Multivariate Bernstein–form polynomials

Jakob Berchtold Irina Voiculescu Adrian Bowyer

{J.Berchtold, I.D.Voiculescu, A.Bowyer}@bath.ac.uk

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Abstract

This report gives algorithms for converting multivariate power–form polynomials into the equivalent Bernstein–basis form. Properties of the Bernstein representation are enounced. Although the univariate conversion formula is well–known, its generalisation is not immediate. Both two– and three–variate examples are given. The resulting Bernstein form of polynomials are more complicated than the power form ones. The former are computed with the intent of using this representation (instead of the latter) in a CSG modeller. Further work needs to be carried out in order to find out whether this replacement would present any advantage.

1 Introduction

The field of computer aided geometric design involves the design of curves and surfaces which are commonly represented by polynomials. For practical and historical reasons these are usually parametric polynomials, but the most familiar and, in certain respects, the simplest representation of a polynomial is the implicit power form. A polynomial of degree $n \in \mathcal{N}$ in the variable x is defined by:

$$p(x) = \sum_{k=0}^{n} a_k x^k \tag{1}$$

where $a_k \in \mathcal{R}$.

Another possible representation of polynomials is in terms of the Bernstein basis. The Bernstein basis has – from a geometrical point of view – some interesting advantages.

1.1 Definition

For a given $n \in \mathcal{N}$ the corresponding Bernstein functions of degree n on the unit interval $[0, 1]$ are defined by:

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad \forall x \in [0,1], \qquad k = 0, 1, \ldots, n. \tag{2}$$

The Bernstein functions $B(x)$ are linearly independent and form a basis of the n–degree polynomial space.

Sometimes it is more convenient to consider the unit interval $[0, 1]$ as the area of interest. However, this is not a real restriction as the generalisation to an arbitrary interval is quite easy. On the interval $[\underline{x}, \overline{x}]$ with $\underline{x}, \overline{x} \in \mathcal{R}$ the Bernstein basis of degree $n \in \mathcal{N}$ is defined by

$$B_k^n(x) = \binom{n}{k} \frac{(x-\underline{x})^k (\overline{x}-x)^{n-k}}{(\overline{x}-\underline{x})^n}, \qquad k = 0, 1, \ldots, n \tag{3}$$

An arbitrary polynomial can be expressed in terms of the Bernstein basis. A Bernstein–form polynomial of degree n has the following form:

$$p(x) = \sum_{k=0}^{n} m_k^n B_k^n(x)$$

where m_k^n are the Bernstein coefficients corresponding to the degree-n bases.

Compared to the power form, the Bernstein form has some special geometric properties, some of which are mentioned in the next section.

1.2 Properties

In this section only an overview of some properties of Bernstein polynomials is given. More detailed information can be found in the work of Bowyer and Woodwark [2] and Farouki and Rajan [3, 4].

1. Bernstein polynomials are invariant under affine transformations.
2. A recursive generation of the n^{th} order basis from the $(n-1)^{th}$ order basis is possible. For the Bernstein basis on the unit interval $[0, 1]$ the recursive generation is defined by

   $$B_k^n(x) = (1-x)B_k^{n-1}(x) + xB_{k-1}^{n-1}(x), \qquad k = 0, 1, \ldots, n.$$

3. All the terms of the Bernstein basis are positive on the interval where they are defined and their sum equals 1:

   $$B_k^n(x) \geq 0, \quad k = 0, 1, \ldots, n \quad \textit{and} \quad \sum_{k=0}^{n} B_k^n(x) = 1.$$

   Farouki and Rajan [4] point out that this gives a bound on the polynomial $p(x)$:

   $$\min_{0 \leq k \leq n} m_k^n \leq p(x) \leq \max_{0 \leq k \leq n} m_k^n.$$

   A tighter bound is given by the convex hull which is determined by the Bernstein coefficients. In two dimensions the convex hull is given by a polygon; in three dimensions it is represented by a convex polyhedron.

4. A polynomial $p(x)$ of degree n can be represented in terms of the Bernstein basis of degree $n+1$ by a procedure known as degree elevation. If m_k^n are the Bernstein coefficients in the degree-n basis, the coefficients m_k^{n+1} in the next higher basis are given by:

   $$m_k^{n+1} = \omega_k m_{k-1}^n + (1-\omega_k) m_k^n \qquad \textit{where} \quad \omega_k = \frac{k}{n+1}$$

   for $k = 1, 2, \ldots, n$, and $m_0^{n+1} \equiv m_0^n$, $m_{n+1}^{n+1} \equiv m_n^n$.

5. Farouki and Rajan [4, 5] show that a Bernstein–form polynomial is always better conditioned[1] than a polynomial in the power form for the determination of simple roots on the unit interval $[0, 1]$. Also for roots on an arbitrary interval $[\underline{x}, \overline{x}]$, the root condition number is smaller in the Bernstein basis on this interval.

6. The Bernstein basis has a better numerical stability than the power form. Spencer in [8] gave the following definition: *Numerical stability is a property of an algorithm which measures its propensity for generation and propagating roundoff error and inherent errors in the input data.* However, if the conversion between the two forms is done frequently numerical instabilities can be re–introduced and the property is obviously lost.

   One way to measure this property is to perturb the coefficients of two representations of the same polynomial. Then for a polynomial in Bernstein form the value of this polynomial at a point has a smaller error bound than the error bound generated by the power form, often by many orders of magnitude.

[1]For a polynomial $p(x)$ in Bernstein from the root condition number is smaller than in the power form.

2 The basis conversion

Most human and computer algebra systems use the power form representation of a polynomial, which was given in Equation 1. When the Bernstein form is used a conversion between the two bases is often necessary, through it should not be used with impunity as, as was mentioned above, conversion re–introduces the numerical instabilities that the Bernstein basis was intended to remove. Whereas the conversion can be done easily in the univariate case, the multivariate cases is not that obvious.

2.1 The univariate case

2.1.1 The unit interval $[0, 1]$

Consider a polynomial $p(x)$ of degree $n \in \mathcal{N}$. Its equivalent power and Bernstein forms are:

$$p(x) = \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} m_k^n B_k^n(x). \tag{4}$$

Each set of coefficients (a_k or m_k^n respectively) can be computed from the others. For example:

$$\begin{aligned} a_k &= \sum_{j=0}^{k} (-1)^{(k-j)} \binom{n}{k} \binom{k}{j} m_j^n && (5) \\ m_k^n &= \sum_{j=0}^{k} \frac{\binom{k}{j}}{\binom{n}{j}} a_j . && (6) \end{aligned}$$

Formula 6 provides the conversion of a univariate polynomial from its power form into the Bernstein form. Here is an example:

$$p(x) = x^3 - 5x^2 + 2x + 4 = 4B_0^3 + \frac{14}{3}B_1^3 + \frac{11}{3}B_2^3 + 2B_3^3$$

where $B_0^3 = (1-x)^3$, $B_1^3 = 3x(1-x)^2$, $B_2^3 = 3x^2(1-x)$ and $B_3^3 = x^3$.

Consider the same polynomial as Equation 4. The calculation done above can also be written as a matrix multiplication. In the following, a formula equivalent to Equation 6 will determine the Bernstein coefficients matrix M in terms of the power form coefficients matrix A.

Two other ways of writing the polynomial $p(x)$ are:

$$\begin{aligned} p(x) &= \sum_{k=0}^{n} a_k x^k = \begin{pmatrix} 1 & x & \ldots & x^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = XA \\ p(x) &= \sum_{k=0}^{n} m_k^n B_k^n(x) = \begin{pmatrix} B_0^n(x) & B_1^n(x) & \ldots & B_n^n(x) \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_n \end{pmatrix} = B_X M. \end{aligned}$$

Rewriting the vector B_X of Bernstein polynomials in terms of matrix multiplication:

$$B_X = \begin{pmatrix} B_0^n(x) & \ldots & B_n^n(x) \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} \binom{n}{0}(1-x)^n & \dots & \binom{n}{n}x^n \end{pmatrix} \\
&= \begin{pmatrix} \binom{n}{0}\left(1+\binom{n}{1}(-x)+\dots+\binom{n}{n}(-x)^n\right) & \dots & \binom{n}{n}x^n \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} 1 & x & \dots & x^n \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} 1 & & & O \\ \binom{n}{0}\binom{n}{1}(-1)^1 & \binom{n}{1}\binom{n-1}{0}(-1)^0 & & \\ \vdots & & \ddots & \\ \binom{n}{0}\binom{n}{n}(-1)^n & \binom{n}{1}\binom{n-1}{n-1}(-1)^{n-1} & \dots & \binom{n}{n}\binom{n-n}{0}(-1)^0 \end{pmatrix}}_{U_X} \\
&= XU_X, \qquad \forall x \in [0,1].
\end{aligned}
$$

So

$$p(x) = B_X M = XU_X M.$$

Now compute the Bernstein coefficients matrix M.

$$
\begin{aligned}
XA &= XU_X M \\
M &= (U_X)^{-1} A
\end{aligned}
$$

where $(U_X)^{-1}$ is the inverse matrix of U_X.

2.1.2 A general interval $[\underline{x}, \overline{x}]$

The constraint $x \in [0,1]$ can be removed by extending the domain of the Bernstein polynomials to $[\underline{x}, \overline{x}]$ as already shown in Equation 3:

$$B_k^n(x) = \binom{n}{k}\left(\frac{x-\underline{x}}{\overline{x}-\underline{x}}\right)^k \left(1-\frac{x-\underline{x}}{\overline{x}-\underline{x}}\right)^{n-k}, \qquad \forall x \in [\underline{x}, \overline{x}].$$

As above a polynomial $p(x)$ is written in Bernstein form as:

$$p(x) = B_X M$$

where B_X is the vector of Bernstein polynomials and M is the Bernstein coefficient matrix. This time the variable $x \in [\underline{x}, \overline{x}]$.

