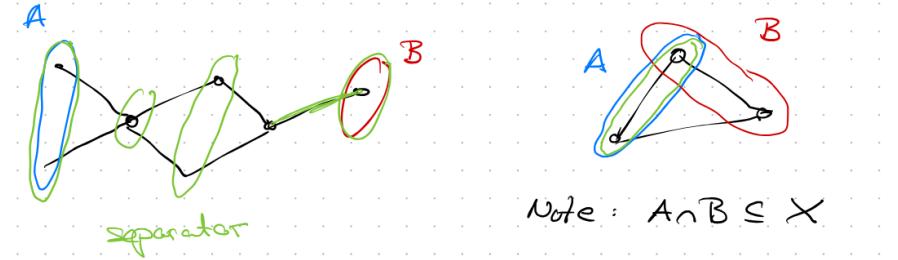


Figure 4.2: Illustration of separation

**Theorem 4.1.3** (Theorem 5: Menger's theorem). *Let  $G = (V, E)$  be a graph and  $A, B \subseteq V$ . Then the minimal cardinality of a vertex-separator of  $A, B$  equals the greatest cardinality of a set of disjoint  $A$ - $B$  paths.*

*Remark 4.1.4.* Note that by disjoint here we mean that the nodes do not share any vertices, not even endpoints.

**Corollary 4.1.5.** *Let  $a, b$  be distinct vertices of  $G$ . If  $ab \notin E$  then the minimum number of vertices not equal to  $a, b$  separating  $a, b$  in  $G$  is equal to the number of independent paths from  $a$  to  $b$ .*

*Proof.* Set  $A = N(a)$  and  $B = N(b)$ , and apply theorem 4.1.3 (independent paths between  $a, b$  are in bijection with disjoint  $A$ - $B$  paths).

To see that they are in bijection: Note that any independent  $\{a\}$ - $\{b\}$  path have to go through one neighbor of  $a$  and one neighbor of  $b$ . This picks out exactly one path between  $N(a)$  and  $N(b)$  (since then no other  $N(a)$ - $N(b)$  path can contain the vertices on this path). On the other hand, any  $N(a)$ - $N(b)$  path gives us an  $\{a\}$ - $\{b\}$  path by adding the edges from the element in  $N(a)$  to  $a$  and the element in  $N(b)$  to  $b$ .  $\square$

**Theorem 4.1.6** (Theorem 6). *Let  $G$  be a graph. Then the following are equivalent:*

- (i)  $G$  is  $k$ -connected.
- (ii) Between any two vertices in  $G$  there are  $k$  independent paths.

*Proof.* See lecture 4.  $\square$

**Corollary 4.1.7** (Corollary 5). *Let  $G$  be a graph with more than one node. Then the following are equivalent.*

- (i)  *$G$  is 2-connected.*
- (ii) *Any two nodes of  $G$  lie on a cycle contained in  $G$ .*

*Proof.* For (ii) to (i) just note that between any pair of nodes there are 2 independent paths which then gives a cycle.  $\square$

## 4.2 Constructing 2-connected graphs

**Definition 4.2.1** ( $H$ -path). A path  $P$  is a  $H$ -path if the only nodes on  $P$  in common with the graph  $H$  are its endpoints.

**Proposition 4.2.2** (Proposition 8). *A graph is two-connected  $\Leftrightarrow$  it can be successively constructed from a cycle by adding  $H$ -paths to graphs  $H$  already constructed.*

We now prove Menger's theorem 4.1.3.

*Remark 4.2.3.* Observe that there are several proofs, and the proof below corresponds to [Die17, p. 67, second proof].

*Proof. Notation:*

- If  $P = v_0v_1 \dots v_t$  is a path, then  $Pv_i = v_0v_1 \dots v_i$ ,  $v_iP = v_iv_{i+1} \dots v_t$  and  $\dot{v}_iP = v_{i+1} \dots v_t$ .
- $k = k(G, A, B) = \min\{|X| : X \subset V(G) \text{ and } X \text{ separates } A \text{ and } B\}$ .
- We interpret  $\bigcup \mathcal{P}$  to mean that we form the set of all vertices occurring on some path in  $\mathcal{P}$ .

*Remark 4.2.4.* Note that the statement is trivially true when  $A$  and  $B$  are disconnected, so we may assume the existence of *at least* one  $A$ - $B$  path.

If  $G$  contained more than  $k$  disjoint  $A$ - $B$  paths, then any removal of  $|X| = k$  nodes from  $G$  can only remove at most  $k$  disjoint  $A$ - $B$  paths from  $G$ . But if  $G$  contained  $> k$  disjoint  $A$ - $B$  paths, this would still leave us with at least one  $A$ - $B$  path. But  $k = k(G, A, B)$  by definition implies the existence of a separator  $X$  of  $k$  nodes, contradiction! Therefore, there can *at most* be  $k$  disjoint  $A$ - $B$  paths. We need to show that there can not be fewer than  $k$ , i.e. that the maximal number of disjoint  $A$ - $B$  paths are precisely  $k$ .

We prove the following statement: For any set  $\mathcal{P}$  of less than  $k$  disjoint  $A$ - $B$  paths in  $G$ , there exists a set  $\mathcal{Q}$  containing  $|\mathcal{P}| + 1$  disjoint  $A$ - $B$  paths, whose endpoints contains all of the endpoints of the paths in  $\mathcal{P}$ . To prove this claim, we apply induction on

$$|\bigcup \mathcal{P}|,$$

i.e. the number of vertices occurring on some path in  $\mathcal{P}$ , while keeping  $G$  and  $A$  fixed, and letting  $B$  vary.

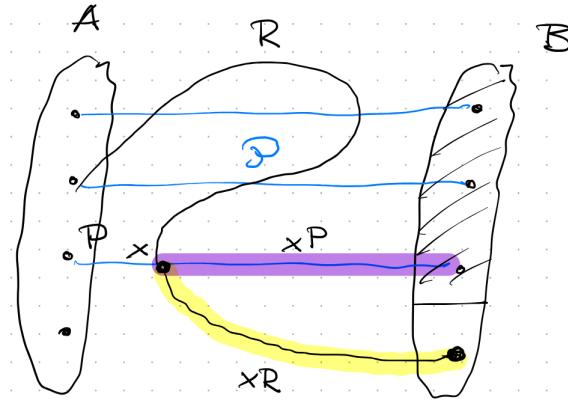


Figure 4.3: Illustration: Second proof of Menger's theorem

Let  $R$  be an  $A$ - $B$  path that avoids the (fewer than  $k$  [since  $\mathcal{P}$  by assumption has less than  $k$  disjoint  $A$ - $B$  paths]) vertices of  $B$  that lie on a path in  $\mathcal{P}$ . Such a path  $R$  exists, since if not, then  $B \cap (\bigcup \mathcal{P})$  would be a separator with less than  $k$  elements (since  $\mathcal{P}$  is a set of less than  $k$   $A$ - $B$  paths, and there is exactly one unique node in  $B$  associated with each path), contradicting the minimality of  $k$ .

If  $R$  is disjoint from all paths  $P$  in  $\mathcal{P}$ , then  $\mathcal{Q} := \mathcal{P} \cup \{R\}$  fulfills the criterion we wanted to show. In particular, this happens when  $\mathcal{P}$  is empty (since we assumed the existence of at least one  $A$ - $B$  path by remark 4.2.4), so the induction starts.

If  $R$  and  $\bigcup \mathcal{P}$  have vertices in common, then let  $x$  be the *last* vertex in  $R$  that lies on some  $P \in \mathcal{P}$ . Define

$$B' := B \cup \underbrace{V(xP)}_{\text{purple}} \cup \underbrace{V(xR)}_{\text{yellow}} \quad \text{and} \quad \mathcal{P}' := (\mathcal{P} \setminus \{P\}) \cup \{Px\}.$$

We find that  $|\mathcal{P}'| = |\mathcal{P}|$  but

$$|\bigcup \mathcal{P}'| < |\bigcup \mathcal{P}| \tag{4.2.1}$$

since  $Px$  is a proper initial segment of  $P$ , since  $x$  can not be the endpoint  $b$  on  $P$  by assumption, i.e.  $V(Px) \subsetneq V(P)$ . Furthermore, we claim that  $k(G, A, B') \geq k(G, A, B)$ . Why? Because since  $B \subseteq B'$ , any separator  $Y$  of  $A$  and  $B'$ , is a separator of  $A$  and  $B$ , i.e.

$$\begin{aligned} Y &\in \{X : X \subset V(G) \text{ and } X \text{ separates } A \text{ and } B'\} \\ &\Rightarrow Y \in \{X : X \subset V(G) \text{ and } X \text{ separates } A \text{ and } B\} \\ \Rightarrow \min\{|X| : X \subset V(G) \text{ and } X \text{ separates } A \text{ and } B'\} &\geq \min\{|X| : X \subset V(G) \text{ and } X \text{ separates } A \text{ and } B\} \\ &\Rightarrow k(G, A, B') \geq k(G, A, B). \end{aligned}$$

Therefore, we have that  $|\mathcal{P}'| < k(G, A, B')$ .

Since  $\mathcal{P}'$  has less than  $k$  disjoint  $A$ - $B'$  paths (by construction) and by 4.2.1, by induction it follows that there is a set  $\mathcal{Q}'$  of  $|\mathcal{P}'| + 1 = |\mathcal{P}| + 1$  disjoint  $A$ - $B'$  paths whose endpoints contains all of the endpoints of the paths in  $\mathcal{P}'$ . In particular,  $\mathcal{Q}'$  contains a path  $S$  whose endpoint is  $x$  (since  $Px \in \mathcal{P}'$ ) and a unique path  $S'$  whose last vertex  $y$  is not one of the endpoints of a path in  $\mathcal{P}'$ . Then by construction, since the end points of paths in  $\mathcal{P}'$  lie in  $B \cup \{x\}$ , we must have that  $y \in B' \setminus \{x\}$  (since we know that  $Px$  is a path in  $\mathcal{P}'$  with endpoint  $x$ ).

We distinguish between two cases:

$y \notin V(xP)$ : If  $y \notin V(xP)$ , then (by definition of  $B'$ ) we must have that  $y$  is in  $B$  or  $V(xR)$ . if  $y \notin B$ , then  $y$  is in  $V(xR)$ , i.e.  $y$  lies on the path  $R$ . Therefore,  $yR$  makes sense, and so we may create an  $A$ - $B$  path  $S' \cup yR$ , and we may create another  $A$ - $B$  path by adding  $xP$  to  $S$ . Notice that  $S$  and  $S'$  are disjoint from any other path in  $\mathcal{Q}'$ , and since the paths in  $\mathcal{Q}'$  are  $A$ - $B'$  paths, their *internal* vertices avoids  $B \cup V(xP) \cup V(xR)$ . Thus if we extend  $S$  by  $xP$  then none of the internal vertices we add to  $S$  lies on any other path  $P \in \mathcal{Q}'$ . Let's also extend  $S'$  with  $yR$ . Since  $S'$  share no internal vertices with  $B \cup V(xP) \cup V(xR)$ , and  $x$  is the last node of  $R$  that lies on some path  $P$  in  $\mathcal{P}'$ , we have that  $S' \cup yR$  is such that since  $yR \subset xR$  it follows that  $S' \cup yR$  does not have any internal nodes shared with the other paths. Lastly, nothing changes about the startnodes in  $A$  when we create  $S \cup xP$  and  $S' \cup yR$  does not share any starting points in  $A$  with any other path in  $\mathcal{Q}'$ . Lastly, since all other paths besides  $S'$  in  $\mathcal{Q}'$  have endpoints among the endpoints of  $\mathcal{P}'$  (which are those of  $\mathcal{P}$  together with  $x$ ), which  $R$  avoided, we see that  $S' \cup yR$  does not share its endpoint in  $B$  with any other path in  $\mathcal{Q}'$ , and  $S \cup xP$  is by construction such that its endpoint, the endpoint of  $P$ , does not share an endpoint in  $B$  with anything from  $\mathcal{P}'$ , by construction (and neither with  $S' \cup yR$  by definition of  $R$ ). Therefore, if we set

$$\mathcal{Q} := (\mathcal{Q}' \setminus (S \cup S')) \cup ((S \cup xP) \cup (S' \cup yR))$$

we get a set of  $|\mathcal{P}| + 1$  disjoint  $A$ - $B$  paths such that all the endpoints of  $\mathcal{P}$  are a subset of the endpoints of  $\mathcal{Q}$  both with respect to  $A$  and  $B$ .

If on the other hand  $y \in B$ , then we may just remove  $S$  from  $\mathcal{Q}'$  and add  $S \cup xP$ . There is no intersection happening internally between  $S \cup xP$  and any of the other paths in  $\mathcal{Q}$  the endpoint of  $P$  is not the endpoint in any other path from  $\mathcal{Q}$  (by construction and since  $y \notin V(xP)$ ).

$y \in V(xP)$ : We see that in fact  $y \neq x$  (since this is the endpoint of  $S$ ) so  $y \in V(\hat{x}P)$ . We let  $\mathcal{Q}$  be obtained from  $\mathcal{Q}'$  by adding  $xR$  to  $S$  and  $yP$  to  $S'$ , i.e.

$$\mathcal{Q} := (\mathcal{Q}' \setminus (S \cup S')) \cup ((S \cup xR) \cup (S' \cup yP)).$$

since  $x$  is the last point of  $R$  that lies on a path in  $\mathcal{P}$  and since  $x \neq p$  we find that  $xR$  and  $yP$  have no node in common (since otherwise we would get a contradiction to our choice of  $x$ ). Furthermore, since  $yP \subset xP$  we find that  $S' \cup yP$  has internal nodes disjoint from the other paths in  $\mathcal{Q}$ , and similarly for  $S \cup xR$  since the internal nodes from  $\mathcal{Q}'$  are disjoint from  $V(xR)$ . Lastly, the endpoint of  $S \cup xR$  is the endpoint of  $R$ , which can not be the endpoint of  $S' \cup yP$ , i.e. the endpoint of  $P$ . All other endpoints of paths in  $\mathcal{Q}$  are those of endpoints of  $\mathcal{P}'$  which are those of  $\mathcal{P} \setminus \{P\}$ , and those are all distinct from the endpoint of  $R$ .  $\square$

## Chapter 5

# Lecture 5: Eulerian Walks/tours, Hamiltonian Paths

Problem: Can you walk through the city crossing each bridge exactly once (Königsberg). Euler found (1776) that this is not possible (1736).

Today: Eulerian walks/tours, Hamiltonian paths/cycles:

- Characterizations.
- Restricted to simple graphs.

**Definition 5.0.1** (Walk). A **walk** of length  $k$  in a graph  $G$  is a non-empty alternating sequence  $v_0e_0\dots e_{k-1}v_k$  of vertices and edges of  $G$  such that  $e_i = \{v_i, v_{i+1}\}$ . If  $v_0 = v_k$  then the walk is **closed**, otherwise it is **open**.

**Definition 5.0.2** (Eulerian Walk). A walk is called **Eulerian** if it traverses every edge of  $G$  exactly once. A *closed* Eulerian walk is called an **Euler tour**.

**Definition 5.0.3** (Eulerian graph). A graph  $G$  is called **Eulerian** if it contains an Eulerian tour.

**Theorem 5.0.4** (Theorem 7). Let  $G$  be a connected graph. Then the following are equivalent:

- $G$  is Eulerian.
- The degree of every node in  $G$  is even.
- $G$  can be written as a disjoint (edge-wise) union of cycles.

*Proof.* See lecture. □

**Theorem 5.0.5** (Theorem 8). Let  $G$  be a connected graph. The following are equivalent:

- $G$  has an open Eulerian walk.
- There are exactly two vertices  $v$  and  $w$  with odd degree.

*Proof.* Follows from theorem 5.0.4. Consider taking away  $e = vw$  or adding it:

(a) implies (b): Let  $ve_0\dots e_kw$  be the Eulerian walk. If we add the edge  $e$  then  $G \cup e$  then  $G$  admits an Eulerian tour  $ve_0\dots e_kwvw$ , which by 5.0.4 implies that the degree of every node in  $G$  is even. If

we remove  $e$  then the degree of  $v$  and  $w$  decreases by one, and become uneven, while any other nodes degree stays even.

(b) implies (a): Assume that there are exactly two vertices  $v$  and  $w$  with odd degree. Assume first that  $e$  is not in  $E(G)$ . If we add the edge  $e$  then every node has even degree, and the graph  $G$  is still connected, so it admits an Eulerian tour  $v_0e_0\dots e_kv_0$  that uses every edge exactly once, so in particular, it uses  $e$ . Thus we have an Euler tour  $ve_\ell v_1\dots e_kwv$ . If we remove  $e$  then we get an open Eulerian walk  $ve_\ell v_1\dots e_kw$ .

If  $e$  is in  $E(G)$ , we remove  $e$  and then get a graph  $G - e$  with all nodes even which then admits an Eulerian tour. Since  $G$  is connected, either  $H := G - e$  has one or two components. If  $H$  has one component, then  $H$  has an Euler tour  $T$ . Pick the first occurrence of  $v$  in  $T$ , and let  $P_1$  be the subpath in  $T$  from this  $v$  to the first occurrence of  $w$  in  $T$  after  $v$ . Then let  $P_2$  be the subpath in  $T$  from  $w$ :s first occurrence after  $v$  to the first occurrence of  $v$  in  $T$ . Then  $eP_1P_2 : w \rightarrow v$  is an open Eulerian walk. I.e. if  $T = v_0e_0\dots e_{k-1}ve_k\dots e_{\ell-1}we_\ell\dots e_{q-1}v_q$  where  $w$  is the first occurrence of  $w$  after the first occurrence of  $v$ , then  $P_1 = ve_k\dots e_{\ell-1}w$  and  $P_2 = we_\ell\dots e_{q-1}v_qv_0e_0\dots e_{k-1}v$ .

If  $H$  has two components  $A \ni v$  and  $B \ni w$  (i.e.  $e$  is a bridge) then  $T_A$  is an Euler tour in  $A$  and  $T_B$  is an Euler tour in  $B$  by theorem 5.0.4. Then  $T_AeT_B : v \rightarrow w$  is an Eulerian walk from  $v$  to  $w$  in  $G$ .  $\square$

**Definition 5.0.6** (Hamiltonian path/cycle). A path in a connected graph  $G$  is called a **Hamiltonian** path if it contains every vertex in  $G$  exactly once. A cycle is **Hamiltonian** if it contains every vertex exactly once. A graph is called **Hamiltonian** if it contains a Hamiltonian cycle.

**Definition 5.0.7** (Articulation point). A vertex in a graph  $G$  is an **articulation point** (or **cut vertex**) if  $G \setminus v$  has more connected components.

**Proposition 5.0.8** (Proposition 10). *Let  $G$  be a Hamiltonian graph. Then  $G$  does not contain any bridges or articulation points.*

*Proof.*  $G$  has a Hamilton cycle  $C$ . Either an edge  $e$  is on  $C$  or not. If it is it can not be a bridge. If it is not, and  $e = vw$  then still it holds that the nodes  $v$  and  $w$  lie on  $C$ . But then there is a path  $P = ve_1\dots e_kw$  from  $v$  to  $w$  in  $C$ , not containing  $e$ . Then  $C' = eP_1 : w \rightarrow w$  is a cycle in  $G$  containing  $e$ , so  $e$  can not be a bridge.

Assume  $v \in V(G)$  is an articulation point and let  $H_1, H_2$  be two different components in  $G \setminus v$  with vertices  $v_1, v_2$ . By assumption  $G$  has a hamiltonian cycle  $C$  that contains the vertices  $v, v_1, v_2$ , so there are atleast two paths  $P_1, P_2$  in  $G$  connecting  $v_1, v_2$ . But since this cycle is hamiltonian each vertex occurs exactly once, and so only one of the paths  $P_1$  or  $P_2$  can contain  $v$ ; assume  $v \in P_1$ . Then  $v \notin P_2$  and so  $P_2$  is still a path in  $G \setminus v$  between  $v_1$  and  $v_2$ , contradicting that  $G \setminus v$  has two components.  $\square$

**Theorem 5.0.9** (Dirac). *Let  $G = (V, E)$  be a graph with at least three nodes. If the degree of each node  $v \in G$  is at least  $\frac{|V|}{2}$  then  $G$  is Hamiltonian.*

*Proof.* Assume that  $G$  was disconnected, so that there was at least two components  $H_1, H_2$  with vertex sets  $V_1, V_2$ . By assumption we have that  $d(v) \geq \frac{|V|}{2}$  for any  $v \in V_i$  so there needs to be at least  $\frac{|V|}{2} + 1$  nodes in any component  $H_i$ . We then have that

$$\begin{aligned} |V| &\geq |V_1| + |V_2| \\ &\geq 2\left(\frac{|V|}{2} + 1\right) \\ &= |V| + 2, \end{aligned}$$

where we used that  $V_i \cap V_j = \emptyset$ . This is a contradiction, so  $G$  is connected. From now on let  $|V| = n$ .

Let  $P = v_0 \dots v_k$  be a path of maximal length in  $G$ . Then all neighbors of  $v_0$  and  $v_k$  lie on  $G$ , otherwise we could extend  $P$  to  $P' = wv_0 \dots v_k$  or  $P' = v_0 \dots v_kw'$ . Hence there must be at least  $\frac{n}{2} + 1$  nodes on  $P$ , and since  $P$  is a tree there must be at least  $\frac{n}{2}$  edges on  $P$ .

Assume there was no index  $i$  between 0 and  $k-1$  such that  $v_i$  is a neighbor to  $v_k$  and  $v_{i+1}$  is a neighbor to  $v_0$ . Let  $A = \{j : v_j \in N(v_0)\}$  and  $B = \{\ell : v_\ell \in N(v_k)\}$ . Then for each index  $i$  between 0 and  $k-1$ ,  $i+1$  can be in  $A$  or  $i$  can be in  $B$  but not both. Therefore, with  $\chi_A, \chi_B$  indicator functions, we have that  $\chi_A(i+1) + \chi_B(i) \leq 1$  for  $0 \leq i \leq k-1$ . Thus

$$\sum_{i=0}^{k-1} (\chi_A(i+1) + \chi_B(i)) \leq k.$$

Notice that

$$\sum_{i=0}^{k-1} \chi_B(i) = |B|$$

and

$$\begin{aligned} \sum_{i=0}^{k-1} \chi_A(i+1) &= \sum_{j=1}^k \chi_A(j) \\ &= |A| \end{aligned}$$

since  $\chi_A(0) = 0$  and  $\chi_B(k) = 0$  ( $G$  simple). This means that  $|A| + |B| \leq k$ . But we also know that  $|A| \geq \frac{n}{2}$  and  $|B| \geq \frac{n}{2}$  so that

$$n \leq |A| + |B| \leq k.$$

This means that the length of the path  $P$  is greater than or equal to the number of nodes in  $G$ , which is impossible, since there can be at most  $n$  nodes present in  $P$  (since this is the number of nodes of  $G$ ), and such a path needs to have  $n-1$  or less edges. We conclude that there must be an index  $i$  such that  $v_i \in N(v_k)$  and  $v_{i+1} \in N(v_0)$ .

If we remove the edge  $v_iv_{i+1}$  from  $P$  and add  $v_iv_k$  and  $v_0v_{i+1}$ , then we get a cycle

$$C = v_0v_1 \dots v_iv_kv_{k-1} \dots v_{i+1}v_0$$

of length  $k+1$ , since this is a cycle with  $k+1$  edges (removing the last vertex gives us a tree with  $k+1$  distinct vertices and so  $k$  edges).

We claim that this cycle is Hamiltonian. If not then there is some vertex  $v$  in  $G$  not in  $P$  (since the cycle  $C$  contains the same vertices as  $C$ ). Since  $G$  is connected, there is some path  $P'$  from  $v$  to some vertex  $v'$  on  $C$  (assume that  $v'$  is the first vertex on  $C$  on the path  $P'$ ). If we then take the part of the path  $P'$  from  $v$  to  $v'$ , and then walk along  $C$  from  $v'$  to the last vertex  $u_\ell$  before  $v'$ , we get a path of length  $|E(P')| + k \geq k+1 > k$ , contradicting that  $P$  is maximal. Thus  $C$  must be Hamiltonian.  $\square$

Strengthening:

**Theorem 5.0.10** (Pósa). *Let  $G$  be a connected graph and set  $n := |V(G)| \geq 3$ . If for every integer  $1 \leq r < \frac{n}{2}$  we have*

$$|\{v \in V(G) : d(v) \leq r\}| < r,$$

*then  $G$  is Hamiltonian.*

**Theorem 5.0.11** (Ore). *Let  $G = (V, E)$  with  $|V| \geq 3$  and  $d(v) + d(w) \geq |V|$  for any  $v, w$  such that  $vw \notin E$ . Then  $G$  is Hamiltonian.*

*Proof.* Let  $|V| = n$ . Let's assume that  $G$  is not Hamiltonian. Then, by theorem ?? there is some  $r < \frac{n}{2}$  and  $r$  vertices of degree atmost  $r$ . Let  $V_1 = \{v_1, \dots, v_r\}$  with  $d(v_i) \leq r$  and  $V_2 = V \setminus V_1$ . For each  $v, w \in V_1$ , we have that  $d(v) + d(w) \leq 2r < n$ . Notice that if  $vw \notin E$  implies that  $d(v) + d(w) \geq n$  by hypothesis, so the contrapositive gives that if  $d(v) + d(w) < n$  then  $vw \in E$ , so  $v, w$  are neighbors in  $G$ . Since this holds for any  $v, w \in V_1$  we have that  $G[V_1]$  is a complete graph. Since  $d(v_i) \leq r$  for all  $v_i \in V_1$  and we saw from above that  $d(v_i) \geq r - 1$  we have that  $r - 1 \leq d(v_i) \leq r$  for all  $v_i$  in  $V_1$ . Therefore, from each node  $v_i$  there can at most be one edge to  $V$ . Furthermore, we have that  $|V_2| = n - r > \frac{n}{2} > r$  (since  $r < \frac{n}{2}$ ) and so there must exist at least one vertex  $w \in V_2$  such that  $v_iw \notin E$  for some  $v_i \in V_1$ , and so by assumption  $d(v_i) + d(w) \geq n$ . Since  $d(v_i) \leq r$  we have that  $d(w) \geq n - d(v_i) \geq n - r$ . But  $|V_2| = n - r$  and all possible neighbors of  $w$  are in  $V_2$ , so  $d(w) \leq n - r - 1$ , contradiction! So  $G$  must be Hamiltonian.  $\square$

# Chapter 6

## Lecture 6

Today: Characterization of bipartite graphs and matchings in bipartite graphs.

**Definition 6.0.1** (Bipartite graph). A graph  $G = (V, E)$  is **bipartite** if there are disjoint sets  $S, T \subseteq V$  such that  $S \sqcup T = V(G)$  and so that

$$\begin{aligned}|S \cap e| &= |T \cap E| \\ &= 1\end{aligned}$$

for all edges  $e$  in  $G$ .

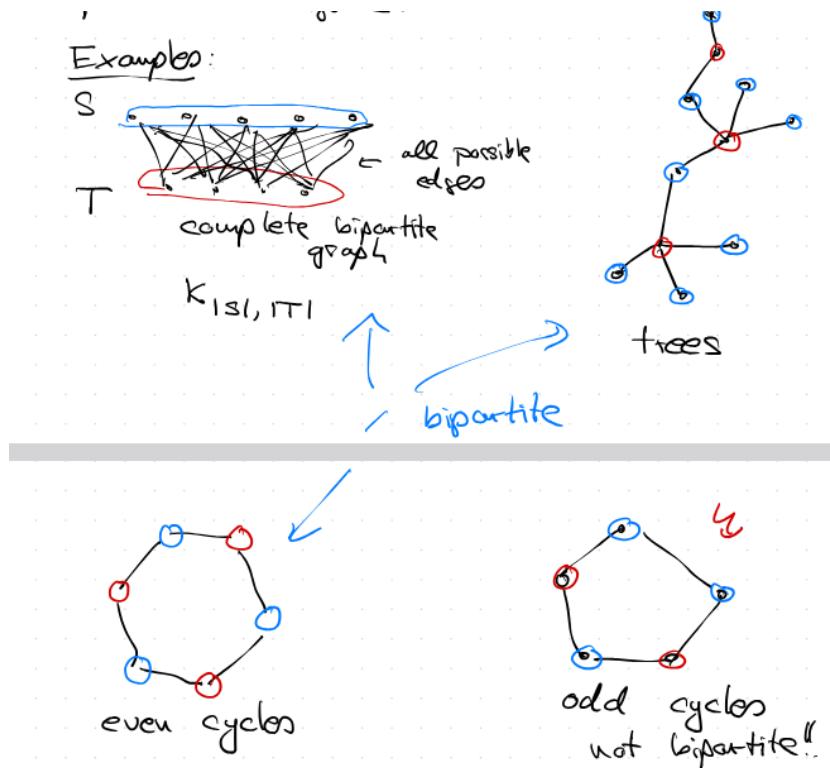


Figure 6.1: Illustration of bipartite graphs (and one non-bipartite graph)

**Proposition 6.0.2** (Proposition 1.1). *Let  $G = (V, E)$  be a graph with  $|V(G)| \geq 2$ . Then the following are equivalent:*

- (i)  $G$  is bipartite.
- (ii)  $G$  contains no odd cycles (or if you want; all cycles in  $G$  have even length).

