Ab(-) is an additive functor.

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Problem

Show that $\mathrm{Ab}(-)$ is an additive functor.

Let $G_1, G_2 \in \mathbf{Grp}$, then we want to show that $\mathrm{Ab}(-) : \mathbf{Grp} \to \mathbf{Ab}$ is an **additive** functor. Note that $*$ is the **coproduct** in **Grp** and that $\oplus$ is the **coproduct** in **Ab**. So what we want to show, I believe, is that $\mathrm{Ab}(G_1 * G_2) \cong \mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2)$.

Recall that since $\oplus$ is the **coproduct** in **Ab**, we have the following diagram:

$$\begin{tikzcd}
 & Y & \\
\mathrm{Ab}(G_1) \arrow[ur, "f_1"] \arrow[r, "\iota_1"] & \mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2) \arrow[u, dashed, "\exists! f"] & \mathrm{Ab}(G_2) \arrow[ul, "f_2"'] \arrow[l, "\iota_2"']
\end{tikzcd} \tag{1.1}$$

That is, $\mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2)$ satisfies the **universal property** that for *any* other object $Y \in \mathbf{Ab}$ and morphisms $f_1 : \mathrm{Ab}(G_1) \to Y$ and $f_2 : \mathrm{Ab}(G_2) \to Y$ there exists a **unique** morphism $f : \mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2) \to Y$ such that

$$f \circ \iota_i = f_i,$$

for $i = 1, 2$. Furthermore, if there is any other object $(A; \iota_1 : \mathrm{Ab}(G_1) \to A, \iota_2 : \mathrm{Ab}(G_2) \to A)$ in **Ab** with the same property, then there is a **unique** isomorphism $A \cong \mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2)$.

We aim to show that $\mathrm{Ab}(G_1 * G_2)$ satisfies *the same universal property* as $\mathrm{Ab}(G_1) \oplus \mathrm{Ab}(G_2)$, hence giving us a unique isomorphism between the objects. We have the following diagram.

$$\begin{tikzcd}
G_i \arrow[r, "\iota_i"] \arrow[rr, bend left, "\pi^* \circ \iota_i"] \arrow[d, "\pi_i"'] \arrow[dr, "f_i \circ \pi_i"] & G_1 * G_2 \arrow[r, "\pi^*"] \arrow[d, dashed, "\exists! \Phi"] & \mathrm{Ab}(G_1 * G_2) \arrow[dl, dashed, "\exists! \Psi"] \\
\mathrm{Ab}(G_i) \arrow[r, "f_i"'] \arrow[urr, dotted, bend right=60, "\exists! \Gamma_i"'] & Y &
\end{tikzcd}$$

We assume that we have maps $f_i : \mathrm{Ab}(G_i) \to Y$, where $Y \in \mathbf{Ab}$ is arbitrary. We get the following maps:

- From the **characteristic property of the free product**, we get a unique map $\Phi : G_1 * G_2 \to Y$ such that
$$\Phi \circ \iota_i = f_i \circ \pi_i. \tag{1.2}$$

- From the **characteristic property of the abelianization** we get a unique map Ψ such that
$$\Psi \circ \pi^* = \Phi. \tag{1.3}$$

- From the **characteristic property of the abelianization** we get a unique map Γ_i such that
$$\Gamma_i \circ \pi_i = \pi^* \circ \iota_i. \tag{1.4}$$

We then have that

$$\begin{aligned}
(\Psi \circ \Gamma_i) \circ \pi_i &= \Psi \circ (\Gamma_i \circ \pi_i) \\
&\overset{1.4}{=} \Psi \circ (\pi^* \circ \iota_i) \\
&= (\Psi \circ \pi^*) \circ \iota_i \\
&\overset{1.3}{=} \Phi \circ \iota_i \\
&\overset{1.2}{=} f_i \circ \pi_i.
\end{aligned}$$

Since π_i is an **epimorphism** $\implies \Psi \circ \Gamma_i = f_i$. Hence Γ_i is our maps into $\mathrm{Ab}(G_1 * G_2)$ and Ψ is the sought after map for our specific Y. Note here that Γ_i did not depend on f_i, but only on π^* and ι_i. Hence Γ_i correspond to ι_i in 1.1, and Ψ corresponds to f in 1.1.