Goldstein Mechanics Notes

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1 Survey of the Elementary Principles

1.1 Mechanics of a particle

The vector velocity $\mathbf{v}$ is defined as

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \tag{1.1}$$

where $\mathbf{r}$ is the radius vector of a particle (which can be thought of as the particle's position) The linear momentum $\mathbf{p}$ is defined as

$$\mathbf{p} = m\mathbf{v} \tag{1.2}$$

Newton's second law of motion states that in inertial frames, the motion of the particle can be described by the differential equation

$$\boxed{\mathbf{F} = \dot{\mathbf{p}}} \tag{1.3}$$

When the mass of the particle is constant, this reduces to

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{a}$$

where $\mathbf{a}$ is the vector acceleration and is defined by

$$\mathbf{a} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \tag{1.4}$$

From the equation $\mathbf{F} = \dot{\mathbf{p}}$, we can derive a useful conservation theorem:

Conservation Theorem for the Linear Momentum of a Particle: If the total force, $\mathbf{F}$, is zero, then $\dot{\mathbf{p}} = \mathbf{0}$ is zero and the linear momentum $\mathbf{p}$, is conserved.

The angular momentum of the particle about point O, denoted by $\mathbf{L}$ is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1.5}$$

where $\mathbf{r}$ is the radius vector **from** O to the particle. The torque about O is defined as

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} \tag{1.6}$$

we now take the time derivative of the angular momentum

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times m\mathbf{v}) = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v})$$

the first term vanishes, leaving us with

$$\boxed{\dot{\mathbf{L}} = \mathbf{r} \times \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = \mathbf{r} \times \mathbf{F} = \mathbf{N}} \tag{1.7}$$

this yields another conservation theorem:

Conservation Theorem for the Angular Momentum of a Particle: If the total torque, $\mathbf{N}$. is zero then $\dot{\mathbf{L}} = \mathbf{0}$, and the angular momentum $\mathbf{L}$ is conserved.

The work done by an external force $\mathbf{F}$ going from point 1 to point 2 is defined to be

$$W_{12} = \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{s} \tag{1.8}$$

For a particle with constant mass, the integral becomes

$$\int \mathbf{F} \cdot \mathrm{d}\mathbf{s} = m \int \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \mathbf{v}\mathrm{d}t = \frac{m}{2} \int \frac{\mathrm{d}}{\mathrm{d}t}(v^2)\mathrm{d}t$$

where we have used the vector identity $\frac{\mathrm{d}}{\mathrm{d}x}(f^2) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f} \cdot \mathbf{f}) = 2\mathbf{f} \cdot \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}$. We can now see

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2)$$

Defining $mv^2/2$ to be the kinetic energy of the particle T, we can rewrite this expression as

$$W_{12} = T_2 - T_1 \tag{1.9}$$

we now analyze a special case where the force $\mathbf{F}$ is conservative. That is,

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{s} = 0$$

according to Strokes' theorem,

$$\int (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{a} = 0$$

one possible solution for $\mathbf{F}$ is when $\mathbf{F}$ is expressed as a gradient of a scalar function. We choose this function to be

$$\mathbf{F} = -\nabla V(\mathbf{r})$$

it is obvious that this fits the original condition because the curl of a gradient is always zero.

According to the fundamental theorem for gradients, the potential change oer a differential path length we have

$$\mathbf{F} \cdot \mathrm{d}\mathbf{s} = -\mathrm{d}V$$

using the definition of the work, we can see that

$$W_{12} = V_1 - V_2 \tag{1.10}$$

for a conservative system.

combining equation 1.9 with 1.10, we have the result

$$\boxed{T_1 + V_1 = T_2 + V_2} \tag{1.11}$$

in words, this states

Energy Conservation Theorem for a Particle: If the force acting on a particle are conservative, then the total energy of the particle, T+V is conserved.