*Proof.* Without loss of generality we assume that  $G$  is connected since both (i) and (ii) are satisfied if they are satisfied for all components of  $G$  (note that any cycle has to be contained in some component).

(i) implies (ii): Let  $V = S \sqcup T$  such that every edge in  $G$  links a node in  $S$  to a node in  $T$ . Let  $C = v_1v_2, \dots, v_nv_1$  be a cycle. Assume without loss of generality that  $v_1 \in S$ , so that  $v_2 \in T, v_3 \in S$  etc; i.e. we see that if  $v_i$  has odd index then  $v_i \in S$  and if  $v_i$  has even index then  $v_i \in T$ . Since  $v_nv_1$  is an edge in  $G$ , we need  $v_n \in T$ , so that  $n$  is even. A cycle on  $n$  distinct nodes has  $n$  edges since by removing  $v_nv_1$  we get a path which is a tree so has  $|E| = |V(C)| - 1$  edges, i.e.  $n - 1$  edges. Adding back  $v_nv_1$  gives us back  $C$  with  $|E| + 1 = n$  edges.

(ii) implies (i): Since  $G$  is by assumption connected, it has a (minimal) spanning tree by Kruskal's algorithm, so it has at least one. Let  $R$  be an arbitrary spanning tree of  $G$  and let  $v \in R$  be an arbitrary node. Let

$$\begin{aligned} S &:= \{v \in V : \text{the unique path in } R \text{ linking } u \text{ to } v \text{ has even length}\} \\ T &:= \{v \in V : \text{the unique path in } R \text{ linking } u \text{ to } v \text{ has odd length}\}. \end{aligned}$$

If  $v, w \in S$  (or  $v, w \in T$ ) then we claim that  $vw \notin E$ : Let  $P = v_0 \dots v_n$  be the unique path in  $R$  starting at  $v$  and ending at  $v = v_n$  and let  $P' = w_0 \dots w_m$  be the unique path starting at  $w$  and ending at  $w = w_m$ . Let  $v_i = w_i$  be the last vertex that  $P$  and  $P'$  have in common; i.e.  $i = \max\{j : v_j = w_j\}$ . Note that the unique path in  $R$  from  $v$  to  $w$  must be by going from  $v$  to  $v_i = w_i$  and then from  $w_i$  to  $w$ . Since the subpath from  $v = v_n$  to  $v_i$  is a tree with  $n - i$  edges (to see this one can reindex by sending  $j \mapsto j - i + 1$  for  $i, i+1, \dots, n$  and noting that a path is a tree and a tree has one less edge than there are nodes, and the last node ends up after reindexing at place  $v_{n-i+1}$  and the first at  $v_1$ , so  $n - i + 1$  nodes and  $n - i$  edges). Similarly, there are  $m - i$  edges in the subpath from  $v_i = w_i$  to  $w$ . So all in all the length of the unique  $v - w$  path is  $n - i + m - i$ . If  $vw \in E(R)$  then this must have length one. Hence  $(n - i) + (m - i) = 1$ . Since both  $n - i$  and  $m - i$  are positive, it follows that either  $n - i = 0$  and  $m - i = 1$  or  $n - i = 1$  and  $m - i = 0$ . In the first case we have that  $m = n + 1$  and in the latter case  $n = m + 1$ . Either way  $v$  is in  $S$  and  $w$  is in  $T$  or  $w$  is in  $S$  and  $v$  is in  $T$ , since one must have even distance from  $u$  and the other one odd. This contradicts our assumption that  $v, w \in S$ , hence  $vw \notin E(R)$ . So the only other option is that  $vw \in E(G) \setminus E(R)$ .

Assume this is the case. Then we may form the cycle  $w_iPv \cup vw \cup wP'w_i$ , which is a cycle in  $G$ . The length of  $w_iPv$  and  $wP'w_i$  are  $n - i$  and  $m - i$ . If we assume that  $v, w \in S$  we have that  $n, m$  are even, and so the added together length is, with  $2\ell = n + m$ ,  $\underbrace{2\ell - 2i + 1}_{\geq 0} = 1 \pmod{2}$ , i.e. of odd length.

If instead  $v, w \in T$  then both are of  $n, m$  are of odd length but then  $n + m$  is still of even length. But by assumption all cycles were even, contradiction! We conclude that  $vw \notin E$  whenever  $v, w \in S$  or  $v, w \in T$ .

Since  $S \sqcup T = V$  the conclusion follows.  $\square$

**Definition 6.0.3** (Matching). A set  $M$  pairwise non-intersecting (no endpoints in common) edges in a graph  $G = (V, E)$  is called a **matching**.

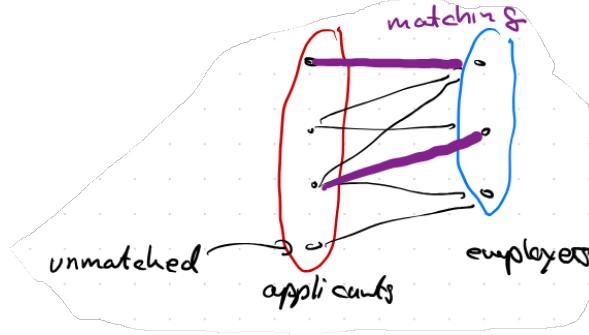


Figure 6.2: Illustration of a matching with  $|M| = 2$  and unmatched node

**Definition 6.0.4** (Matching number  $m(G)$ ; maximal matching). The **matching number**  $m(G)$  of a graph  $G$  is the maximal number of edges a matching in  $G$  can have. A matching  $M$  is a **maximal matching** if  $|M| = m(G)$ . A matching  $M$  is a matching of  $U \subseteq V$  if every vertex  $v \in U$  is contained in an edge of  $M$ . A vertex that is an endpoint of an edge in  $M$  is called **matched** and otherwise **unmatched**.

For a bipartite graph  $G = (S \sqcup T, E)$  and  $A \subseteq S$ , let

$$N(A) = \{u \in T : uv \in E \text{ for some } v \in A\}$$

be the neighbors of  $A$ . A necessary condition for a matching of  $S$  is that

$$|N(A)| \geq |A|$$

for all subsets  $A \subseteq S$  (“marriage condition”). Note that each matching of  $A \subseteq S$  gives an injection from  $A$  to  $T$ , so that  $|N(A)| \geq |A|$  must hold, since if  $|A| > |N(A)|$  then there is no injection from  $A$  to  $T$ , and so there can not be a matching of  $A$ .

**Theorem 6.0.5** (Theorem 12; Hall’s marriage theorem). *Let  $G = (S \sqcup T, E)$  be a bipartite graph. The following are then equivalent:*

- (i)  $G$  contains a matching of  $S$ .
- (ii)  $m(G) = |S|$ .
- (iii)  $|N(A)| \geq |A|$  for all  $A \subseteq S$ .

*Proof.* (ii)  $\Leftrightarrow$  (i): We have  $m(G) \leq |S|$  since every edge must contain a vertex in  $S$ , and we also have  $m(G) \geq |S|$  since there is a matching of  $S$  by assumption.

(i)  $\Rightarrow$  (ii): Any matching of  $S$  in  $G$  induces a matching of  $A \subseteq S$  (just pick out those edges that are associated with  $a \in A$ ). If  $|A| > N(A)$ , then since this was a necessary condition for a matching, there can not be a matching of  $A$ ; this contradicts there being a matching of  $S$ , which we assumed existed, so  $|A| \leq |N(A)|$ .

(iii)  $\Rightarrow$  (i): “Intermezzo”:

**Definition 6.0.6** (Alternating path). A path in  $G$  with a matching  $M$  is called **alternating** if it starts with an unmatched vertex in  $S$ , and then, alternating between edges in  $M$  and  $E \setminus M$ . An alternating path  $P$  that ends in an unmatched vertex of  $T$  is called an **augmenting** path, since we can use it to turn  $M$  into a larger matching.

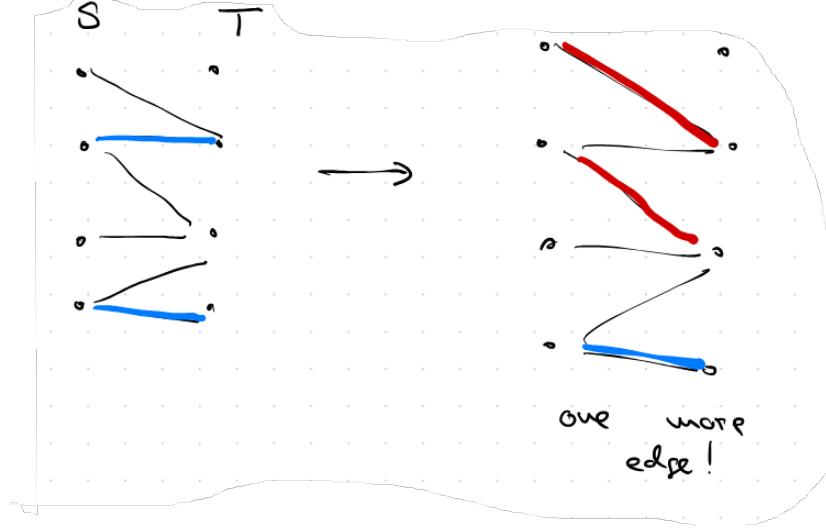


Figure 6.3: Illustration

Proof idea: Show that for every matching  $M$  of  $G$  that leaves a vertex  $v \in S$  unmatched, there is an augmenting path with respect to  $M$ .

Let  $v \in S$  be an unmatched vertex. Let  $S' \subseteq S$  be the set of vertices that can be reached from  $v$  by a non-trivial alternating path, and let  $T' \subseteq T$  be the set of penultimate vertices of such paths (i.e. if  $P = v_0 \dots v_{k-1}v_k$  is such a path then we pick out  $v_{k-1}$ ). Since the last edge of each such path is in  $M$ , we have  $|S'| = |T'|$ . By (iii) there is an edge between a vertex  $a \in A := S' \cup \{v\}$  and some vertex  $b \in T \setminus T'$ . Since  $a \in S' \cup \{v\}$  there is an alternating path from  $v$  to  $a$ . In the case that  $v = a$ , then since  $v$  is unmatched, we have that  $vb$  is an edge not in  $M$ . If  $b$  was matched, then  $a'b$  would be an edge in  $M$  (where  $a' \neq v$  since  $v$  is unmatched). Hence  $v - b - a'$  would be an alternating path to  $a'$ , so that  $b \in T'$ , contradiction!

Otherwise, let  $P'$  be the path  $P$  from  $v$  to  $a$  prolonged by the edge  $ab$ . Then since the last edge  $e = b'a$  into  $a$  on  $P$  needs to be in  $M$ , so this means that  $ab$  can not be in  $M$  (every vertex is incident to at most one edge of  $M$ ), so indeed this path  $P'$  is still alternating.

Furthermore,  $P'$  is augmenting because if  $b$  were matched, say  $a'b \in M$  then prolongating  $P'$  with  $a'b$  yields an alternating path from  $v$  to  $a'$  and thus we would have  $b \in T'$ , contradiction! Thus  $b$  is unmatched and  $P'$  is augmenting.  $\square$

*Remark 6.0.7.* Why does an **augmenting path** yield a larger matching? Let  $P = v_0 \dots v_k$  be an augmenting path for a matching  $M$ . By definition,  $v_0 \in S$  and  $v_k \in T$  are then unmatched vertices. Define  $M'$  as the symmetric difference of  $M$  and  $E(P)$ . Observe that every internal vertex of  $P$  is incident to one edge in  $M$  and one edge not in  $M$ . **Claim:**  $M'$  is a matching.

Vertices not on  $P$ : Vertices not on  $P$  are not incident to  $E(P)$  so they are not incident with any edge in  $E(P) \setminus M$ . Hence they can have share at most one edge with  $M \setminus E(P)$ .

Internal vertices of  $P$ : Each internal node on  $P$  is incident to one edge in  $M$  and one edge not in  $M$ . We find that  $M \setminus E(P)$  removes the edge in  $M$ . On the other hand in  $E(P) \setminus M$ , we still have the edge not incident to  $M$ .

Endpoints: Since the endpoints  $v_0, v_k$  were unmatched, they are not incident to  $M$ , which means that the edge-set  $M \setminus E(P)$  does not contain any edges incident to  $v_0, v_k$ . On the other hand, in  $E(P) \setminus M$  we get one edge incident to  $v_0, e_1$  and one edge incident to  $v_k, e_{k-1}$ .

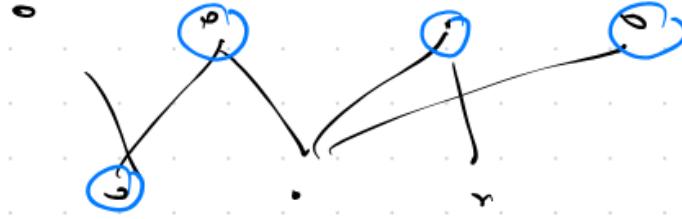


Figure 6.4: Vertex cover illustration

From the above we see that any vertex  $v$  in  $G$  is incident to at most one edge of  $M'$ , so  $M'$  is a matching.

**Claim:**  $M'$  is a strictly larger matching than  $M$ .

*Proof.* Notice that  $M' = (M \setminus E(P)) \sqcup (E(P) \setminus M)$ . Furthermore, we have that  $M = (M \cap E(P)) \sqcup (M \setminus E(P)) \Leftrightarrow |M| - |M \cap E(P)| = |M \setminus E(P)|$ . Adding  $|E(P) \setminus M|$  to both sides gives us that

$$\begin{aligned} |M \setminus E(P)| + |E(P) \setminus M| &= |M| - |M \cap E(P)| + |E(P) \setminus M| \\ &= |M'|. \end{aligned}$$

Since  $P = v_0e_1v_1e_2 \dots v_{k-1}e_kv_k$ , there are  $k$  edges and  $k+1$  vertices. Notice then that  $e_1 \notin M$  and  $e_2 \in M$ , etc and so on (alternating) so that  $e_i \in M \Leftrightarrow i$  even and  $e_i \notin M \Leftrightarrow i$  odd. We have that  $e_{2i}$  for  $i = 1, \dots, \frac{k-1}{2}$  are even, and  $e_{2i-1}$  for  $i = 1, \dots, \frac{k+1}{2}$  are odd.

From, this, we see that  $|E(P) \setminus M| = \frac{k+1}{2}$  and  $|E(P) \cap M| = \frac{k-1}{2}$ . Therefore, we have that

$$\begin{aligned} |M'| &= |M| - \left(\frac{k-1}{2}\right) + \frac{k+1}{2} \\ &= |M| + \frac{-k+1+k+1}{2} \\ &= |M| + \frac{2}{2} \\ &= |M| + 1. \end{aligned}$$

□

**Definition 6.0.8** (Vertex cover). A set  $U \subseteq V$  is a **vertex cover** of  $E$  if every edge is incident with a vertex in  $U$ . More formally, given a graph  $G = (V, E)$ , we have that  $U \subseteq V$  is a vertex cover if for every edge  $e \in E$  it holds that  $|U \cap e| \geq 1$ .

**Theorem 6.0.9** (Theorem 13: König). *Let  $G$  be a bipartite graph. Then the maximum cardinality of a matching is the minimal cardinality of a vertex cover.*

*Proof.* Special case of Menger's theorem:

$$\# \max\{\text{vertex-disjoint } S-T \text{ paths}\} = \min\{|U| : U \text{ separates } S, T\}$$

We have that  $V = S \sqcup T$  by assumption. We show the following:

Cardinality of disjoint  $S$ - $T$  paths are in correspondence with cardinality of matchings: Given  $k$  vertex-disjoint  $S$ - $T$  paths  $P_1, \dots, P_k$ , replace each  $P_i$  with the first edge  $s_i t_i$ . Since the paths are vertex-disjoint, they must be edge-disjoint. Thus we get  $k$  edges  $s_1 t_1, \dots, s_k t_k$  with  $s_i \neq s_j$  and  $t_i \neq t_j$  whenever  $i \neq j$ . This gives a matching  $\{s_1 t_1, \dots, s_k t_k\}$  of size  $k$ . On the other hand, given a matching  $\{s_1 t_1, \dots, s_k t_k\}$  (note that it needs to take this form since  $G$  is presumed to be bipartite, so all edges go between  $S$  and  $T$ ) is a family of  $S$ - $T$  paths of length one that are disjoint (no endpoints in common). Hence each such family of cardinality  $k$  gives  $k$  vertex disjoint  $S$ - $T$  paths. This gives a correspondence in size between the two sets, but note that it is not literally a bijection, since we made an arbitrary choice in our choice of  $s_1 t_1, \dots, s_k t_k$  from  $P_1, \dots, P_k$  (since  $G$  is bipartite any edge in any of the paths is of this form).

$U \subseteq V$  separating sets are in bijection with vertex covers:  $U$  is a separator of  $S$  and  $T$  if there is no  $S$ - $T$  path in  $G - U$ . But  $G$  is bipartite, so all edges in  $G$  are  $S$ - $T$  paths of length one, and so that  $U$  separates  $S$  and  $T$  means that  $E(G - U) = \emptyset$ . This means that for every edge  $e \in E$  we must have that  $|U \cap e| \geq 1$ , so  $U$  must be a vertex cover of  $G$ .

On the other hand, if  $U$  is a vertex cover of  $G$ , then  $|U \cap e| \geq 1$  for each  $e \in E$ . Thus  $G - U$  contains no edges, so in particular contains no  $S$ - $T$  paths, which means that any  $S$ - $T$  path has to go through  $U$ , so that  $U$  is an  $S$ - $T$  separator.

If we now apply Mengers theorem, we see that the maximal cardinality  $m(G)$  of a matching  $M$  in  $G$ , is equal to the minimal cardinality of a vertex covering of  $G$ , which is what we wanted to show.  $\square$

## 6.1 Stable matching

**Definition 6.1.1** (Preferences and stable matching). A family of linear orderings  $(\preceq_v)_{v \in V}$  of edges incident to  $v$  is called a set of **preferences**. A matching  $M$  is called **stable** if for every  $e \in E \setminus M$  there is an edge  $f \in M$  such that  $e$  and  $f$  share a vertex  $v$  for which  $e \prec_v f$ .

**Theorem 6.1.2** (Gale-Shapley's theorem; stable marriage theorem). *For every set of preferences, a stable matching always exists.*

Given a matching  $M$ , we call a vertex  $a \in S$  **acceptable** to a vertex  $b \in T$  if  $a$  is a neighbor of  $b$ ,  $ab \notin M$ , and  $b$  is either unmatched, or if there exists a vertex  $c$  with  $cb \in M$  then  $c \prec_b a$ .

We call  $a \in S$  **happy** with respect to  $M$  if  $a$  is unmatched, or its matching edge  $f = ca$  in  $M$  is such that  $c \succ_a b$  for all  $b \in T$  such that  $a$  is acceptable to  $b$ .

*Proof.* Start with the empty matching  $M = \emptyset$ , i.e. where everyone is “single”.

- (1) If there is no unmatched  $a$  in  $S$  that some  $b$  in  $T$  would accept, then **stop**.
- (2) Otherwise, pick such an unmatched  $a$ . Let  $a$  choose the *best* (with respect to  $\prec_a$ ) among those  $b$ :s that would accept  $a$ .
- (3) Match  $a$  with this  $b$  from step (2), i.e. add the edge  $e = ab$  to  $M$ . If  $b$  had a previous partner  $c$ , remove this edge  $e' = bc$  from  $M$ , so that  $c$  becomes unmatched.

We claim that  $a \in S$  are all **happy** with the resulting matching.

Base case  $M_0 = \emptyset$ : When  $M = \emptyset$ , we have that  $a \in S$  are all unmatched, hence happy.

Assume it holds for  $k < n$  that all  $a \in S$  are happy with  $M_k$ .

Inductive step: Consider the step  $M_{n-1} \rightarrow M_n$ . We pick an unmatched  $a \in S$  and take its most preferred  $b \in T$  among those  $b$  in  $T$  that would accept  $a$ . Then it is clear that  $a$  is happy with  $M_n$  (it got to pick its best option among those  $b$  that accepted  $a$ ). If  $b$  had a partner  $c$  in  $S$  before, then  $c$

is now unmatched, hence happy. If  $a'$  is any other vertex in  $S$ , then either  $a'$  is unmatched still, and then  $a'$  is happy. If not, then  $a'$  was matched with the best  $b'$  that found  $a'$  acceptable, in an earlier iteration of the algorithm. If  $b$  accepted  $a'$  in the iteration where we matched  $a'$ , then we must have that  $b \prec_{a'} b'$  so that  $a'$  is still happy with its current partner. If  $b$  did not accept  $a'$  in the iteration where we paired off  $a'$ , then this must have been because either there is no edge in  $G$  between  $a'$  and  $b$ , or because the earlier partner of  $b$ , call it  $c$ , was such that  $a' \prec_b c$ . But we know that  $c \prec_b a$  since  $b$  chose  $a$ . So by transitivity, we have that  $a' \prec_b a$ , and so  $b$  does not accept  $a'$  even after the new matching  $M_n$  is constructed. For any other  $b' \neq b$ ,  $b'$  is still matched with its partner from  $M_{n-1}$ . If it was the case that  $a'$  preferred  $b'$  over its current partner in  $M_n$ , and  $b'$  accepted  $a'$ , then this would already be true in  $M_{n-1}$ , contradicting our inductive assumption, i.e. that every  $a' \in S$  is happy with  $M_{n-1}$ . Therefore,  $a' \neq a$ ,  $a'$  is still happy with  $M_n$ . We conclude that  $a \in S$  are happy with  $M_n$ , so it holds that  $a \in S$  are happy in each iteration  $M_k$  for  $k \in \mathbb{N}$ .

The algorithm terminates: In each step of the algorithm, either a  $b$  in  $T$  goes from unmatched to matched, or it *improves* its partner. Since each  $b \in B$  only has a finite number of improvements to make, and a  $b$  can not become unmatched, after a finite number of steps there are no further improvements to make for any  $b \in T$ , i.e. after a finite number of steps no  $b$  can improve its partner (observe also that no  $b$  can become unmatched when it has been matched). So, consider step (1) in the algorithm - after a finite number of steps no  $b$  in  $T$  will accept any  $a \in S$  since they have all reach their best possible option, i.e. they can't make further improvements, so we can not pick an unmatched  $a$  to pair off with some  $b \in T$ .

The resulting matching  $M$  is stable: Another formulation of a **stable** matching, in terms of vertices, is that  $M$  is stable iff there is no edge  $ab$  in  $G$  with

$$a \succ_b M(b) \quad \text{and} \quad b \succ_a M(a)$$

where  $M(b)$  is the node (different from  $b$ ) incident to the unique edge containing  $b$  in  $M$ , and similarly for  $M(a)$ .

Suppose on the contrary that there is such an edge  $ab$  in  $G$ . That  $a \succ_b M(b)$  means that  $b$  would accept  $a$ . If  $a$  was unmatched, then the algorithm could not have stopped, so this is not possible. If instead  $a$  was already matched to some  $b' \in T$ , then since  $b \succ_a M(a)$ , this would contradict  $a$  being happy. But we showed that in each step of the algorithm,  $a$  was happy, and the algorithm terminated in a finite number of steps, so this is not possible. Hence we are forced to conclude that no such edge  $ab$  can exist, and so  $M$  is stable.  $\square$

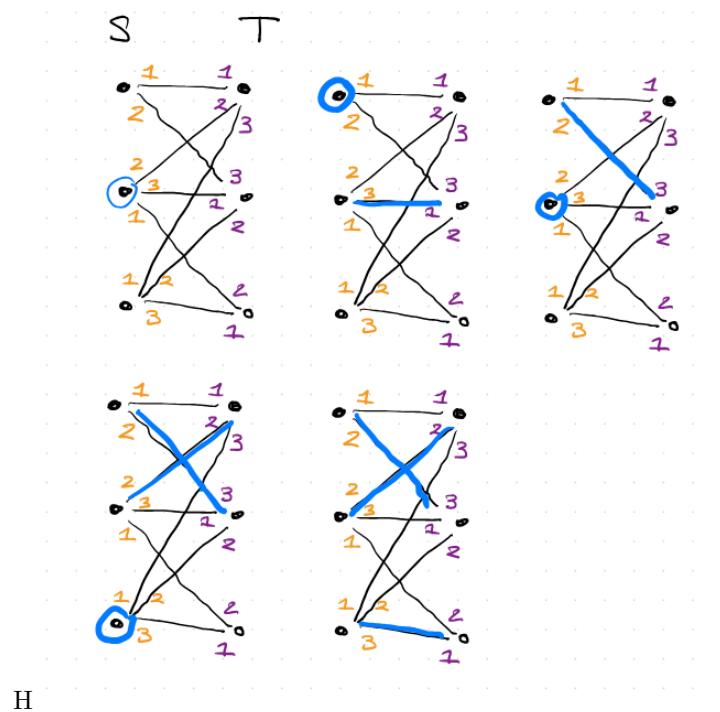


Figure 6.5: Illustration of Gale-Shapley's algorithm

# Chapter 7

## Lecture 7

Today:

- Planar graphs = graphs that can be drawn in the plane.
- Properties and Characterization.

Recall: A **graph drawing** (informally) is an illustration of a graph in the plane such that:

- Each vertex is represented by a point.
- Each edge by a curve (i.e. the image of an injective continuous map  $[0, 1] \rightarrow \mathbb{R}^2$ ) connecting the points corresponding to its endpoints.
- The reader can unambiguously recover  $G$  (i.e. each vertex is represented by a unique point, curves only pass through they are meant to connect).

**Definition 7.0.1** (Planar drawing). A **drawing** is **planar** if two curves representing two edges at most at common endpoints.



Figure 7.1: Illustration of planar and non-planar drawing of  $K_4$ , the complete graph on four vertices

**Definition 7.0.2** (Planar graph). If an (abstract) graph  $G = (V, E)$  admits a planar drawing, then the graph is called **planar**.

*Remark 7.0.3.* A planar drawing is also called a **plane graph**.

**Definition 7.0.4** (Faces; Outer face; Inner faces). If  $G$  is a plane graph then  $\mathbb{R}^2 \setminus G$  consists of finitely many regions called **faces**. One face is unbounded, called the **outer face**. The other faces are *bounded* and called **inner faces**.

**Example 7.0.5.** In the illustration below, for the leftmost planar graph we have that  $X_1$  and  $X_2$  are its *inner* faces and  $X_3$  is its *outer face*, while on the right hand side it only has an *outer face*,  $X$ .

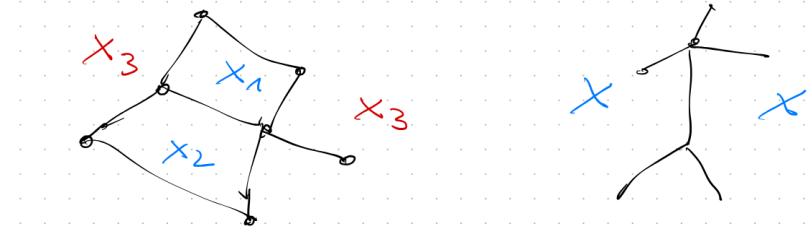


Figure 7.2: Illustration of the concept of outer face and inner faces

**Definition 7.0.6** (Boundary). The **boundary** of a region is a subgraph of all vertices and edges in its closure.

*Remark 7.0.7.* For our purposes here, we may think of region as as a face of a planar graph, i.e. a connected component of  $\mathbb{R}^2 \setminus G$ : Recall that  $\mathbb{R}^2$  is locally path-connected, so is locally connected, and any open subset of a locally connected space, is locally connected. Since  $G$  is the union of its edges  $\gamma_{uv}([0, 1])$ , (each image is the continuous image of a compact set, so compact, and hence closed since it is a subset of a Hausdorff space) and possibly isolated points (which are closed as subsets of Hausdorff space) we have that  $G$  is the union of a finite set of closed sets, so closed, hence  $\mathbb{R}^2 \setminus G$  is open.

Disclaimer: In the following, we will explicitly distinguish between plane graphs and (abstract) planar graphs. We will rely on our geometric intuition and we will freely make use of the following theorem.

**Theorem 7.0.8** (Jordan's curve theorem). *Every Jordan curve (i.e. every image of an injective, continuous map  $S^1 \rightarrow \mathbb{R}^2$ ) has a complement consisting of two regions.*

**Theorem 7.0.9** (Euler's formula). *Let  $G = (V, E)$  be a (non-trivial) connected planar graph with  $f$  faces. Then*

$$|V| - |E| + f = 2.$$

(**Important**): In particular, the number of faces is independent of the drawing.

*Proof.* Base case: We do induction on  $|E|$ . If  $|E| = 0$  then (since non-trivial)  $|V| = 1$  and  $f = 1$  (draw a picture), so it holds.

Induction  $|E| \geq 1$ : Suppose  $G$  does not contain a cycle. Then by definition (2.0.9)  $G$  is a tree and  $|V| = |E| + 1$ . Since  $f = 1$  (draw a picture) it follows that

$$\begin{aligned} |V| - |E| + f &= (|E| + 1) - |E| + 1 \\ &= 2, \end{aligned}$$

so that it holds. If  $G$  has a cycle, then let  $e$  be an edge on the cycle. The graph  $G' = G \setminus e$  is also connected and planar. Thus, by induction, we have that

$$|V(G')| - |E(G')| + f(G') = 2$$

The edge  $e$  is in the boundary of two regions of  $G$  which one joins by removing  $e$ . We have that  $|V(G)| = |V(G')|$ ,  $|E(G)| = |E(G')| + 1$  and  $f(G) = f(G') + 1$ , so that

$$\begin{aligned} |V(G)| - |E(G)| + f(G) &= |V(G')| - (|E(G')| + 1) + (f(G') + 1) \\ &= 2. \end{aligned}$$

□

*Remark 7.0.10* (Our comment). Note that this last part of the proof, where we state that  $f(G) = f(G') + 1$ , is a little less obvious than one might think, and relies on Jordan Curve theorem together with several other results, see [Die17, Chap. 4.1-2].

