Supplemental Material

(Optimization problems (19), (20), (22), and (24) in [1])

Takao Murakami, AIST (April 10, 2017)

In this material, we explain how to solve the optimization problems in (19), (20), (22), and (24) in [1] in details. They can be expressed in the following general form:

$$\begin{aligned}
&\operatorname*{arg\,min}_{\mathbf{z}\geq 0} \sum_{i=1}^{d} (\mu_i z_i^2 + \nu_i z_i + \xi_i) + \beta ||\mathbf{z}||_q && (25)\\
=&\operatorname*{arg\,min}_{\mathbf{z}\geq 0} \sum_{i=1}^{d} \mu_i (z_i - \lambda_i)^2 + \beta ||\mathbf{z}||_q, && (26)
\end{aligned}$$

where $\mu_i \in \mathbb{R}_+$, $\nu_i, \xi_i \in \mathbb{R}$, $\mathbf{z} = (z_1, \cdots, z_d)^T \in \mathbb{R}^d$, $\lambda_i = -\frac{\nu_i}{2\mu_i}$, and $\mu_i > 0$. For example, (19) in [1] can be written as follows:

$$\begin{aligned}
&\operatorname*{arg\,min}_{\tilde{\mathbf{u}}_{g,k}^{(a)}\geq 0} \sum_{\mathbb{D}_g^2} (a_{n,i,j} - \hat{a}_{n,i,j})^2 + \beta ||\tilde{\mathbf{u}}_{g,k}^{(a)}||_q && (27)\\
=&\operatorname*{arg\,min}_{\tilde{\mathbf{u}}_{g,k}^{(a)}\geq 0} \sum_{i\in M_g} \sum_{\mathbb{D}_i^2} (\gamma_i - u_{i,k}^{(a)} v_{j,k}^{(a)})^2 + \beta ||\tilde{\mathbf{u}}_{g,k}^{(a)}||_q, && (28)
\end{aligned}$$

where $\mathbb{D}_i^2 = \{(n,j) | \sum_{j'=1}^{M} a_{n,i,j'} \geq 1\}$ and

$$\gamma_i = a_{n,i,j} - \hat{a}_{n,i,j} + u_{i,k}^{(a)} v_{j,k}^{(a)} \tag{29}$$

(note that γ_i does not depend on $u_{i,k}^{(a)}$). Thus, (19) in [1] can be expressed in the form of (25), where

$$\begin{aligned}
d =& M_g && (30)\\
z_i =& u_{i,k}^{(a)} && (31)\\
\mu_i =& \sum_{\mathbb{D}_i^2} (v_{j,k}^{(a)})^2 \quad (> 0) && (32)\\
\nu_i =& -2 \sum_{\mathbb{D}_i^2} \gamma_i v_{j,k}^{(a)} && (33)\\
\xi_i =& \sum_{\mathbb{D}_i^2} \gamma_i^2. && (34)
\end{aligned}$$

Similarly, (20), (22), and (24) can also be expressed in the form of (25).

Therefore, we explain how to solve (26). Kim *et al.* [2] considers the case when $\mu_i = 1/2$ $(1 \leq i \leq d)$. We show that (26) can be solved as well in the general case when $\mu_i > 0$

$(1 \leq i \leq d)$. Let $\mathbf{z}^* = (z_1^*, \cdots, z_d^*)^T$ be the minimizer of (26). If $\lambda_i < 0$, then $z_i^* = 0$. Thus, (26) is equivalent to the following optimization problem:

$$\arg\min_{\mathbf{z} \geq 0} \sum_{i=1}^{d} \mu_i (z_i - [\lambda_i]_+)^2 + \beta ||\mathbf{z}||_q, \tag{35}$$

where $[x]_+ = \max(x, 0)$. We then consider the following optimization problem:

$$\arg\min_{\mathbf{z}} \sum_{i=1}^{d} \mu_i (z_i - [\lambda_i]_+)^2 + \beta ||\mathbf{z}||_q. \tag{36}$$

Since the minimizer of (36) satisfies nonnegativity (i.e., $z_i \geq 0$), (35) is equivalent to (36). (36) can solved via Fenchel duality. More specifically, the following problem is dual to (36).

$$\arg\min_{\mathbf{s}} \sum_{i=1}^{d} (s_i - 2\mu_i [\lambda_i]_+)^2 \quad \text{s.t.} \quad ||\mathbf{s}||_{q^*} \leq \beta, \tag{37}$$

where $\mathbf{s} = (s_1, \cdots, s_d)^T \in \mathbb{R}^d$ is a vector such that

$$z_i = [\lambda_i]_+ - \frac{s_i}{2\mu_i}. \tag{38}$$

$|| \cdot ||_{q^*}$ is the dual norm of $|| \cdot ||_q$. For example, if $q = 2$, then $q^* = 2$. If $q = \infty$, then $q^* = 1$.

The optimization problem (26) can be solved by computing the minimizer $\mathbf{s}^* = (s_1^*, \cdots, s_d^*)^T$ of (37), and then computing the minimizer $\mathbf{z}^* = (z_1^*, \cdots, z_d^*)^T$ of (26) by (38). For example, if $q = 2$, (37) becomes

$$\arg\min_{\mathbf{s}} \sum_{i=1}^{d} (s_i - 2\mu_i [\lambda_i]_+)^2 \quad \text{s.t.} \quad ||\mathbf{s}||_2 \leq \beta. \tag{39}$$

This can be easily solved by normalization as follows. Let $\mathbf{s}' = (s'_1, \cdots, s'_d)^T$ be a vector such that $s'_i = 2\mu_i [\lambda_i]_+$. If $||\mathbf{s}'||_2 \leq \beta$, then $\mathbf{s}^* = \mathbf{s}'$ (and $\mathbf{z}^* = (0, \cdots, 0)^T$, which is a group sparse solution). If $||\mathbf{s}'||_2 > \beta$, then $s_i^* = (\beta / ||\mathbf{s}'||_2) s'_i$.

References

1. T. Murakami, A. Kanemura, and H. Hino, “Group Sparsity Tensor Factorization for Re-identification of Open Mobility Traces,” IEEE Trans. Information Forensics and Security, Vol.12, No.3, pp.689-704, 2017.

2. J. Kim, R. D. Monteiro, and H. Park, “Group sparsity in nonnegative matrix factorization,” in *Proc. SDM’12*, pp. 851–862, 2012.