Following a similar sequence of steps as above the vector B_X can be rewritten itself as a matrix multiplication[2]:

$$
\begin{aligned}
B_X &= \begin{pmatrix} B_0^n(x) & B_1^n(x) & \dots & B_n^n(x) \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{x-\underline{x}}{\overline{x}-\underline{x}} & \dots & (\frac{x-\underline{x}}{\overline{x}-\underline{x}})^n \end{pmatrix} U_X \\
&= \begin{pmatrix} 1 & x-\underline{x} & \dots & (x-\underline{x})^n \end{pmatrix} \underbrace{\begin{pmatrix} \frac{1}{(\overline{x}-\underline{x})^0} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{(\overline{x}-\underline{x})^n} \end{pmatrix}}_{V_X} U_X \\
&= \begin{pmatrix} 1 & x-\underline{x} & \dots & (x-\underline{x})^n \end{pmatrix} V_X U_X \\
&= \begin{pmatrix} \sum_{i=0}^{0}\binom{0}{i}x^i(-\underline{x})^{0-i} & \sum_{i=0}^{1}\binom{1}{i}x^i(-\underline{x})^{1-i} & \dots & \sum_{i=0}^{n}\binom{n}{i}x^i(-\underline{x})^{n-i} \end{pmatrix} V_X U_X \\
&= \underbrace{\begin{pmatrix} 1 & x & \dots & x^n \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} 1 & \binom{1}{0}(-\underline{x})^1 & \binom{2}{0}(-\underline{x})^2 & \dots & \binom{n}{0}(-\underline{x})^n \\ 0 & \binom{1}{1}(-\underline{x})^{1-1} & \binom{2}{1}(-\underline{x})^{2-1} & \dots & \binom{n}{1}(-\underline{x})^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \binom{n}{n}(-\underline{x})^{n-n} \end{pmatrix}}_{W_X} V_X U_X.
\end{aligned}
$$

[2]Note that U_X is a scalar matrix, so it stays the same as above, since it does not depend on the variable X.

Hence

$$B_X = X W_X V_X U_X. \tag{7}$$

The Bernstein coefficients matrix M for a general interval $[\underline{x}, \overline{x}]$ can be determined in the following manner:

$$\begin{aligned} XA &= X W_X V_X U_X M \\ M &= (U_X)^{-1}(V_X)^{-1}(W_X)^{-1}A. \end{aligned}$$

2.2 The bivariate case

The implicit expression of a bivariate polynomial in the power basis can also be rewritten by means of matrix multiplication:

$$p(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + \ldots + a_{mn}x^m y^n = XAY,$$

where

$$X = \begin{pmatrix} 1 & x & \ldots & x^m \end{pmatrix} \qquad Y = \begin{pmatrix} 1 \\ y \\ \vdots \\ y^n \end{pmatrix} \qquad A = \begin{pmatrix} a_{00} & \ldots & a_{0n} \\ \vdots & & \vdots \\ a_{m0} & \ldots & a_{mn} \end{pmatrix}.$$

By analogy with the univariate case,

$$p(x,y) = XAY = B_X M B_Y$$

where B_X and B_Y are Bernstein vectors in the variables $x \in [\underline{x}, \overline{x}]$ and $y \in [\underline{y}, \overline{y}]$. These vectors can be decomposed as shown in Section 2.1.

In the case of the Bernstein vector corresponding to the variable Y the factors U, V and W in Equation 7 will appear in reverse order. This happens because B_Y is a column vector (as opposed to B_X which is a row vector).

Hence

$$\begin{aligned} XAY &= X W_X V_X U_X \; M \; U_Y V_Y W_Y Y \\ M &= (U_X)^{-1}(V_X)^{-1}(W_X)^{-1} \; A \; (W_Y)^{-1}(V_Y)^{-1}(U_Y)^{-1} \\ & \qquad \forall x \in [\underline{x}, \overline{x}] \\ & \qquad \forall y \in [\underline{y}, \overline{y}]. \end{aligned}$$

2.3 The trivariate case

By analogy with the univariate and bivariate cases, the implicit expression of a trivariate polynomial in the power basis can also be rewritten by means of matrix multiplication:

$$p(x,y,z) = a_{000} + a_{100}x + a_{010}y + a_{001}z + a_{110}xy + a_{101}xz + a_{011}yz + \ldots + a_{mnl}x^m y^n z^l = Y \otimes_y (X \otimes_x A) \otimes_z Z.$$

where $A_{m\times n\times l}$ is the three–dimensional coefficient tensor, and X, Y and Z are chosen such that the tensor multiplications are well–defined.

The following types of tensor multiplication have been chosen :

$$\begin{aligned} \otimes_x &: \quad Q_{q\times m} \otimes_x A_{m\times n\times l} = R_{q\times n\times l} \\ \otimes_y &: \quad Q_{q\times n} \otimes_y A_{m\times n\times l} = R_{m\times q\times l} \\ \otimes_z &: \quad A_{m\times n\times l} \otimes_z Q_{l\times q} = R_{m\times n\times q}. \end{aligned}$$

If B_X, B_Y and B_Z are Bernstein vectors in the respective variables, the Bernstein form of the polynomial $p(x,y,z)$ is:

$$p(x,y,z) = Y \otimes_y (X \otimes_x A) \otimes_z Z = B_Y \otimes_y (B_X \otimes_x M) \otimes_z B_Z.$$

The Bernstein vectors can be decomposed as shown previously (Equation 7). When the power form is made equal to the Bernstein form, the following relation is obtained:

$$Y \otimes_y (X \otimes_x A) \otimes_z Z = Y W_Y V_Y U_Y \otimes_y (X \otimes_x (W_X \otimes_x (V_X \otimes_x (U_X \otimes_x M)))) \otimes_z U_Z V_Z W_Z Z.$$

In this equation the three–dimensional tensor M is being multiplied consecutively by each of the two–dimensional factors. At each stage another three–dimensional tensor is produced. After the $\otimes_x$–multiplication with the vector X, the three–dimensional tensor is reduced to two dimensions. The rest of the multiplications are the usual two–dimensional ones.

Hence, the Bernstein coefficients tensor M can be calculated by:

$$M = \underbrace{(U_Y)^{-1} \otimes_y (V_Y)^{-1} \otimes_y (W_Y)^{-1} \otimes_y \underbrace{((U_X)^{-1} \otimes_x (V_X)^{-1} \otimes_x \underbrace{(W_X)^{-1} \otimes_x A}_{\longleftarrow})}_{\longleftarrow} \otimes_z (W_Z)^{-1} \otimes_z (V_Z)^{-1} \otimes_z (U_Z)^{-1}}_{\longrightarrow}$$

$$\forall x \in [\underline{x}, \overline{x}], \forall y \in [\underline{y}, \overline{y}], \forall z \in [\underline{z}, \overline{z}].$$

As above, in this equation the order of the multiplications is starting from the tensor A outwards (according to the orientation of the arrows).

2.4 Another multivariate approach

Zettler and Garloff [9] give an equivalent formula for the calculation of the coefficients for an n–variate Bernstein–form polynomial.

Let $l \in \mathcal{N}$ be the number of variables and $\mathbf{x} = (x_1, \ldots, x_l) \in \mathcal{R}^l$. A multi–index I is defined as $I = (i_1, \ldots, i_l) \in \mathbf{N}^l$. For two given multi–indices $I, J \in \mathbf{N}^l$ the following conventions are made:

Notation. Write $I \leq J$ for the case where $0 \leq i_1 \leq j_1, \ldots, 0 \leq i_l \leq j_l$.

Notation. Denote the product $\binom{i_1}{j_1} \cdots \binom{i_l}{j_l}$ by $\binom{I}{J}$.

Notation. Denote by the product $x_1^{i_1} \cdots x_l^{i_l}$ $\mathbf{x}^I$.

Let $p(\mathbf{x})$ be a multivariate polynomial in l variables with real coefficients.

Definition. $D = (d_1, \ldots, d_l)$ is the tuple of maximum degrees so that d_k is the maximum degree of x_k in $p(\mathbf{x})$, for $k = 1, \ldots, l$.

Definition. The set $S = \{I \in N^l : I \leq D\}$ contains all the tuples from $\mathcal{R}^l$ which are "smaller than or equal to" the tuple D of maximum degrees.

Then an arbitrary polynomial $p(\mathbf{x})$ can be written as :

$$p(\mathbf{x}) = \sum_{I \in S} a_I \mathbf{x}^I$$

where $a_I \in \mathcal{R}$ represents the corresponding coefficient[3] to each $\mathbf{x}^I \in \mathcal{R}^l$.

As before, a univariate Bernstein polynomial of degree n on the unit interval $[0,1]$ is defined by:

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad k = 0, \ldots, n;\ x \in [0,1].$$

For the multivariate case consider, without loss of generality, a unit box $U = [0,1]^l$ and the I^{th} Bernstein polynomial of degree D is defined by:

$$B_I^D(\mathbf{x}) = B_{i_1}^{d_1}(x_1) \cdot\ \ldots\ \cdot B_{i_l}^{d_l}(x_l) \qquad \mathbf{x} \in \mathcal{R}^l.$$

The Bernstein coefficients $B_I(U)$ of p over the unit box $U = [0,1]^l$ are given by:

$$B_I(U) = \sum_{J \leq I} \frac{\binom{I}{J}}{\binom{D}{J}} a_J \qquad I \in S.$$

And so the Bernstein form of a multivariate polynomial p is defined by:

$$p(\mathbf{x}) = \sum_{I \in S} B_I(U) B_I^D(\mathbf{x}).$$

For a given uni–, bi– or trivariate polynomial this formula and the formulae given throughout Section 2 generate the same Bernstein form.