**Definition 7.0.11** (Subdivisions). A graph  $G'$  is a **subdivision** of a graph  $G$  if we obtain  $G'$  by replacing some edges with paths, so that none of these paths have inner vertices in  $V(G)$  or another such path (informally: Insert new vertices on those edges).

*Remark 7.0.12.* E.g. start with a graph  $G = (V, E)$  and let  $e = uv$  be an edge of  $G$ , then form  $uw$  and  $wu$  with  $w \notin G$  and replace  $uv$  with  $uw$  and  $wv$ , and let this be  $G' = (V(G) \cup \{w\}, (E(G) \setminus \{uv\}) \cup (\{uw\} \cup \{wv\}))$ . This is one subdivision of  $G$ .

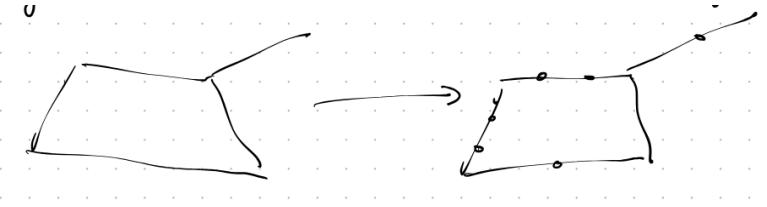


Figure 7.3: Illustration of subdivisions

#### Observations:

- A graph is planar iff all subdivisions of it are planar.
- A graph is planar iff all subgraphs are planar (it seems we must allow subgraphs to be non-proper here, i.e. we include  $G \subset G$  as a subgraph).

**Proposition 7.0.13** (Proposition 12). *Let  $G$  be a 2-connected planar graph. Then any face (*in any planar drawing*) is the inner or outer region of a cycle.*

*Proof.* If  $G$  is itself a cycle, then this holds by Jordans curve theorem. If not then  $G$  since  $G$  is 2-connected it can successively be constructed by adding  $H$ -paths to a cycle. And since  $G$  is plane graph its subgraphs are plane. Hence  $G = H \cup P$  where  $P$  is some  $H$ -path. Let  $f'$  be the face of  $H$  that contains  $\overset{\circ}{P}$ . Let  $C$  be the (inductively known) cycle that bounds the face  $f'$  in  $H$ . Fix a face  $f$  of  $G$ .

Let  $G[f]$  be the subgraph whose point set is the frontier  $\partial f$  of a face  $f$  of  $G$ .

Case 1:  $G[f] \subset H$ : If  $G[f] \subset H$  then  $\partial f \subset H$  so that  $f$  is also a face of  $H$ . By induction, it follows that  $f$  is bounded by a cycle.

Case 2:  $G[f] \not\subset H$ : In the case that  $G[f] \not\subset H$  then since  $G[f] \subset G = H \cup P$  we find that  $G[f] \cap (P \setminus H) \neq \emptyset$ . It follows that  $P \subseteq G[f]$ . Let  $f'$  be the  $H$ -face with boundary cycle  $C$  and  $\overset{\circ}{P} \subset f'$ . We find that  $f \subseteq f'$  since otherwise  $f$  would have to cross  $C$  but this is not possible since  $f \subset \mathbb{R}^2 \setminus G$  and  $C \subset G$ .

If  $e \subset G$  is an edge such that  $e \subset H$  that belong to the boundary of  $f$ , i.e.  $G[f]$ , then since  $f \subseteq f'$  we must have that  $e \subset C$ , and so  $G[f] \subset C \cup P$ . We find that  $f$  is (by [Die17, Lemma 4.2.1.(ii)]) is a face of  $C \cup P$  (since  $G[f] = \partial f \subset C \cup P$  and the latter is a subgraph of  $G$ ). Since  $\overset{\circ}{P} \subset f'$  we have that  $P \subset \overline{f'} = f' \cup C$ . We have that  $H \cap \overline{f'} = C$ . We know that  $P \cap H = \{u, v\}$  where  $u, v$  are the endpoints of  $P$ , and so  $\{u, v\} \subset C$ , i.e. the endpoints of  $P$  lie on  $C$ . Hence we may split  $C$  into two internally disjoint paths  $C_1, C_2$  that goes from  $u$  to  $v$ . Together with  $P$ , we then have three arcs that are disjoint (since  $P$  does not lie on  $H$ ) with the same endpoints. Note that  $C_1 \cup C_2 = C$ . By [Die17, Lemma 4.2.1(i)] we have that  $\mathbb{R}^2 \setminus (C \cup P)$  has three faces, whose frontiers are  $C_1 \cup P, C_2 \cup P$  and  $C_1 \cup C_2 = C$ , all of which are cycles. Since  $f$  is one of those faces, its frontier is a cycle.

In both cases, adding edges does not destroy cycles, and since  $f$  is already a face in  $G$  in both cases, anything attached happens on the outside of the boundary of  $\partial f$  in  $H$ , hence does not affect  $f$ :s boundary in  $G$  (the distance from an inner point  $x$  of a newly added edge  $e$  to  $\partial f$  is nonzero), hence it still holds that  $f$  is bounded by a cycle in  $G$ .

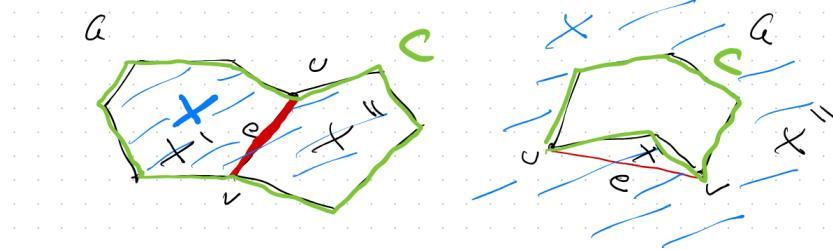


Figure 7.4: Illustration: Proof of proposition 12, lecture 7

□

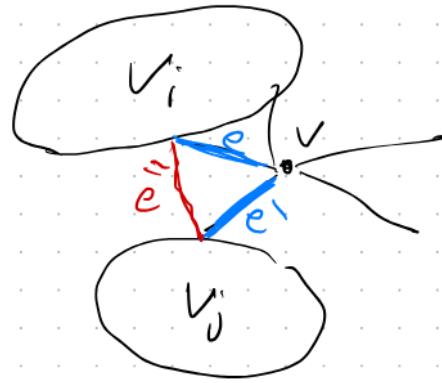
**Definition 7.0.14** (Maximal planar graph). A graph  $G$  is called **maximal planar** if adding any non-edge  $uv \notin E$  with  $u, v \in V$  to  $G$  yields a non-planar graph.

**Theorem 7.0.15** (Theorem 16). *Let  $G$  be a planar graph with atleast three nodes. Then*

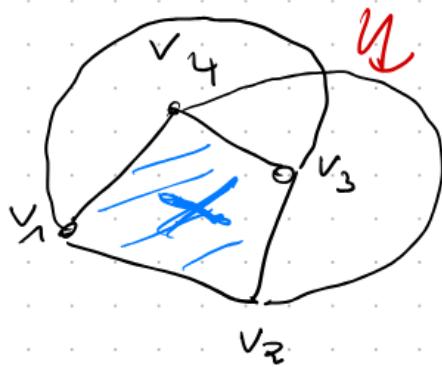
1.  $|E| \leq 3|V| - 6$  with equality if and only if  $G$  is maximal planar. In particular,  $G$  has a vertex of degree at most 5.
2. If  $G$  does not contain  $K_3$  as a subgraph, then  $|E| \leq 2|V| - 4$ . In particular,  $G$  has a vertex of at most degree 3.

*Proof.* (1): Since every planar graph is contained in a maximal planar graph, and  $|E(H)|$  is monotonically increasing with respect to such graphs, it is enough to prove this for  $G$  maximal planar. We will show that very face is bounded by a  $K_3$ . One may show (with some work) that if  $G$  was disconnected, it could not be maximal. Therefore,  $G$  is maximal.

Suppose  $G$  is not 2-connected. Then there is a vertex  $v$  such that  $G \setminus v$  is disconnected, with components  $C_1, \dots, C_k$  for  $k \geq 2$  (since  $|V| \geq 3$ ). Let  $e$  and  $e'$  be edges from two components  $C, C'$  to  $v$  that are consecutive (i.e. there are a finite number of edges from the components  $C_i$  to  $v$ ). Then we “complete” the triangle with  $e''$  as in the illustration below. Here we take it that  $e''$  is drawn inside the face  $F$  of the drawing of  $G$  between two edges  $e, e'$  of  $G$ . Each point on the interior of  $e$  and  $e'$  belongs to  $F$  on one side, so we may start at the endpoint of  $e'$  in  $C_j$ , move directly into the face  $F$  (the part visible of  $F$  is the sector between  $e$  and  $e'$  and then move along  $F$  (it is path-connected) until we come to the endpoint, say  $y$  of  $e$ ).

Figure 7.5: Illustration but with  $V_j, V_i$  instead of  $C_i, C_j$ 

By proposition 7.0.13 we have that every face of  $G$  is bounded by a cycle. Suppose that  $G$  has a face  $\textcolor{blue}{X}$  bounded by a cycle of length  $t \geq 4$  with nodes  $v_1, \dots, v_t$ . If  $v_1v_3 \notin E$  then we could add  $v_1v_3$  (written inside of  $\textcolor{blue}{X}$ ) without violating planarity. Since we know that  $G$  is maximal, we must conclude that  $v_1v_3$  must be in  $E$ , and hence written *outside* of  $\textcolor{blue}{X}$ . Similarly, if  $v_2v_4$  was not in  $E$  then we could add it so to the inside of  $\textcolor{blue}{X}$ , so  $v_2v_4 \in E$  but written outside of  $\textcolor{blue}{X}$ .

Figure 7.6: Illustration; face  $\textcolor{blue}{X}$  bound by cycle  $v_1v_2v_3v_4$  and added edges outside of  $\textcolor{blue}{X}$ 

By the same argument as for  $v_1v_3$ , we must have that  $v_2v_4$  is an edge of  $E$  drawn outside of  $\textcolor{blue}{X}$ . By [Die17, Lemma 4.1.2.(ii)] we have that  $v_1v_3$  and  $v_2v_4$  meet (take  $P_1 = v_1v_2 \cup v_2v_3$ ,  $P_2 = v_1v_3$ ,  $P_3 = v_1v_4 \cup v_4v_3$  and  $P = v_2v_4$  in the lemma), so the graph could not be planar in this case (it violates (iv) in the definition of planar graph [Die17, p. 92]). Contradiction! Thus, we must have that  $t = 3$  (minimal number of nodes needed for a cycle).

Since  $G$  is 2-connected, every edge lies on a cycle. By [Die17, Lemma 4.2.2.(ii)]  $e$  lies on the *frontier*

(boundary) of exactly two faces of  $G$ . By double-counting, we get that:

$$\begin{aligned}
2|E| &= \sum_{e \in E} |\{X \text{ face} : e \text{ is in the boundary of } X\}| \\
&= \sum_{X \text{ face}} |\{e : e \text{ is in the boundary of } X\}| \quad (\text{change of order: finite sum}) \\
&= \sum_{X \text{ face}} 3 \\
&= 3f.
\end{aligned}$$

Where we in the next-to-last equality used that each face had boundary a cycle consisting of three edges.

By Euler's formula (7.0.9) we find that  $3f =$

$$\begin{aligned}
2|E| &= 6 - 3|V| + 3|E| \\
\Leftrightarrow 3|V| - 6 &= |E|.
\end{aligned}$$

Assume that all vertices of  $G$  have degree atleast 6 (the contrapositive of the second claim in 1). Then

$$\begin{aligned}
6|V| &\leq \sum_{v \in V} d(v) \\
&= 2|E| \\
&= 6|V| - 12,
\end{aligned}$$

contradiction! Therefore, there is atleast one node  $v$  in  $G$  of at most degree 5.

2: We may assume that  $G$  is maximal  $K_3$  and planar, i.e. adding an edge either creates a  $K_3$  or breaks planarity.

We claim that  $G$  is 2-connected: Suppose that  $G$  was not 2-connected. Then there is a vertex  $v$  such that  $G - v$  has components  $C_1, \dots, C_k$  with  $k \geq 2$ . We distinguish different cases.

Every component  $C_i$  is an isolated vertex: If all vertices are isolated in  $G - v$ , then it follows that  $G$  is a tree: If not then there is a cycle  $C$  in  $G$ , but a cycle must involve an edge  $uw$  where  $u, w \neq v$ . But this contradicts  $\{u\}$  and  $\{w\}$  being components in  $G - v$ . Since  $G$  is also connected,  $G$  is a tree. If  $G$  is a tree then  $|E| = |V| - 1$  we have that

$$\begin{aligned}
|V| - 1 &\leq 2|V| - 4 \\
\Leftrightarrow |V| &\leq 3
\end{aligned}$$

which hold by assumption.

Every component  $C_i$  is not isolated vertex: By assumption there is a component  $C_i$  with  $|V(C_i)| \geq 2$  in  $G' = G - v$ . We know that there is atleast one edge from each  $C_i$  to  $v$  in  $G$ . So in particular there is an edge  $v_1v$  from  $v_1 \in V_1$  to  $v$ . Let  $C_i$  for  $i > 1$  be the component in  $G'$  with the closest edge  $e_i = vv_i$  entering  $v$  (in some small disc around  $v$  in  $\mathbb{R}^2$ ). It seems to us that with a “consecutive spokes” argument for suitably small disks  $D$  around  $v$  and  $v_1$ , together with some added work, one may show that there is a face  $X$  of  $G'$  that contains as its boundary both vertices of  $C_1$  and  $C_i$  for  $i > 1$ , together with some edge  $v_1v_2$  of  $C_1$  with  $v_2$  not having an edge to  $v$ . Then we see that we may add an edge  $e'$  from  $v$  to a node  $v_i$  (where the edge  $e'$ 's interior goes through the face  $X$  in  $G'$ ). This is still planar since there are no edges in  $X$ , and since  $v_2$  has no edge to  $v$  this does not create a  $K_3$  (since this would require  $v \neq v$  connected to both  $C_1$  and  $C_i$ , but this would contradict  $C_1, C_i$  being components in  $G'$ ). But this contradicts maximal planar with no  $K_3$ , and so  $G$  must be 2-connected.

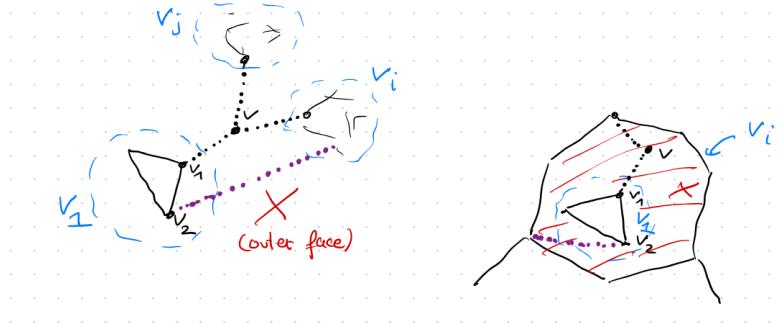


Figure 7.7: Illustration of proof of Theorem 16, part 2.

By proposition 7.0.13 every face of  $G$  is bounded by a cycle. Every such cycle has atleast length 4 (since otherwise it contains a  $K_3$ ). By double counting (as in 1), and using 2-connected (so that every edge is on a cycle and hence on the boundary of exactly two faces) we get that  $2|E| \geq 4f$ .

By Euler's formula 7.0.9 we get that

$$\begin{aligned} |E| &= |V| + f - 2 \\ &\leq |V| + \frac{|E|}{2} - 2 \\ \Leftrightarrow |E| &\leq 2|V| - 4. \end{aligned}$$

In particular, if all vertices had atleast degree four (contrapositive) then

$$\begin{aligned} 4|V| &\leq \sum_{v \in V} d(v) \\ &= 2|E| \\ &\leq 4|V| - 8, \end{aligned}$$

contradiction!

□

**Theorem 7.0.16** (Theorem 17).  $K_5$  and  $K_{3,3}$  are not planar.

*Proof.* Suppose  $K_5$  were planar. Then by theorem 7.0.15(i) it would follow that

$$\begin{aligned} |E(K_5)| &\leq 15 - 6 \\ &= 9. \end{aligned}$$

But  $|E(K_5)| = \frac{1}{2} \sum_{v \in K_5} d(v) = \frac{4 \cdot 5}{2} = 10$ , contradiction!

Suppose that  $K_{3,3}$  were planar. Then note that  $K_{3,3}$  is bipartite so all cycles are of even length  $\neq 3$ , and we obtain

$$\begin{aligned} 9 &= |E(K_{3,3})| \\ &\leq 2|V(K_{3,3})| - 4 \\ &= 8, \end{aligned}$$

contradiction!

□

**Definition 7.0.17** (Topological minor). If a graph  $G$  contains a subdivision of a graph  $X$  as a subgraph then  $X$  is said to be a **topological minor** of  $G$ .

**Theorem 7.0.18** (Kuratowski). *A graph  $G$  is planar if and only if it does not contain  $K_5$  or  $K_{3,3}$  as a topological minor.*

# Chapter 8

## Lecture 8

Today: Graph colorings.

**Definition 8.0.1** (Graph Coloring). Given a graph  $G = (V, E)$  a map  $c : V \rightarrow S$  is called a (proper vertex) **coloring** of  $G$  if  $c(u) \neq c(v)$  for all edges  $uv \in E$ , where  $S$  is a set of colors.

**Definition 8.0.2** (Chromatic number). The (vertex)-**chromatic number** of a coloring  $c : V \rightarrow S$  of a graph  $G = (V, E)$ , denoted  $\chi(G)$ , is the *smallest*  $k \in \mathbb{N}$  such that  $G$  has a coloring with  $|S| = k$ .

**Definition 8.0.3** ( $k$ -colorable). If  $\chi(G) \leq k$  for  $k \in \mathbb{N}$ , then  $G$  is called  **$k$ -colorable**.

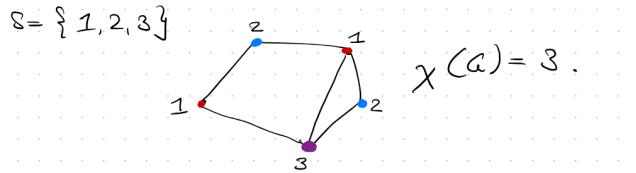


Figure 8.1: Example of a graph  $G$  with chromatic number  $\chi(G) = 3$ .

**Example 8.0.4.** Note here that we need atleast three colours for the  $K_3$  contained in  $G$  above (so  $\chi(G) \geq 3$ ), and it is then enough that we find a colouring with 3 colours as above to conclude that  $\chi(G) = 3$ .

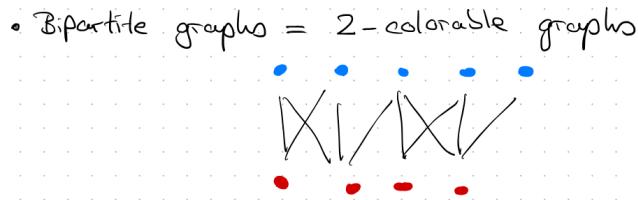


Figure 8.2: Bipartite graphs = 2-colorable graphs

**Example 8.0.5.**

More generally, a subset of vertices  $V' \subset V$  is called **independent** or **stable** if no two of its vertices are adjacent, with respect to a graph  $G = (V, E)$ . Thus, a  $k$ -coloring  $c : V \rightarrow S$  ( $|S| = k$ ) is precisely

a partition of the vertices  $V$  into  $k$  independent sets, also called **color classes**.

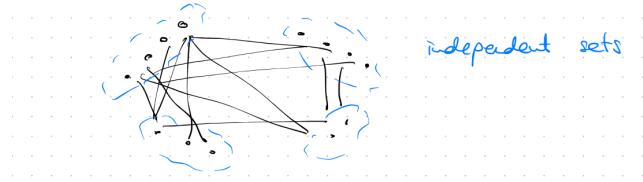


Figure 8.3: Illustration of independent sets of vertices.

- Committee scheduling problem.
  - Committees  $C_1, \dots, C_n$  with politicians
  - $k$  time slots for meetings.

Task: Device a meeting schedule for all committees such that no two committees with a common member meet at the same time.

Consider  $G = (V, E)$  with  $V = \{C_1, \dots, C_n\}$  and  $E = \{C_i C_j : C_i \cap C_j \neq \emptyset\}$ .

The scheduling problem is equivalent to finding a  $k$ -coloring as two committees can have simultaneous meetings iff no senator is a member of both.

**Theorem 8.0.6** (Theorem 19: Four-Color-theorem). *Every planar graph is 4-colorable.*

*Remark 8.0.7.* Proved by Appel and Haken in 1976. The only known proofs rely on computers.

**Theorem 8.0.8** (Theorem 20: Five-Color-Theorem). *For any planar graph  $G$ ,  $\chi(G) \leq 5$ .*

*Proof.* We use induction on the number of vertices  $|V(G)|$  of  $G$ . The statement is clear for  $|V(G)| \leq 5$ , so assume that  $|V(G)| \geq 6$ , that  $|E(G)| = m$  and consider a particular planar drawing of  $G$ . By 7.0.15.1 there is a vertex  $v \in V(G)$  of degree at most 5. Let  $H := G \setminus v$ . Then  $H$  is planar, and by induction has a coloring  $c : V(H) \rightarrow \{1, 2, \dots, 5\}$ . If the neighbors of  $v$  are colored by at most four colors  $s_1, s_2, s_3, s_4$ , then we extend this to a coloring of  $G$  by letting  $\bar{c} : G \rightarrow \{1, 2, \dots, 5\}$  be an extension of  $c$  with  $\bar{c}(v) = s_5$  and  $\bar{c}|_H = c$ . If not, then  $v$  has exactly 5 neighbors (since  $d(v) \leq 5$ ) all with different colors. Pick any  $v_1, \dots, v_5$  of these neighbors with different colors, with indices cyclic, such as in the picture:

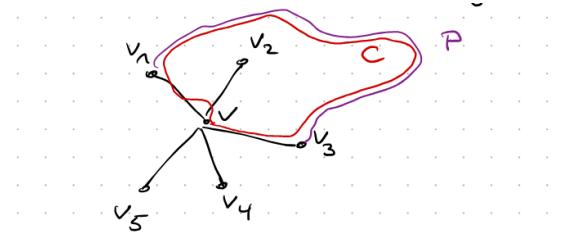


Figure 8.4: Cyclic ordering  $v_1, \dots, v_5$

This is possible since, identifying  $G$  with its planar embedding  $P(G)$ , we have that  $G \subseteq \mathbb{R}^2$  and so we may pick a small disk around  $v$ , and choose the ordering based on the “spokes” entering this disk. Let  $V_{1,3}$  be the set of all vertices that are colored with colors  $c(v_1)$  and  $c(v_3) \sim v_1, v_3 \in V_{1,3}$ . Suppose there is no path  $P_{1,3}^H : v_1 \rightsquigarrow v_3$  in  $H$  that only uses vertices in  $V_{1,3}$  and let  $V'_{1,3}$  be the set of vertices

that can be reached by paths from  $v_1$  by only using vertices in  $V_{1,3}$ . Then  $v_3 \notin V'_{1,3}$ , and we can define a new 5-coloring of  $H$  by setting

$$c'(v) = \begin{cases} c(v), & \text{if } v \notin V_{1,3}, \\ c(v_3), & \text{if } v \in V'_{1,3} \text{ and } c(v) = c(v_1), \\ c(v_1), & \text{if } v \in V'_{1,3} \text{ and } c(v) = c(v_3) \end{cases},$$

i.e. we permuted the colors in  $V'_{1,3}$  between  $c(v_1)$  and  $c(v_3)$ .

*Remark 8.0.9.* If both endpoints of an edge are in  $V'_{1,3}$  then one of them is of color  $c(v_1)$  and one of color  $c(v_3)$ . After applying  $c'$  they just swap colors. If  $e$  is an edge such that one of its vertices are in  $V'_{1,3}$ , then one vertex has color  $c(v_1)$  or  $c(v_3)$ , while the other vertex must have a color different from  $c(v_1)$  and  $c(v_3)$ . Hence after applying  $c'$  one of the vertices is either equal to  $c(v_1)$  or  $c(v_3)$ , while the other vertex is still of the same color different from  $c(v_1)$  and  $c(v_2)$ . If both endpoints of an edge  $e$  are not in  $V'_{1,3}$  then they are different with respect to  $c$  and are the same after applying  $c'$ . Hence  $c'$  is indeed a coloring of  $H$ .

In this new coloring, we see that  $c'(v_1) = c'(v_3) = c(v_3)$ . If we extend  $c'$  to  $G$  by letting  $c'(v)$  be the color not used on  $v_1, \dots, v_5$  with respect to the coloring  $c'$  of  $H$ , we see that  $vv_1, \dots, vv_5$  are the only edges that can break this being a coloring, but the endpoints are all of different colors so this becomes a coloring of  $G$ .

It remains to consider the case when  $v_3 \in V'_{1,3}$ , i.e. when there is a path in  $H$  between  $v_1$  and  $v_3$  only using edges of  $V_{1,3}$  (i.e. vertices in  $H$  colored by  $c$  with the colors  $c(v_1)$  or  $c(v_3)$ ).

Define  $V_{2,4}$  similarly to  $V_{1,3}$ , i.e. as the vertices in  $H$  colored (with respect to  $c$ ) with the colors  $c(v_2)$  or  $c(v_4)$  and observe that  $V_{1,3} \cap V_{2,4} = \emptyset$ . Let  $P_{1,3}^H : v_1 \rightsquigarrow v_3$  be a path in  $V_{1,3}$  from  $v_1$  to  $v_3$ , and let  $C$  be the cycle  $vv_1 \cup P \cup v_3v$ . Then we claim that  $v_2$  and  $v_4$  lie in different regions defined by  $C$ , one inside and one on the outside: Take a tiny open disk  $D$  centered around  $v$ , such that  $D$  in the drawing meets only the 5 edges  $vv_i$  for  $i = 1, \dots, 5$  (this is possible since the set, say  $S$  of every node and edge not equal to  $v_1v_i$  and  $v$  is closed and so have positive distance to  $v$  in the planar drawing). See the picture below:

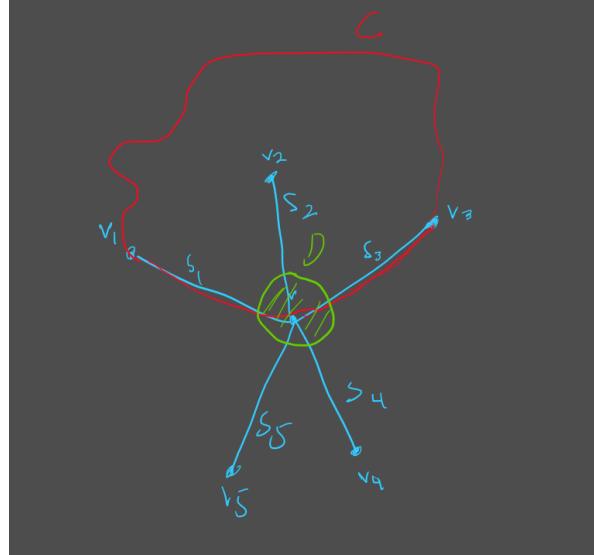


Figure 8.5: Illustration; proof of theorem 20.

In the above,  $s_i$  should be understood as the segment of  $vv_i$  that lies in  $D$ , i.e.  $s_i := vv_i \cap D$ .

Consider the two regions of  $D \setminus (s_1 \cup s_3)$ . One of the regions meet  $s_4$  and the other meets  $s_2$ . We observe that  $C \cap D \subset s_1 \cup s_3$ . Let the two regions of  $D \setminus (s_1 \cup s_3)$  be  $R_1, R_2$ . We note that so that

$$\begin{aligned} R_i \cap C &= R_i \cap (C \cap D) \\ &\subseteq R_i \cap (s_1 \cup s_3) \\ &= \emptyset. \end{aligned}$$

Hence  $R_i \subset \mathbb{R}^2 \setminus C$ . Furthermore, since  $R_i$  is connected, it lies inside a single face of  $C$ . We claim that  $R_1$  and  $R_2$  lies inside distinct faces of  $C$ : If not, i.e.  $R_1, R_2$  are inside the same face of  $C$ , then  $D \cap (\mathbb{R}^2 \setminus C) = R_1 \cup R_2$  are in the same face, call it  $F$ . Let  $F'$  be other face of  $C$  and observe that  $C = \partial F = \partial F'$ , that  $F, F'$  are open in  $\mathbb{R}^2 \setminus C$  and disjoint. Since  $v \in \partial F'$  any neighborhood of  $v$  must intersect  $F'$ . In particular, since  $D$  is an open neighborhood of  $v$ , and we have that

$$D = (D \cap (\mathbb{R}^2 \setminus C)) \cup (D \cap C)$$

where neither  $D \cap \mathbb{R}^2 \setminus C$  or  $D \cap C$  intersects  $F'$ , we find that  $D$  does not intersect  $F'$ , contradicting that  $v \in \partial F'$ . Thus  $R_1$  and  $R_2$  are contained in different faces of  $C$ , and then so are  $s_2 \subset vv_2$  and  $s_4 \subset vv_4$ . The cycle  $C$  meets the edges  $vv_2$  and  $vv_4$  only in  $v$  since it is a planar drawing, and  $C$  uses neither of the edges  $vv_2$  nor  $vv_4$ . It follows that  $K_i := (vv_i \setminus \{v\}) \subset \mathbb{R}^2 \setminus C$  for  $i = 2, 4$ . Furthermore,  $K_i$  is connected (it is the continuous image of a connected set). Since  $\mathbb{R}^2 \setminus C = F \sqcup F'$  and  $F, F'$  are open and disjoint, this forces either  $K_i \subset F$  or  $K_i \subset F'$  ([Lee11, Theorem 4.9.(a)]). We see from this that  $K_1 \subset F$  and  $K_2 \subset F'$  or the other way around from what we showed before, so that indeed  $v_2$  and  $v_4$  are in different components of  $\mathbb{R}^2 \setminus C$ .