1.2 Mechanics of a system of particles

Let us analyze a system of particles, in which the particles experience external forces and internal forces. Newton's second law for the ith particle is written as

$$\sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)} = \dot{\mathbf{p}}_i$$

summed over all particles,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_i m_i \mathbf{r}_i = \sum_i \mathbf{F}_i^{(e)} + \sum_{i,j} \mathbf{F}_{ji}$$

where $\mathbf{F}_i j$ is the force acting on particle j by particle i. The second term vanishes because of **the weak law of action and reation** (which is the same as Newton's third law of motion), which states

$$\mathbf{F}_{ij} + \mathbf{F}_{ji} = \mathbf{0} \tag{1.12}$$

and because $\mathbf{F}_{i,i}$ is naturally zero. the center of mass is defined to be the average of the radii vectors of the particles, weighted in proportion to their mass.

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{M} \tag{1.13}$$

the equation of motion summed over all particles becomes

$$M \frac{\mathrm{d}^2 \mathbf{R}}{\mathrm{d}t^2} = \mathbf{F}^{(e)} \tag{1.14}$$

where $\mathbf{F}^{(e)}$ is the net external force acting on our system of particles.

The total linear momentum of the system of particles is

$$\boxed{\mathbf{P} = \sum m_i \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = M\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}} \tag{1.15}$$

equation 1.23 can be restated as a conservation theorem;

Conservatino Theorem for the Linear Momentum of a System of Particles: If the total external force is zero, the total linear momentum is conserved.

the total angular momentum can be obtained by finding the individual angular momenta and summing them over i.

$$\sum_i(\mathbf{r}_i \times \dot{\mathbf{p}}_i) = \sum_i \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_i \times \mathbf{p}_i) = \dot{\mathbf{L}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{\substack{i,j \\ i\neq j}} \mathbf{r}_i \times \mathbf{F}_{ji}$$

the last term can be written as

$$\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_J \times \mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji}$$

but since the vector $\mathbf{r}_i - \mathbf{r}_j$ is identical to the vector $\mathbf{r}_{ij}$ from j to i, the entire term vanishes. This is known as the **strong law of action and reaction**, which states that the internal forces between two particles are equal and opposite and lie along the line joining the particles. Using this,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{N}^{(e)} \tag{1.16}$$

corresponding to the equation is a conservation theorem:

Conservation Theorem for Total Angular Momentum: $\mathbf{L}$ is constant in time if the applied (external) torque is zero.

This conservation theorem assumes the weak law of action and reaction is valid. However, **it is not always valid** (system involving magnetic charges. for example). However, there is an alternate conservation law involving the angular momentum (for magnetic charges, the sum of the angular momentum and the electromagnetic angular momentum of the field are conserved)

To find an equation relating the total angular momentum to the center of mass as we did in equation 1.15, we start by finding the total angular momentum of the system with respect to the origin.

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

we define the relative position/velocity to be

$$\mathbf{r}_i = \mathbf{r}_i' + \mathbf{R} \tag{1.17}$$
$$\mathbf{v}_i = \mathbf{v}_i' + \mathbf{v} \tag{1.18}$$

where

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}$$

and

$$\mathbf{v}'_i = \frac{\mathrm{d}\mathbf{r}'_i}{\mathrm{d}t}$$

where $\mathbf{r}'_i$ is the position of the particle with respect to the center of mass and $\mathbf{v}'_i$ is the velocity of the particle in the center of mass reference frame.

plugging in equation 1.17 and 1.18, the total angular momentum becomes

$$\mathbf{L} = \sum_i (\mathbf{r}'_i + \mathbf{R}) \times m_i(\mathbf{v}'_i + \mathbf{v})$$

expanding this expression,

$$\mathbf{L} = \sum_i \mathbf{R} \times m_i \mathbf{v} + \sum_i \mathbf{r}'_i \times m_i \mathbf{v}'_i + \left(\sum_i m_i \mathbf{r}'_i\right) \times \mathbf{v} + \mathbf{R} \times \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i \mathbf{r}'_i$$

where we've used the fact $\mathbf{v}$ and $\mathbf{R}$ is constant when summing over $\mathbf{r}'_i$. We also notice the last two terms cancel out because

$$\sum_i m_i \mathbf{r}'_i = \sum_i m_i \mathbf{r}_i - \sum_i m_i \mathbf{R} = M\mathbf{R} - M\mathbf{R} = \mathbf{0}$$

therefore, the total angular momentum about O is

$$\boxed{\mathbf{L} = \mathbf{R} \times M\mathbf{v} + \sum_i \mathbf{r}'_i \times \mathbf{p}'_i} \tag{1.19}$$

in words, the total angular momentum is the angular momentum about the center of mass plus the angular momentum of the center of mass.