3 Applications in the unit box

3.1 Some two–dimensional examples

Example 1

The power form of a circle centred at $(\frac{1}{2}, \frac{1}{2})$ of radius $\frac{2}{5}$ is

$$pf = x^2 + y^2 - x - y + \frac{17}{50}.$$

The corresponding Bernstein form in the rectangular box $[0,1] \times [0,1]$ is

$$\begin{aligned} bf = & \left(\frac{17}{50}(1-x)^2 - \frac{8}{25}x(1-x) + \frac{17}{50}x^2\right)(1-y)^2 \\ & + 2\left(-\frac{4}{25}(1-x)^2 - \frac{33}{25}x(1-x) - \frac{4}{25}x^2\right)y(1-y) \\ & + \left(\frac{17}{50}(1-x)^2 - \frac{8}{25}x(1-x) + \frac{17}{50}x^2\right)y^2. \end{aligned}$$

Example 2

The power form of the polynomial is

$$p = -2x^3 + 7x^2y - 4xy^2 - y^3$$

and its Bernstein form is

$$\begin{aligned} bf = & -2x^3(1-y)^3 + 3\left(\frac{7}{3}x^2(1-x) + \frac{1}{3}x^3\right)y(1-y)^2 + 3\left(-\frac{4}{3}x(1-x)^2\right. \\ & \left. + 2x^2(1-x) + \frac{4}{3}x^3\right)y^2(1-y) + \left(-(1-x)^3 - 7x(1-x)^2 - 4x^2(1-x)\right)y^3. \end{aligned}$$

[3] Please note that some of the a_I may be 0.

Example 3

The polynomial

$$p \quad = \quad x^9 - 1.25x^7y + 3x^2y^6 - y^3 + y^5 + 1.11y^4x - 4.05y^4x^3$$

has following Bernstein form:

$$\begin{aligned}
bf \quad = \quad & x^9(1-y)^6 + 6(-0.2084x^7(1-x)^2 - 0.4168x^8(1-x) + 0.7916x^9)y(1-y)^5 + 15(-0.4169x^7(1-x)^2 - 0.8333x^8(1-x) \\
& +0.5834x^9)y^2(1-y)^4 + 20(-\frac{1}{20}(1-x)^9 - \frac{9}{20}x(1-x)^8 - \frac{9}{5}x^2(1-x)^7 - \frac{21}{5}x^3(1-x)^6 - \frac{63}{10}x^4(1-x)^5 - \frac{63}{10}x^5(1-x)^4 \\
& -\frac{21}{5}x^6(1-x)^3 - 2.425x^7(1-x)^2 - 1.700x^8(1-x) + 0.3250x^9)y^3(1-y)^3 + 15(-\frac{1}{5}(1-x)^9 - 1.726x(1-x)^8 \\
& -6.610x^2(1-x)^7 - 15.00x^3(1-x)^6 - 22.68x^4(1-x)^5 - 24.07x^5(1-x)^4 - 18.06x^6(1-x)^3 - 10.02x^7(1-x)^2 \\
& -4.495x^8(1-x) - 0.2294x^9)y^4(1-y)^2 + 6(-\frac{1}{3}(1-x)^9 - 2.630x(1-x)^8 - 9.040x^2(1-x)^7 - 18.98x^3(1-x)^6 \\
& -29.38x^4(1-x)^5 - 36.34x^5(1-x)^4 - 34.28x^6(1-x)^3 - 22.93x^7(1-x)^2 - 10.22x^8(1-x) - 1.355x^9)y^5(1-y) \\
& +(1.110x(1-x)^8 + 11.88x^2(1-x)^7 + 48.03x^3(1-x)^6 + 100.9x^4(1-x)^5 + 122.0x^5(1-x)^4 + 86.18x^6(1-x)^3 \\
& +32.04x^7(1-x)^2 + 3.078x^8(1-x) - 0.19x^9)y^6.
\end{aligned}$$

Example 4

This example will illustrate how a coefficient perturbation in the power form effects a considerable change of shape and/or location; conversely, the same sort of a coefficient perturbation in the Bernstein form has hardly any effect on the surface.

Example 4.0

Here are the initial power form of the polynomial and its Bernstein equivalent:

$$\begin{aligned}
p \quad &= \quad 5x^5y - 9xy^4 + 2x^2y^3 + 5xy^3 \\
bf \quad &= \quad 5x^5y(1-y)^3 + 15x^5y^2(1-y)^2 + 4\left[\frac{5}{4}x(1-x)^4 + \frac{11}{2}x^2(1-x)^3 + 9x^3(1-x)^2 + \frac{13}{2}x^4(1-x) + \frac{11}{2}x^5\right] \\
& \quad\; y^3(1-y) + \left[-4x(1-x)^4 - 14x^2(1-x)^3 - 18x^3(1-x)^2 - 10x^4(1-x) + 3x^5\right]y^4
\end{aligned}$$

Example 4.1

The following example uses the same polynomial as before with a change in the second coefficient ($-9xy^4$ to $-7xy^4$), giving the following power form of the new polynomial:

$$p \quad = \quad 5\,x^5\,y\underline{-7\,x\,y^4} + 2\,x^2\,y^3 + 5\,x\,y^3$$

and its Bernstein form:

$$\begin{aligned}
bf \quad = \quad & 5x^5y(1-y)^3 + 15x^5y^2(1-y)^2 + 4\left[\frac{5}{4}x(1-x)^4 + \frac{11}{2}, x^2(1-x)^3 + 9x^3(1-x)^2 + \frac{13}{2}x^4(1-x) + \frac{11}{2}x^5\right] \\
& y^3\,(1-y) + \left[-2x(1-x)^4 - 6x^2(1-x)^3 - 6x^3(1-x)^2 - 2x^4(1-x) + 5x^5\right]y^4.
\end{aligned}$$

Example 4.2

In this example one coefficient of the initial Bernstein form, given in Example 3.0, is changed ($9x^3(1-x)^2$ to $7x^3(1-x)^2$). Although it changes the power form of the polynomial radically, this coefficient perturbation has hardly any noticeable effect on the shape. The power form of the new polynomial is

$$p \;=\; 5x^5y - 8x^5y^3 + 8x^5y^4 + 5xy^3 - 9xy^4 + 2x^2y^3 - 8y^3x^3 + 8y^4x^3 + 16y^3x^4 - 16y^4x^4$$

and the slightly changed Bernstein form

$$\begin{aligned} bf \;=\; & 5x^5y(1-y)^3 + 15x^5y^2(1-y)^2 + 4\left[\frac{5}{4}x(1-x)^4 + \frac{11}{2}x^2(1-x)^3 \underline{+7x^3(1-x)^2} + \frac{13}{2}x^4(1-x) + \frac{11}{2}x^5\right] \\ & y^3(1-y) + \left[-4x(1-x)^4 - 14x^2(1-x)^3 - 18x^3(1-x)^2 - 10x^4(1-x) + 3x^5\right] y^4 \end{aligned}$$

3.2 A three–dimensional example

Section 2.3 shows how three–dimensional polynomials can be represented in Bernstein form.

The following three–dimensional polynomial p represents a torus with the centre in the origin and the radii $\frac{3}{4}$ and $\frac{1}{5}$.

$$p \;=\; x^4 + 2x^2y^2 + 2x^2z^2 - \frac{241}{200}x^2 + y^4 + 2y^2z^2 - \frac{241}{200}y^2 + z^4 + \frac{209}{200}z^2 + \frac{43681}{160000}$$

The Bernstein form corresponding to the part of the torus lying in the unit box is:

$$\begin{aligned} bf \;=\; & ((1-y)^4(\frac{43681}{160000}(1-x)^4 + \frac{43681}{40000}x(1-x)^3 + \frac{34643}{80000}x^2(1-x)^2 - \frac{52719}{40000}x^3(1-x) + \frac{10881}{160000}x^4) + 4y(1-y)^3(\frac{43681}{160000}(1- \\ & +\frac{43681}{40000}x(1-x)^3 + \frac{34643}{80000}x^2(1-x)^2 - \frac{52719}{40000}x^3(1-x) + \frac{10881}{160000}x^4) + 6y^2(1-y)^2(\frac{34643}{480000}(1-x)^4 + \frac{34643}{120000}x(1-x)^3 \\ & -\frac{105271}{240000}x^2(1-x)^2 - \frac{174557}{120000}x^3(1-x) + \frac{32081}{160000}x^4) + 4y^3(1-y)(\frac{-52719}{160000}(1-x)^4 - \frac{52719}{40000}x(1-x)^3 - \frac{174557}{80000}x^2(1-x) \\ & -\frac{69119}{40000}x^3(1-x) + \frac{74481}{160000}x^4) + y^4(\frac{10881}{160000}(1-x)^4 + \frac{10881}{40000}x(1-x)^3 + \frac{96243}{80000}x^2(1-x)^2 + \frac{74481}{40000}x^3(1-x) + \frac{298081}{160000}x^4 \\ & (1-z)^4 + 4((1-y)^4(\frac{43681}{160000}(1-x)^4 + \frac{43681}{40000}x(1-x)^3 + \frac{34643}{80000}x^2(1-x)^2 - \frac{52719}{40000}x^3(1-x) + \frac{10881}{160000}x^4) + 4y(1-y)^3 \\ & (\frac{43681}{160000}(1-x)^4 + \frac{43681}{40000}x(1-x)^3 + \frac{34643}{80000}x^2(1-x)^2 - \frac{52719}{40000}x^3(1-x) + \frac{10881}{160000}x^4) + 6y^2(1-y)^2(\frac{34643}{480000}(1-x)^4 \\ & +\frac{34643}{120000}x(1-x)^3 - \frac{105271}{240000}x^2(1-x)^2 - \frac{174557}{120000}x^3(1-x) + \frac{32081}{160000}x^4) + 4y^3(1-y)(-\frac{52719}{160000}(1-x)^4 - \frac{52719}{40000}x(1-x) \\ & -\frac{174557}{80000}x^2(1-x)^2 - \frac{69119}{40000}x^3(1-x) + \frac{74481}{160000}x^4) + y^4(\frac{10881}{160000}(1-x)^4 + \frac{10881}{40000}x(1-x)^3 + \frac{96243}{80000}x^2(1-x)^2 \\ & +\frac{74481}{40000}x^3(1-x) + \frac{298081}{160000}x^4))z(1-z)^3 + 6((1-y)^4(\frac{214643}{480000}(1-x)^4 + \frac{214643}{120000}x(1-x)^3 + \frac{434729}{240000}x^2(1-x)^2 \\ & +\frac{5443}{120000}x^3(1-x) + \frac{92081}{160000}x^4) + 4y(1-y)^3(\frac{214643}{480000}(1-x)^4 + \frac{214643}{120000}x(1-x)^3 + \frac{434729}{240000}x^2(1-x)^2 + \frac{5443}{120000}x^3(1-x \\ & +\frac{92081}{160000}x^4) + 6y^2(1-y)^2(\frac{434729}{1440000}(1-x)^4 + \frac{434729}{360000}x(1-x)^3 + \frac{101843}{80000}x^2(1-x)^2 + \frac{47129}{360000}x^3(1-x) + \frac{1099529}{1440000}x^4) \\ & +4y^3(1-y)(\frac{5443}{480000}(1-x)^4 + \frac{5443}{120000}x(1-x)^3 + \frac{47129}{240000}x^2(1-x)^2 + \frac{12081}{40000}x^3(1-x) + \frac{547043}{480000}x^4) + y^4(\frac{92081}{160000}(1-x) \\ & +\frac{92081}{40000}x(1-x)^3 + \frac{1099529}{240000}x^2(1-x)^2 + \frac{547043}{120000}x^3(1-x) + \frac{1297843}{480000}x^4))z^2(1-z)^2 + 4((1-y)^4 \\ & (\frac{127281}{160000}(1-x)^4 + \frac{127281}{40000}x(1-x)^3 + \frac{365443}{80000}x^2(1-x)^2 + \frac{110881}{40000}x^3(1-x) + \frac{254481}{160000}x^4) + 4y(1-y)^3(\frac{127281}{160000}(1-x)^4 \\ & +\frac{127281}{40000}x(1-x)^3 + \frac{365443}{80000}x^2(1-x)^2 + \frac{110881}{40000}x^3(1-x) + \frac{254481}{160000}x^4) + 6y^2(1-y)^2(\frac{365443}{480000}(1-x)^4 + \frac{365443}{120000}x(1-x \end{aligned}$$