From this, we claim that there can not be a path  $v_2 \rightsquigarrow v_4$  in  $H$  that uses only vertices in  $V_{2,4}$ , since otherwise there exists a vertex on this path that lies on  $C$ : If not then since we have a planar drawing, the path  $v_2 \rightsquigarrow v_4$  can only intersect vertices and not edges. If it does not intersect any vertex on  $C$  then this is a path in  $\mathbb{R}^2 \setminus C$  from  $v_2$  to  $v_4$ . But since  $\mathbb{R}^2$  is locally path-connected, and  $\mathbb{R}^2 \setminus C$  is an open subset, we have that  $\mathbb{R}^2 \setminus C$  is locally path connected [Lee11, Prop. 4.26.(b)] and so the components  $F$  and  $F'$  coincide with its path-components ([Lee11, Prop. 4.26.(d)]) which means that there can be no path from  $F$  to  $F'$  in  $\mathbb{R}^2 \setminus C$ . Therefore indeed any path from  $v_2$  to  $v_4$  needs to contain a vertex on  $C$ . Since the path  $v_2 \rightsquigarrow v_4$  is in  $H$  it must hit something inside  $V_{1,3}$  (by construction of  $C$  all other vertices beside  $v$  are in  $V_{1,3}$ ). Thus  $V_{1,3} \cap V_{2,4} \neq \emptyset$ , contradiction! Now we are in the same situation as before, but with respect to  $V_{2,4}$  and  $v_2, v_4$  instead of  $v_1, v_3$  and  $V_{1,3}$ , and the same argument goes to show that we can find a 5-coloring of  $G$ .  $\square$

**Theorem 8.0.10** (Grötsch). *Every planar graph not containing a triangle is 3-colorable.*

### 8.0.1 Relation to other graph-invariants

**Definition 8.0.11** ( $\Delta(G)$ ). Let

$$\Delta(G) := \max\{d(v) : v \in V(G)\}$$

be the **maximum degree of  $G$**

**Proposition 8.0.12** (Proposition 13). *Let  $G = (V, E)$  be a graph. Then*

$$\chi(G) \leq \Delta(G) + 1.$$

*If  $G$  is a complete graph or a cycle of odd length, then equality holds.*

**Remark 8.0.13.** Observe that by a cycle being of *odd order*, we mean that the cycle  $C_n$  has an odd order of vertices, i.e. if  $C_n : v_1, v_2, \dots, v_n$  the  $n = 2k + 1$  for  $k \in \mathbb{N}$ .

*Proof.* We choose an ordering  $v_1, \dots, v_n$  of the vertices of  $G$ . We construct inductively a coloring: We color  $v_i$  with the smallest number  $c(v_i) \in \mathbb{N}$  such that no neighbor  $v_j$  of  $v_i$  with  $j < i$  was colored with it. By construction, we have that

$$c(v_i) \leq \Delta(G) + 1.$$

The induction stops after a finite number of steps.

To do the induction a bit more formally: Let the colors start from 1, i.e. we choose colors in  $\mathbb{N}_{\geq 1}$

Base case  $v_1$ : Then there are no  $j < 1$  so we may choose  $c(v_1) = 1$ , so indeed  $c(v_1) \leq \Delta(G) + 1$ .

Inductive hypothesis: After coloring  $v_1, \dots, v_i$  the colors used are in  $T := \{1, 2, \dots, \Delta(G) + 1\}$ .

Inductive step: Consider  $v_{i+1}$ . The only forbidden colors for  $v_{i+1}$  are those among its already colored neighbors for  $j < i$ . There are at most  $\Delta(G)$  such neighbors. Furthermore, by induction the colors used are in  $T$ . Since  $|T| = \Delta(G) + 1$  there is at least one free color. Choose the smallest such color from  $T$ . Then we still have a coloring of  $v_1, \dots, v_{i+1}$  in the colors of  $T$ . By induction it follows that we get a coloring of  $G$  in  $T$  (not necessarily using all colors of  $T$ ). Hence  $\chi(G) \leq \Delta(G) + 1$ . We claim that  $\chi(K_n) = n = \Delta(K_n) + 1$  so we have equality for a complete graph  $K$  on  $n$  vertices.

If  $C_n$  is a cycle and  $n = 2k$  for  $k \geq 2$ , then it contains no sub-cycles, so in particular no odd cycle. By [Die17, Prop. 1.6.1]  $C_n$  is then bipartite. A bipartite graph is 2-colorable, and every vertex has two neighbors in a cycle, so  $\chi(G) = 2 < \Delta(G) + 1 = 3$ . An alternative easier way is just to let say the vertices with odd index have one color and the even vertices have the other color. Then  $v_{2n}$  will have a color different from  $v_1$ , and so there is no cause for concern.

On the other hand, if  $C_n$  is a cycle with  $n = 2k + 1$  for  $k \in \mathbb{N}_{\geq 1}$  then assume on the contrary that there was a two-coloring of  $C_n$ . Then let vertices with odd index have the color **red** and vertices with even index have the color **blue**. Then  $v_{2k+1}$  must have the color **red**, but also  $v_1$  has the color **red**, which is impossible. Hence by the upper bound and the fact that  $\Delta(G) = 2$  we must have that  $\chi(C_n) = 3 = \Delta(G) + 1$ .  $\square$

Stronger:

**Theorem 8.0.14** (Brooks, 1941). *Let  $G$  be a connected graph. If  $G$  is neither complete nor an odd cycle, then  $\chi(G) \leq \Delta(G)$ .*

*Proof.* See [Die17, Theorem 5.2.4].  $\square$

Observation: The proof of proposition 13 gives an algorithm which allows for improvements:

- The color of  $v_i$  only depends on the neighbors in  $\{v_1, \dots, v_{i-1}\}$  and not on those in  $\{v_{i+1}, \dots, v_n\}$ .
- Choose an ordering such that  $v_i$  has only few neighbors in  $\{v_1, \dots, v_{i-1}\}$  for all  $i$ .

**Definition 8.0.15** (Coloring number). The **coloring number**  $\text{col}(G)$  is the smallest number  $k$  such that  $G$  has an ordering of its vertices  $v_1, \dots, v_n$  such that for any  $i$ , the vertex  $v_i$  has fewer than  $k$  neighbors in  $\{v_1, \dots, v_{i-1}\}$ .

**Proposition 8.0.16** (Proposition 14). *Let  $G = (V, E)$  be a graph. Then  $\chi(G) \leq \text{col}(G) = \max\{\delta(H) : H \text{ subgraph of } G\} + 1$ . In particular,  $G$  has a subgraph  $H$  with  $\delta(H) \geq \chi(G) - 1$ .*

*Proof.*  $\chi(G) \leq \text{col}(G)$ : Proceed in the same way as in the proof of 8.0.12 with respect to an ordering  $\mathcal{O}$  of  $V(G)$  that witnesses  $k = \text{col}(G)$  to construct a coloring  $k$ -coloring. Hence by definition we must have that  $\chi(G) \leq k = \text{col}(G)$ .

$\text{col}(G) = \max\{\delta(H) : H \text{ subgraph of } G\} + 1$ : Choose an ordering of the vertices  $V$  of  $G$  in the following way: Let  $v_n$  be a vertex with *minimal degree*  $\delta(G)$ , let  $v_{n-1}$  be a vertex with minimal degree in  $G \setminus v_n$ , and so on. Let  $H_i := G[v_1, \dots, v_i]$  be the induced subgraph on  $v_1, \dots, v_i$  of  $G$ .

*Remark 8.0.17.* Observe that  $G \setminus \{v_{i+1}, \dots, v_n\} = H_i = G[v_1, \dots, v_i]$ .

Then

$$\begin{aligned}\text{col}(G) &\leq \max_{1 \leq i \leq n} \left\{ \underbrace{\delta(H_i)}_{=d_{H_i}(v_i)} \right\} + 1 \\ &\leq \max\{\delta(H) : H \text{ subgraph of } G\} + 1.\end{aligned}$$

The second inequality is clear from definition. For the first inequality: By construction, each  $v_i$  has fewer than  $\delta(H_i) + 1$  neighbors (it has exactly  $\delta(H_i)$  neighbors) in  $\{v_1, \dots, v_{i-1}\}$ . Hence if we take  $\ell := \max_{1 \leq i \leq n} \delta(H_i) + 1$  then each  $v_i$  has fewer than  $\ell$  neighbors in  $\{v_1, \dots, v_{i-1}\}$ . Therefore, under the identification

$$\text{col}(G) = \min \left\{ \begin{array}{l} k(\mathcal{O}_{V(G)}) : \mathcal{O}_{V(G)} \text{ is an ordering of the vertices } V(G) \text{ of } G \\ \text{and } k(\mathcal{O}_{V(G)}) \text{ is the least number } p \text{ such that every vertex } v_i \text{ has fewer than } p \text{ neighbors in } \{v_1, \dots, v_{i-1}\} \end{array} \right\}$$

then if we call the ordering we choose for  $\mathcal{O}'_{V(G)}$  we see that  $k(\mathcal{O}'_{V(G)})$  occurs as an element in the set over which we take the minimum, hence, and  $k(\mathcal{O}'_{V(G)}) \leq \ell$ , hence  $\text{col}(G) \leq \ell$ .

It remains to show the  $\geq$  direction: Take any ordering of  $V(G)$  such that witnesses  $\text{col}(G) = k$  (every vertex has fewer than  $k$  earlier neighbors). Restrict the ordering of  $G$  to the vertices of  $H$ . Then the number of earlier neighbors for vertices in  $H$  is less than or equal to the number of earlier neighbors in  $G$ , hence still  $< k$ , so that this induced order of  $V(H)$  witnesses  $\text{col}(H) \leq k$ . Hence  $\text{col}(H) \leq \text{col}(G)$ . For any subgraph  $H \subseteq G$  and any ordering  $v_1, \dots, v_m$  of the vertices  $V(H)$ , we have that  $v_m$  has  $d_H(v_m) \geq \delta(H)$  neighbors in  $\{v_1, \dots, v_{m-1}\}$ . Hence we must have that  $\text{col}(H) \geq \delta(H) + 1$ , since for a  $k \leq \delta(H)$  would fail for  $v_m$ , since we would get  $k \leq \delta(H) \leq d_H(v_m)$  so that the vertex  $m$  does *no* have fewer than  $k$  neighbors in  $\{v_1, \dots, v_{m-1}\}$ . Summarizing, we have found that

$$\text{col}(G) \geq \text{col}(H) \quad \text{and} \quad \text{col}(H) \geq \delta(H) + 1$$

so that  $\text{col}(G) \geq \delta(H) + 1$  for any subgraph  $H$ . It follows that  $\text{col}(G) \geq \max\{\delta(H) : H \text{ subgraph of } G\} + 1$ . We conclude that

$$\text{col}(G) = \max\{\delta(H) : H \text{ subgraph of } G\} + 1.$$

For the last statement, we just pick the subgraph  $H$  with maximal minimal degree over all subgraphs of  $G$ . Then we have that  $\delta(H) + 1 \geq \chi(G) \Leftrightarrow \delta(H) \geq \chi(G) - 1$  (well, that the maximum is well-defined requires that the number of subgraphs  $H$  of a simple finite graph, is a finite number, which we have not shown).

□

### 8.0.2 Edge colorings

**Definition 8.0.18** (Line graph). The **line graph**  $L(G)$  of a graph  $G = (V, E)$  is the graph with vertex set  $E$  where two edges  $e \neq f$  (i.e. vertices in  $L(G)$ ) form an edge in  $L(G)$  if  $e \cap f \neq \emptyset$ , i.e. they have exactly one adjacent vertex in common.

**Definition 8.0.19** (Edge coloring). An **edge coloring** of a graph  $G = (V, E)$  is a map  $c : E \rightarrow S$  with  $c(e) \neq c(f)$  whenever  $e \cap f \neq \emptyset$ , i.e.  $c$  is a *vertex coloring* of the line graph  $L(G)$ .

**Definition 8.0.20** (Edge-chromatic number  $\chi'(G)$  of  $G$ ). The **edge-chromatic number**  $\chi'(G) = \chi(\mathcal{L}(G))$  of  $G$ , is the smallest number  $k \in \mathbb{N}$  such that there is a  $k$ -coloring of  $\mathcal{L}(G)$ . Or that is, the smallest number  $k$  for which an  $k$ -edge-coloring of  $G$  exists.

**Theorem 8.0.21** (Theorem 23; Vizing). *For all graphs  $G = (V, E)$ , the claim is that*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

*Proof.* See [Die17, Theorem 5.3.2].  $\rightsquigarrow$  partition of graphs into two classes, e.g. □

**Proposition 8.0.22** (Proposition 15). *For any bipartite graph  $G$ , we have that  $\chi'(G) = \Delta(G)$ .*

*Proof.* For any graph  $G$  pick a vertex  $v$  with maximal degree  $\Delta := \Delta(G)$ . If  $\chi'(G) < \Delta$  then there would be too few colors to color the edges incident with  $v$ , hence  $\chi'(G) \geq \Delta$ . Therefore, it is enough to show that  $\chi'(G) \leq \Delta(G)$ .

We proceed by induction on  $|E|$ . □

Base case: If  $|E| = 0$ , then  $L(G) = 0$  is the empty graph, and so  $\chi'(G) = 0 = \Delta(G)$  since there are no nodes to color in  $L(G)$  and since  $d(v) = 0$  for all nodes  $v$  in  $G$ .

Induction: Let now  $|E| > 0$ . Let  $e = xy \in E$ . By induction, there is an edge-coloring  $c$  of  $H := G \setminus e$  with at most  $\Delta(H) \leq \Delta(G)$  colors. Both  $x$  and  $y$  are endpoints of at most  $\Delta(G) - 1$  edges of  $H$ . Thus, there exists a color  $\alpha \in \{1, \dots, \Delta(G)\}$  that is not attained by any edge incident to  $x$ , and a color  $\beta \in \{1, \dots, \Delta(G)\}$  not taken by any edge having  $y$  as an endpoint in  $H$ . If  $\alpha = \beta$  then we may extend the edge-coloring of  $H$  to  $G$  by setting  $c(e) = \beta$ . We check that this is indeed an edge-coloring of  $G$ : If  $h, f$  are edges of  $G$  such that neither  $h$  nor  $f$  equals  $e = xy$  then we already know that if  $h \cap f \neq \emptyset$  then  $c(h) \neq c(f)$ . Now assume we have  $e$  and  $f$  such that  $e \cap f \neq \emptyset$ . If  $e \cap f = \{y\}$  then we know that  $e$  is assigned a color different from  $f$  by construction, and similarly if  $e \cap f = \{x\}$ . Furthermore, by construction, the coloring uses *at most*  $\Delta(G)$  colors, so indeed  $\chi'(G) \leq \Delta(G)$ .

We proceed to treat the case when  $\alpha \neq \beta$ : In this case, we can assume that there is an edge  $f$  with endpoint  $x$  and color  $c(f) = \beta$  and an edge incident to  $y$  with color  $\alpha$  since if not then we can choose  $\alpha = \beta$  and either way this simplifies to the previous case.

We extend  $f$  to a maximal trail  $T$  such that the edges in  $T$  are colored in an alternating fashion by  $\beta$  and  $\alpha$ . Since this is induced from our coloring of  $H$ , we see that there can be no repetition of any vertex, so this is indeed a path. Suppose  $T$  contains  $y$ . Then since  $y$  is not incident with an edge of color  $\beta$ , we must have that  $T$  ends in  $y$ , with last color  $\alpha$ . Since odd edge-numbers in the cycle (if we count from one) have color  $\beta$ , and even  $\alpha$ , we find that  $T$  must have even length. But then  $T \cup \{xy\}$  form an odd cycle in  $G$ , contradicting that  $G$  is bipartite. Therefore,  $T$  does not end in  $y$ . Thus  $y$  is not contained in  $T$ . We now swap the colors of the edges along  $T$ . By the choice of  $\alpha$  and maximality of  $T$ , this is a coloring of  $G \setminus e$ . In this new colouring, we are still only using colors in  $\{1, \dots, \Delta(G)\}$ , and now neither  $x$  nor  $y$  are incident with an edge with color  $\beta$ . We may then color  $e = xy$  by  $\beta$ , and get an edge-coloring of  $G$ .

# Chapter 9

## Lecture 9

Today:

- Perfect graphs.
- Counting graph coloring.

**Definition 9.0.1** (Clique number). The **clique number**  $\omega(G)$  of a graph  $G$ , is the *greatest* integer  $r$  such that  $K_r$  is a subgraph of  $G$ .

We have that  $\chi(G) \geq \omega(G)$ , since any (vertex)-coloring of  $G \supset K_{\omega(G)}$  requires at least  $\omega(G)$  colors.

**Definition 9.0.2** (Girth). The **girth**  $g(G)$  of a graph  $G$ , is the *minimum* length of a cycle in  $G$ .

*Remark 9.0.3.* By definition,  $g(G) \geq 3$  if  $G$  has at least one cycle.

**Theorem 9.0.4** (Theorem 24: Erdős). *For every integer  $k$ , there exists a graph  $G$  with girth  $g(G) > k$  and  $\chi(G) > k$ .*

*Proof.* Later (probabilistic). □

**Q:** When is  $\chi(G) = \omega(G)$ ?

**Definition 9.0.5** (Perfect graph). A graph is **perfect** if for every induced subgraph  $H \subseteq G$ ,

$$\chi(H) = \omega(H).$$

Many interesting graphs are perfect:

- Complete graphs.
- Empty graphs.
- Bipartite graphs.
- Line graphs of bipartite graphs.
- Comparability graphs of posets.
- Interval graphs.
- Chordal graphs.

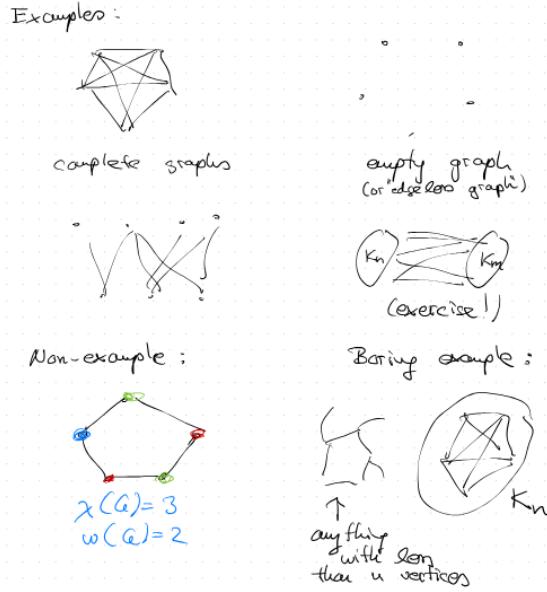


Figure 9.1: Examples

**Definition 9.0.6** (Chordal graph). A graph is called **chordal** (or **triangulated**) if it does not contain induced cycles of length  $\geq 4$ .

*Remark 9.0.7.* Observe that a chordal graph  $G$  may contain a cycle  $C$  of length  $\geq 4$ , but it must then hold that there are extra edges in  $G$  between non-consecutive vertices on  $C$ . This is saying the same thing as “there is no vertex-set  $S \subset V$  such that the *induced* graph  $G[S] \subseteq G$  is exactly a cycle of length  $\geq 4$ ”.

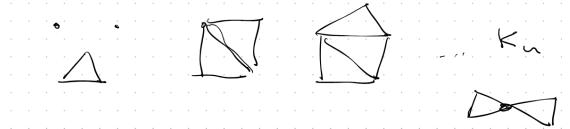


Figure 9.2:

*Remark 9.0.8.* We claim that an induced subgraph of a chordal graph, is chordal. This is immediate from the fact that if  $G' := G[S]$  is induced from  $G$ , and  $G'[V]$  is induced from  $G'$ , then  $G'[V] = G[V]$ .

Given two graphs  $G, H$ , let

$$G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$$

and

$$G \cap H = (V(G) \cap V(H), E(G) \cap E(H)).$$

**Definition 9.0.9** (Pasting). If  $G$  is a graph with *induced* subgraphs  $G_1, G_2$  and  $S$  such that  $G_1 \cup G_2 = G$  and  $S = G_1 \cap G_2$ , we say that  $G$  *arises* from  $G_1$  and  $G_2$  by **pasting** these graphs together along  $S$ .

**Proposition 9.0.10** (Proposition 16). *A graph  $G$  is chordal  $\Leftrightarrow G$  can be constructed recursively by pasting along complete graphs, starting from complete graphs.*

*Remark 9.0.11.* That is, we initialize with the set  $\mathcal{C} = \{K_m : m \geq 1\}$  and then recursively we add  $G_1 \cup G_2 = G$  to  $\mathcal{C}$  whenever  $G_1 \cap G_2$  is a complete subgraph.

*Proof.* [Die17, Proposition 5.5.1]. □

**Theorem 9.0.12** (Theorem 25). *If a graph  $G$  is chordal, then it is perfect.*

*Proof.* We proceed by induction on the statement, applied to the process described in proposition 16.

If  $G = K_n$  is complete then  $\chi(G) = n$ , and  $\omega(G) = n$ , so indeed  $\omega(G) = \chi(G)$ . Since any induced subgraph on  $G$  is also complete, it holds that  $G$  is perfect.

If not then by proposition 16  $G$  must have been recursively constructed by pasting along complete graphs, starting from complete graphs. So let  $G_1, G_2$  be chordal graphs, such that  $G = G_1 \cup G_2$  and  $S = G_1 \cap G_2$  is complete. Let  $H$  be an arbitrary induced subgraph. It suffices to show that  $\chi(H) \leq \omega(H)$ . Let  $H_1 = H \cap G_1$ ,  $H_2 = H \cap G_2$  and  $T = S \cap H$ . Let  $u, v \in V(T) = V(H) \cap V(S)$ . Then since  $u, v$  are in  $V(S)$  and  $S$  is complete, we have that  $e = uv \in E(S) \subseteq E(G_i) \subseteq E(G)$  (for  $i = 1, 2$ ). Since  $H$  is an induced subgraph, and  $u, v$  are in  $V(H)$ , it follows that  $uv \in E(H)$  since  $uv$  in  $E(G)$ . So  $uv \in E(T) = E(H) \cap E(S)$ . Thus  $T$  is complete.

We observe that  $H_1 \cap H_2 = T$ , and that

$$\begin{aligned} H_1 \cup H_2 &= (H \cap G_1) \cup (H \cap G_2) \\ &= H \cap (G_1 \cup G_2) \\ &= H \cap G \\ &= H. \end{aligned}$$

By assumption,  $G_i$  is an induced subgraph of  $G$ . Let  $u, v \in V(H_i) \subset V(H)$  such that  $e = uv \in E(G_i)$ . Since  $E(G_i) \subset E(G)$  we see that  $e \in E(G)$ . Since  $H$  is induced from  $G$  it follows that  $e \in E(H)$ . Hence  $e \in E(H) \cap E(G_i) = E(H_i)$ . Therefore,  $H_i$  is an induced subgraph of  $G_i$  for  $i = 1, 2$ . By the inductive assumption,  $G_i$  is chordal and so perfect. Thus, we have that  $\omega(H_i) = \chi(H_i)$ , i.e.  $H_i$  is perfect, and so can be colored with  $\omega(H_i)$  many colors. Since  $T$  is complete we need  $|V(T)|$  many colors to color it. Since  $H_i$  is an induced subgraph of  $G_i$  and  $G_i$  is induced from  $G$ , it follows that  $H_i$  is an induced subgraph of  $G$ . Since  $H_i \subset H$  and  $H$  is an induced subgraph of  $G$  it follows that if  $e = uv \in H$  then  $e \in E(G)$ . If  $u, v \in V(H_i)$  then  $e \in E(H_i)$  since  $H_i$  is induced. Therefore,  $\omega(H_i) \leq \omega(H)$ . Let  $S = \{1, \dots, \omega\}$  where  $\omega := \max\{\omega(H_1), \omega(H_2)\}$ , and consider a coloring  $c_1 : V(H_1) \rightarrow S$  of  $H_1$  and  $c_2 : V(H_2) \rightarrow S$  of  $H_2$ . Assume without loss of generality that  $\omega = \omega(H_1)$ . If needed, permute the coloring of  $c_2$  so that it agrees with  $c_1$  on  $T$ . Then since there are no edges from  $H_1 \setminus T$  to  $H_2 \setminus T$ , by “gluing”  $c_1$  and  $c_2$  along their coloring at  $T$  we get a proper coloring of  $H$  (check!). This coloring uses at most  $\omega \leq \omega(H)$  colors, so indeed  $\chi(H) \leq \omega(H)$ .

□

**Definition 9.0.13** (Complement of a graph;  $\overline{G}$ ). Let  $G = (V, E)$  be a graph. Then its **complement**,  $\overline{G}$ , is defined as

$$\overline{G} = \left( V, \binom{V}{2} \setminus E \right).$$

I.e., if  $|V(G)| = n$ , then  $\overline{G} = K_n \setminus E$ .

**Theorem 9.0.14** (Theorem 26; Weak perfect graph theorem, 1972). *A graph is perfect iff its complement is perfect.*

*Remark 9.0.15.* Observe that it is enough to show that if a graph  $G$  is perfect, then  $\overline{G}$  is perfect, since then applying the same theorem to the graph  $\overline{G}$  we get that  $\overline{\overline{G}} = G$  is perfect.

**Theorem 9.0.16** (Strong perfect graph theorem; 2006). *A graph is perfect iff it does not contain any holes or anti-holes.*

**Definition 9.0.17** (Hole). Let  $G = (V, E)$  be a graph. Then  $G$  **has a hole** if there is a subset  $S \subseteq V$  such that the induced subgraph  $G[S]$  on the vertex-set  $S$  is an (odd;  $(\star)$ ) cycle of length at least 5.

*Remark 9.0.18.* Seems like conventions differ on whether or not it needs to be odd  $(\star)$ .

**Definition 9.0.19** (Antihole). Let  $G = (V, E)$  be a graph. Then  $G$  has an **antihole** if its complement  $\overline{G}$  has a hole.

### 9.0.1 Counting graph colorings

**Q:** Given a graph  $G$  and  $k \in \mathbb{N}$ , how many colorings of  $G$  with colors in  $\{1, \dots, k\}$  are there?

**Notation:** Let  $\chi_G(k)$  be the number of ways to color  $G$  with colors in  $\{1, \dots, k\}$ .

**Example 9.0.20.** • Paths  $P_n$ : Let  $P_n$  be the path  $v_1v_2 \dots v_n$  on  $n$  vertices. Recursively, for  $P_1$  there are  $k$  choices. For  $P_{n+1}$  there are the number of choices for  $P_n$  together with  $k - 1$  choices for the added vertex; i.e. we get the recurrence relation  $c_{n+1} = (k - 1)c_n$ . We claim that  $c_n = k(k - 1)^{n-1}$ . For  $n$  we have that  $c_1 = k(k - 1)^{1-1} = k$ . Assume it holds for  $c_{\ell-1}$ , we want to show it holds for  $c_\ell$ . We then have that

$$\begin{aligned} c_\ell &= c_{\ell-1}(k - 1) \\ &= k(k - 1)^{(\ell-1)-1}(k - 1) \\ &= k(k - 1)^{\ell-1}, \end{aligned}$$

where we used the recurrence relation between  $c_\ell$  and  $c_{\ell-1}$ . Thus,  $\chi_{P_n}(k) = k(k - 1)^{n-1}$ .

- For a complete graph  $G = K_n$  on  $n$  vertices, a  $k$ -coloring corresponds to an injective function  $f : V = \{1, \dots, n\} \rightarrow S = \{1, \dots, k\}$ . Injective functions  $f$  then correspond precisely to subsets  $A \subset S$  of cardinality  $|A| = n$ , multiplied by  $n!$ , since for each subset  $A$  we then have  $n!$  choices of which vertex goes to which color (this is the same as the # of bijections from  $V$  to  $A$ ; but this is the same as the number of permutations of  $n$  elements, i.e.  $|S_n| = n!$ ). It is then clear that if  $n > k$  there are no colorings, and if  $n \leq k$  then this is the same as

$$\begin{aligned} n! \cdot \binom{k}{n} &= \frac{k!}{(k-n)!} \\ &= k(k-1)\cdots(k-n+1) \\ &= \prod_{i=1}^n (k-i+1) \\ &= k(n), \end{aligned}$$

where  $(k_n)$  is called the **falling factorial**. That is,  $\chi_{K_n}(k) = k(n)$ .