To calculate the energy of a system of particles, we start off by calculating the work done on a single particle:

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot \mathrm{d}\mathbf{s}_i = \sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot \mathrm{d}\mathbf{s}_i + \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \mathbf{F}_{ji} \cdot \mathrm{d}\mathbf{s}_i \tag{1.20}$$

the second term can be rewritten as

$$\sum_i \int_1^2 \mathbf{F}_i \cdot \mathrm{d}\mathbf{s} = \sum_i \int_1^2 m_i \dot{\mathbf{v}}_\mathbf{i} \cdot \mathbf{v}_i \mathrm{d}t = \sum_i \int_1^2 \mathrm{d}\left(\frac{1}{2} m_i v_i^2\right)$$

therefore, the work done can be written as the difference between the final and initial kinetic energies

$$W_{12} = T_2 - T_1$$

where the kinetic energy of the system is

$$T = \frac{1}{2}\sum_i m_i v_1^2 \tag{1.21}$$

in the center of mass frame of reference,

$$\begin{aligned}
T &= \frac{1}{2}\sum_i m_i(\mathbf{v}+\mathbf{v}'_i)\cdot(\mathbf{v}+\mathbf{v}'_i) \\
&= \frac{1}{2}\sum_i m_i v^2 + \frac{1}{2}\sum_i m_i v_i^2 + \mathbf{v}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i m_i \mathbf{r}'_i\right)
\end{aligned}$$

since the last term vanishes,

$$\boxed{T = \frac{1}{2}Mv^2 + \frac{1}{2}\sum_i m_i v_i'^2} \tag{1.22}$$

when the external forces are conservative, the first term on the right side ($\sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot \mathrm{d}\mathbf{s}_i$) becomes[1]

$$-\sum_i \int_1^2 \nabla_i V_i \cdot \mathrm{d}\mathbf{s}_i = -\sum_i V_i \Big|_1^2$$

where we have used the fundamental theorem for gradients.

to satisfy the strong law of action and reaction, we define a new function of the distance between the particles, that is $V_{ij} = V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$. The forces acting on the particles are equal and opposite:

$$\mathbf{F}_{ji} = -\nabla_i V_{ij} = \nabla_j V_{ij} = -\mathbf{F}_{ij} \tag{1.23}$$

because

$$\begin{aligned}
\frac{\partial V_{ij}}{\partial \mathbf{r}_i} &= \frac{\partial V_{ij}}{\partial(\mathbf{r_i}-\mathbf{r_j})}\frac{\partial(\mathbf{r_i}-\mathbf{r_j})}{\partial \mathbf{r}_i} = (1-\delta_{ji})\nabla_{i-j}V_{ij} \\
\frac{\partial V_{ij}}{\partial \mathbf{r}_j} &= \frac{\partial V_{ij}}{\partial(\mathbf{r_i}-\mathbf{r_j})}\frac{\partial(\mathbf{r_i}-\mathbf{r_j})}{\partial \mathbf{r}_j} = (\delta_{ij}-1)\nabla_{i-j}V_{ij}
\end{aligned}$$

and

$$\nabla_{\mathbf{r_i}} r_j = \nabla r_j \nabla r_i = \nabla_{\mathbf{r}_j} r_i$$

and lie along the line joining the two particles,

$$\nabla V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) = (\mathbf{r}_i - \mathbf{r}_j) f \tag{1.24}$$

[1] here we use spacial gradients because the author defined the force field as the spacial gradient of the potential. As far as I know, they are similar to the gradient as $\nabla_{\mathbf{r}} V = \nabla V \cdot \nabla |r| = \nabla V$, and the equations still make sense if the spacial gradients are replaced with gradients.