$$
\begin{aligned}
&+\frac{1127129}{240000}x^2(1-x)^2+\frac{132081}{40000}x^3(1-x)+\frac{907043}{480000}x^4)+4y^3(1-y)(\frac{110881}{160000}(1-x)^4+\frac{110881}{40000}x(1-x)^3+\frac{396243}{80000}x^2(1-\\
&+\frac{174481}{40000}x^3(1-x)+\frac{398081}{160000}x^4)+y^4(\frac{254481}{160000}(1-x)^4+\frac{254481}{40000}x(1-x)^3+\frac{907043}{80000}x^2(1-x)^2+\frac{398081}{40000}x^3(1-x)\\
&+\frac{701681}{160000}x^4))z^3(1-z)+((1-y)^4(\frac{370881}{160000}(1-x)^4+\frac{370881}{40000}x(1-x)^3+\frac{1176243}{80000}x^2(1-x)^2+\frac{434481}{40000}x^3(1-x)+\frac{658081}{160000}\\
&+4y(1-y)^3(\frac{370881}{160000}(1-x)^4+\frac{370881}{40000}x(1-x)^3+\frac{1176243}{80000}x^2(1-x)^2+\frac{434481}{40000}x^3(1-x)+\frac{658081}{160000}x^4)+6y^2(1-y)^2\\
&(\frac{392081}{160000}(1-x)^4+\frac{392081}{40000}x(1-x)^3+\frac{3799529}{240000}x^2(1-x)^2+\frac{1447043}{120000}x^3(1-x)+\frac{2197843}{480000}x^4)+4y^3(1-y)(\frac{434481}{160000}(1-x\\
&+\frac{434481}{40000}x(1-x)^3+\frac{1447043}{80000}x^2(1-x)^2+\frac{578081}{40000}x^3(1-x)+\frac{881681}{160000}x^4)+y^4(\frac{658081}{160000}(1-x)^4+\frac{658081}{40000}x(1-x)^3\\
&+\frac{2197843}{80000}x^2(1-x)^2+\frac{881681}{40000}x^3(1-x)+\frac{1265281}{160000}x^4))z^4.
\end{aligned}
$$

4 Applications in a general box

The examples shown in Section 3 deal with polynomials passing through the unit box $[0,1]\times[0,1]$. However, as said in the introductory section, it is not necessary that the area of interest be restricted in this way. Any arbitrary box $[\underline{x},\overline{x}]\times[\underline{y},\overline{y}]$ can be chosen as the area of interest. In the following examples the polynomial and/or the box are being translated further away from the origin. Formula 3 is used to calculate the Bernstein polynomials corresponding to the general box $[\underline{x},\overline{x}]\times[\underline{y},\overline{y}]$.

4.1 Study of a circle

Using the same circle as in Section 3 Example 1 but moving the area of interest away from the origin (so that it only contains part of the circle) similar results can be obtained. Since the Bernstein base depends on the box in which it is considered, the Bernstein–form polynomial in this case is not identical to the one given in Section 3 Example 1. The Bernstein form in the first case is computed in the box $[\frac{1}{2},\frac{3}{2}]\times[\frac{1}{2},\frac{3}{2}]$ in terms of the Bernstein polynomials $B_0^2(x)=(\frac{3}{2}-x)^2$, $B_1^2(x)=2(x-\frac{1}{2})(\frac{3}{2}-x)$ and $B_2^2(x)=(x-\frac{1}{2})^2$. The power form of the circle is:

$$p \;=\; x^2-x+\frac{17}{50}+y^2-y$$

and its Bernstein form in the box $[\frac{1}{2},\frac{3}{2}]\times[\frac{1}{2},\frac{3}{2}]$.

$$
\begin{aligned}
bf \;=\; &(-\frac{4}{25}(\frac{3}{2}-x)^2-\frac{8}{25}(x-\frac{1}{2})(\frac{3}{2}-x)+\frac{21}{25}(x-\frac{1}{2})^2)(\frac{3}{2}-y)^2+\\
&2(-\frac{4}{25}(\frac{3}{2}-x)^2-\frac{8}{25}(x-\frac{2}{2})(\frac{3}{2}-x)+\frac{21}{25}(x-\frac{1}{2})^2)(y-\frac{1}{2})(\frac{3}{2}-y)+\\
&(\frac{21}{25}(\frac{3}{2}-x)^2+\frac{42}{25}(x-\frac{1}{2})(\frac{3}{2}-x)+\frac{46}{25}(x-\frac{1}{2})^2)(y-\frac{1}{2})^2.
\end{aligned}
$$

The circle is then translated a little further from the unit box, and enclosed once again in a box of unit–length edges. The power form of the translated circle is:

$$p \;=\; x^2-\frac{3}{2}x+\frac{193}{200}+y^2-\frac{3}{2}y$$

and its Bernstein form in the box $[\frac{1}{4},\frac{5}{4}]\times[\frac{1}{4},\frac{5}{4}]$:

$$bf \;=\; (\frac{17}{50}(\frac{5}{4}-x)^2-\frac{8}{25}(x-\frac{1}{4})(\frac{5}{4}-x)+\frac{17}{50}(x-\frac{1}{4})^2)(\frac{5}{4}-y)^2+$$

$$2(-\frac{4}{25}(\frac{5}{4}-x)^2-\frac{33}{25}(x-\frac{1}{4})(\frac{5}{4}-x)-\frac{4}{25}(x-\frac{1}{4})^2)(y-\frac{1}{4})(\frac{5}{4}-y)+$$
$$(\frac{17}{50}(\frac{5}{4}-x)^2-\frac{8}{25}(x-\frac{1}{4})(\frac{5}{4}-x)+\frac{17}{50}(x-\frac{1}{4})^2)(y-\frac{1}{4})^2$$

If the circle is translated even further away from the origin the following power is obtained:

$$p \quad = \quad x^2-2x+\frac{46}{25}+y^2-2y$$

and its Bernstein form in the box $[\frac{1}{2},\frac{3}{2}]\times[\frac{1}{2},\frac{3}{2}]$:

$$\begin{aligned} bf \quad = \quad & (\frac{17}{50}(\frac{3}{2}-x)^2-\frac{8}{25}(x-\frac{1}{2})(\frac{3}{2}-x)+\frac{17}{50}(x-\frac{1}{2})^2)(\frac{3}{2}-y)^2+ \\ & 2(-\frac{4}{25}(\frac{3}{2}-x)^2-\frac{33}{25}(x-\frac{1}{2})(\frac{3}{2}-x)-\frac{4}{25}(x-\frac{1}{2})^2)(y-\frac{1}{2})(\frac{3}{2}-y) \\ & +(\frac{17}{50}(\frac{3}{2}-x)^2-\frac{8}{25}(x-\frac{1}{2})(\frac{3}{2}-x)+\frac{17}{50}(x-\frac{1}{2})^2)(y-\frac{1}{2})^2 \end{aligned}$$

The circle and the bounding box are translated even further. The circle is now given by:

$$p \quad = \quad x^2-11x+\frac{5117}{50}+y^2-17y$$

and its Bernstein form in the box $[5,6]\times[8,9]$ is:

$$\begin{aligned} bf \quad = \quad & (\frac{17}{50}(6-x)^2-\frac{8}{25}(x-5)(6-x)+\frac{17}{50}(x-5)^2)(9-y)^2+ \\ & 2(-\frac{4}{25}(6-x)^2-\frac{33}{25}(x-5)(6-x)-\frac{4}{25}(x-5)^2)(y-8)(9-y)+ \\ & (\frac{17}{50}(6-x)^2-\frac{8}{25}(x-5)(6-x)+\frac{17}{50}(x-5)^2)(y-8)^2. \end{aligned}$$

In the following example the circle centred at $(\frac{31}{2},\frac{31}{2})$ and of radius 1 is considered in the larger box $[13,18]\times[14,19]$. Its power form is:

$$p \quad = \quad x^2-31\,x+\frac{959}{2}+y^2-31\,y$$

and the corresponding Bernstein form is:

$$\begin{aligned} bf \quad = \quad & (\frac{15}{2}\,(\frac{18}{5}-\frac{1}{5}\,x)^2-10\,(\frac{1}{5}\,x-\frac{13}{5})\,(\frac{18}{5}-\frac{1}{5}\,x)+\frac{15}{2}\,(\frac{1}{5}\,x-\frac{13}{5})^2)\,(\frac{19}{5}-\frac{1}{5}\,y)^2 \\ & -50\,(\frac{1}{5}\,x-\frac{13}{5})\,(\frac{18}{5}-\frac{1}{5}\,x)\,(\frac{1}{5}\,y-\frac{14}{5})\,(\frac{19}{5}-\frac{1}{5}\,y) \\ & +(\frac{35}{2}\,(\frac{18}{5}-\frac{1}{5}\,x)^2+10\,(\frac{1}{5}\,x-\frac{13}{5})\,(\frac{18}{5}-\frac{1}{5}\,x)+\frac{35}{2}\,(\frac{1}{5}\,x-\frac{13}{5})^2)\,(\frac{1}{5}\,y-\frac{14}{5})^2 \end{aligned}$$