- For cycles  $C_n$ , this depends on whether neighboring vertices have equal colors or not.

**Proposition 9.0.21** (Proposition 17).

$$\chi_{C_n}(k) = (k - 1)^n + (-1)^n(k - 1).$$

*Proof.* We perform induction on  $n$ . Since the smallest cycle is  $K_3$ , our induction begins on  $n = 3$ . Then we have that  $\chi_{C_n}(k) = k(k - 1)(k - 2)$  since this just reduces to the complete-graph case (see the previous example). But

$$\begin{aligned} k(k - 1)(k - 2) &= (k - 1)^3 - (k - 1) \\ &= (k - 1)^3 + (-1)^3(k - 1). \end{aligned}$$

Hence our base-case is proved.

For  $n > 3$ : Label the vertices of the cycle  $C_n$  successively as  $v_1, \dots, v_n$  and consider the graph  $C' := C_n \setminus e$  with  $e = v_1v_n$ . Then  $C'$  is the graph  $P_n$ . We partition its colorings into two cases: Those where  $v_1$  and  $v_n$  have the same color, and those colorings where they don't. One may show that the colorings of  $P_n$  with  $v_1$  and  $v_n$  having different colors are in bijection with  $C_n$ . When  $v_1$  and  $v_n$  have the same color in a coloring for  $P_n$ , this is the same as a coloring of  $C_{n-1}$ . Therefore, we have that (“Addition principle”)

$$\begin{aligned} \chi_{P_n}(k) &= \chi_{C_n}(k) + \chi_{C_{n-1}}(k) \\ \Leftrightarrow k(k - 1)^{n-1} &= \chi_{C_n}(k) + (k - 1)^{n-1} + (-1)^{n-1}(k - 1) \\ \Leftrightarrow \chi_{C_n}(k) &= k(k - 1)^{n-1} - (k - 1)^{n-1} + (-1)^n(k - 1) \\ &= (k - 1)^{n-1}(k - 1) + (-1)^n(k - 1) \\ &= (k - 1)^n + (-1)^n(k - 1). \end{aligned}$$

□

**Definition 9.0.22** (Edge contraction). An edge  $e = uv$  in  $G$  is removed and its two incident vertices  $u, v$  are merged into a new vertex  $w$ ; edges incident to  $u$  or  $v$  yield edges incident to  $w$ ; multiple edges are removed. The resulting graph is denoted  $G/e$ .

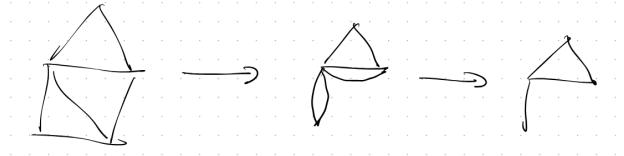


Figure 9.3: Illustration of edge-contraction along the diagonal followed by edge-deletion

**Definition 9.0.23** (Minor). A graph  $H$  is called a **minor** of a graph  $G$  if  $H$  can be obtained from  $G$  by successively deleting or contracting edges and isolated vertices.

**Theorem 9.0.24** (Theorem 28; Wagner). *A graph is planar iff it does not contain  $K_{3,3}$  or  $K_5$  as a minor (9.0.23).*

**Theorem 9.0.25** (Theorem 29). *For any graph  $G$  with edge  $e$ ,*

$$\chi_G(k) = \chi_{G/e}(k) - \chi_{G\setminus e}(k).$$

*Proof.* Proceed as in proposition 17. □

*Remark 9.0.26.* Observe that  $G\setminus e$  is the  $G$  with the edge  $e$  deleted (as compared to  $G/e$ , which is the edge-contraction described earlier).

**Theorem 9.0.27** (Theorem 30). *Let  $G$  be a graph on  $n$  vertices and  $c$  connected components  $G_1, \dots, G_c$ . Then  $\chi_G(k)$  is a polynomial in  $k$  called the **chromatic polynomial**, which has the following properties:*

- 1) The coefficients of  $k^0, \dots, k^{c-1}$  are zero.
- 2) The coefficients of  $k^c, k^{c+1}, \dots, k^n$  are non-zero and alternating in sign.
- 3) The coefficient of  $k^n$  is 1.
- 4) The coefficient of  $k^{n-1}$  is  $-|E(G)|$ .

*Proof.* We proceed by induction on  $|E(G)| = \ell$ . If  $\ell = 0$  then  $G$  has  $n$  isolated vertices and hence  $n$  components, so that  $\chi_G(k) = k^n$ . If  $\ell = 1$  then there is one edge, say  $e = uv$ . Hence for the other (isolated) vertices there are  $k^{n-2}$  choices, and for each such choice there are  $k(k-1)$  choices for our  $K_2$ , hence there are

$$\begin{aligned}\chi_G(k) &= k^{n-2} \cdot k(k-1) \\ &= k^{n-1}(k-1) \\ &= k^n - k^{n-1}\end{aligned}$$

choices. We see that the coefficient of  $k^n$  is 1, the coefficient of  $k^{n-1}$  is  $-|E(G)| = -1$ , that the coefficients of  $k^0, \dots, k^{n-2}$  are zero (note that there are  $n-2+1 = n-1$  components). Also the coefficients of  $k^{n-1}, k^n$  are non-zero and alternating in sign.

Assume by induction this holds for  $|E(G)| = 0, 1, \dots, \ell-1$ . Let  $e$  be an edge of  $G$  for  $G$  such that  $|E(G)| = \ell$ . In  $G \setminus e$  there are clearly  $\ell-1$  edges, so we may apply the induction hypothesis to  $G \setminus e$  to get that

$$\chi_{G \setminus e}(k) = a_n k^n - a_{n-1} k^{n-1} \dots$$

For  $G/e$  we have that  $|E(G/e)| \leq |E(G)| - 1$  since we *at least* remove the edge  $e$  (and possibly more if there are more than one edge between two vertices in the resulting graph obtained by collapsing two vertices into one). Hence the induction hypothesis also holds for  $\chi_{G/e}(k)$ , giving us

$$\chi_{G/e}(k) = b_{n-1} k^{n-1} - b_{n-2} k^{n-2} \dots$$

*Remark 9.0.28.* Note that the degree of  $\chi_{G/e}(k)$  is  $n-1$  since  $|V(G/e)| = |V(G)| - 1$ .

By induction, we have

- $a_n = b_{n-1} = 1$ .
- Note that deleting an edge can only in the “worst case” scenario increase the number of components by one (if  $e$  is a bridge), but never decrease it. If we contract along  $e = uv$  and  $e$  is not a bridge (so  $e$  is an edge between two vertices in the same component), then there is still a path between the new vertex  $w$  that  $u, v$  collapsed to and every other vertex in the component that  $u, v$  originally belonged to, while other components are unchanged, hence there is still the same number of components (as far as we can tell). If  $e$  is a bridge, then this does not change anything from the case when  $e$  is not a bridge - everything stays connected. This means that the number of components for  $G \setminus e$  and  $G/e$  is in  $\{c, c+1\}$ . Therefore, we have that  $a_0 = \dots = a_{c-1} = b_{c-1} = \dots = b_0 = 0$ .
- By the previous bullet point, we have that  $a_{c+1}, a_{c+2}, \dots, a_n$  and  $b_c, b_{c+1}, \dots, b_n$  are non-zero and that  $a_i, b_i = 0$  for  $0 \leq i \leq c$  is zero.
- By induction,  $a_{n-1} = -|E(G \setminus e)| = -(|E(G)| - 1) = 1 - |E(G)|$ .

By theorem 9.0.25 we have that

$$\begin{aligned}\chi_G(k) &= (a_n k^n - a_{n-1} k^{n-1} + \dots) - (b_{n-1} k^{n-1} - b_{n-2} k^{n-2} + \dots) \\ &= a_n k^n + \sum_{s=1}^n (-1)^s (a_{n-s} + b_{n-s}) k^{n-s}\end{aligned}$$

We observe that  $a_{n-s} + b_{n-s} > b_{n-s} > 0$  for  $1 \leq s \leq n-c$  (since then  $c \leq n-s$ ), and that  $a_{n-s} + b_{n-s} = 0$  for  $s > n-c \Leftrightarrow c > n-s$ .

By the first and last bullet points above we see that  $-(a_{n-1} + b_{n-1}) = -|E(G)|$  and that  $a_n = 1$ .  $\square$

# Chapter 10

## Lecture 10

Today: Networks and flows.

**Definition 10.0.1** (Directed graph). A **directed graph** (or digraph) is a pair  $D = (V, A)$  where  $V$  is a finite set, called *vertices*, and  $A \subseteq V \times V$  is a subset of ordered pair of vertices, called *directed edges* (or *arcs*). For every  $a = (u, v) \in V \times V$ , we call  $u$  the *initial* (or *start*) vertex and  $v$  the *terminal* (or *end*) vertex, and draw

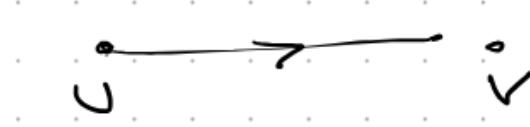


Figure 10.1: Directed edge  $(u, v) \in A$ .

**Definition 10.0.2** (Network). A **network**  $(D, s, t, c)$  is a directed graph  $D = (V, A)$ , with

- $s, t \in V$  special vertices, called **source** and **sink**.
- A **capacity function**  $c : A \rightarrow \mathbb{R}_{\geq 0}$ . The value  $c(u, v)$  is called the **capacity** of the arc  $(u, v)$ .

*Remark 10.0.3.* We will assume that (as before with undirected graphs)  $D$  is simple, i.e. there are no parallel directed edges or self-loops.

**Definition 10.0.4** ((Network) flow). A (network) **flow** is a function  $f : A \rightarrow \mathbb{R}_{\geq 0}$  such that

- $0 \leq f(a) \leq c(a)$  for all  $a \in A$ .
- The **Kirschhoff condition** (“What comes in goes out”):

$$\sum_{u:(u,v) \in A} f(u, v) = \sum_{u:(v,u) \in A} f(u, v)$$

is satisfied for all  $v \in V \setminus \{s, t\}$ , i.e. all vertices except the source and sink.

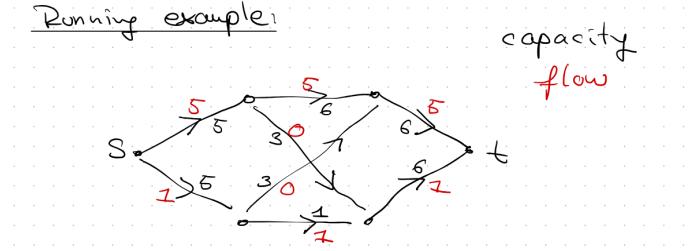


Figure 10.2: Capacity; flow.

For any function  $f : A \rightarrow \mathbb{R}$  and any  $v \in V$ , let

$$\partial f(v) = \sum_{u:(u,v) \in A} f(u,v) - \sum_{u:(v,u) \in A} f(v,u).$$

In particular, if  $f$  is a flow, then for any  $v \in V \setminus \{s, t\}$ , we have that  $\partial(f)(v) = 0$ .

**Definition 10.0.5** ((Total) value of a flow). The (total) **value** of a flow is

$$\begin{aligned} \text{value}(f) &= -\partial(s) \\ &= \sum_{u:(s,u) \in A} f(s,u) - \sum_{u:(u,s) \in A} f(u,s). \end{aligned}$$

*Remark 10.0.6.* Informally, the (total) value of a flow is “What flows out of the source into the network (minus what went in)”.

Maxflow problem: Given a network  $(D, s, t, c)$ , find a flow  $f : A \rightarrow \mathbb{R}_{\geq 0}$  of maximum value.

We have a trivial upper bound for any flow  $f : A \rightarrow \mathbb{R}_{\geq 0}$ :

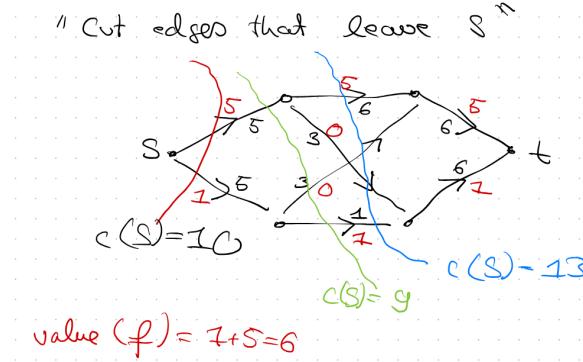
$$\begin{aligned} \text{value}(f) &= \sum_{u:(s,u) \in A} f(s,u) - \sum_{u:(u,s) \in A} f(u,s) \leq \sum_{u:(s,u) \in A} f(s,u) \\ &\leq \sum_{u:(s,u) \in A} c(u), \end{aligned}$$

where we used that  $f$  is non-negative and that  $f \leq c$ .

More generally:

**Definition 10.0.7** (Cut in a network; capacity of cut). A **cut in a network** is a set  $S \subset V$  with the source  $s \in S$  and sink  $t \notin S$ . The **capacity of the cut**  $S$  is

$$c(S) = \sum_{\substack{(u,v) \in A \\ u \in S, v \notin S}} c(u,v)$$

Figure 10.3: "Cut edges that leave  $S$ "

**Proposition 10.0.8** (Proposition 18). *Let  $(D, s, t, c)$  be a network. For any flow  $f : A \rightarrow \mathbb{R}_{\geq 0}$  and any cut  $S \subset V$ ,*

$$\text{value}(f) \leq c(S).$$

*Furthermore,  $\text{value}(f) = c(S)$  if and only if the following two conditions are satisfied:*

- 1)  $f(u, v) = c(u, v)$  for all  $(u, v) \in A$  such that  $u \in S$  and  $v \notin S$ .
- 2)  $f(v, u) = 0$  for all  $(v, u) \in A$  such that  $u \in S$  and  $v \notin S$ .

*Proof.* Let  $f$  be a flow and let  $S \subset V$  be a cut. Then  $f$  satisfies the Kirschhoff condition, i.e. for any vertex  $u \in V \setminus \{s, t\}$  we have that  $\partial(f)(u) = 0$ . Hence

$$\begin{aligned} \sum_{v \in S} -\partial(f)(v) &= -\partial(f)(s) + \underbrace{\sum_{v \in S \setminus \{s\}} -\partial(f)(v)}_{=0} \\ &= -\partial(f)(s) \\ &= \text{value}(f) \quad (\star), \end{aligned}$$

where we used that  $t \notin S$  since  $S$  is a cut.

By definition, we have that

$$\begin{aligned} \text{value}(f) &= \sum_{u \in S} -\partial(f)(u) \\ &= \sum_{u \in S} \left( \sum_{(u,v) \in A} f(u, v) - \sum_{(v,u) \in A} f(v, u) \right) \\ &= \sum_{\substack{(u,v) \in A \\ u \in S, v \notin S}} f(u, v) - \sum_{\substack{(v,u) \in A \\ u \in S, v \notin S}} f(v, u) \quad (\text{terms } f(u, v) \text{ with } u, v \in S \text{ cancel}) \\ &\stackrel{\text{if 2) holds}}{\leq} \sum_{\substack{(u,v) \in A \\ u \in S, v \notin S}} f(u, v) \\ &\stackrel{\text{if 1) holds}}{\leq} \sum_{\substack{(u,v) \in A \\ u \in S, v \notin S}} c(u, v) \\ &= c(S). \end{aligned}$$

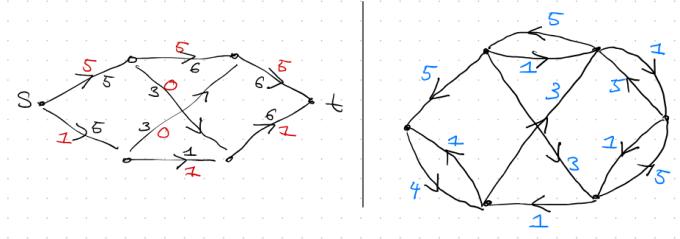


Figure 10.4:

□

**Definition 10.0.9** (Residual graph associated with a network  $(D, s, t, c)$  and flow  $f$ ). Given a network  $(D, s, t, c)$  and a flow  $f : A \rightarrow \mathbb{R}_{\geq 0}$ , we let  $D_f = (V, A_f)$  be the **residual graph** (directed) with arc-set

$$A_f := \{(u, v) : (u, v) \in A, f(u, v) < c(u, v)\} \sqcup \{(v, u) : (u, v) \in A, 0 < f(u, v)\}.$$

*Remark 10.0.10.* Observe that if  $D$  is originally a simple digraph then there can be at most two parallel edges  $u \Rightarrow v$  for  $u, v \in V$ , that is, when  $(u, v), (v, u) \in A$  such that  $f(u, v) < c(u, v)$  and  $f(v, u) > 0$ .

*Remark 10.0.11.* Note that the residual graph can be a multigraph.

**Theorem 10.0.12** (Theorem 31, Maxflow-mincut theorem; Ford-Fulkerson algorithm). *Let  $(D, s, t, c)$  be a network,  $f : A \rightarrow \mathbb{R}_{\geq 0}$  be a flow of maximum value, and  $S \subset V$  be a cut of minimum capacity. Then*

$$\text{value}(f) = c(S).$$

*Remark 10.0.13.* That  $S$  is a cut of minimum capacity we interpret as saying that  $c(S) = \min_{\substack{S \subset V \\ S \text{ cut}}} c(S)$ .

To prove this, we consider a new network. Consider the residual graph  $A_f$  of  $(D, s, t, c)$  with respect to the given flow  $f$ . We define a capacity  $c_f : A_f \rightarrow \mathbb{R}_{\geq 0}$  as

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v), & \text{if } (u, v) \in A \text{ with } f(u, v) < c(u, v) \\ f(v, u), & \text{if } (v, u) \in A \text{ with } f(v, u) > 0 \end{cases}.$$

We call a vertex  $v \in V$  **reachable** from  $s$  in  $D_f$  if there is a directed path  $s = v_0 v_1 \dots v_k = v$  with  $v_{i-1} v_i \in A_f$ .

**Proposition 10.0.14** (Proposition 19). *Let  $(D, s, t, c)$  be a network and  $f : A \rightarrow \mathbb{R}_{\geq 0}$  a flow. Let  $S \subset V$  be the vertices reachable from  $s$  and assume that  $t \notin S$  (i.e. there is no directed  $s - t$  path in  $D_f$  and  $S$  is a cut). Then  $\text{value}(f) = c(S)$ . In particular,  $f$  is a flow of maximum value.*

*Proof.* For all  $(u, v) \in A$  with  $u \in S$  and  $v \notin S$  we have that  $f(u, v) = c(u, v)$ , since otherwise  $f(u, v) < c(u, v)$  and then  $(u, v) \in A_f$ . But  $u$  is by definition reachable by  $s$ , and since there is then a directed path (edge) from  $u$  to  $v$  in  $A_f$  this means that by concatenation of this path from  $s$  to  $u$  with the edge  $(u, v) \in A_f$  we get a directed path from  $s$  to  $v$ , so that  $v \in S$ , contradiction!

On the other hand, let  $(u, v) \in A$  with  $u \notin S$  and  $v \in S$ , if  $f(u, v) > 0$  then  $(v, u)$  is in  $A_f$ . But then since  $v \in S$  we can reach  $v$  from  $s$ , and then by post-concatenation with the edge  $(v, u)$  we can reach  $u$  from  $s$  so  $u \in S$ , contradiction! Therefore  $f(u, v) = 0$ . Thus both 1) and 2) in 10.0.8 are satisfied,

so that  $\text{value}(f) = c(S)$ . By the same proposition, we see that  $\text{value}(f) > c(S)$  can never happen for any cut  $S \subset V$ . We guess this is the sense in which  $f$  is a flow of maximum value - it is the best possible!  $\square$

We want to prove the Maxflow-mincut theorem (10.0.12).

*Proof.* Let  $f$  be a flow of maximum value. By proposition 18 we know that  $\text{value}(f) \leq c(S)$  for all cuts  $S$ . If there is no  $s - t$ -directed path in  $D_f$  then there is a cut  $S \subset V$  not containing  $t$  such that  $\text{value}(f) = c(S)$ . Therefore  $c(S) \leq c(T)$  for all other cuts  $T$ , so indeed  $S$  is a minimal cut. To this end, we proceed to try and show that there is no directed  $s - t$ -path in  $D_f$ . Assume to the contrary that there is a directed  $s - t$ -path  $s = v_0 v_1 \dots v_k = t$  in  $D_f$ . Let

$$\varepsilon := \min\{c_f(v_{i-1}, v_i) : i = 1, \dots, k\} > 0$$

be the minimal capacity along this path (oftentimes called an **augmenting path**).

Define  $g : A \rightarrow \mathbb{R}$  by

$$g(u, v) = \begin{cases} \varepsilon, & \text{if } (u, v) = v_{i-1} v_i \\ -\varepsilon, & \text{if } (v, u) = v_{i-1} v_i \\ 0, & \text{otherwise} \end{cases}.$$

To clarify the definition above, we are saying that if the edge  $v_{i-1} v_i$  in the path  $s = v_0 \dots v_k = t \subset D_f$  comes from a pair  $(u, v) \in A$  such that  $f(u, v) < c(u, v)$  then set it to  $\varepsilon$ , and if  $v_{i-1} v_i$  comes from  $(v, u)$  such that  $f(v, u) > 0$  then set it to  $-\varepsilon$ , and zero otherwise. Each edge in the path comes from one of these alternatives by construction, *but not both*, because the edge must by construction be induced from one specific case. So for a specific edge  $v_{i-1} v_i$ , it can either be induced by  $(v_{i-1}, v_i) \in A$  such that  $f(v_{i-1}, v_i) < c(v_{i-1}, v_i)$  or it can be induced from  $(v_i, v_{i-1})$  such that  $f(v_i, v_{i-1}) > 0$ .

For  $v \in V \setminus \{s, t\}$  we have that  $\partial(g)(v)$  by definition of  $\partial$  and  $g$  is zero whenever  $v$  is not on the path  $s - t$  path: If  $v = v_i$  for some  $v_i$  on the  $s - t$  path, then

$$\begin{aligned} \partial(g)(v) &= \sum_{(u,v) \in A} g(u, v) - \sum_{(v,u) \in A} g(v, u) \\ &= g(v_{i-1}, v_i) + g(v_{i+1}, v_i) - (g(v_i, v_{i-1}) + g(v_i, v_{i+1})) \\ &= \underbrace{(g(v_{i-1}, v_i) - g(v_i, v_{i-1}))}_{\varepsilon} + \underbrace{(g(v_{i+1}, v_i) - g(v_i, v_{i+1}))}_{-\varepsilon} \\ &= 0. \end{aligned}$$

To explain the steps above: If  $v_{i-1} v_i$  comes from  $(v_{i-1}, v_i)$  then  $g(v_{i-1}, v_i) = \varepsilon$  and  $g(v_i, v_{i-1}) = 0$ . If  $v_{i-1} v_i$  comes from  $(v_i, v_{i-1})$  then  $g(v_i, v_{i-1}) = -\varepsilon$  and so it also becomes  $\varepsilon$  in the sum above. These cover the two possibilities for the edge  $v_{i-1} v_i$ . For the edge  $v_i v_{i+1}$  it either comes from  $(v_i, v_{i+1})$  and then  $g(v_i, v_{i+1}) = \varepsilon$  while  $g(v_{i+1}, v_i) = 0$  and so becomes  $-\varepsilon$  in the sum, or it comes from  $(v_{i+1}, v_i)$  and then  $g(v_{i+1}, v_i) = -\varepsilon$  while  $g(v_i, v_{i+1}) = 0$ .

For the source  $s$ , the edge  $s = v_0 \rightarrow v_1$  is either induced from  $(s, v_1)$  with  $f(s, v_1) < c(s, v_1)$  or it is induced from  $(v_1, s)$  such that  $f(v_1, s) > 0$ . So the only possible non-zero cases are  $g(s, v_1), g(v_1, s)$ , so that we get

$$\partial(g)(s) = g(v_1, s) - g(s, v_1)$$

In the former case above we have that  $g(v_1, s) = 0$  and  $g(s, v_1) = \varepsilon$  so that the sum becomes  $-\varepsilon$ , while in the latter case we have that  $g(v_1, s) = -\varepsilon$  while  $g(s, v_1) = 0$  so the sum becomes  $-\varepsilon$ . That is, we

have  $\partial(g)(s) = -\varepsilon$ . Hence  $\text{value}(g) = -\partial(g)(s) = \varepsilon$ . However, this is not necessarily a flow since we do not know that it is non-negative.

**Claim:**  $f' := f + g$  is a flow.

This is because:

- $f'(a) = f(a)$  for all  $a$  not on the path, so since  $f$  is a flow it holds that  $0 \leq f(a) \leq c(a)$ .
- For  $a \in A$  such that  $f(a) < c(a)$  and  $a$  on the path we have  $f'(a) = f(a) + \varepsilon$ . We have that  $\varepsilon \leq c_f(a) = c(a) - f(a)$  so that  $0 \leq \varepsilon + f(a) \leq c(a)$ .
- $f'(a) = f(a) - \varepsilon$  for all  $a \in A$  such that  $a = (u, v) \in A_f$  is on the path, and such that  $(v, u) \in A$  and  $f(v, u) > 0$ , we have that  $\varepsilon \leq c_f(a) = f(v, u)$  from which it follows that  $c(v, u) \geq f(v, u) - \varepsilon \geq 0$ .

Hence the first condition in flow (10.0.4) is satisfied.

We showed earlier that  $\partial(g)(v) = 0$  for all  $v \in V \setminus \{s, t\}$ . Furthermore,  $f$  is a flow, so it satisfies Kirschhoff's condition. Together with  $\partial(f')(v) = \partial(f)(v) + \partial(g)(v) = \partial(f)(v) = 0$  for all  $v \in V \setminus \{s, t\}$  this shows that  $f'$  satisfies Kirschhoff's condition. We conclude that  $f'$  is a (Network) flow with respect to  $(D, s, t, c)$ .

But, we have that

$$\begin{aligned} \text{value}(f') &= \text{value}(f + g) \\ &= \text{value}(f) + \text{value}(g) \\ &= \text{value}(f) + \varepsilon \\ &> \text{value}(f), \end{aligned}$$

since  $\varepsilon > 0$  by definition. Thus,  $f$  can not be a flow of maximum value, contradicting our hypothesis on  $f$ ! Therefore, there can be no such directed  $s - t$  path in  $D_f$ .  $\square$

The proof gives an algorithm to construct a *sequence* of flows with increasing value.

Algorithm (with notation in the above proof):

- (1) Initialize with the zero flow  $f := 0 : A \rightarrow \mathbb{R}_{\geq 0}$ .
- (2) While  $t$  is reachable from  $s$  in  $D_f$ , set  $f := f + g$ .
- (3) Return  $f$ .

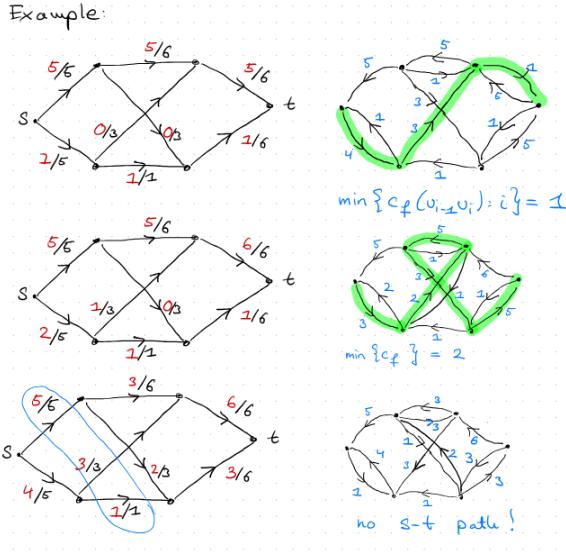


Figure 10.5: Enter Caption

**Corollary 10.0.15** (Corollary 6). *Let  $(D, s, t, c)$  be a network with  $c(a) \in \mathbb{Z}_{\geq 0}$  for all  $a \in A$ . Then there is an integer-valued network flow  $f : A \rightarrow \mathbb{Z}_{\geq 0}$  of maximum value.*

*Proof.* Apply algorithm 4. Since  $c$  has only integer values, and we initialize at  $f = 0$  and update with  $f + g$  in each step (and thinking about how  $g$  is defined in terms of  $\varepsilon$  which must then be integer-valued) it follows that the constructed flow  $f$  in each step is integer-valued. Now recall from the proof earlier that  $\text{value}(f') = \text{value}(f) + \varepsilon$ . Since  $\varepsilon > 0$  is necessarily an integer, it follows that the value of the flow increase by at least one in each step of the algorithm. By proposition 18 we have that  $\text{value}(f) \leq c(S)$  for any cut  $S$ . Therefore, the algorithm must terminate in a finite number of steps (since  $c(S)$  is finite for any cut  $S$ ).  $\square$

*Remark 10.0.16.* • Claim I: Algorithm 4 also terminates if all capacities are rational, i.e.  $c(a) \in \mathbb{Q}_{\geq 0}$  for all  $a \in A$ .

- Claim II: If capacities are irrational then the algorithm might not terminate.

# Chapter 11

## Lecture 11

Today: “Ramsey theory” - Complete order is impossible.

“Among six people, there are three people that know each other, or three people that do not.”