since
$$\nabla V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) = \frac{\partial V_{ij}}{\partial(|\mathbf{r}_i - \mathbf{r}_j|)} \nabla(|\mathbf{r}_i - \mathbf{r}_j|) = f \cdot \frac{\mathbf{r_i} - \mathbf{r_j}}{|\mathbf{r_i} - \mathbf{r_j}|}$$
the same result can be achieved using the definition of a conservative force setting the direction of the force radial.

the second term in equation 1.20 ($\sum_{i.j} \int_1^2 \mathbf{F}_{ji} \cdot \mathrm{d}\mathbf{s}_i$) can be rewritten as a sum over pairs of particles, in the form

$$-\int_1^2 (\nabla_i V_{ij} \cdot \mathrm{d}\mathbf{s}_i + \nabla_j V_{ij} \cdot \mathrm{d}\mathbf{s}_j) = -\int_1^2 \nabla_{ij} V_{ij} \cdot \mathrm{d}\mathbf{r}_{ij}$$

where $\mathbf{r_i} - \mathbf{r_j} = \mathbf{r_{ij}}$ and ∇_{ij} is the gradient with respect to $\mathbf{r}_{ij}$ then the sum over all particles becomes

$$-\frac{1}{2} \sum_{i,j} \int_1^2 \nabla_{ij} V_{ij} \cdot \mathrm{d}\mathbf{r}_{ij} = -\frac{1}{2} \sum_{i,j} V_{ij} \Big|_1^2$$

we can now define a total potential energy of the system,

$$\boxed{V = \sum_i V_i + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} V_{ij}} \tag{1.25}$$

1.3 Constraints

types of constraints:

- Holonomic: when there exists a curve particle is confined to: a bead moving on a wire
- Nonholonomic: when there exists a region particle is confined in. Example: the walls of a gas container
- Rhenomous: when the equations of constraint contain time as an explicit variable
- Scleronomous: when the equations of constraint are time-independent

constraints introduce two types of difficulties in the solution of mechanical problems.

First, the coordinates are no longer all independent.

In the case of holonomic constraints, this is solved by using **generalized coordinates**: that is, $3N - k$ independent variables for a system of N particles and k equations of constraints. One example of generalized coordinates is the angles to the vertical in a double pendulum system.

For nonholonomic constraints, there is no general way to solve for the equations of constraints, making it significantly harder.

Second, having constraints on the equations of the motion implies the existence of unknown forces. To surmount this difficulty, we can formulate the mechanics that the forces of constraint disappear.

1.4 D' Alembert's principle and Lagrange's equations

A **virtual displacement** refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates, consistent with the forces and constraints imposed on the system at the given instant t. If the system is in equilibrium, $\mathbf{F_i} = 0$ for each particle, and the dot product with the virtual displacement $\delta\mathbf{r_i}$, summed over all particles, has to be zero.

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0$$

since the total force is composed of the applied force, $\mathbf{F}_i^{(a)}$, and the force of constraint, $\mathbf{f}_i$,

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i + \sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0$$

in systems where the net virtual work of the forces of constraint is zero (for example, if a particle is confined to move on a surface, the force of constraint is perpendicular to the surface, and the virtual displacement is parallel to the surface. This is **no longer true if sliding friction forces are present**.) Therefore,

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i = 0 \tag{1.26}$$

This is known as the **principle of virtual work**. However, this form is not very useful since it doesn't say anything about the q_i.

The equation of motion can be rewritten as

$$\mathbf{F}_i - \dot{\mathbf{p}}_\mathbf{i} = 0$$

Multiplying by the virtual displacement and summing over i particles,

$$\sum_i (\mathbf{F_i} - \dot{\mathbf{p}}_\mathbf{i}) \cdot \delta\mathbf{r}_i = 0$$

decomposing the net force into applied forces and forces of constraint and setting the work done by forces of constraint to zero,

$$\sum_i (\mathbf{F_i^{(a)}} - \dot{\mathbf{p}}_\mathbf{i}) \cdot \delta\mathbf{r_i} = 0 \tag{1.27}$$

This is called **D'Alembert's principle**.

we start rewriting this equation in terms of the coordinates q_n by the transformation equations,