4.2 Study of a closed curve

Consider the following degree–six curve in the general box $[3,7]\times[1,5]$. Its power form is:

$$\begin{aligned} p \quad = \quad & -32556x-19487y+3x^2y^4+11842xy+6831y^2-2985x^2y+3x^4y^2+13158x^2 \\ & -3060xy^2-60x^3y^2+360x^3y-18x^4y+360xy^3-30xy^4+606x^2y^2-36x^2y^3 \\ & +35592-18y^5-30x^5-1404y^3-2980x^3+399x^4+207y^4+x^6+y^6. \end{aligned}$$

Its Bernstein form in the box $[3,7] \times [1,5]$ is:

$$
\begin{aligned}
bf \quad = \quad & (287\,(\frac{7}{4} - \frac{1}{4}\,x)^6 - 470\,(\frac{1}{4}\,x - \frac{3}{4})\,(\frac{7}{4} - \frac{1}{4}\,x)^5 + 977\,(\frac{1}{4}\,x - \frac{3}{4})^2\,(\frac{7}{4} - \frac{1}{4}\,x)^4 - 500\,(\frac{1}{4}\,x - \frac{3}{4})^3 \\
& (\frac{7}{4} - \frac{1}{4}\,x)^3 + 1297\,(\frac{1}{4}\,x - \frac{3}{4})^4\,(\frac{7}{4} - \frac{1}{4}\,x)^2 - 214\,(\frac{1}{4}\,x - \frac{3}{4})^5\,(\frac{7}{4} - \frac{1}{4}\,x) + 351\,(\frac{1}{4}\,x - \frac{3}{4})^6)(\frac{5}{4} - \frac{1}{4}\,y)^6 \\
& +6(-\frac{259}{3}\,(\frac{7}{4} - \frac{1}{4}\,x)^6 - \frac{2914}{3}\,(\frac{1}{4}\,x - \frac{3}{4})\,(\frac{7}{4} - \frac{1}{4}\,x)^5 + \frac{787}{3}\,(\frac{1}{4}\,x - \frac{3}{4})^2\,(\frac{7}{4} - \frac{1}{4}\,x)^4 - 1716\,(\frac{1}{4}\,x - \frac{3}{4})^3 \\
& (\frac{7}{4} - \frac{1}{4}\,x)^3 + \frac{1427}{3}\,(\frac{1}{4}\,x - \frac{3}{4})^4\,(\frac{7}{4} - \frac{1}{4}\,x)^2 - \frac{2402}{3}\,(\frac{1}{4}\,x - \frac{3}{4})^5\,(\frac{7}{4} - \frac{1}{4}\,x) - \frac{131}{3}\,(\frac{1}{4}\,x - \frac{3}{4})^6) \\
& (\frac{1}{4}\,y - \frac{1}{4})\,(\frac{5}{4} - \frac{1}{4}\,y)^5 + 15(\frac{833}{15}\,(\frac{7}{4} - \frac{1}{4}\,x)^6 + \frac{1238}{15}\,(\frac{1}{4}\,x - \frac{3}{4})\,(\frac{7}{4} - \frac{1}{4}\,x)^5 + \frac{29263}{15}\,(\frac{1}{4}\,x - \frac{3}{4})^2 \\
& (\frac{7}{4} - \frac{1}{4}\,x)^4 - \frac{1028}{5}\,(\frac{1}{4}\,x - \frac{3}{4})^3\,(\frac{7}{4} - \frac{1}{4}\,x)^3 + \frac{30863}{15}\,(\frac{1}{4}\,x - \frac{3}{4})^4\,(\frac{7}{4} - \frac{1}{4}\,x)^2 + \frac{2518}{15}(\frac{1}{4}\,x - \frac{3}{4})^5 \\
& (\frac{7}{4} - \frac{1}{4}\,x) + \frac{1153}{15}\,(\frac{1}{4}\,x - \frac{3}{4})^6)(\frac{1}{4}\,y - \frac{1}{4})^2\,(\frac{5}{4} - \frac{1}{4}\,y)^4 + 20(-\frac{149}{5}\,(\frac{7}{4} - \frac{1}{4}\,x)^6 - 534\,(\frac{1}{4}\,x - \frac{3}{4}) \\
& (\frac{7}{4} - \frac{1}{4}\,x)^5 - \frac{891}{5}\,(\frac{1}{4}\,x - \frac{3}{4})^2\,(\frac{7}{4} - \frac{1}{4}\,x)^4 - 3444\,(\frac{1}{4}\,x - \frac{3}{4})^3\,(\frac{7}{4} - \frac{1}{4}\,x)^3 - \frac{891}{5}\,(\frac{1}{4}\,x - \frac{3}{4})^4\,(\frac{7}{4} - \frac{1}{4}\,x)^2 \\
& -534\,(\frac{1}{4}\,x - \frac{3}{4})^5\,(\frac{7}{4} - \frac{1}{4}\,x) - \frac{149}{5}\,(\frac{1}{4}\,x - \frac{3}{4})^6)(\frac{1}{4}\,y - \frac{1}{4})^3\,(\frac{5}{4} - \frac{1}{4}\,y)^3 + 15(\frac{1393}{15}\,(\frac{7}{4} - \frac{1}{4}\,x)^6 \\
& +\frac{2998}{15}\,(\frac{1}{4}\,x - \frac{3}{4})\,(\frac{7}{4} - \frac{1}{4}\,x)^5 + \frac{30623}{15}\,(\frac{1}{4}\,x - \frac{3}{4})^2\,(\frac{7}{4} - \frac{1}{4}\,x)^4 - \frac{1348}{5}\,(\frac{1}{4}\,x - \frac{3}{4})^3\,(\frac{7}{4} - \frac{1}{4}\,x)^3 \\
& +\frac{29023}{15}\,(\frac{1}{4}\,x - \frac{3}{4})^4\,(\frac{7}{4} - \frac{1}{4}\,x)^2 + \frac{1718}{15}\,(\frac{1}{4}\,x - \frac{3}{4})^5\,(\frac{7}{4} - \frac{1}{4}\,x) + \frac{1073}{15}\,(\frac{1}{4}\,x - \frac{3}{4})^6)(\frac{1}{4}\,y - \frac{1}{4})^4 \\
& (\frac{5}{4} - \frac{1}{4}\,y)^2 + 6(-\frac{35}{3}\,(\frac{7}{4} - \frac{1}{4}\,x)^6 - \frac{2210}{3}\,(\frac{1}{4}\,x - \frac{3}{4})\,(\frac{7}{4} - \frac{1}{4}\,x)^5 + \frac{1331}{3}(\frac{1}{4}\,x - \frac{3}{4})^2(\frac{7}{4} - \frac{1}{4}\,x)^4 \\
& -1844\,(\frac{1}{4}\,x - \frac{3}{4})^3\,(\frac{7}{4} - \frac{1}{4}\,x)^3 + \frac{691}{3}\,(\frac{1}{4}\,x - \frac{3}{4})^4\,(\frac{7}{4} - \frac{1}{4}\,x)^2 - \frac{2722}{3}\,(\frac{1}{4}\,x - \frac{3}{4})^5\,(\frac{7}{4} - \frac{1}{4}\,x) - \frac{163}{3} \\
& (\frac{1}{4}\,x - \frac{3}{4})^6)(\frac{1}{4}\,y - \frac{1}{4})^5\,(\frac{5}{4} - \frac{1}{4}\,y) + (399\,(\frac{7}{4} - \frac{1}{4}\,x)^6 - 118\,(\frac{1}{4}\,x - \frac{3}{4})\,(\frac{7}{4} - \frac{1}{4}\,x)^5 + 1249\,(\frac{1}{4}\,x - \frac{3}{4})^2 \\
& (\frac{7}{4} - \frac{1}{4}\,x)^4 - 692\,(\frac{1}{4}\,x - \frac{3}{4})^3\,(\frac{7}{4} - \frac{1}{4}\,x)^3 + 929\,(\frac{1}{4}\,x - \frac{3}{4})^4\,(\frac{7}{4} - \frac{1}{4}\,x)^2 - 374\,(\frac{1}{4}\,x - \frac{3}{4})^5\,(\frac{7}{4} - \frac{1}{4}\,x) \\
& +335\,(\frac{1}{4}\,x - \frac{3}{4})^6)(\frac{1}{4}\,y - \frac{1}{4})^6.
\end{aligned}
$$

If the same curve is now cropped to a smaller (but still general) box $[5,7] \times [3,4]$ its Bernstein form is different from the previous one because the Bernstein coefficients and polynomials of the basis itself are different.