**Proposition 11.0.1** (Proposition 20). *Any graph with at least 6 vertices contains either  $K_3$  or  $\overline{K}_3$  as an induced subgraph.*

*Remark 11.0.2.* Observe that  $\overline{K}_3$  is just three isolated vertices.

*Proof.* It is sufficient to show it for a graph with 6 vertices. Let  $v$  be a vertex in the graph. There are two cases.

Case 1:  $v$  has at least three neighbors. If any of these three neighbors have an edge between them (i.e. are “adjacent”), then we are done. If we label the neighbors as  $w, z$  then  $G[v, w, z]$  will be exactly a copy of  $K_3$ .

Case 2: If  $v$  does not have at least three neighbors, then there are at least three non-neighbors of  $v$ . If there are at least two of them, say  $w, z$  that do not have an edge between them, then we see that  $G[v, w, z] = \overline{K}_3$ . If there are no two of them that does not have an edge between them, then there are three neighbors  $u, v, w$  such that all have an edge between them. Then  $G[u, v, w] = K_3$  and we are done.  $\square$

More general: We want to find a complete graph (**clique**) or the complement of a complete graph (**independent set**) of a certain size.

Equivalently: We color the edges of  $K_n$  with two colors (typically red/blue) and ask whether there exists monochromatic clique of a certain size, i.e. a clique with all edges red or blue. What is meant here, we think, is that we take a finite simple graph  $G$  on  $n$  vertices, then we form  $K_n$ . If an edge belongs to  $G$ , we color it say blue, and if it does not belong to  $G$ , then we color it blue. Then a clique in  $G$  would correspond to some  $K_k$  with all edges present in the color blue. An independent set in  $G$  would instead correspond to some  $K_\ell$  in the color red.

**Definition 11.0.3** (Ramsey number). For positive integers  $k$  and  $\ell$ , the **Ramsey number**  $R(k, \ell)$  is the *smallest* number  $n$  such that *every* red/blue edge coloring of  $K_n$  contains a red  $K_k$  or a blue  $K_\ell$ .

**Theorem 11.0.4** (Theorem 32). *For any  $k, \ell \geq 1$ , we have that*

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}.$$

In particular,  $R(k, \ell)$  is finite.

*Proof.* We will show the following **Claim**:

$$R(k, \ell) \leq R(k-1, \ell) + R(k, \ell-1).$$

For that, let

$$n := R(k-1, \ell) + R(k, \ell-1) \quad (\star)$$

and consider an arbitrary red/blue coloring of  $K_n$ , and a vertex  $v$  in  $K_n$ . Let  $A$  be the set of vertices linked (adjacent) with  $v$  with a red edge and let  $B$  be the set of vertices linked with  $v$  with a blue edge.

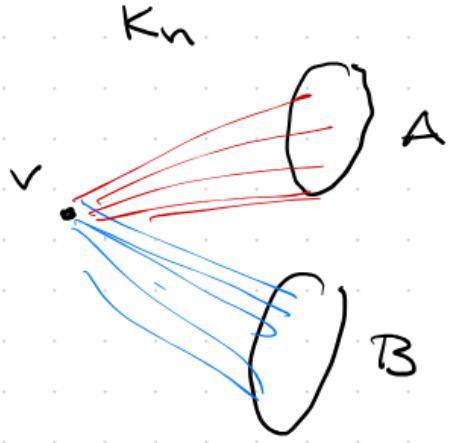


Figure 11.1: Illustration

We must have  $|A| \geq R(k-1, \ell)$  or  $|B| \geq R(k, \ell-1)$  since otherwise

$$\begin{aligned} n - 1 &= |A| + |B| \\ &\leq (R(k-1, \ell) - 1) + (R(k, \ell-1) - 1) \\ &= \underbrace{(R(k-1, \ell) + R(k, \ell-1))}_{n} - 2 \\ &= n - 2. \end{aligned}$$

If  $|A| \geq R(k-1, \ell)$ , then by definition,  $K_n[A]$  has either a red clique of size  $k-1$  which together with  $v$  (who we note is adjacent to all vertices in this red clique with a red edge) forms a red  $K_k$ , or a blue clique/ $K_\ell$  of size  $\ell$ .

If instead  $|B| \geq R(k, \ell-1)$  then  $K_n[B]$  has a red  $K_k$  or together with  $v$  (which is joined by a blue edge to  $B$ ) has a blue  $K_\ell$ .

All together, this shows that  $K_n$  with  $n$  as in  $(\star)$ , has either a red  $K_k$  or a blue  $K_\ell$ , so the inequality in the claim must hold.

We now proceed by induction on  $\ell+k = n$ : If  $k=1$  or  $\ell=1$ , then in both cases the binomial coefficient in the statement becomes 1, and this is always trivially satisfied for any non-empty graph.

$k, \ell > 1$ : We now use the claim:

$$\begin{aligned}
R(k, \ell) &\leq R(k-1, \ell) + R(k, \ell-1) \\
&\leq \binom{(k-1)+\ell-2}{(k-1)-1} + \binom{k+(\ell-1)-2}{k-1} \\
&= \binom{k+\ell-3}{k-2} + \binom{k+\ell-3}{k-1} \\
&= \binom{k+\ell-2}{k-1},
\end{aligned}$$

where we used induction in the second inequality, and Pascal's identity in the last equality.  $\square$

**Corollary 11.0.5.** For  $k \geq 2$ , we have that  $R(k) := R(k, k) \leq 2^{2k-3}$ .

*Remark 11.0.6.*  $R(k)$  are called the **Diagonal** Ramsey numbers .

*Proof.* By theorem (11.0.4), we have that

$$\begin{aligned}
R(k, k) &\leq \binom{2k-2}{k-1} \\
&= \binom{2k-3}{k-1} + \binom{2k-3}{k-2} \\
&\leq \sum_{i=0}^{2k-3} \binom{2k-3}{i} 1^{2k-3-i} 1^i \\
&= (1+1)^{2k-3} \\
&= 2^{2k-3}.
\end{aligned}$$

$\square$

Here are some known values of diagonal Ramsey numbers:

- $R(1) = 1$ .
- $R(2) = 2$ .
- $R(3) = 6$ .
- $R(4) = 18$  (Exercise!).
- For  $k \geq 5$  we only have bounds.

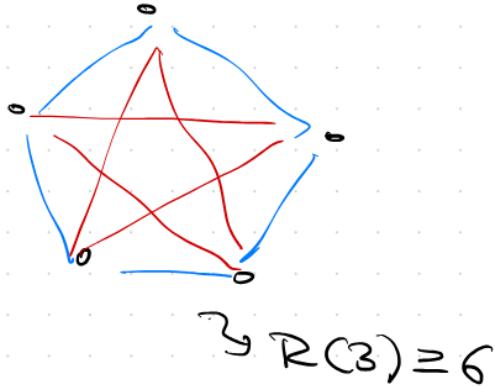


Figure 11.2: Illustration of a red/blue coloring of  $K_5$  with no red or blue  $K_3 \rightsquigarrow R(3) \geq 6$ .

**Definition 11.0.7** (Multicolor Ramsey number  $R_k(s_1, \dots, s_k)$ ). The **multicolor Ramsey number**  $R_k(s_1, \dots, s_k)$  is the least  $n$  such that any coloring of edges of  $K_n$  with  $k$  colors contains a clique of size  $s_i$  in color  $i$  for some  $1 \leq i \leq k$ .

*Remark 11.0.8.*  $R(s_1, \dots, s_k)$  is finite.

**Theorem 11.0.9** (Theorem 33). *For any  $s_1, \dots, s_k \geq 1$  there is  $R_k(s_1, \dots, s_k) < \infty$  such that for any  $k$ -coloring of the edges of  $K_n$ , there is a clique of size  $s_i$  in some color  $i$ , for  $n \geq R_k(s_1, \dots, s_k)$ .*

*Proof.* We prove the following **Claim**:

$$R_k(s_1, \dots, s_k) \leq R_{k-1}(s_1, s_2, \dots, s_{k-2}, R(s_{k-1}, s_k)).$$

Then the theorem will follow by induction, with base case  $R(k, \ell) < \infty$ .

To prove the claim, we consider a graph  $K_n$  on  $n = R_{k-1}(s_1, \dots, s_{k-2}, R(s_{k-1}, s_k))$  vertices, and color the edges with  $k$  colors  $c_1, \dots, c_k$ . Then we go “color blind” and identify the last two colors  $c_{k-1}$  and  $c_k$ . Then the edges of  $K_n$  are colored with  $k-1$  colors, and by definition, there is a clique of size  $s_i$  in color  $c_i$  for  $1 \leq i \leq k-2$ , or a clique of size  $R(s_{k-1}, s_k)$  in color  $c_{k-1} = c_k$ . In the first case, we are done (why?) and in the latter case, we inspect the monochromatic clique of size  $R(s_{k-1}, s_k)$ . We go back to differentiating the colors in this clique, and then by definition there is clique of size  $s_{k-1}$  in color  $c_{k-1}$  or a clique of size  $s_k$  in color  $c_k$ , within that clique.

The induction part would proceed as follows: The base case was taken care of earlier. Assume by induction the statement holds for  $R_{k-1}(t_1, \dots, t_{k-1})$  that it is finite. Then in particular, it holds with  $t_{k-1} = R(s_{k-1}, s_k)$ . By the claim it then follows that it holds for  $R_k(s_1, \dots, s_k)$ .  $\square$

#### Application:

**Theorem 11.0.10** (Theorem 34; Schur 1916). *For any  $k \geq 2$ , there exists an  $n$  such that for any  $k$ -coloring of  $[n] = \{1, \dots, n\}$  there are integers  $1 \leq x, y, z \leq n$  of the same color such that  $x + y = z$ .*

*Remark 11.0.11.* Observe that a  $k$ -coloring of  $[n]$  is “just” a function  $c : [n] \rightarrow [k]$ .

*Proof.* We choose  $n = R_k(3, 3, \dots, 3)$ , i.e. so that any  $k$ -coloring of  $K_n$  contains some triangle  $K_3$  all in the same color. Let  $c : [n] \rightarrow [k]$  be a  $k$ -coloring of  $[n]$ . We define a  $k$ -coloring  $\chi$  of the edges of  $K_n$

by

$$\chi : \binom{[n]}{2} \rightarrow [k], \quad (i, j) \mapsto c(|i - j|).$$

By Ramsey's theorem, there exists a triangle  $K_3$  in  $K_n$ , with all edges the same color, say with vertices  $\{i, j, k\}$  with  $i < j < k$ .

Consider

$$\begin{aligned} x &= j - i \\ y &= k - j \\ z &= k - i \\ &= (j - i) + (k - j) \\ &= x + y. \end{aligned}$$

Observe that  $x, y, z \in [n]$ . By definition of the previous paragraph, the edges in the  $K_3$  with vertices  $\{i, j, k\}$  are exactly  $\{i, j\}, \{j, k\}, \{i, k\}$  with colors under  $\chi$  as  $c(x) = c(y) = c(z)$ .  $\square$

Notation:

$$\binom{[n]}{k}$$

denotes the set of all subsets of  $[n]$  of size  $k$ , i.e.

$$\binom{[n]}{k} = \{A \subset [n] : |A| = k\}.$$

**Theorem 11.0.12** (Theorem 35; Erdős). *For all  $k \geq 2$ ,*

$$R(k) \geq 2^{\frac{k}{2}}.$$

*Proof.* We assume the results stated earlier, i.e. that  $R(2) = 2 \geq 2^1$  and  $R(3) = 6 \geq 2^{\frac{4}{2}} = 4 \geq 2^{\frac{3}{2}}$ . Therefore, assume that  $k \geq 4$ . Let  $n < 2^{\frac{k}{2}}$  and consider a random red/blue coloring of the edges of  $K_n$  such that each edge is red with probability  $\frac{1}{2}$  and each edge is blue with probability  $\frac{1}{2}$ , and the colors of the edges are independent of each other. Let  $S \in \binom{[n]}{k}$  be a subset of the vertices of size  $k$ , and let  $A_S$  be the event that all edges between vertices in  $S$  are colored in red. Then, with  $\mu$  the probability measure, we have that  $\mu(A_S) = \left(\frac{1}{2}\right)^{\binom{k}{2}}$ , the reason being that there are  $\binom{k}{2}$  edges in the induced (complete) subgraph  $K_n[A_S]$ , each with probability  $\frac{1}{2}$  of having color red, and the probabilities are independent.

There are  $\binom{n}{k}$  subsets of  $[n]$  of size  $k$ . Therefore, the probability that any of these subsets is all red is

bounded by (using that  $\mu(\cdot)$  is a measure so countably sub-additive)

$$\begin{aligned}
\mu \left( \bigcup_{S \in \binom{[n]}{k}} A_S \right) &\leq \sum_{S \in \binom{[n]}{k}} \mu(A_S) \\
&= \binom{n}{k} \left( \frac{1}{2} \right)^{\binom{k}{2}} \\
&\leq \frac{n^k}{k!} \left( \frac{1}{2} \right)^{\binom{k}{2}} \\
&\leq \frac{2^{\frac{k^2}{2}}}{k!} \left( \frac{1}{2} \right)^{\binom{k}{2}} \\
&= \frac{2^{\frac{k^2}{2}}}{k!} \left( \frac{1}{2} \right)^{\frac{k(k-1)}{2}} \\
&= \frac{2^{\frac{k^2}{2}}}{k!} \left( \frac{1}{2} \right)^{\frac{k^2-k}{2}} \\
&= \frac{2^{\frac{k^2}{2}}}{k!} 2^{-\frac{k^2}{2} + \frac{k}{2}} \\
&= \frac{2^{\frac{k}{2}}}{k!}.
\end{aligned}$$

For  $k \geq 4$ , we **claim** that  $k! \geq 2^k$ : For the base case  $k = 4$  we have that  $4! = 24 \geq 2^4 = 16$ . Assume it holds for  $m$ . Then

$$\begin{aligned}
(m+1)! &= (m+1)m! \\
&\geq (m+1)2^m \\
&\geq 2^{m+1} \quad (\text{since } m \geq 2).
\end{aligned}$$

Therefore, it follows that  $\frac{1}{k!} \leq \frac{1}{2^k}$  for  $k \geq 4$  and so

$$\begin{aligned}
\frac{2^{\frac{k}{2}}}{k!} &\leq \frac{2^{\frac{k}{2}}}{2^k} \\
&= \frac{1}{2^{\frac{k}{2}}} \\
&\leq \frac{1}{2^{\frac{4}{2}}} \\
&= \frac{1}{4} \\
&< \frac{1}{2}.
\end{aligned}$$

That is, the probability that there is a red clique of size  $k$  is less than  $\frac{1}{2}$ . The same calculation would have worked with red exchanged with blue (since we choose the probabilities to be the same for an edge being of color red/blue). In particular, the probability that there is a clique of one color is smaller than  $\frac{1}{2} + \frac{1}{2} = 1$ . Therefore, there must exist a coloring of  $K_n$  with  $k$ -colors with no red or blue cliques of size  $k$ . Hence  $R(k) \geq 2^{\frac{k}{2}}$ .  $\square$

*Remark 11.0.13.* One choice in the proof that might seem arbitrary, is when we choose the probability of an edge to be either of color blue or red (in the proof:  $\frac{1}{2}$ ). Could not another choice of assignment

of probabilities have given us another result? No, and in fact, this is the theme of the probabilistic method . The “philosophy” behind the method is the following basic **lemma**:

*Lemma 11.0.14. Let  $(X, \mu)$  be a probability space (i.e. so that  $\mu(X) = 1$  and  $\mu(\emptyset) = 0$ , and  $\mu$  is a measure). If  $E \subset X$  for some event  $E$  is such that  $\mu(E) > 0$ , then  $E \neq \emptyset$ .*

*Proof.* Follows directly from the fact that if  $E = \emptyset$  then  $\mu(E) = 0$ . □

How does this work in our setting? We have a set  $X$  of combinatorial objects ( $k$ -colorings of  $K_n$ ) and we showed that the set of  $k$ -colorings  $G$  where  $K_n$  has a  $K_k$  all in blue or all in red had the property that  $\mu(G) < 1$ , then  $\mu(X \setminus G) > 0$  since  $\mu(X) = \mu(G \sqcup (X \setminus G)) = \mu(G) + \mu(X \setminus G) = 1$ . Therefore, by the lemma above, we have that, with  $E = X \setminus G$ ,  $\mu(E) > 0$  so  $E \neq \emptyset$ , i.e. there exists at least one  $k$ -coloring of  $K_n$  who does not have a clique of size  $k$  all in one color red or blue.

# Chapter 12

## Lecture 12

Today:

- Erdős-Rényi graphs.
- Theorem 24: Existence of graphs with high chromatic number and high girth.
- Properties of almost all graphs.

**Definition 12.0.1** (Erdős-Rényi graphs; random graph  $G(n, p)$ ). Given an integer  $n \geq 1$  and a real number  $0 \leq p \leq 1$ , the **random graph**  $G(n, p)$  is the graph with vertex set  $[n] = \{1, \dots, n\}$  in which each of the  $\binom{n}{2}$  possible edges is present with probability  $p$ , *independently* of the other edges.

*Remark 12.0.2.* In particular, for any graph  $H$  with vertex set  $[n]$ , we have that

$$\mu(G(n, p) = H) = p^{|E(H)|} \cdot (1-p)^{\binom{n}{2} - |E(H)|}.$$

Here we used that each edge in  $E(H)$  is present with probability  $p$ , and the  $\binom{n}{2} - |E(H)|$  edges not in  $E(H)$  but in  $K_n$  are then not present with probability  $1-p$ .

**Example 12.0.3.** There are three labeled (!) graphs on  $[3] = \{1, 2, 3\}$  that are isomorphic to a path  $P_2$ :

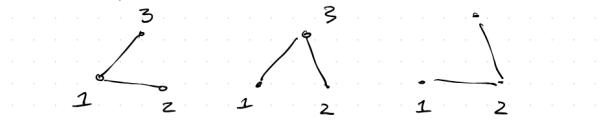


Figure 12.1: Three graphs on  $[3]$  isomorphic to a path of length 2,  $P_2$

This implies that  $\mu(G(3, p))$  is isomorphic to  $P_2 = 3p^2(1-p)$  (where we used that a measure is countably additive on disjoint events).

**Definition 12.0.4** (Erdős-Rényi-model). The probability space of graphs on  $[n]$  with probabilities given as above is called the **Erdős-Rényi-model**, denoted by  $\mathcal{G}(n, p)$ .

*Remark 12.0.5.* There are different notations:

$$G \in \mathcal{G}(n, p), \quad G = G(n, p), \quad G \sim G(n, p), \quad G_{n,p}.$$

“Strategies” - first moment methods:

- Let  $(\Omega, \mathcal{A}, \mu)$  be a probability space with  $\mu : \mathcal{A} \rightarrow [0, 1]$ .
- $A$  event, with  $\mu(A) > 0 \Rightarrow A \neq \emptyset$ .
- $\mu(A_1 \cup \dots \cup A_n) \leq \sum_{i=1}^n \mu(A_i)$  (finitely subadditive).
- For  $X$  a **random variable** with finite outcome set  $x = \{x_1, \dots, x_k\} \subset \mathbb{R}$ ,

$$\mathbb{E}(X) = \sum_{i=1}^k x_i \mu(X = x_i),$$

called the **expectation** or **first moment**. Observe that a random variable  $X$  is a  $\mu$ -measurable function  $X : \Omega \rightarrow \mathbb{R}$  where  $\Omega$  is the sample space, and  $\mathcal{A} \subset \mathcal{P}(\Omega)$  is the set of  $\mu$ -measurable events. So “finite outcome set”  $x$  means that the *range*  $x$  of  $X$ , as above, is finite.

- $\mu(X \geq \mathbb{E}(X)) > 0$  and  $\mu(X \leq \mathbb{E}(X)) > 0$ . Observe that  $X \geq \mathbb{E}(X)$  is shorthand for

$$\{A \in \Omega : X(A) \geq \mathbb{E}(X)\}$$

and likewise for  $X \leq \mathbb{E}(X)$ . Since  $X$  is  $\mu$ -measurable, sets of the above form are in  $\mathcal{A}$ . Let  $m = \min\{x_i : \mu(X = x_i) > 0\}$  and let  $M = \max\{x_i : \mu(X = x_i) > 0\}$ . Since  $\sum_{i=1}^k \mu(X = x_i) = 1$  it follows that  $m \leq \mathbb{E}(X) \leq M$ . If we set  $\mathbb{E}(X) = c$ , then we may observe that

$$\{A : X(A) \geq c\} = \bigcup_{x_i \geq c} \{X = x_i\}.$$

It follows that

$$\begin{aligned} \mu(X \geq c) &= \sum_{x_i \geq c} \mu(X = x_i) \\ &\geq \mu(X = M) \\ &> 0, \end{aligned}$$

since  $\mu(X = M) > 0$  by definition, and since  $M \geq c$ . Similarly, we may show that  $\mu(X \leq c) \geq \mu(X = m) > 0$ .

- $X_1, \dots, X_k$  random variables and  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ , then

$$\mathbb{E}\left(\sum_{i=1}^k \lambda_i X_i\right) = \sum_{i=1}^k \lambda_i \mathbb{E}(X_i) \quad (\text{linearity}).$$

- **Markov's inequality:** For a random variable  $X : \Omega \rightarrow \mathbb{R}_{\geq 0}$  taking only non-negative values, and any  $t > 0$ , we have that

$$\mu(X \geq t) \leq \frac{\mathbb{E}(X)}{t}$$

**Theorem 12.0.6** (Theorem 24; Erdős). *For  $k \geq 2$ , there exists a graph  $G$  with chromatic number  $\chi(G) > k$  and girth  $g(G) > k$ .*

General idea:

- Consider  $G \in \mathcal{G}(n, p)$  with  $p$  carefully chosen.
- Show that:

1.  $\mu(g(G) \leq k) < \frac{1}{2}$ .
2.  $\mu(\chi(G) \leq k) < \frac{1}{2}$ .

- This implies that  $\mu(g(G) > k \wedge \chi(G) > k) > 0$ . So by the “probabilistic method” (recall from last lecture), such an object must exist.
- Not exactly ... but something similar.

*Remark 12.0.7.* Observe that  $\mu(g(G) \leq k \vee \chi(G) \leq k) \leq \mu(g(G) \leq k) + \mu(\chi(G) \leq k) < 1$ . Therefore,

$$\begin{aligned} \mu(g(G) > k \wedge \chi(G) > k) &= 1 - \mu(g(G) \leq k \vee \chi(G) \leq k) \\ &\geq 1 - (\mu(g(G) \leq k) + \mu(\chi(G) \leq k)) \\ &> 0. \end{aligned}$$

*Proof.* Let  $G \in \mathcal{G}(n, p)$ . First consider the chromatic number: Let  $\alpha := \alpha(G)$  be the number of elements in the largest *independent* set (= independence number).<sup>1</sup> We have  $\chi(G) \cdot \alpha(G) \geq |V(G)| = n$ , since in any color class, all vertices are independent: Take any  $\chi(G)$ -coloring  $C$  of  $G$ . Then  $C$  partitions the vertex set  $V(G)$  into  $\chi(G)$  independent sets  $C_1 \sqcup C_2 \dots \sqcup C_{\chi(G)}$ . We then have

$$\begin{aligned} n &= |V(G)| \\ &= \sum_{i=1}^{\chi(G)} |C_i| \\ &\leq \sum_{i=1}^{\chi(G)} \alpha(G) \\ &= \chi(G) \cdot \alpha(G). \end{aligned}$$

Thus, if  $\alpha$  is small in comparison to  $n$ , then  $\chi(G)$  must be large, which is what we desire.

For any  $r$ -subset  $S \in \binom{V(G)}{r}$  of vertices from  $G$ , let  $A_S$  be the event that the vertices in  $S$  are independent. That is,  $A_S = \{G \in \mathcal{G}(n, p) : S \text{ is independent in } G\}$ . Then

$$\mu(A_S) = (1-p)^{\binom{r}{2}}$$

and (with  $\{\alpha(G) \geq r\} = \{G \in \mathcal{G}(n, p) : \alpha(G) \geq r\}$ ) since  $\{\alpha(G) \geq r\} = \bigcup_{S \in \binom{V}{r}} A_S$ ,

$$\begin{aligned} \mu(\{\alpha(G) \geq r\}) &= \mu\left(\bigcup_{S \in \binom{V}{r}} A_S\right) \\ &\leq \sum_{S \in \binom{V}{r}} \mu(A_S) \\ &= \binom{n}{r} (1-p)^{\binom{r}{2}} \\ &\leq n^r (1-p)^{\frac{r^2-r}{2}} \\ &= \left(n(1-p)^{\frac{r-1}{2}}\right)^r \\ &\leq \left(n(e^{-p})^{\frac{r-1}{2}}\right)^r \quad (\star), \end{aligned}$$

---

<sup>1</sup>Recall that a set  $I \subset V(G)$  of vertices of a graph is **independent** if there are no edges between any of the vertices in the set, in  $G$ . Equivalently, the *induced subgraph* on  $I$ ,  $G[I]$ , consists of isolated vertices

where we in the last step used that  $(1 - p) \leq e^{-p}$  (look at the function  $f(x) = e^x - x - 1$  and show that  $e^x \geq 1 + x$  for all  $x$ ; then set  $x = -p$ ).

For fixed  $k > 0$  we choose

$$p := n^{-\frac{k}{k+1}}.$$

**Claim:** For large enough  $n$ , we have that  $\mu(\{\alpha(G) \geq \frac{2n}{k}\}) < \frac{1}{2}$ .

To see this, we observe that  $n^{\frac{1}{k+1}} \geq 6k \log n$  for large enough  $n$ . Therefore,

$$\begin{aligned} \frac{n^{\frac{1}{k+1}}}{n} &= n^{-\frac{k}{k+1}} \\ &\geq \frac{6k \log n}{n}. \end{aligned}$$

Set  $r := \lceil \frac{n}{2k} \rceil$ . This gives  $p \cdot r \geq 3 \log n$ . Therefore, we find that

$$\begin{aligned} n \cdot e^{-\frac{p(r-1)}{2}} &= n \cdot e^{-\frac{pr}{2}} \cdot e^{\frac{p}{2}} \\ &\leq n \cdot e^{-\frac{3 \log n}{2}} e^{\frac{1}{2}} \\ &= n^{-\frac{1}{2}} e^{\frac{1}{2}} \\ &= \left(\frac{e}{n}\right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Therefore, by our previous computation  $(\star)$ , we find that

$$\begin{aligned} \mu\left(\alpha(G) \geq \left\lceil \frac{n}{2k} \right\rceil\right) &= \mu\left(\alpha(G) \geq \frac{n}{2k}\right) \\ &\leq \left(\left(\frac{e}{n}\right)^{\frac{1}{2}}\right)^{\lceil \frac{n}{2k} \rceil} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Therefore, there is an  $n_1 \in \mathbb{N}_{\geq 1}$  such that if  $n \geq n_1$  then  $\mu(\alpha(G) \geq \frac{n}{2k})$ . Why did we want this? Well, recall that  $\chi(G) \cdot \alpha(G) \geq n$  (where  $|V| = n$ ). If  $\chi(G) \leq k$  then  $\alpha(G) \geq \frac{n}{k} \geq \frac{n}{2k}$ . It follows that  $\mu(\alpha(G) \geq \frac{n}{2k}) \geq \mu(\chi(G) \leq k) < \frac{1}{2}$  for  $n \geq n_1$ , which is the claim we wanted (roughly; recall the proof strategy).

Now, let's have a look at the girth  $g(G)$ : For a given  $k$ , we first show that there are not too many cycles of length  $\leq k$  for  $n$  large enough. For any  $3 \leq i \leq k$  and any subset  $S \subseteq V$  with  $i$  elements, the number of cycles with vertex set  $S$

□

# Chapter 13

## Lecture 13

Today: Matroids - abstract study of dependence/independence.

**Definition 13.0.1** (Matroid). A **matroid**  $M$  is a pair  $M = (E, \mathcal{I})$  consisting of a finite set  $E$  and a collection  $\mathcal{I}$  of subsets of  $E$  having the following three properties:

- (i)  $\emptyset \in \mathcal{I}$ .
- (ii) If  $I \in \mathcal{I}$  and  $I' \subseteq I$  then  $I' \in \mathcal{I}$ .
- (iii) If  $I, I' \in \mathcal{I}$  and  $|I| < |I'|$ , then there is an element  $e \in I' \setminus I$  such that  $I \cup \{e\} \in \mathcal{I}$ .

The elements  $I \in \mathcal{I}$  are called **independent sets** of  $M$ , and  $E$  is called the **ground set** of  $M$ . A subset  $A \subset E$  that is not in  $\mathcal{I}$  is called **dependent**.

*Remark 13.0.2.* Property (ii) is called the hereditary property, and (iii) is called the independence augmentation property.

- Matroids  $\leadsto$  Vector configuration/matrices.

**Proposition 13.0.3** (Proposition 22). *Let  $E$  be the set of column labels of a  $m \times n$  matrix  $A$  over a field  $F$ , and let  $\mathcal{I}$  be the set of subsets  $X$  of  $E$  for which the columns labeled by  $X$  is a linearly independent set in the  $F$ -vector space  $F^m$ . Then  $(E, \mathcal{I})$  is a matroid.*

*Remark 13.0.4.* As  $A$  is an  $m \times n$ -matrix, each column is a vector in  $F^m$ , which motivates  $m$  in  $F^m$  above.