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \cdots, q_n, t)$$

using the chain rules of partial differentiation, $\mathbf{v}_i$ is expressed as

$$\mathbf{v}_i \equiv \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \tag{1.28}$$

similarly, the virtual displacement can be written as

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j \tag{1.29}$$

In terms of the generalized coordinates, the virtual work becomes

$$\begin{aligned}\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i &= \sum_{i,j} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j \\ &= \sum_j Q_j \delta q_j\end{aligned}$$

where Q_j is called the components of the **generalized force**, defined as

$$\boxed{Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}} \tag{1.30}$$

the second term in equation 1.27 can be written as

$$\sum_i \dot{\mathbf{p}}_\mathbf{i} \cdot \delta \mathbf{r}_\mathbf{i} = \sum_i m_i \ddot{\mathbf{r}}_\mathbf{i} \cdot \delta \mathbf{r}_\mathbf{i}$$

using equation 1.29,

$$\sum_{i,j} m_i \ddot{\mathbf{r}}_\mathbf{i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j$$

consider now the relation

$$\sum_i m_i \ddot{\mathbf{r}}_\mathbf{i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \dot{\mathbf{r}}_\mathbf{i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\right) - m_i \dot{\mathbf{r}}_\mathbf{i} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right)\right] \tag{1.31}$$

the last term of equation 1.31 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\mathbf{r}}_\mathbf{i}}{\partial q_j} = \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t} = \frac{\partial \mathbf{v}_i}{\partial q_j}$$

we can also see that

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{1.32}$$

which can be obtained by taking the partial derivative of equation 1.28. Substituting these changes in equation 1.31 leads to the result that

$$\sum_i m_i \ddot{\mathbf{r}}_{\mathbf{i}} \cdot \frac{\partial r_i}{\partial q_j} = \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right]$$

and the second term on the left hand side of equation 1.27 can be expanded into

$$\sum_j \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i^2 v_i^2 \right) - Q_j \right\} \delta q_j$$

since $\sum_i \frac{1}{2} m_i v_i^2$ is the kinetic energy of the system, D'Alembert's principle becomes

$$\sum_j \left\{ \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0 \tag{1.33}$$

If the constraints are holonomic, it is possible to find independent coordinates that contain the constraint conditions implicitly. Therefore, the only way 1.33 to hold is for the individual coefficients to vanish:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \tag{1.34}$$

When the forces are derivable from a scalar potential function V, the generalized forces can be written as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = - \sum_i \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

which is the same as the partial derivative of -V with respect to q_j:

$$Q_j = -\frac{\partial V}{\partial q_i}$$

If the potential V does not depend on the generalized velocities, equation 1.34 can be rewritten as

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0} \tag{1.35}$$

where L, the Lagrangian, is

$$L = T - V \tag{1.36}$$

1.5 Velocity-Dependent potentials and the dissipitation function

For a velocity-dependent potential, Lagrange's equations can be put in the form 1.35 if the generalized forces are obtained from the equation

$$Q_j = -\frac{\partial U}{\partial q_i} + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial U}{\partial \dot{q}_j} \right) \tag{1.37}$$

Here U may be called a **generalized coordinates**. This kind of field applies to the electromagnetic forces on moving charges.
Both **E** and **B** are continuous functions of time and position derivable from a scalar potential and a vector potential:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

The generalized potential of the system is then

$$U = q\phi - q\mathbf{A}\cdot\mathbf{v}$$

so the Lagrangian is

$$L = \frac{1}{2}mv^2 - q\phi + q\mathbf{A}\cdot\mathbf{v}$$

considering just the x-component of Lagrange's equations gives

$$m\ddot{x} = q\left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right) - q\left(\frac{\partial \phi}{\partial x} + \frac{\mathrm{d}A_x}{\mathrm{d}t}\right)$$

The total time derivative of A_x is

$$\begin{aligned}\frac{\mathrm{d}A_x}{\mathrm{d}t} &= \frac{\partial A_x}{\partial t} + \mathbf{v}\cdot\nabla A_x \\ &= \frac{\partial A_x}{\partial t} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}\end{aligned}$$

The relation between the magnetic field and its vector potential gives

$$(\mathbf{v}\times\mathbf{B})_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)$$

combining these expressions gives

$$m\ddot{x} = q[\mathbf{E}_x + (\mathbf{v}\times\mathbf{B})_x] \tag{1.38}$$

Now we turn our attention to cases when there are forces not arising from a potential. In this case, we rewrite Lagrange's equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