$$
\begin{aligned}
bf_{[5,7]\times[3,4]} \quad = \quad & (-(\frac{7}{2} - \frac{1}{2}\,x)^6 - 6\,(\frac{1}{2}\,x - \frac{5}{2})\,(\frac{7}{2} - \frac{1}{2}\,x)^5 - 3\,(\frac{1}{2}\,x - \frac{5}{2})^2\,(\frac{7}{2} - \frac{1}{2}\,x)^4 + 28\,(\frac{1}{2}\,x - \frac{5}{2})^3\,(\frac{7}{2} - \frac{1}{2}\,x)^3 \\
& +9\,(\frac{1}{2}\,x - \frac{5}{2})^4\,(\frac{7}{2} - \frac{1}{2}\,x)^2 - 54\,(\frac{1}{2}\,x - \frac{5}{2})^5\,(\frac{7}{2} - \frac{1}{2}\,x) + 27\,(\frac{1}{2}\,x - \frac{5}{2})^6)\,(4 - y)^6 + 6 \\
& (-(\frac{7}{2} - \frac{1}{2}\,x)^6 - \frac{26}{3}\,(\frac{1}{2}\,x - \frac{5}{2})\,(\frac{7}{2} - \frac{1}{2}\,x)^5 - \frac{43}{3}\,(\frac{1}{2}\,x - \frac{5}{2})^2\,(\frac{7}{2} - \frac{1}{2}\,x)^4 + \frac{28}{3}\,(\frac{1}{2}\,x - \frac{5}{2})^3 \\
& (\frac{7}{2} - \frac{1}{2}\,x)^3 - \frac{17}{3}\,(\frac{1}{2}\,x - \frac{5}{2})^4\,(\frac{7}{2} - \frac{1}{2}\,x)^2 - \frac{178}{3}\,(\frac{1}{2}\,x - \frac{5}{2})^5\,(\frac{7}{2} - \frac{1}{2}\,x) + \frac{79}{3}\,(\frac{1}{2}\,x - \frac{5}{2})^6) \\
& (y - 3)\,(4 - y)^5 + 15(-\frac{4}{5}\,(\frac{7}{2} - \frac{1}{2}\,x)^6 - \frac{152}{15}\,(\frac{1}{2}\,x - \frac{5}{2})\,(\frac{7}{2} - \frac{1}{2}\,x)^5 - \frac{364}{15}\,(\frac{1}{2}\,x - \frac{5}{2})^2\,(\frac{7}{2} - \frac{1}{2}\,x)^4 \\
& -\frac{176}{15}\,(\frac{1}{2}\,x - \frac{5}{2})^3\,(\frac{7}{2} - \frac{1}{2}\,x)^3 - \frac{356}{15}\,(\frac{1}{2}\,x - \frac{5}{2})^4\,(\frac{7}{2} - \frac{1}{2}\,x)^2 - \frac{952}{15}\,(\frac{1}{2}\,x - \frac{5}{2})^5\,(\frac{7}{2} - \frac{1}{2}\,x) \\
& +\frac{412}{15}\,(\frac{1}{2}\,x - \frac{5}{2})^6)(y - 3)^2\,(4 - y)^4 + 20(-\frac{2}{5}\,(\frac{7}{2} - \frac{1}{2}\,x)^6 - \frac{52}{5}\,(\frac{1}{2}\,x - \frac{5}{2})\,(\frac{7}{2} - \frac{1}{2}\,x)^5 - \frac{164}{5} \\
& (\frac{1}{2}\,x - \frac{5}{2})^2\,(\frac{7}{2} - \frac{1}{2}\,x)^4 - \frac{176}{5}\,(\frac{1}{2}\,x - \frac{5}{2})^3\,(\frac{7}{2} - \frac{1}{2}\,x)^3 - \frac{226}{5}\,(\frac{1}{2}\,x - \frac{5}{2})^4\,(\frac{7}{2} - \frac{1}{2}\,x)^2 - \frac{332}{5} \\
& (\frac{1}{2}\,x - \frac{5}{2})^5\,(\frac{7}{2} - \frac{1}{2}\,x) + \frac{152}{5}\,(\frac{1}{2}\,x - \frac{5}{2})^6)\,(y - 3)^3(4 - y)^3 + 15(-\frac{32}{3}\,(\frac{1}{2}\,x - \frac{5}{2})\,(\frac{7}{2} - \frac{1}{2}\,x)^5
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{632}{15}\,(\frac{1}{2}\,x-\frac{5}{2})^2\,(\frac{7}{2}-\frac{1}{2}\,x)^4-\frac{928}{15}\,(\frac{1}{2}\,x-\frac{5}{2})^3\,(\frac{7}{2}-\frac{1}{2}\,x)^3-\frac{1024}{15}\,(\frac{1}{2}\,x-\frac{5}{2})^4\,(\frac{7}{2}-\frac{1}{2}\,x)^2\\
&-\frac{992}{15}\,(\frac{1}{2}\,x-\frac{5}{2})^5\,(\frac{7}{2}-\frac{1}{2}\,x)+\frac{536}{15}\,(\frac{1}{2}\,x-\frac{5}{2})^6)\,(y-3)^4\,(4-y)^2+6\,(-\frac{40}{3}\,(\frac{1}{2}\,x-\frac{5}{2})\,(\frac{7}{2}-\frac{1}{2}\,x)^5\\
&-\frac{170}{3}\,(\frac{1}{2}\,x-\frac{5}{2})^2\,(\frac{7}{2}-\frac{1}{2}\,x)^4-\frac{280}{3}\,(\frac{1}{2}\,x-\frac{5}{2})^3\,(\frac{7}{2}-\frac{1}{2}\,x)^3-\frac{268}{3}\,(\frac{1}{2}\,x-\frac{5}{2})^4\,(\frac{7}{2}-\frac{1}{2}\,x)^2-\frac{176}{3}\\
&(\frac{1}{2}\,x-\frac{5}{2})^5\,(\frac{7}{2}-\frac{1}{2}\,x)+\frac{134}{3}\,(\frac{1}{2}\,x-\frac{5}{2})^6)\,(y-3)^5\,(4-y)+(-16\,(\frac{1}{2}\,x-\frac{5}{2})(\frac{7}{2}-\frac{1}{2}\,x)^5\\
&-68\,(\frac{1}{2}\,x-\frac{5}{2})^2\,(\frac{7}{2}-\frac{1}{2}\,x)^4-112\,(\frac{1}{2}\,x-\frac{5}{2})^3\,(\frac{7}{2}-\frac{1}{2}\,x)^3-88\,(\frac{1}{2}\,x-\frac{5}{2})^4\,(\frac{7}{2}-\frac{1}{2}\,x)^2\\
&-32\,(\frac{1}{2}\,x-\frac{5}{2})^5\,(\frac{7}{2}-\frac{1}{2}\,x)+60\,(\frac{1}{2}\,x-\frac{5}{2})^6)\,(y-3)^6
\end{aligned}
$$

4.3 Study of another curve

Consider the cardioid curve in the box $[2,6]\times[4,9]$:

$$
\begin{aligned}
p \;=\; & -67608\,x-98820\,y-6300\,x^2\,y+30699\,y^2+984\,x^2\,y^2+31968\,x\,y-6336\,x\,y^2-36\,x^4\,y\\
&+3\,x^4\,y^2-24\,x\,y^4+600\,x\,y^3-75\,x^2\,y^3+576\,x^3\,y-48\,x^3\,y^2+3\,x^2\,y^4+18243\,x^2+345\,x^4\\
&-2960\,x^3+585\,y^4-5448\,y^3+x^6-24\,x^5+y^6-36\,y^5+143019
\end{aligned}
$$

and its corresponding Bernstein form in that box:

$$
\begin{aligned}
bf \;=\; & (439\,(\frac{3}{2}-\frac{1}{4}\,x)^6-102\,(\frac{1}{4}\,x-\frac{1}{2})\,(\frac{3}{2}-\frac{1}{4}\,x)^5+1017\,(\frac{1}{4}\,x-\frac{1}{2})^2\,(\frac{3}{2}-\frac{1}{4}\,x)^4-980\\
&(\frac{1}{4}\,x-\frac{1}{2})^3\,(\frac{3}{2}-\frac{1}{4}\,x)^3+1017\,(\frac{1}{4}\,x-\frac{1}{2})^4\,(\frac{3}{2}-\frac{1}{4}\,x)^2-102\,(\frac{1}{4}\,x-\frac{1}{2})^5\,(\frac{3}{2}-\frac{1}{4}\,x)+439\\
&(\frac{1}{4}\,x-\frac{1}{2})^6)(\frac{9}{5}-\frac{1}{5}\,y)^6+6(-171\,(\frac{3}{2}-\frac{1}{4}\,x)^6-1042\,(\frac{1}{4}\,x-\frac{1}{2})\,(\frac{3}{2}-\frac{1}{4}\,x)^5+187\,(\frac{1}{4}\,x-\frac{1}{2})^2\\
&(\frac{3}{2}-\frac{1}{4}\,x)^4-1980(\frac{1}{4}\,x-\frac{1}{2})^3\,(\frac{3}{2}-\frac{1}{4}\,x)^3+187\,(\frac{1}{4}\,x-\frac{1}{2})^4\,(\frac{3}{2}-\frac{1}{4}\,x)^2-1042\,(\frac{1}{4}\,x-\frac{1}{2})^5\\
&(\frac{3}{2}-\frac{1}{4}\,x)-171\,(\frac{1}{4}\,x-\frac{1}{2})^6)(\frac{1}{5}\,y-\frac{4}{5})\,(\frac{9}{5}-\frac{1}{5}\,y)^5+15(144\,(\frac{3}{2}-\frac{1}{4}\,x)^6+688\,(\frac{1}{4}\,x-\frac{1}{2})\\
&(\frac{3}{2}-\frac{1}{4}\,x)^5+2992\,(\frac{1}{4}\,x-\frac{1}{2})^2\,(\frac{3}{2}-\frac{1}{4}\,x)^4+800\,(\frac{1}{4}\,x-\frac{1}{2})^3\,(\frac{3}{2}-\frac{1}{4}\,x)^3+2992\,(\frac{1}{4}\,x-\frac{1}{2})^4\\
&(\frac{3}{2}-\frac{1}{4}\,x)^2+688\,(\frac{1}{4}\,x-\frac{1}{2})^5\,(\frac{3}{2}-\frac{1}{4}\,x)+144\,(\frac{1}{4}\,x-\frac{1}{2})^6)(\frac{1}{5}\,y-\frac{4}{5})^2\,(\frac{9}{5}-\frac{1}{5}\,y)^4+20\\
&(-141\,(\frac{3}{2}-\frac{1}{4}\,x)^6-1362\,(\frac{1}{4}\,x-\frac{1}{2})\,(\frac{3}{2}-\frac{1}{4}\,x)^5-2643\,(\frac{1}{4}\,x-\frac{1}{2})^2\,(\frac{3}{2}-\frac{1}{4}\,x)^4-6940\,(\frac{1}{4}\,x-\frac{1}{2})^3\\
&(\frac{3}{2}-\frac{1}{4}\,x)^3-2643\,(\frac{1}{4}\,x-\frac{1}{2})^4\,(\frac{3}{2}-\frac{1}{4}\,x)^2-1362\,(\frac{1}{4}\,x-\frac{1}{2})^5\,(\frac{3}{2}-\frac{1}{4}\,x)-141\,(\frac{1}{4}\,x-\frac{1}{2})^6)\\
&(\frac{1}{5}\,y-\frac{4}{5})^3\,(\frac{9}{5}-\frac{1}{5}\,y)^3+15(324\,(\frac{3}{2}-\frac{1}{4}\,x)^6+1608\,(\frac{1}{4}\,x-\frac{1}{2})\,(\frac{3}{2}-\frac{1}{4}\,x)^5+6332\,(\frac{1}{4}\,x-\frac{1}{2})^2\\
&(\frac{3}{2}-\frac{1}{4}\,x)^4+6000\,(\frac{1}{4}\,x-\frac{1}{2})^3\,(\frac{3}{2}-\frac{1}{4}\,x)^3+6332\,(\frac{1}{4}\,x-\frac{1}{2})^4\,(\frac{3}{2}-\frac{1}{4}\,x)^2+1608\,(\frac{1}{4}\,x-\frac{1}{2})^5\\
&(\frac{3}{2}-\frac{1}{4}\,x)+324\,(\frac{1}{4}\,x-\frac{1}{2})^6)(\frac{1}{5}\,y-\frac{4}{5})^4\,(\frac{9}{5}-\frac{1}{5}\,y)^2+6(-486\,(\frac{3}{2}-\frac{1}{4}\,x)^6-3852\,(\frac{1}{4}\,x-\frac{1}{2})\\
&(\frac{3}{2}-\frac{1}{4}\,x)^5-5658\,(\frac{1}{4}\,x-\frac{1}{2})^2\,(\frac{3}{2}-\frac{1}{4}\,x)^4-8680\,(\frac{1}{4}\,x-\frac{1}{2})^3\,(\frac{3}{2}-\frac{1}{4}\,x)^3-5658\,(\frac{1}{4}\,x-\frac{1}{2})^4\\
&(\frac{3}{2}-\frac{1}{4}\,x)^2-3852\,(\frac{1}{4}\,x-\frac{1}{2})^5\,(\frac{3}{2}-\frac{1}{4}\,x)-486\,(\frac{1}{4}\,x-\frac{1}{2})^6)(\frac{1}{5}\,y-\frac{4}{5})^5\,(\frac{9}{5}-\frac{1}{5}\,y)+(1404\\
&(\frac{3}{2}-\frac{1}{4}\,x)^6+2808\,(\frac{1}{4}\,x-\frac{1}{2})\,(\frac{3}{2}-\frac{1}{4}\,x)^5+7812\,(\frac{1}{4}\,x-\frac{1}{2})^2\,(\frac{3}{2}-\frac{1}{4}\,x)^4+8720\,(\frac{1}{4}\,x-\frac{1}{2})^3
\end{aligned}
$$