**Example 13.0.5.** Let  $A \in \mathbb{R}^{2 \times 5}$  be the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then  $E = \{1, 2, 3, 4, 5\}$  denotes the 5 columns of  $A$ , and

$$\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}.$$

We now want to prove proposition 22.

*Proof.* We need to check axiom (i)-(iii) in the definition of matroid (13.0.1).

Since  $\emptyset \in \mathcal{I}$ , (i) holds. Since any subset of a linearly independent set is linearly independent, (ii) holds. It remains to show that (iii) holds. So let  $I, I' \in \mathcal{I}$  with  $|I| < |I'|$ . Let  $W = \text{span}_F(I \cup I')$ . Then

$\dim W \geq |I'|$  (since  $I' \subset W$ ). Suppose that  $I \cup \{e\}$  is linearly dependent for all  $e \in I' \setminus I$ . This means that for any  $e \in I' \setminus I$  there is some  $c_i \in F$  not all zero such that

$$\left( \sum_i c_i v_i \right) + c_k e = 0$$

for some  $i \in I$ . Since the  $v_i$  are linearly independent, we must have that  $c_k \neq 0$ , and so for any  $e \neq 0$  we then find that

$$-\frac{1}{c_k} \left( \sum_i c_i v_i \right) = e$$

so that  $e$  is in the span of the vectors (columns of the matrix  $A$ ) associated with  $I$ . Furthermore,  $e \neq 0$  by definition, since  $a \cdot 0 = 0$  so that if  $A$  were to contain some zero-column, this column would not show up in  $I$  (this is also illustrated in example 13.0.5, where  $A$  contains a zero-column). But this means that  $W \subset \text{span}(v_i : i \in I)$ . Since we already have that  $\text{span}(v_i : i \in I) \subset W$  it follows that

$$|I'| \leq \dim W = |I| < |I'|,$$

contradiction! Therefore, there exists some  $e \in I' \setminus I$  such that  $I \cup \{e\}$  is a linearly independent set. Then by definition  $I \cup \{e\} \in \mathcal{I}$ , which is what we wanted to show.  $\square$

**Definition 13.0.6** (Vector matroid  $M[A]$ ). Given a matrix  $A \in M_{n \times m}(F)$ , the matroid obtained as in proposition 22 is denoted as  $M[A] = (E, \mathcal{I})$ , and is called the **vector matroid** of  $A$ .

**Definition 13.0.7.** The **uniform matroid**  $U_{m,n}$  for integers  $m \leq n$  is the matroid on  $E = [n] = \{1, \dots, n\}$  and

$$\mathcal{I}(U_{m,n}) = \{X \subset E : |X| \leq m\}.$$

**Example 13.0.8** ( $U_{2,4}$ ). For example,  $U_{2,4}$  is the (uniform) matroid  $(E, \mathcal{I})$  with

$$E = [4] = \{1, 2, 3, 4\}$$

and

$$\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

We claim that the same independent sets are given by the matrix  $A \in \mathbb{R}^{2 \times 4}$ , with

$$A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

. Given two matroids  $M_1 = (E_1, \mathcal{I}_1)$  and  $M_2 = (E_2, \mathcal{I}_2)$ , we say that they are **isomorphic as matroids**, if there is a bijection

$$\psi : E(M_1) \rightarrow E(M_2)$$

with  $E(M_i) = E_i$ , such that for all  $X \subseteq E(M_1)$ , we have that

$$\psi(X) \in \mathcal{I}(M_2) \Leftrightarrow X \in \mathcal{I}(M_1)$$

**Definition 13.0.9** (Matroid  $M$  representable over some field  $F$ ). If  $M = (E, \mathcal{I})$  is a matroid such that  $M$  is the vector matroid for some matrix  $A \in M_{m \times n}(F)$ , i.e.  $M = M[A]$ , then we say that  $M$  is **representable over  $F$** .

Exercise:  $U_{2,4}$  is realizable over any field  $F$  with more than two elements.

**Example 13.0.10.**  $U_{m,n}$  is representable over  $\mathbb{R}$  for any all  $m \leq n$ . Take  $n$  vectors in  $\mathbb{R}^m$  in **general position**, i.e. such that any  $m$  of them are linearly independent. This is always possible: Consider

$$A[x] = \begin{pmatrix} x_{1,1} & \dots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \dots & x_{m,n} \end{pmatrix}.$$

If we fix a size of  $m$  columns of  $A[x]$ , we get a “submatrix”  $B[x]$  of matrix-dimensions  $m \times m$ . We have that the  $m$  columns in  $B[x]$  are linearly independent iff  $\det B[x] \neq 0$ . Consider  $B$  as a subset of size  $m$  of  $[n]$ , and let  $\pi_B : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times m}$  be the “projection” map that takes a matrix  $A$  to the  $m \times m$  submatrix  $B$  of  $A$ . Define  $f_B : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$  explicitly as  $\det \circ \pi_B$ . Then let  $S_B = \{x \in \mathbb{R}^{m \times n} : f_B(x) = 0\} = f_B^{-1}(0)$ . Observe that  $f_B$  is a polynomial in the entries  $x_{1,i_1}, x_{2,i_1}, \dots, x_{1,i_m}, \dots, x_{m,i_m}$  and so is continuous. Since  $\{0\}$  is closed in the (Hausdorff) space  $\mathbb{R}$  it follows that  $S_B$  is closed in  $\mathbb{R}^{m \times n}$ . Since for the identity matrix  $I_m$  one has that  $f_B(I_m) \neq 0$ , it follows that  $f_B$  can not be the zero-polynomial  $\rightsquigarrow S_B \neq \mathbb{R}^{m \times n}$ .

**Lemma 13.0.11.** *If  $p : \mathbb{R}^N \rightarrow \mathbb{R}$  is not the zero polynomial, then  $p^{-1}(\{0\})$  has empty interior.*

*Proof.* This proof was provided to us by GPT-5.2, we will only give a sketch: For  $N = 1$  this reduces to the fact that any non-zero polynomial have only finitely many values, and so  $p^{-1}(\{0\})$  can not contain an open interval (open ball) and so it can not have any interior points. Then prove it for general  $N$  by induction, and rewriting  $p \in \mathbb{R}[y_1, \dots, y_{N-1}, t]$  on the form  $p = \sum_{k=0}^d a_k(y) t^k$  with  $a_k(y)$  a polynomial in  $\mathbb{R}[y_1, \dots, y_{N-1}]$ . Then assume by contradiction that  $p^{-1}(\{0\})$  in the inductive step had interior points, so contains some ball  $B'$ , the project this ball to  $\mathbb{R}^{N-1}$ . Then  $t \mapsto p(y, t)$  vanishes on an interval and so must be the zero-polynomial which means that  $a_k(y) = 0$  for all  $k = 0, \dots, d$ . Since the projection to  $\mathbb{R}^{N-1}$  is an open map, it holds that the image in  $\mathbb{R}^{N-1}$  of  $B'$  is open, and so  $a_k(y)$  is open on an (non-empty?) open set, therefore  $a_k(y) \equiv 0$  for all  $k = 0, \dots, d$  but then  $p = 0$ .  $\square$

By the lemma it follows that each  $S_B$  has empty interior, i.e. is nowhere dense and closed. It follows that its complement, which we may denote  $U_B$ , defined by  $\mathbb{R}^{m \times n} \setminus S_B$  is open and dense in  $\mathbb{R}^{m \times n}$ . There are only finitely ( $\binom{n}{m}$ ) such subsets  $B$ , and so

$$U := \bigcap_{B \in \binom{[n]}{m}} U_B$$

is still open, and dense (a finite intersection of dense open sets is dense). Since  $U$  is dense,  $U$  is non-empty. Then any  $A \in U \neq \emptyset$  does the job, i.e.  $U_{m,n} = A$  for some  $A \in U$ .

**Example 13.0.12** (Graphic matroids  $M(G)$ ). For a graph  $G = (V, E)$  we have a matroid with groundset  $E$  (i.e. the ground set is the same set as the *edge-set* of  $G$ ), and with

$$\mathcal{I} = \{H \subseteq E : (V, H) \text{ is a forest}\},$$

i.e.  $\mathcal{I}$  is the edge-sets of forest subgraphs of  $G$ . Let's denote such matroids (induced from a graph  $G = (V, E)$  as  $M(G)$ ). Observe that  $(V, \emptyset)$  is forest, since there are no edges, so surely no cycles, hence  $M(G) = (E, \mathcal{I})$  fulfills criteria (i). If  $H \in \mathcal{I}$  then  $H$  is a forest  $\rightsquigarrow$  if  $H' \subseteq H$  then  $(V, H')$  is also a forest, since otherwise  $H$  would have a cycle  $\Rightarrow H' \in \mathcal{I}$ , and so (ii) is satisfied. Lastly, for criterium (iii):

**Lemma 13.0.13** (c.f. lemma 3.0.3). *Let  $T = (V, E)$  and  $T' = (V, E')$  be forests with  $|E| < |E'|$ . Then there exists  $e \in E' \setminus E$  such that  $T \cup \{e\} = (V, T \cup \{e\})$  is a forest.*

*Proof.* Recall (see lecture 3) that for any forest  $T = (V, E)$  it holds that  $|E| = |V| - k$  where  $k$  is the number of components in  $T$ . Since  $|E| < |E'|$ , it follows that  $|V| - k < |E'|$ . Consider the components  $T_1, \dots, T_k$  of  $T$ , and suppose that for every  $e \in E' \setminus E$ , we have that  $e$  has both of its endpoints in some  $T_i$ . Then it follows that all of the edges in  $E'$  are in the components  $T_i$  for  $i = 1, \dots, k$ , that is,  $E'$  does not connect different components of  $T$ . But this means that intuitively, when we add the edges  $e$  in  $E' \setminus E$  to  $T$  to get  $T'$ , we would still not have any paths between  $T_1, \dots, T_k$ . This means that  $T_1, \dots, T_k$  must all be contained in different components of  $T'$ . This in turn means that  $T'$  have atleast  $k$  different components, which implies that  $|E'| \leq |V| - k$ . But then

$$\begin{aligned}|E'| &\leq |V| - k \\ &= |E|\end{aligned}$$

contradicting that  $|E| < |E'|$ . Therefore, there is some  $e \in E' \setminus E$  such that  $e$  has endpoints in different components  $T_i, T_j$  of  $T$ . This means that  $T$  has strictly more components than  $T \cup \{e\} \Leftrightarrow e$  is a bridge  $\Leftrightarrow e$  does not lie on any cycle in  $T \cup \{e\}$ . By assumption there are no cycles in  $T$ . If there was a cycle in  $T \cup \{e\}$  it would have to be a cycle  $C = f_1, \dots, f_\ell$  contained in  $T$ , contradicting that  $T$  is a forest. Therefore, there are no cycles in  $T \cup \{e\}$  and so  $T \cup \{e\}$  is a forest.  $\square$

This works more generally with multigraphs with loops.

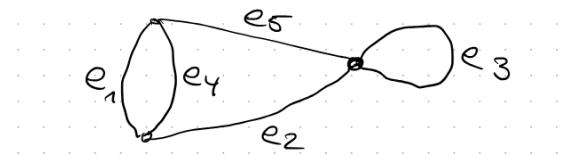


Figure 13.1: Graphic matroid  $M(G)$  of multigraph  $G$  with loops.

**Example 13.0.14** (Graphic matroid of multigraph with loops). Consider the graph  $G$  as depicted above. We then have

$$E = \{e_1, e_2, e_3, e_4, e_5\}$$

and

$$\mathcal{I} = \{\emptyset, \{e_1\}, \{e_2\}, \{e_4\}, \{e_5\}, \{e_1, e_2\}, \{e_1, e_5\}, \{e_2, e_4\}, \{e_2, e_5\}, \{e_4, e_5\}\}.$$

If we compare this with example 13.0.5 we claim that we have a matroid isomorphism from the matroid  $M$  in 13.0.5 to the matroid given above, by  $i \mapsto e_i$  for  $i = 1, \dots, 5$ .

**Definition 13.0.15** (Graphic matroids). Matroids that are isomorphic to matroids given by graphs, are called **graphic**.

**Theorem 13.0.16.** *Every graphic matroid is representable over every field  $F$ .*

*Proof.* Check: Oxley, Chap. 5.  $\square$

### 13.0.1 Transversal matroids

For a finite set  $S$ , a **family** of subsets of  $S$  is a finite sequence  $(A_1, \dots, A_m)$  such that  $A_j \subseteq S$  for all  $j \in [m]$ . If  $\mathcal{J} = [m]$ , then we abbreviate  $(A_1, \dots, A_m)$  as  $(A_j : j \in \mathcal{J})$ .

**Definition 13.0.17** (Transversal; partial transversal). Let  $S$  be a set and let  $A_j \subseteq S$  for  $j \in \mathcal{J} = [m]$ . A **transversal**, or system of **distinct representatives** is a subset  $\{e_1, \dots, e_m\} \subset S$  such that  $e_j \in A_j$  for all  $j$ , and all  $e_1, \dots, e_m$  are distinct (and so that  $|\{e_1, \dots, e_m\}| = m = \text{length of (finite) sequence } (A_1, \dots, A_m)$ ).

A subset  $X \subseteq S$  is a **partial transversal** of  $(A_j : j \in \mathcal{J})$  if  $X$  is a transversal of  $(A_j : j \in K)$  for some subset  $K \subseteq \mathcal{J}$ .

Another perspective: Bipartite graphs! If  $\mathcal{A}$  is a family  $(A_1, \dots, A_m)$  of subsets of  $S$  and  $\mathcal{J} = [m]$ , then

$$\Delta[\mathcal{A}]$$

is the bipartite graph with vertex-set  $S \cup \mathcal{J}$  and edge-set

$$E = \{\{x, j\} : x \in S, j \in \mathcal{J} \text{ and } x \in A_j\}.$$

**Example 13.0.18.**  $S = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ ,  $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ , and

$$\begin{aligned} A_1 &= \{x_1, x_2, x_6\}, \\ A_2 &= \{x_3, x_4, x_5, x_6\}, \\ A_3 &= \{x_2, x_3\}, \\ A_4 &= \{x_2, x_4, x_6\}. \end{aligned}$$

Then  $\Delta[\mathcal{A}] = (V, E)$  with  $V = \{x_1, x_2, \dots, x_6\} \sqcup \{1, 2, 3, 4\}$  and

$$E = \{\{x_1, 1\}, \{x_2, 1\}, \{x_6, 1\}, \{x_3, 2\}, \{x_4, 2\}, \{x_5, 2\}, \{x_6, 2\}, \{x_2, 3\}, \{x_3, 3\}, \{x_2, 4\}, \{x_4, 4\}, \{x_6, 4\}\},$$

as illustrated below.

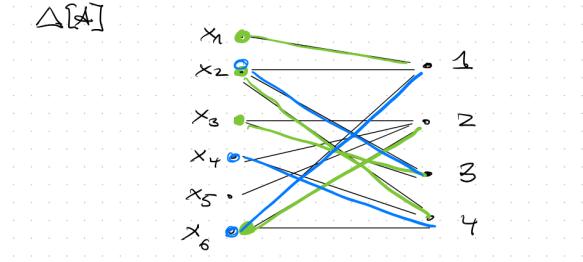


Figure 13.2: Illustration of  $\Delta[\mathcal{A}]$ .

Observe that  $x_1 \in A_1, x_2 \in A_4, x_3 \in A_3, x_6 \in A_2$ . Then we see that  $\{x_1, x_2, x_3, x_6\}$  is a **transversal** (c.f. figure 13.2), and that  $\{x_6, x_2, x_4\}$  is a **partial transversal** since it is a transversal of  $(A_1, A_3, A_4)$ .

$X \subseteq S$  is a partial transversal of  $\mathcal{A}$  if there is a *matching* in  $\Delta[\mathcal{A}]$  in which every edge has an endpoint in  $X$ .

Observe: Transversals encode endpoints of matchings and not *actual* matchings.

**Theorem 13.0.19** (Theorem 37). *Let  $\mathcal{A}$  be a family  $(A_1, \dots, A_m)$  of subsets of a set  $S$ . Let  $\mathcal{I}$  be the set of partial transversals of  $\mathcal{A}$ . Then  $\mathcal{I}$  is the collection of independent sets of a matroid on  $S$ .*

*Proof.* We proceed by checking the three axioms (i)-(iii) for a matroid (13.0.1).

(i): Take  $K = \emptyset \subset [m]$ . Then  $(A_j : j \in K)$  is an empty subfamily of  $(A_j : j \in [m])$  and  $\emptyset \subseteq S$  is vacuously a transversal for this (empty) subfamily  $(A_j : j \in K)$ .

(ii): If  $I \subseteq S$  is a partial transversal,  $I' \subseteq I$ , then  $I'$  is still a partial transversal by *restricting*  $(A_j : j \in K_I)$  to  $(A_j : j \in K_{I'})$  where  $K_I$  is the indexing-set for  $I$  and  $K_{I'}$  is the restriction to the indexing-set where  $e_j \in I'$  shows up.

(iii): Suppose that  $I_1, I_2 \subseteq S$  are partial transversals of  $\mathcal{A}$  with  $|I_1| < |I_2|$ . Then in  $\Delta[\mathcal{A}]$  there are matchings  $W_1, W_2$  with endpoints in  $I_1, I_2$ , respectively. We color the edges in  $W_1 \setminus W_2, W_2 \setminus W_1$  and  $W_1 \cap W_2$  in **red**, **blue** and **purple**, respectively. Since  $|I_2| > |I_1|$ , there are more blue than red:  $|W_i| = |I_i|$  since each edge of  $W_i$  uses exactly one vertex of  $I_i$ . Then

$$\begin{aligned} |W_1| &= |W_1 \setminus W_2| + |W_1 \cap W_2| \\ |W_2| &= |W_2 \setminus W_1| + |W_1 \cap W_2| \\ \Rightarrow |W_2 \setminus W_1| - |W_1 \setminus W_2| &= (|W_2| - |W_1 \cap W_2|) - (|W_1| - |W_1 \cap W_2|) \\ &= |W_2| - |W_1| \\ &= |I_2| - |I_1| \\ &> 0, \end{aligned}$$

i.e.  $|W_2 \setminus W_1| > |W_1 \setminus W_2|$ .

Let  $W$  be the subgraph of  $\Delta[\mathcal{A}]$  consisting of the edges in **red** and **blue**. Since  $W_1$  and  $W_2$  are both matchings, every vertex has degree 1 or 2. Therefore, every connected component  $C_i$  of  $W$  is either a path or a cycle: Start at any vertex  $v_i$  in a component  $C_i$ , and traverse edges in  $C_i$ . Observe first that this process must come to an end since there only a finite number of edges in  $C_i$ . If  $C_i$  has no vertex of degree 1, then all vertices are of degree 2. Start at some vertex  $v_i \in C_i$  and take a *maximal* path  $P : v_i - v_{i+1} - \dots - v_{i+\ell}$  in  $C_i$  that starts at  $v_i$ . Since  $\deg(v_\ell) = 2$  we must have (by maximality) that the unused edge incident with  $v_\ell$  is one of the neighbors  $v_i, \dots, v_{i+\ell-1}$ ; call this neighbor  $v_{i+k}$ . But then this gives a cycle  $v_{i+k} - v_{i+k+1} - \dots - v_{i+\ell}$ . If there was some vertex  $u \notin P$  then since  $C_i$  is connected there is a (minimal) path  $P'$  from  $u$  to  $P$ . Then the endpoint of  $P'$  must have at least degree three, contradiction! Therefore  $u$  is in this cycle, for all  $u \in C_i$ . Since all vertices are of degree 2, it must be all of  $C_i$ .

On the otherhand, if there is at least one vertex of degree 1, in  $C_j$ , then take a maximal path  $P : v_j \dots v_\ell$  starting at  $v_j$  for  $\deg_W(v_j) = 1$ . We claim that  $P = C_j$ . First, observe that  $\deg(v_\ell) = 1$  since otherwise  $P$  is not maximal or some  $v_k$  in  $P$  has degree at least three if  $v_k \neq v_j$  and otherwise  $v_j$  has degree at least 2, again a contradiction. We claim that  $P = C_j$ . If not then there is some edge  $e = \{x, y\}$  not on  $P$ . Since  $C_j$  is connected there is a path from either  $x$  or  $y$  to  $P$ , say  $P' : w_0 = x - w_1 - \dots - w_r$  with  $r$  minimal. Hence  $e' = \{w_{r-1}, w_r\}$  is not on  $P'$ . If  $w_r$  is an internal vertex of  $P$  then this gives  $\deg_W(w_r) \geq 3$ , contradiction. If  $w_r$  is an endpoint of  $P$  then since  $w_{r-1} \notin P$  this would give an extension  $P \cup e'$  of  $P$  so that  $P$  can not be maximal, contradiction! Therefore  $C_j = P$ .

Notice that if  $C_i$  is a cycle for some component  $C_i$  then it must be an *even* cycle. Since  $W$  has more blue than red edges, and in every cycle and path the red and blue edges alternate, there must be a path of odd length whose first and last edge is blue: For even cycles  $\mathcal{C} : e_1 - \dots - e_n$ , with  $n = 2k$ , we have that  $e_{2i-1}$  for  $i = 1, \dots, k$  is of one color, and  $e_{2i}$  for  $i = 1, \dots, k$  is the other color, so the red and blue edges contribute the same number of edges to  $W$ . On the other hand if  $C_i = P$  is a path for some component  $C_i$  then (without investigating this to carefully; prompt by GPT-5.2) that the same idea should work as for an even cycle. If all components  $C_i$  were either an even cycle or an even path then there could not be more **blue** than **red** edges in  $W$  and so since all components are either a path or a cycle, and they can not be an odd cycle, there must be an odd path: For an odd path (with alternating colors on edges)  $P = e_1 - \dots - e_{2k+1}$  we have that  $e_{2i-1}$  for  $i = 1, \dots, k+1$  are of one

color and  $e_{2i}$  for  $i = 1, \dots, k$  are of the other color. Hence there must be an odd path starting and ending with a blue edge. Call this path  $P$ .

Let  $\{v_1, \dots, v_k\}$  be the vertices occurring in  $P$ . If  $v_1 \in S$  then  $v_{2i+1} \in S$  and  $v_{2i} \in [m]$ , for  $i = 1, \dots, k$ . This is interchanged if  $v_1 \in [m]$ ; therefore it seems to us that  $v_1 \in S$  and  $v_{2k} \in [m]$  or the other way around. WLOG assume that  $v_1 \in S$ . Since the path starts with a blue edge, and  $v_1 \in S$  it follows that  $v_1 \in I_2 \setminus I_1$  since  $e = \{v_1, v_2\} \in W_2 \setminus W_1$ .

We interchange the colors of the edges in  $P$  while leaving the rest of the colors of  $W$  the same. Now the component corresponding to  $P$  has one more red edge than blue edge, which means that this coloring of  $W$  has one more red edge than the previous coloring. Indeed, every vertex in  $I_1 \cup \{v_1\}$  is the endpoint of a red or purple edge, and those edges form a matching in  $\Delta[\mathcal{A}]$ , and  $I_1 \cup \{v_1\} \subset S$  so we claim that this is a partial transversal of  $\mathcal{A}$ , so indeed  $I_1 \cup \{v_1\} \in \mathcal{I}$ .

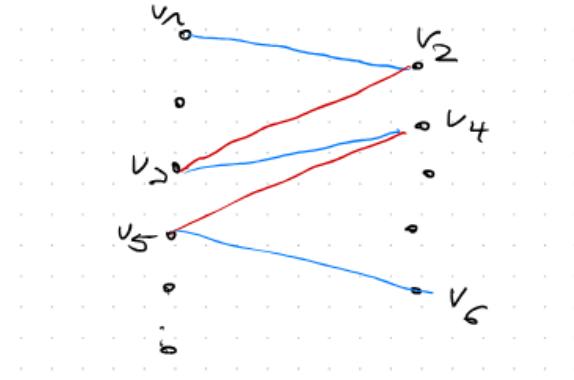


Figure 13.3: Illustration; proof of 13.0.19

□

# Chapter 14

## Lecture 14

Today:

- Different characterizations of matroids.
- Greedy Algorithm/Kruskal's algorithm revisited.

Recall the definition of a matroid (13.0.1)  $M = (E, \mathcal{I})$ . The *independent sets* are specified by (inclusion)-maximal independent sets by (i) and (ii). A maximal independent set is called a **basis**. That is,

$B \in \mathcal{I}$  is a basis if whenever  $I \in \mathcal{I}$  such that  $B \subseteq I$ , then  $B = I$ .

**Lemma 14.0.1.** *If  $B_1$  and  $B_2$  are bases of a matroid  $M$ , then  $|B_1| = |B_2|$ .*

*Proof.* Suppose that  $|B_1| < |B_2|$ . Since  $B_1, B_2 \in \mathcal{I}$  there is  $e \in B_2 \setminus B_1$  such that  $B_1 \cup \{e\} \in \mathcal{I}$ , by (iii). But then  $B_1 \subsetneq B_1 \cup \{e\} \in \mathcal{I}$ , contradicting maximality. Therefore,  $|B_1| \leq |B_2|$ . Similarly, one can show that  $|B_2| \leq |B_1|$  so that  $|B_1| = |B_2|$ .  $\square$

### 14.0.1 Characterization of matroid via basis axiom

**Theorem 14.0.2** (Theorem 38). *Let  $\mathcal{B}$  be a set of subsets of  $E$ . Then  $\mathcal{B}$  is the collection of bases of a matroid on  $E$  if and only if*

- (1)  $\mathcal{B}$  is non-empty.
- (2) *If  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \setminus B_2$ , then there is a  $y \in B_2 \setminus B_1$  such that*

$$(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}.$$

*Proof.*  $\square$

*Remark 14.0.3.* Many characterizations of matroids, e.g.

- Circuits: Minimal dependent sets  $\rightsquigarrow$  circuit axioms.
- Rank functions:  $\rightsquigarrow$  dimension of linear span of vectors.
- Greedy algorithm: Generalization of Kruskal's algorithm.

Recall: The basic idea is that in every step, we add an edge of minimal weight that does not form a cycle with the previously chosen edges.

More general optimization problem:

- $\mathcal{I}$  collection of subsets of  $E$  that satisfy (i) and (ii).
- weight-function  $w : E \rightarrow \mathbb{R}$ .
- For any  $X \subseteq E$ ,  $w(X) = \sum_{x \in X} w(x)$  with  $w(\emptyset) = 0$ .

Optimization problem: Find an inclusion-maximal  $B \in \mathcal{I}$  of maximum weight  $(\star)$ . Such a  $B$  is called a **solution**.

The greedy algorithm for a pair  $(\mathcal{I}, w)$  (Algorithm 5):

1. Set  $X_0 = \emptyset$  and  $j = 0$ . While  $E \setminus X_j$  contains an element  $e$  such that  $X_j \cup \{e\} \in \mathcal{I}$ , choose such an element  $e_{j+1}$  of *maximal* weight.
2. Set  $X_{j+1} = X_j \cup \{e_{j+1}\}$  and  $j = j + 1$ .
3. Return  $B_G = X_j$ .

**Proposition 14.0.4** (Proposition 23). *If  $(E, \mathcal{I})$  is a matroid, then  $B_G$  is a solution of the optimization problem  $(\star)$ .*

*Proof.* Let  $B_G = \{e_1, \dots, e_r\}$  with  $w(e_1) \geq w(e_2) \geq \dots \geq w(e_r)$ . Then  $B_G$  is inclusion-maximal, since otherwise the algorithm would not have stopped at  $B_G$ . Let  $B$  be another basis. Then by lemma 14.0.1 B(?) we have that  $|B| = |B_G|$ . Let  $B = \{f_1, \dots, f_r\}$  with  $w(f_1) \geq w(f_2) \geq \dots \geq w(f_r)$ .

Claim: For  $1 \leq j \leq r$ , we have  $w(e_j) \geq w(f_j)$  (\*). To prove the claim (\*), assume to the contrary that there is some  $j$  such that  $w(e_j) < w(f_j)$ . Let  $k$  be the least index such that this holds. Let  $I_1 = \{e_1, \dots, e_{k-1}\}$  and  $I_2 = \{f_1, \dots, f_k\}$ . Then  $|I_1| < |I_2|$  so there is some  $f_t \in I_2 \setminus I_1$  (by (iii) - not sure why) such that  $I_1 \cup \{f_t\} \in \mathcal{I}$ . But then  $w(f_t) \geq w(f_k) > w(e_k)$  by assumption, so  $w(f_t) > w(e_k)$ , and so the greedy algorithm would have chosen  $f_t$  over  $e_k$  in step  $k$ , contradiction! Thus, we conclude that  $w(e_j) \geq w(f_j)$  for  $1 \leq j \leq r$ . In particular, this means that

$$\begin{aligned} w(B_G) &= \sum_{j=1}^r w(e_j) \\ &\geq \sum_{j=1}^r w(f_j) \\ &= w(B). \end{aligned}$$

By construction,  $B_G$  is inclusion maximal, and for any other inclusion-maximal set  $B$  in  $\mathcal{I}$  (i.e. basis) its weight is less than or equal to  $B_G$ .  $\square$

This shows that the algorithm always gives an optima for matroids. Even stronger, this property characterizes matroids!

## 14.0.2 Characterizations of matroids in terms of the greedy algorithm

**Theorem 14.0.5** (Theorem 39; characterization of matroids in terms of the greedy algorithm). *Let  $\mathcal{I}$  be a collection of subsets of a finite set  $E$ . Then  $(E, \mathcal{I})$  is a matroid iff*

- (i)  $\emptyset \in \mathcal{I}$ .
- (ii) If  $I \in \mathcal{I}$  and  $I' \subseteq I$  then  $I' \in \mathcal{I}$ .