When the frictional force is proportional to the velocity of the particle, frictional forces can be derived in terms of a function $\mathcal{F}$, also known as **Rayleigh's dissipiation function**, defined as

$$\mathcal{F} = \frac{1}{2}\sum_i\left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right) \tag{1.39}$$

From this, it is clear that the friction force is the gradient of the dissipation function.

$$\mathbf{F}_f = -\nabla_v \mathcal{F}$$

The work done against friction is

$$\mathrm{d}W_f = -\mathbf{F}_f \cdot \mathrm{d}\mathbf{r} = -\mathbf{F}_f \cdot \mathbf{v}\mathrm{d}t = \left(k_x v_x^2 + k_y v_y^2 + k_z v_z^2\right)\mathrm{d}t = 2\mathcal{F}\mathrm{d}t$$

The generalized force resulting from the force of friction is given by

$$\begin{aligned} Q_j = \sum_i \mathbf{F}_{f_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} &= -\sum \nabla_v \mathcal{F} \cdot \frac{\partial \mathbf{r}_i}{\partial q_i} \\ &= -\sum \nabla_v \mathcal{F} \cdot \frac{\partial \dot{\mathbf{r}}_\mathbf{i}}{\partial \dot{q}_j} \\ &= -\frac{\partial \mathcal{F}}{\partial \dot{q}_j} \end{aligned}$$

The Lagrange equations with dissipation become

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0 \tag{1.40}$$

1.6 Simple applications of the lagrangian formulation

We have greatly reduced the work by eliminating the forces of constraint from the equations of motion, and replacing the many vector forces and accelerations with two scalar functions, T and V. A straightforward procedure can now be established for all problems of mechanics to which the Lagrangian formulation is applicable. We only have to apply the transformation equations and find the Lagrangian.

Motion of one particle: using polar coordinates. Here, the equations of transformations are simply

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

using equation 1.28, the velocities are given by

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned}$$

The kinetic energy then reduces to

$$T = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2\right]$$

since there are two generalized coordinates, there are two Lagrange equations: plugging them into equation 1.35, we get

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 &= F_r \\ \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} &= rF_\theta \end{aligned}$$

the second equation is identical to the torque equation.

1.7 Derivations

1. Show that for a single particle with constant mass,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{v}$$

and if the mass varies time, the corresponding equation is

$$\frac{\mathrm{d}(mT)}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{p}$$

Using the definition of the kinetic energy for a single particle,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m(\mathbf{v}\cdot\mathbf{v})\right) = \frac{1}{2}m\left(2\mathbf{v}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v}$$

and for the variable mass case,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m^2v^2\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\cdot\mathbf{p}) = \dot{\mathbf{p}}\cdot\mathbf{p} = \mathbf{F}\cdot\mathbf{p}$$

2. Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2$$

using $\mathbf{r}_{ij} = \mathbf{r_i} - \mathbf{r_j}$, we see that

$$\begin{aligned}
M^2R^2 = M\mathbf{R}\cdot M\mathbf{R} &= \sum_i m_i\mathbf{r}_i \cdot \sum_j m_j\mathbf{r}_j \\
&= \sum_{i,j} m_i m_j \mathbf{r_i}\cdot\mathbf{r_j} \\
&= \sum_{i,j} m_i m_j \mathbf{r_i}\cdot(\mathbf{r_i} - \mathbf{r_{ij}}) \\
&= \sum_{i,j} m_i m_j r_i^2 - \sum_{i,j} m_i m_j \mathbf{r_i}\cdot\mathbf{r_{ij}} \\
&= \sum_j m_j \sum_i m_i r_i^2 - \sum_{i,j} m_i m_j \mathbf{r_{ij}}\cdot\frac{1}{2}((\mathbf{r_i}+\mathbf{r_j}) + \mathbf{r_{ij}}) \\
&= M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j(\mathbf{r_{ij}}\cdot\mathbf{r_{ij}}) - \frac{1}{2}\sum_{i,j} m_i m_j (r_i^2 - r_j^2) \\
&= M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2
\end{aligned}$$