$$
(\frac{3}{2}-\frac{1}{4}\,x)^3+7812\,(\frac{1}{4}\,x-\frac{1}{2})^4\,(\frac{3}{2}-\frac{1}{4}\,x)^2+2808\,(\frac{1}{4}\,x-\frac{1}{2})^5\,(\frac{3}{2}-\frac{1}{4}\,x)+1404\,(\frac{1}{4}\,x-\frac{1}{2})^6)
$$
$$
(\frac{1}{5}\,y-\frac{4}{5})^6
$$

Again, the original box is cropped to a smaller area of interest. For the same reason as above the Bernstein form in the smaller box $[3,5]\times[7,8]$ is different from the previous one:

$$
\begin{aligned}
bf_{[3,5]\times[7,8]} \;=\;& (-2\,(\frac{5}{2}-\frac{1}{2}\,x)^6-12\,(\frac{1}{2}\,x-\frac{3}{2})\,(\frac{5}{2}-\frac{1}{2}\,x)^5+18\,(\frac{1}{2}\,x-\frac{3}{2})^2\,(\frac{5}{2}-\frac{1}{2}\,x)^4-8\,(\frac{1}{2}\,x-\frac{3}{2})^3\,(\frac{5}{2}-\frac{1}{2}\,x)^3\\
&+18\,(\frac{1}{2}\,x-\frac{3}{2})^4\,(\frac{5}{2}-\frac{1}{2}\,x)^2-12\,(\frac{1}{2}\,x-\frac{3}{2})^5\,(\frac{5}{2}-\frac{1}{2}\,x)-2\,(\frac{1}{2}\,x-\frac{3}{2})^6)\,(8-y)^6+6(-\frac{5}{2}\\
&(\frac{5}{2}-\frac{1}{2}\,x)^6-17\,(\frac{1}{2}\,x-\frac{3}{2})\,(\frac{5}{2}-\frac{1}{2}\,x)^5+\frac{37}{2}\,(\frac{1}{2}\,x-\frac{3}{2})^2\,(\frac{5}{2}-\frac{1}{2}\,x)^4+2\,(\frac{1}{2}\,x-\frac{3}{2})^3\,(\frac{5}{2}-\frac{1}{2}\,x)^3\\
&+\frac{37}{2}\,(\frac{1}{2}\,x-\frac{3}{2})^4\,(\frac{5}{2}-\frac{1}{2}\,x)^2-17\,(\frac{1}{2}\,x-\frac{3}{2})^5\,(\frac{5}{2}-\frac{1}{2}\,x)-\frac{5}{2}\,(\frac{1}{2}\,x-\frac{3}{2})^6)\,(y-7)\,(8-y)^5\\
&+15(-\frac{13}{5}\,(\frac{5}{2}-\frac{1}{2}\,x)^6-22\,(\frac{1}{2}\,x-\frac{3}{2})\,(\frac{5}{2}-\frac{1}{2}\,x)^5+\frac{93}{5}\,(\frac{1}{2}\,x-\frac{3}{2})^2\,(\frac{5}{2}-\frac{1}{2}\,x)^4+12\,(\frac{1}{2}\,x-\frac{3}{2})^3\\
&(\frac{5}{2}-\frac{1}{2}\,x)^3+\frac{93}{5}\,(\frac{1}{2}\,x-\frac{3}{2})^4\,(\frac{5}{2}-\frac{1}{2}\,x)^2-22\,(\frac{1}{2}\,x-\frac{3}{2})^5\,(\frac{5}{2}-\frac{1}{2}\,x)-\frac{13}{5}\,(\frac{1}{2}\,x-\frac{3}{2})^6)\,(y-7)^2\\
&(8-y)^4+20(-\frac{29}{20}\,(\frac{5}{2}-\frac{1}{2}\,x)^6-\frac{237}{10}\,(\frac{1}{2}\,x-\frac{3}{2})\,(\frac{5}{2}-\frac{1}{2}\,x)^5+\frac{477}{20}\,(\frac{1}{2}\,x-\frac{3}{2})^2\,(\frac{5}{2}-\frac{1}{2}\,x)^4+\\
&\frac{141}{5}(\frac{1}{2}\,x-\frac{3}{2})^3\,(\frac{5}{2}-\frac{1}{2}\,x)^3+\frac{477}{20}\,(\frac{1}{2}\,x-\frac{3}{2})^4\,(\frac{5}{2}-\frac{1}{2}\,x)^2-\frac{237}{10}\,(\frac{1}{2}\,x-\frac{3}{2})^5\,(\frac{5}{2}-\frac{1}{2}\,x)-\frac{29}{20}\\
&(\frac{1}{2}\,x-\frac{3}{2})^6)(y-7)^3\,(8-y)^3+15(\frac{14}{5}\,(\frac{5}{2}-\frac{1}{2}\,x)^6-\frac{68}{5}\,(\frac{1}{2}\,x-\frac{3}{2})\,(\frac{5}{2}-\frac{1}{2}\,x)^5+\frac{258}{5}\,(\frac{1}{2}\,x-\frac{3}{2})^2\\
&(\frac{5}{2}-\frac{1}{2}\,x)^4+72\,(\frac{1}{2}\,x-\frac{3}{2})^3\,(\frac{5}{2}-\frac{1}{2}\,x)^3+\frac{258}{5}\,(\frac{1}{2}\,x-\frac{3}{2})^4\,(\frac{5}{2}-\frac{1}{2}\,x)^2-\frac{68}{5}\,(\frac{1}{2}\,x-\frac{3}{2})^5\,(\frac{5}{2}-\frac{1}{2}\,x)\\
&+\frac{14}{5}\,(\frac{1}{2}\,x-\frac{3}{2})^6)\,(y-7)^4(8-y)^2+6(14\,(\frac{5}{2}-\frac{1}{2}\,x)^6+28\,(\frac{1}{2}\,x-\frac{3}{2})\,(\frac{5}{2}-\frac{1}{2}\,x)^5+146\,(\frac{1}{2}\,x-\frac{3}{2})^2\\
&(\frac{5}{2}-\frac{1}{2}\,x)^4+200\,(\frac{1}{2}\,x-\frac{3}{2})^3\,(\frac{5}{2}-\frac{1}{2}\,x)^3+146\,(\frac{1}{2}\,x-\frac{3}{2})^4\,(\frac{5}{2}-\frac{1}{2}\,x)^2+28\,(\frac{1}{2}\,x-\frac{3}{2})^5\,(\frac{5}{2}-\frac{1}{2}\,x)\\
&+14\,(\frac{1}{2}\,x-\frac{3}{2})^6)\,(y-7)^5(8-y)+(40\,(\frac{5}{2}-\frac{1}{2}\,x)^6+144\,(\frac{1}{2}\,x-\frac{3}{2})\,(\frac{5}{2}-\frac{1}{2}\,x)^5+408\,(\frac{1}{2}\,x-\frac{3}{2})^2\\
&(\frac{5}{2}-\frac{1}{2}\,x)^4+544\,(\frac{1}{2}\,x-\frac{3}{2})^3\,(\frac{5}{2}-\frac{1}{2}\,x)^3+408\,(\frac{1}{2}\,x-\frac{3}{2})^4\,(\frac{5}{2}-\frac{1}{2}\,x)^2+144\,(\frac{1}{2}\,x-\frac{3}{2})^5\,(\frac{5}{2}-\frac{1}{2}\,x)\\
&+40\,(\frac{1}{2}\,x-\frac{3}{2})^6)\,(y-7)^6.
\end{aligned}
$$