(G) For all weight-functions  $w : E \rightarrow \mathbb{R}$ , the greedy algorithm produces an inclusion-maximal member of  $\mathcal{I}$  of maximum weight.

*Proof.*  $\Rightarrow$ : If  $(E, \mathcal{I})$  is a matroid then (i) and (ii) holds by definition, and (G) holds by proposition 23.

$\Leftarrow$ : (i) and (ii) are the same axioms as the axioms for a matroid, so we are left with showing that (iii) in definition 13.0.1 holds. Suppose to the contrary that there are  $I_1, I_2 \in \mathcal{I}$ ,  $|I_1| < |I_2|$  such that  $I_1 \cup \{e\} \notin \mathcal{I}$  for all  $e \in I_2 \setminus I_1$ . Since  $|I_1| < |I_2|$ , and we have

$$\begin{aligned} |I_1| &= |I_1 \setminus I_2| + |I_1 \cap I_2| \\ |I_2| &= |I_2 \setminus I_1| + |I_1 \cap I_2| \\ \Rightarrow |I_2| - |I_1| &= |I_2 \setminus I_1| - |I_1 \setminus I_2| \\ &> 0 \\ \Rightarrow |I_2 \setminus I_1| &> |I_1 \setminus I_2| \\ \Rightarrow 1 &> \frac{|I_1 \setminus I_2|}{|I_2 \setminus I_1|}. \end{aligned}$$

Therefore, there is some real number  $\varepsilon > 0$  such that  $1 > \varepsilon > \frac{|I_1 \setminus I_2|}{|I_2 \setminus I_1|}$ . Define  $w : E \rightarrow \mathbb{R}$  by

$$w(e) = \begin{cases} 1, & \text{if } e \in I_1 \\ \varepsilon, & \text{if } e \in I_2 \setminus I_1 \\ 0, & \text{otherwise} \end{cases}.$$

Then the greedy algorithm picks all elements in  $I_1$  first, since these have the highest weight. By assumption, it can not pick elements  $e \in I_2 \setminus I_1$  afterwards. Therefore, the remaining elements are in  $E \setminus (I_1 \cup I_2 \setminus I_1)$  and so have weight zero, so that  $w(B_G) = w(I_1) = |I_1|$ . On the other hand,  $I_2$  is contained in an inclusion-maximal  $I'_2 \in \mathcal{I}$  (one may show that any set  $I \in \mathcal{I}$  is contained in some inclusion-maximal element; it is bounded above by cardinality of the finite set  $E$ ). Since  $I_2 \subseteq I'_2$ , we find that

$$\begin{aligned} w(I'_2) &\geq w(I_2) \\ &= |I_1 \cap I_2| + \varepsilon |I_2 \setminus I_1| \\ &> |I_1 \cap I_2| + \frac{|I_1 \setminus I_2|}{|I_2 \setminus I_1|} |I_2 \setminus I_1| \\ &= |I_1| \\ &= w(B_G). \end{aligned}$$

Therefore,  $w(I'_2) > w(B_G)$ . Since  $I'_2$  was inclusion-maximal, this contradicts (G). Therefore, we conclude that  $(E, \mathcal{I})$  also satisfy (iii), and so  $(E, \mathcal{I})$  is a matroid.  $\square$

### 14.0.3 Application to transversal matroids

Let  $S$  = jobs for one worker,  $Y$  = workers, let  $\mathcal{A} = (A_y : y \in S)$  where  $A_y \subseteq S$  are jobs that workers  $y$  can perform, and let  $p : S \rightarrow \mathbb{R}$  be a priority function (lower value = higher priority). A **job assignment** is a basis of the transversal matroid  $M[\mathcal{A}]$ . A basis  $B = \{x_1, \dots, x_r\}$  with  $p(x_1) \leq \dots \leq p(x_r)$  is *optimal* if for any other job assignment  $B' = \{z_1, \dots, z_r\}$  with  $p(z_1) \leq \dots \leq p(z_r)$  it holds that  $p(x_i) \leq p(z_i)$  for  $1 \leq i \leq r$ .

If  $\mathcal{I}$  is the set of (partial) transversals of  $\mathcal{A}$ , then by proposition 23 and (\*), the greedy algorithm applied to  $(S, \mathcal{I})$  finds an optimal job assignment (recall theorem 37).

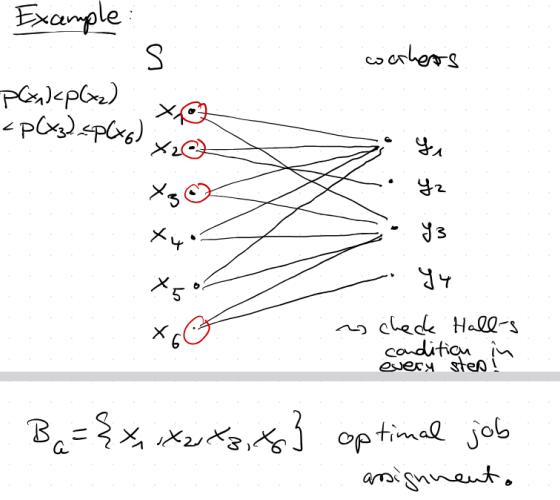


Figure 14.1: Example job assignment

**Example 14.0.6.**

**Definition 14.0.7** (Transversals; partial transversals). Let  $S$  be a *finite set*, and let  $\mathcal{A} := (A_j : j \in J)$  be a collection of subsets  $A_j \subset S$ . A **transversal**  $T$  of  $\mathcal{A}$  is a set  $T \subset S$  such that there is a bijection  $\psi : J \rightarrow T$  such that  $\psi(j) = t \in A_j$  for all  $j \in J$ . A **partial transversal**  $P$  of  $\mathcal{A}$  is a set  $P \subset S$  such that there is a subset  $K \subset J$  and a bijective function  $\psi' : K \rightarrow P$  such that  $\psi'(k) = t \in A_k$  for all  $k \in K$ .

We may form the bipartite graph  $\Delta[\mathcal{A}]$  associated with the collection  $\mathcal{A}$  as follows: Take as vertex set  $S \cup J$ , with edge-set  $E = \{xj : x \in S, j \in J, \text{ and } x \in A_j\}$ .

**Lemma 14.0.8.** *A subset  $X \subset S$  is a partial transversal of  $\mathcal{A} \Leftrightarrow$  there is a matching in  $\Delta[\mathcal{A}]$  in which every edge has one endpoint in  $X$ , and the matching saturates  $X$ , i.e. every  $x \in X$  meets some edge of the matching.*

*Proof.*  $\Rightarrow$ : Assume that  $X$  is a partial transversal of  $\mathcal{A}$ . Then there is a subset  $K \subset J$  and a bijective function  $\psi : K \rightarrow X$  such that  $\psi(k) = t \in A_k$  for all  $k \in K$ . Then  $xk$  is an edge in  $\Delta[\mathcal{A}]$  for each  $k \in K$ , hence the pairing  $\{xk\}$  coming from this bijection, gives an edge-set in  $\Delta[\mathcal{A}]$  in which each  $x$  and each  $k$  is distinct, so this is a matching.

$\Leftarrow$ : assume that there is a matching  $M$  in  $\Delta[\mathcal{A}]$  with one endpoint of each edge  $e \in M$  meeting  $X$  and so that  $M$  saturates  $X$ . Then by picking out from  $M$  those edges that meet  $X$ , we get a bijection  $\psi : K \subset J \rightarrow X$ , such that  $\psi(k) = x \in A_k$  for all  $k \in K$  (by which edges are allowed in  $\Delta[\mathcal{A}]$ ).  $\square$

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# Appendix A

## Exercise session 1

### Problem 1

Let  $T$  be a tree and  $v \in V(T)$ . Show that  $T$  has at least  $\deg(v)$  many leaves.

If we remove  $v$  together with its edges, we get a new tree  $T - \{v\}$ . Assume that  $\deg(v) = n$  and that we have that  $v$  is adjacent to  $\{u_1, \dots, u_n\}$ . Then note that  $u_i$  is not adjacent to  $u_j$  for  $i \neq j$  since this would create a cycle in  $T$ . Since the unique path between any  $u_i$  and any  $u_j$  in  $T$  must be by  $v$ , we see that there are no paths between any  $u_i$  and  $u_j$  in  $T - v$ . Therefore we have that  $u_1, \dots, u_n$  are all in different components  $C_1, \dots, C_n$  in the graph  $T - v$ . Each component is connected and acyclic since  $T$  is acyclic, hence is a tree. If  $C_i$  consists of more than one node, then by proposition 2.0.12 we see that it has a leaf  $\ell_i$ . The only way  $\ell_i$  is not a leaf in  $T$  is if  $\ell_i$  is adjacent to  $v$ , but this would give us a cycle  $u, v, \ell_i, u$ , hence is impossible. Therefore  $\ell_i$  is a leaf in  $T$ .

If  $C_1$  only consists of one node,  $u_i$ , then  $u_i$  is a leaf in  $T$ .

Thus we see that each component  $C_i$  gives us a distinct leaf in  $T$  different from any other leaf (since the components are disconnected in  $T - v$ ). Therefore the number of leafs in  $T$  is at least as many as the number of components  $n = \deg(v)$ .

### Problem 2

Prove (without induction) that every tree with no vertices of degree two has more leaves than other vertices.

The intuition here is that “there are not enough” vertices for all the edges if the number of vertices  $v$  of  $\deg(v) \geq 3$  are greater than the number of leaves  $v$  of  $\deg(v) = 1$ .

Let  $G = (V, E)$  be a simple graph.

Recall that  $2|E| = \sum_{v \in V} \deg(v) \Leftrightarrow |E| = \sum_{v \in V} \frac{\deg(v)}{2}$ .

Let  $|V| = m$  and assume that the number  $n$  of leaves is less than the number  $k$  of vertices of degree greater than or equal to 3. Since this is a tree, we have that  $|E| = |V| - 1$ .

By the formulas above, we must have that

$$\begin{aligned}
2|E| &= \sum_{v \in V} d(v) \\
&= \left( \sum_{\ell \text{ leaf}} d(\ell) \right) + \left( \sum_{v \text{ not leaf}} d(v) \right) \\
&\geq n + 3k \\
\Rightarrow |E| &\geq \frac{n + 3k}{2} \\
\Rightarrow |E| - 1 &\geq \frac{n + 3k}{2} - 1 \\
\Rightarrow m - 1 &\geq \frac{n + 3(m - n)}{2} - 1 \\
\Rightarrow m - 1 &\geq \frac{3m}{2} - n - 1 \\
\Rightarrow n &\geq \frac{m}{2}.
\end{aligned}$$

But  $m = n + k > 2n$  which implies that  $\frac{m}{2} > n$ , contradiction!

### Problem 3

A *cut vertex* in a graph  $G$  is a vertex  $v$  such that  $G \setminus v$ , the induced subgraph of  $G$  on vertex set  $V(G) \setminus v$ , has strictly more components than  $G$ . Show that a connected graph  $G$  has exactly  $|V(G)| - 2$  cut vertices if and only if  $G$  is a path.

*Remark A.0.1.* The original problem says  $|G| - 2$  but we believe they mean  $|V(G)| - 2$  (we guess that the former is a shorthand for the latter when the context is clear; however, it is according to us not really clear here).

Recall that if  $G \setminus v$  has more components than  $G$ , then  $v$  is a bridge, which is equivalent to  $v$  not lying on any cycle in  $G$ .

$\Rightarrow$ :  $G$  is connected, so by proposition 2.0.23 it has a spanning tree  $T$ . Let  $x, y$  be the non-cut vertices and let  $L = \{y_1, \dots, y_m\}$  be the cut-vertices of  $G$ .

**Claim:** Any  $y \in L$  is a cut-vertex in  $T$ .

*Proof.* We know that  $G \setminus y$  has at least two components,  $C_1$  and  $C_2$ , which are connected only by  $y$ . Take  $u \in C_1$  and  $v \in C_2$ . Then the only path in  $G$  between  $u$  and  $v$  is through  $y$ . Since  $T \subset G$ , it follows that any path in  $T$  between  $u$  and  $v$  is through  $y$ . Therefore,  $T \setminus y$  is disconnected, since there will not be a path between  $u$  and  $v$  (since it would have to go through  $y$ ).  $\square$

It follows from the claim that the number of cut-vertices of  $T$  is *at least* as many as the cut-vertices of  $G$ , i.e.  $|V(G)| - 2$ .

**Lemma A.0.2.** *Any tree such that  $|V(T)| \geq 2$  has two leaves.*

*Proof.* Take a maximal path  $P$  in  $T$ . Then its endpoints must be of degree one.  $\square$

Since  $|V(T)| = |V(G)|$  and  $G$  has  $|V(G)| - 2$  cut vertices, and it can not have a *negative number of cut vertices*, it follows that  $|V(T)| \geq 2$ . Thus  $T$  has two leaves. Leaves can not be cut-vertices, hence  $T$  has two non-cut vertices, which must then be precisely  $x$  and  $y$ .

**Claim:** Any tree  $T$  with exactly two leaves is a path.

*Proof.* Assume there was a node with  $\deg(v) \geq 3$ , and assume that  $x, y$  are the leaves. Since  $T$  is a tree, we have that

$$2(|V| - 1) = \sum_{v \in V(T)} d(v).$$

But

$$\begin{aligned} 2|E(T)| &= \sum_{v \in V(T)} d(v) \\ &= 2 + \sum_{v \in V(T) \setminus \{x, y\}} d(v) \geq 2 + 2(|V| - 3) + 3 \\ &= 2 + 2|V(T)| - 6 + 3 \\ &= 2|V(T)| - 1 \\ &> 2|V(T)| - 2 \\ &= 2(|V(T)| - 1) \\ &= 2|E(T)| \end{aligned}$$

contradiction! □

Thus  $T$  is a path.

Let  $e = uv \in E(G) \setminus E(T)$ . Then  $T \cup \{e\}$  contains a cycle, which must include  $e$ . Notice that  $e$  can not by definition be nodes that are adjacent in the path  $T$ . Therefore, there must be some  $z$  such that there is a subpath  $P_{u,v}$  in  $T$  that has at least one inner node  $z$ . Thus  $z$  can not be one of the leafs of  $T$ , hence  $z$  must be a cut-vertex of  $T \Leftrightarrow$  must be a cut vertex of  $G$ . When we remove  $z$  from  $G$ , we still have the edge  $e$ , and what is left from  $P_{u,v}$  is the “broken path”  $u - v_1 - v_2 - \dots - v_{i-1}$  and  $v_{i+1} - v_{i+2} - \dots - u$ , where  $v_i = z$ . But in  $G \setminus z$ , we then have that  $u, v_1, \dots, v_{i-1}$  and  $v_{i+1}, \dots, u$  are all connected since we can cross over from anything on one side of the “broken path” to the other by the edge  $e$ . Furthermore, if the original path  $T$  extended on the “other side” of  $u$  (i.e. not the  $u - v_1 - v_2 - \dots$  direction but the other), then such nodes are still connected to everything else through the edge with  $u$ , and the same holds for anything extending to the “right” of  $v$  through  $v$ . This is a contradiction, since  $z$  was presumed to be a cut-vertex in  $T$  and so also a cut vertex in  $G$ . Therefore,  $E(G) \setminus E(T) = \emptyset \Rightarrow E(G) = E(T)$ . Hence  $G = T$  and  $G$  is a path.

$\Leftarrow$ : If  $G$  is a path  $P = v_0, \dots, v_k$ , then it has two leaves,  $v_0$  and  $v_k$ , and all other nodes have degree  $d(v_i) = 2$  for  $i = 1, \dots, k-1$ . Therefore, if we remove the endpoints, we still get a path, hence not a cut vertex. If we remove  $v_1$  or  $v_{k-1}$ , we get an isolated point and a subpath, i.e. one component more than  $G$ . If we remove  $v_i$  for  $i$  greater than 1 and less than  $k-1$ , we get two subpaths, i.e. two components. Thus we see that there are precisely  $|V(G)| - 2$  cut-vertices on  $G$ .

**Problem 4 (Lemma 3 and 4)**

Let  $G$  be a graph with vertex set  $V = \{v_1, \dots, v_n\}$  and edge-set  $E = \{e_1, \dots, e_m\}$  and connected components  $G_1, \dots, G_t$ . Let  $L \in \mathbb{R}^{n \times n}$  denote the Laplacian matrix of  $G$  and  $N \in \mathbb{R}^{n \times m}$  its oriented vertex-edge incidence matrix.

(a) Show that  $L = NN^T$ .

(b) For  $1 \leq i \leq t$ , let  $\chi^i \in \mathbb{R}^n$  be the characteristic vector of the component  $G_i$ , defined by

$$\chi_j^i := \begin{cases} 1, & \text{if } v_j \in G_i, \\ 0, & \text{otherwise} \end{cases}.$$

Show that  $\chi^1, \dots, \chi^t$  form a basis for  $\ker L$ .

(a) Note that the matrix-element  $(NN^T)_{ij}$  equal the scalar-product of the  $i^{\text{th}}$  row in  $N$  with the  $j^{\text{th}}$  row of  $N$ , i.e.

$$\begin{aligned} (NN^T)_{ij} &= \sum_{k=1}^m N_{ik}(N^T)_{kj} \\ &= \sum_{k=1}^m N_{ik}N_{jk}. \end{aligned}$$

By definition 3.1.14, if  $i = j$ , then  $(NN^T)_{ii} = \sum_{k=1}^m (N_{ik})^2$ . We see that we get a 1 for each edge  $e_i$  such that  $v_i$  is one of its vertices, and zero otherwise, i.e.  $(NN^T)_{ii} = \deg(v_i)$ . This is equal to  $L(G)_{ii}$  by definition 3.1.5. If  $i \neq j$  and  $v_i$  and  $v_j$  are adjacent, so that there is a unique edge  $e_q$  such that  $e_q = \{v_i, v_j\}$ . Then  $N_{ik}N_{jk}$  is zero whenever  $k \neq q$  for  $1 \leq k \leq m$  (can be seen by inspection the definition of the oriented vertex-edge incidence matrix). Therefore,

$$\sum_{k=1}^m N_{ik}N_{jk} = N_{iq}N_{jq}.$$

Regardless of whether  $i < j$  or  $j > i$ , one of the factors equals  $-1$  and the other  $1$ , thus

$$(NN^T)_{ij} = -1$$

whenever  $i \neq j$  and  $v_i, v_j$  are adjacent. If  $i \neq j$  and  $v_i, v_j$  are not adjacent, then  $N_{ik}N_{jk}$  must be zero for all  $1 \leq k \leq m$  since if not, there would be an edge  $e_k$  between  $v_i$  and  $v_j$ . By comparing  $(NN^T)_{ij}$  with  $(L(G))_{ij}$  element-wise, we see that they are identical, i.e.  $L(G) = NN^T$ .

(b): To show that  $\chi^1, \dots, \chi^t$  is a basis for  $\ker L$ , we need to show that  $\text{span}\{\chi^1, \dots, \chi^t\} = \ker L$  and that  $\chi^1, \dots, \chi^t$  is a linearly independent set.

$\text{span}\{\chi^1, \dots, \chi^t\} = \ker L$ : Notice that for any  $i = 1, \dots, t$ ,  $L\chi^i$  is such that if  $\ell$  is a row of  $L$  corresponding to a vertex  $v$  that is not in  $G_i$ , then whenever there is a non-zero element in column  $p$  of this row, then  $\chi^i$  has a zero there. Thus the scalar product of this row with  $\chi^i$  equals zero. If we take a row  $q$  of  $L$  corresponding to a vertex  $v$  in component  $G_i$ , then there are ones in  $\chi^i$  in the same columns as the non-zero entries of this row. But the rows of  $L$  sum to zero. Thus this also equals 0. Hence  $\chi^i \in \ker L$ . Since  $\ker L$  is a linear subspace, it follows that the span of  $\chi^1, \dots, \chi^t$  are in  $\ker L$ . It remains to show that  $\ker L \subset \text{span}\{\chi^1, \dots, \chi^t\}$ . This follows from  $x^T L x = \sum_{\{u,v\} \in E} (x_u - x_v)^2$ . So if  $Lx = 0$  then  $x^T L x = 0$  and so if  $u, v$  are connected then  $x_u = x_v$  since the sum is non-negative. So, we know that  $G$  has a partition into components  $G_1, \dots, G_t$ . And so we know that  $x = (x_1, \dots, x_n)$  must be such that it is equal on the components.

$\chi^1, \dots, \chi^t$  linearly independent: This follows from the fact that the components  $G_1, \dots, G_t$  partition  $V(G)$ . Thus, if  $\chi_j^i = 1$ , then no other  $\chi_j^\ell = 1$  for  $\ell \neq i$ . I.e.  $\chi_j^\ell = 0$  for  $\ell \neq i$ . And we know that (again, since it is a partition) that for each  $1 \leq j \leq n$  there is some  $i$  such that  $\chi_j^i = 1$ . Therefore, if  $\alpha_1\chi^1 + \dots + \alpha_t\chi^t = 0$  then this forces  $\alpha_1 = \dots = \alpha_n = 0$ , since

$$\alpha_1\chi^1 + \dots + \alpha_t\chi^t = (\alpha_1, \dots, \alpha_t)$$

(c): The case when  $G$  was connected was dealt with in lemma 3.1.10. So assume that  $G$  is disconnected.

#### Problem 5:

Count the number of spanning trees of  $K_{2,n}$

- (a) using Kirschoff's matrix tree theorem.

(a): We need to find the Laplacian matrix associated with  $G = K_{2,n}$ . There are  $2+n$  vertices. We let  $v_1, v_2$  be the vertices that connects to the  $n$  others, and  $v_3, \dots, v_n$  the ones that only connect with  $v_1, v_2$ . Then it is clear that the diagonal of  $L(G)$  is on the form  $L(G)_{ii} = n$  if  $i = 1, 2$  and  $L(G)_{ii} = 2$  if  $3 \leq i \leq n$ .

On off-diagonal elements, we have that  $v_1$  and  $v_2$  have ones in columns three to  $n$ , and zero in column two, respectively column 1. For  $v_3, \dots, v_{n+2}$ . The off-diagonal elements are 1 in column one and two, and zero otherwise.

Thus we get that

$$L(G) = \begin{pmatrix} n & 0 & -1 & -1 & \dots & -1 \\ 0 & n & -1 & -1 & \dots & -1 \\ -1 & -1 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & 0 & 0 & \dots & 2 \end{pmatrix}$$

By Kirschoff's matrix theorem, the number of spanning trees is  $t(G) = \frac{1}{n+2}\lambda_1\lambda_2 \cdots \lambda_{n+1}$ . We may see by the proof for the connected case that  $\mathbf{1}$ , the  $n+2 \times 1$  column vector consisting of ones, is an eigenvector to  $L(G)$  with associated eigenvalue zero. By 4.(b) this vector spans the null-space of  $L(G)$ , hence the geometric multiplicity of is one. Notice that  $L(G)$  is *diagonalizable*. It follows that the algebraic multiplicity is one, for  $\lambda = 0$ .

But we are mainly interested in the *non-zero* eigenvalues of  $L(G)$ , since it is these eigenvalues that shows up in the formula for  $t(G)$ , i.e.  $\lambda_1, \dots, \lambda_{n+1}$ .

First, notice that the vector  $(1, -1, 0, 0, \dots, 0)^T$  is taken to  $(n, -n, 0, 0, \dots, 0)^T = n(1, -1, 0, 0, \dots, 0)^T$  under left-multiplication by  $L(G)$ . Thus it is an eigenvector of  $L(G)$  with eigenvalue  $n$ .

From now on, let's write  $L$  for  $L(G)$ . Generally, for any vector  $x = (x_a, x_b, y_1, \dots, y_n)$ , we have that

$$\begin{aligned} (Lx)_a &= nx_a - \sum_{i=1}^n y_i \\ (Lx)_b &= nx_b - \sum_{i=1}^n y_i \\ (Lx)_i &= -(x_a + x_b) + 2y_i. \end{aligned}$$

Consider the set  $S = \{x = (0, 0, y_1, \dots, y_n) : \sum_{i=1}^n y_i = 0\}$ . We see that any such vector is such that  $Lx = (0, 0, 2y_1, \dots, 2y_n) = 2(0, 0, y_1, \dots, y_n)$ . In fact, this is a subspace of  $\mathbb{R}^{n+2}$ , and we claim it has basis  $\Delta = \{e_3 - e_4, e_4 - e_5, \dots, e_{n+1} - e_{n+2}\}$  of dimension  $n - 1$ . This gives us the eigenvalue  $\lambda = 2$  with geometric multiplicity  $n - 1$ . We are missing one eigenvalue of geometric multiplicity one.

Consider the family of vectors  $L = \{x = (h, h, \ell, \ell, \dots, \ell)\}$ . We then get the equations

$$\begin{aligned}(Lx)_a &= hn - n\ell \\ &= n(h - \ell) \\ &= L(x)_b\end{aligned}$$

and

$$\begin{aligned}(Lx)_i &= -2h + 2\ell \\ &= 2(\ell - h)\end{aligned}$$

where we want

$$\begin{aligned}n(h - \ell) &= \lambda h \\ 2(\ell - h) &= \lambda \ell \\ \Leftrightarrow nh - n\ell - \lambda h &= 0 \quad \text{and} \quad 2\ell - 2h - \lambda \ell = 0.\end{aligned}$$

This is the same as the matrix-equation

$$\begin{pmatrix} n - \lambda & -n \\ -2 & 2 - \lambda \end{pmatrix} \begin{pmatrix} h \\ \ell \end{pmatrix} = 0.$$

By linear algebra theory, that this has a non-trivial solution is equivalent to the  $2 \times 2$ -matrix on the left-hand side to have determinant equal to zero.

We compute:

$$\begin{aligned}\begin{vmatrix} n - \lambda & -n \\ -2 & 2 - \lambda \end{vmatrix} &= 0 \\ \Leftrightarrow (n - \lambda)(2 - \lambda) - ((-n)(-2)) &= 0 \\ \Leftrightarrow 2n - n\lambda - 2\lambda + \lambda^2 - 2n &= 0 \\ \Leftrightarrow \lambda^2 - \lambda(n + 2) &= 0 \\ \Leftrightarrow \lambda(\lambda - (n + 2)) &= 0.\end{aligned}$$

The only non-zero solution is  $\lambda = n + 2$ . Since  $\lambda = 0$  and  $\lambda = 2$  had geometric multiplicity 1 and  $n - 1$ , respectively, adding up to  $n$  so their algebraic multiplicity must also equal at least  $n$ , we must have that  $n$  and  $n + 2$  have geometric multiplicity 1 so also their algebraic multiplicity must be one (we want to claim that this holds for diagonalizable matrices), thus their algebraic multiplicities of all the eigenvalues sum to  $n + 2$ . Thus we have found all the eigenvalues of  $L(G)$ . We compute

$$t(G) = \frac{2^{n-1} \cdot n \cdot (n + 2)}{n + 2} = 2^{n-1}n.$$

(b): We want to show that  $t(G) = 2^{n-1}n$  with a direct combinatorial argument:

Denote the side with two vertices in  $K_{2,n}$  by  $x_a, x_b$  and the other side by  $y_1, \dots, y_n$ . Notice that if  $\deg(y_i) = 0$  then this can not be a tree, hence  $\deg(y_i) \geq 1$  for all  $y_i$ . The same holds for  $x_a, x_b$ . Assume

that there are at least two different  $y_i, y_j$  of degree two. Then there is a cycle  $x_a, y_i, x_b, y_j, x_a$ , hence there can not be a tree. So there can only be at most one node in the set  $L = \{y_1, \dots, y_n\}$  with degree two. Assume there was no such node, i.e.  $\deg(y_i) = 1$  for all  $y_i$ . But then there is no path from  $x_a$  to  $x_b$ , since such a path would need to have a subpath on the form  $x_a, y_i, x_b$  or  $x_b, y_i, x_a$ , hence such a graph can not be a tree, since it is not connected. Therefore we need there to be exactly one such node. There are  $n$  such choices. Thus every other node has to be adjacent to either  $x_a$  or  $x_b$ , but not both. This gives  $2^{n-1}$  choices. Each such choice is acyclic, since if not, then we would have (without loss of generality) a cycle  $C = x_a v_i, x_b, \dots, x_a v_j x_b$ , but this would require at least two nodes in  $L$  of degree two, which is impossible.

Furthermore, each such choice is connected: If we take  $x_a$  and  $x_b$ , then the path goes by the unique choice of degree two vertex  $v_i$  in  $L$ . Take two elements  $y_i, y_j$  in  $L$ . If both are adjacent to  $x_a$  or  $x_b$ , then we are done. If, without loss of generality,  $y_i$  is connected to  $x_a$  and  $y_j$  is connected to  $x_b$ , then we may take the path from  $y_i$  to  $x_a$  and from  $x_a$  the unique node  $y_\ell$  adjacent to both  $x_a$  and  $x_b$ , and from  $y_\ell$  to  $x_b$  and then from  $x_b$  to  $y_j$ .

For any  $x_a, y_i$ , either  $x_a$  and  $y_i$  are adjacent, or if not (so that  $y_i$  must be adjacent to  $x_b$ , since it has degree one), take the path  $y_i, x_b, y_\ell, x_a$ . Thus indeed each choice is a tree. Furthermore, each choice is unique.