3. Suppose a system of two particles is known to obey the equations of motion

$$M\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2} = \sum_i \mathbf{F}_i^{(e)} \equiv \mathbf{F}^{(e)}$$

and

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{N}^{(e)}$$

from the equations of the motion show that the internal forces satisfy both the weak and the strong law of action.

the equations of motion for the system of particles is

$$\begin{aligned}
&M\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2} = \mathbf{F}_i^{(e)} + \mathbf{F}_j^{(e)} \\
&\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_i \times m_i\mathbf{v}_i + \mathbf{r}_j \times m_j\mathbf{v}_j) = \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \mathbf{r}_j \times \mathbf{F}_j^{(e)}
\end{aligned}$$

the equations of motion for the individual particles is

$$\begin{aligned}
&m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i^{(e)} + \mathbf{F}_{ji} \\
&m_j\ddot{\mathbf{r}}_j = \mathbf{F}_j^{(e)} + \mathbf{F}_{ij} \\
&\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_i \times m_i\mathbf{v}_i) = \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \mathbf{r}_i \times \mathbf{F}_{ji} \\
&\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_j \times m_j\mathbf{v}_j) = \mathbf{r}_j \times \mathbf{F}_j^{(e)} + \mathbf{r}_j \times \mathbf{F}_{ij}
\end{aligned}$$

solving for the external forces and plugging them into the equations of motion for the system,

$$\begin{aligned}
&M\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2} = m_i\ddot{\mathbf{r}}_i + m_j\ddot{\mathbf{r}}_j - (\mathbf{F}_{ij} + \mathbf{F}_{ji}) \\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}(m_i\mathbf{r}_i + m_j\mathbf{r}_j) - (m_i\ddot{\mathbf{r}}_i + m_j\ddot{\mathbf{r}}_j) = (\mathbf{F}_{ij} + \mathbf{F}_{ji}) \\
&\mathbf{F}_{ij} + \mathbf{F}_{ji} = 0
\end{aligned}$$

which is identical to the weak law of action and reaction. The second second equation of motion for the system of particles is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_i \times m_i\mathbf{v}_i + \mathbf{r}_j \times m_j\mathbf{v}_j) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_i \times m_i\mathbf{v}_i) - \mathbf{r}_i \times \mathbf{F}_{ji} + \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_j \times m_j\mathbf{v}_j) - \mathbf{r}_j \times \mathbf{F}_{ij} \\
&\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = 0
\end{aligned}$$

which is identical to the strong law of action and reaction.

4. The equations of constraint for the rolling disk,

$$\begin{aligned}
\mathrm{d}x - a\sin\theta\mathrm{d}\phi = 0 \\
\mathrm{d}y + a\cos\theta\mathrm{d}\phi = 0
\end{aligned}$$

, are special cases of general linear differential equations of constraint of the form

$$\sum_{i=1}^{n} g_i(x_i, \cdots, x_n)\mathrm{d}x_i = 0$$

a constraint condition of this type is holonomic iff an integrating function can be found that turns it into an exact differential. Clearly the function must be such that

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial(fg_j)}{\partial x_i} \tag{1.41}$$

it is obvious that the coordinates of the system are x, y, θ, ϕ. Using equation 1.41, we get

$$\begin{aligned}
&\frac{\partial f}{\partial \theta} = 0 \\
&\frac{\partial f}{\partial \phi} = -a\frac{\partial f \sin\theta}{\partial x} \\
&a\frac{\partial f \sin\theta}{\partial \theta} = 0
\end{aligned}$$

the first equation means that f does not depend on θ, meaning $f = f(x, \phi)$. In this case, the last equation doesn't always hold true, meaning no integrating factor can be found for the equations of constraint. Similarly, no such integrating factor can be found for the second equation, since replacing the x with y doesn't affect our method

5. Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane. Show that there are two nonholonomic equations of constraint,

$$\begin{aligned}
\cos\theta\mathrm{d}x + \sin\theta\mathrm{d}y &= 0 \\
\sin\theta\mathrm{d}x - \cos\theta\mathrm{d}y &= \frac{1}{2}a(\mathrm{d}\phi + \mathrm{d}\phi')
\end{aligned}$$

and one holonomic equation of constraint,

$$\theta = C - \frac{a}{b}(\phi - \phi')$$

the nonslipping condition states

$$\begin{aligned}
v &= a\dot{\phi} \\
v' &= a\dot{\phi}'
\end{aligned}$$

In the center of axle's frame of reference, the velocities of the wheels must be equal and opposite, since it is the center of mass. Symbolically,

$$\begin{aligned}
\mathbf{v} - \mathbf{v}_m &= -(\mathbf{v}' - \mathbf{v}_m) \\
\mathbf{v}_m &= \frac{1}{2}(\mathbf{v} + \mathbf{v}')
\end{aligned}$$

all the velocities are perpendicular to θ, so we can convert from generalized coordinates to the coordinates of the center of the axle.

$$\dot{x} = |\mathbf{v}_m| \sin\theta$$
$$\dot{y} = -|\mathbf{v}_m| \cos\theta$$

since the wheels are pointing in the same direction, we simply add the velocities when converting from the velocity of the center to the velocity of the wheels.

$$\dot{x} = \frac{1}{2} a \sin\theta(\dot{\phi} + \dot{\phi}')$$
$$\dot{y} = -\frac{1}{2} a \cos\theta(\dot{\phi} + \dot{\phi}')$$

it is obvious that

$$\cos\theta \mathrm{d}x + \sin\theta \mathrm{d}y = 0$$
$$\sin\theta \mathrm{d}x - \cos\theta \mathrm{d}y = \frac{1}{2} a(\mathrm{d}\phi + \mathrm{d}\phi')$$

and in the frame of reference of the center, the rate at which the axle is spinning has to match up with the wheels' speed.

$$\dot{\theta}\frac{b}{2} = (v' - v_m) = \frac{1}{2}(v' - v)$$

using the nonslipping condition again,

$$\mathrm{d}\theta = \frac{a}{b}(\mathrm{d}\phi' - \mathrm{d}\phi)$$

integrating and rearranging,

$$\theta = C - \frac{a}{b}(\phi - \phi')$$

6. A particle moves in the xy plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time $f(t)$. Show that for $f(t)$ is differentiable, but otherwise arbitrary, the constraint is nonholonomic. (abscissa is simply the x value)

Let θ be the angle between the x axis and the line connecting $(f(t), 0)$ and the particle and r be the distance to it. Writing down the transformation equations and taking the derivative with respect to time,

$$x = f + r\cos\theta$$
$$y = r\sin\theta$$
$$\dot{x} = \dot{f} + \dot{r}\cos\theta - r\sin\theta\dot{\theta}$$
$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$$

lastly, making θ point towards the particle,

$$\tan\theta = \dot{y}/\dot{x}$$

solving the equations gives us

$$\begin{aligned}
&\dot{r}\sin\theta + r\cos\theta\dot{\theta} = \tan\theta\dot{f} + \dot{r}\sin\theta - r\frac{\sin^2\theta}{\cos\theta}\dot{\theta} \\
&\tan\theta\dot{f} = r\dot{\theta}\frac{\cos^2\theta + \sin^2\theta}{\cos\theta} = r\dot{\theta}\sec\theta \\
&\sin\theta \mathrm{d}f = r\mathrm{d}\theta
\end{aligned}$$

it is evident this equation is not an exact differential because f is independent of r.

7. Show that Lagrange's equations in the form of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

can also be written as

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j$$

These are sometimes known as the Nielsen form of the Lagrange equations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i\left(m_i\mathbf{v}_i\cdot\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}\right) - \sum_i\left(m_i\mathbf{v}_i\cdot\frac{\partial \mathbf{v}_i}{\partial q_j}\right) \\
&= \sum_i m_i\left(\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t}\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} + \mathbf{v}_i\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} - \mathbf{v}_i\frac{\partial \mathbf{v}_i}{\partial q_j}\right)
\end{aligned}$$

since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \frac{\partial \mathbf{v}_i}{\partial q_j}$$

the last two terms cancel out, leaving us with

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \sum_i m_i\left(\frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t}\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}\right)$$