4.4 Three–dimensional example in a general box

In order to demonstrate how the results presented in Section 2 can be easily used for a general three–dimensional example, a torus centred at $(16,7,4)$ and of radii $\frac{2}{3}$ and $\frac{1}{4}$ is considered:

$$
\begin{aligned}
p \;=\;& x^4+2\,x^2\,y^2+2\,x^2\,z^2+y^4+2\,y^2\,z^2+z^4-64\,x^3-28\,x^2\,y-16\,x^2\,z-64\,x\,y^2-64\,x\,z^2-28\,y^3-16\,y^2\,z-28\,y\,z^2-16\,z^3\\
&+\frac{119879}{72}\,x^2+896\,x\,y+512\,x\,z+\frac{60263}{72}\,y^2+224\,y\,z+\frac{50887}{72}\,z^2-\frac{184604}{9}\,x-\frac{323057}{36}\,y-\frac{46279}{9}\,z+\frac{2130501025}{20736}.
\end{aligned}
$$

The Bernstein form in the box $[15,17]\times[6,8]\times[3,5]$ is:

$$
\begin{aligned}
bf \;=\;& ((4-\frac{1}{2}\,y)^4\,\%4+4\,(\frac{1}{2}\,y-3)\,(4-\frac{1}{2}\,y)^3\,\%5+6(\frac{1}{2}\,y-3)^2\,(4-\frac{1}{2}\,y)^2\\
&(\frac{79585}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4+\frac{31489}{5184}\,\%3+\frac{70753}{3456}\,\%2+\frac{31489}{5184}\,\%1+\frac{79585}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4)\\
&+4\,(\frac{1}{2}\,y-3)^3\,(4-\frac{1}{2}\,y)\,\%5+(\frac{1}{2}\,y-3)^4\,\%4)(\frac{5}{2}-\frac{1}{2}\,z)^4+4((4-\frac{1}{2}\,y)^4\,\%6
\end{aligned}
$$

$$+\,4\,(\frac{1}{2}\,y-3)\,(4-\frac{1}{2}\,y)^3\,\%7+6(\frac{1}{2}\,y-3)^2\,(4-\frac{1}{2}\,y)^2$$
$$(-\frac{5375}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4-\frac{11999}{5184}\,\%3+\frac{41089}{3456}\,\%2-\frac{11999}{5184}\,\%1-\frac{5375}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4)$$
$$+\,4\,(\frac{1}{2}\,y-3)^3\,(4-\frac{1}{2}\,y)\,\%7+(\frac{1}{2}\,y-3)^4\,\%6)(\frac{1}{2}\,z-\frac{3}{2})\,(\frac{5}{2}-\frac{1}{2}\,z)^3+6((4-\frac{1}{2}\,y)^4$$
$$(\frac{30433}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4-\frac{17663}{5184}\,\%3+\frac{21601}{3456}\,\%2-\frac{17663}{5184}\,\%1+\frac{30433}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4)+4$$
$$(\frac{1}{2}\,y-3)\,(4-\frac{1}{2}\,y)^3$$
$$(-\frac{17663}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4-\frac{24287}{5184}\,\%3+\frac{28801}{3456}\,\%2-\frac{24287}{5184}\,\%1-\frac{17663}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4)+$$
$$6(\frac{1}{2}\,y-3)^2\,(4-\frac{1}{2}\,y)^2$$
$$(\frac{21601}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4+\frac{28801}{5184}\,\%3+\frac{86497}{3456}\,\%2+\frac{28801}{5184}\,\%1+\frac{21601}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4)+4$$
$$(\frac{1}{2}\,y-3)^3\,(4-\frac{1}{2}\,y)$$
$$(-\frac{17663}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4-\frac{24287}{5184}\,\%3+\frac{28801}{3456}\,\%2-\frac{24287}{5184}\,\%1-\frac{17663}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4)+$$
$$(\frac{1}{2}\,y-3)^4$$
$$(\frac{30433}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4-\frac{17663}{5184}\,\%3+\frac{21601}{3456}\,\%2-\frac{17663}{5184}\,\%1+\frac{30433}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4))$$
$$(\frac{1}{2}\,z-\frac{3}{2})^2\,(\frac{5}{2}-\frac{1}{2}\,z)^2+4((4-\frac{1}{2}\,y)^4\,\%6+4\,(\frac{1}{2}\,y-3)\,(4-\frac{1}{2}\,y)^3\,\%7+6(\frac{1}{2}\,y-3)^2$$
$$(4-\frac{1}{2}\,y)^2$$
$$(-\frac{5375}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4-\frac{11999}{5184}\,\%3+\frac{41089}{3456}\,\%2-\frac{11999}{5184}\,\%1-\frac{5375}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4)$$
$$+\,4\,(\frac{1}{2}\,y-3)^3\,(4-\frac{1}{2}\,y)\,\%7+(\frac{1}{2}\,y-3)^4\,\%6)(\frac{1}{2}\,z-\frac{3}{2})^3\,(\frac{5}{2}-\frac{1}{2}\,z)+((4-\frac{1}{2}\,y)^4\,\%4$$
$$+\,4\,(\frac{1}{2}\,y-3)\,(4-\frac{1}{2}\,y)^3\,\%5+6(\frac{1}{2}\,y-3)^2\,(4-\frac{1}{2}\,y)^2$$
$$(\frac{79585}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4+\frac{31489}{5184}\,\%3+\frac{70753}{3456}\,\%2+\frac{31489}{5184}\,\%1+\frac{79585}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4)$$
$$+\,4\,(\frac{1}{2}\,y-3)^3\,(4-\frac{1}{2}\,y)\,\%5+(\frac{1}{2}\,y-3)^4\,\%4)(\frac{1}{2}\,z-\frac{3}{2})^4$$
$$\%1:=(\frac{1}{2}\,x-\frac{15}{2})^3\,(\frac{17}{2}-\frac{1}{2}\,x)$$
$$\%2:=(\frac{1}{2}\,x-\frac{15}{2})^2\,(\frac{17}{2}-\frac{1}{2}\,x)^2$$
$$\%3:=(\frac{1}{2}\,x-\frac{15}{2})\,(\frac{17}{2}-\frac{1}{2}\,x)^3$$
$$\%4:=\frac{162145}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4+\frac{58753}{5184}\,\%3+\frac{79585}{3456}\,\%2+\frac{58753}{5184}\,\%1+\frac{162145}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4$$
$$\%5:=\frac{58753}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4-\frac{3167}{5184}\,\%3+\frac{31489}{3456}\,\%2-\frac{3167}{5184}\,\%1+\frac{58753}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4$$
$$\%6:=\frac{21889}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4-\frac{40031}{5184}\,\%3-\frac{5375}{3456}\,\%2-\frac{40031}{5184}\,\%1+\frac{21889}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4$$
$$\%7:=-\frac{40031}{20736}\,(\frac{17}{2}-\frac{1}{2}\,x)^4-\frac{60479}{5184}\,\%3-\frac{11999}{3456}\,\%2-\frac{60479}{5184}\,\%1-\frac{40031}{20736}\,(\frac{1}{2}\,x-\frac{15}{2})^4$$

The Bernstein form of the same torus in the box $[16,17]\times[6,7]\times[\frac{7}{2},\frac{9}{2}]$ is:

$$
\begin{aligned}
&((7-y)^4(\frac{17065}{20736}(17-x)^4+\frac{17065}{5184}(x-16)(17-x)^3+\frac{22201}{3456}\%1\\
&\quad+\frac{32473}{5184}(x-16)^3(17-x)+\frac{68617}{20736}(x-16)^4)+4(y-6)(7-y)^3(\frac{1657}{20736}(17-x)^4\\
&\quad+\frac{1657}{5184}(x-16)(17-x)^3+\frac{3337}{3456}\%1+\frac{6697}{5184}(x-16)^3(17-x)+\frac{32473}{20736}(x-16)^4)\\
&\quad+6(y-6)^2(7-y)^2(\frac{5209}{20736}(17-x)^4+\frac{5209}{5184}(x-16)(17-x)^3+\frac{4585}{3456}\%1\\
&\quad+\frac{3337}{5184}(x-16)^3(17-x)+\frac{22201}{20736}(x-16)^4)+4(y-6)^3(7-y)\%2+(y-6)^4\%2\\
&\quad)(\frac{9}{2}-z)^4+4((7-y)^4(\frac{145}{20736}(17-x)^4+\frac{145}{5184}(x-16)(17-x)^3+\frac{3553}{3456}\%1\\
&\quad+\frac{10369}{5184}(x-16)^3(17-x)+\frac{41329}{20736}(x-16)^4)+4(y-6)(7-y)^3(\\
&\quad-\frac{10079}{20736}(17-x)^4-\frac{10079}{5184}(x-16)(17-x)^3-\frac{10127}{3456}\%1\\
&\quad-\frac{10223}{5184}(x-16)^3(17-x)+\frac{10369}{20736}(x-16)^4)+6(y-6)^2(7-y)^2(\\
&\quad-\frac{3071}{20736}(17-x)^4-\frac{3071}{5184}(x-16)(17-x)^3-\frac{5423}{3456}\%1-\frac{10127}{5184}(x-16)^3(17-x)\\
&\quad+\frac{3553}{20736}(x-16)^4)+4(y-6)^3(7-y)\%3+(y-6)^4\%3)(z-\frac{7}{2})(\frac{9}{2}-z)^3+6(\\
&\quad(7-y)^4(-\frac{2039}{20736}(17-x)^4-\frac{2039}{5184}(x-16)(17-x)^3+\frac{793}{3456}\%1\\
&\quad+\frac{6457}{5184}(x-16)^3(17-x)+\frac{35689}{20736}(x-16)^4)+4(y-6)(7-y)^3(-\frac{10535}{20736}(17-x)^4\\
&\quad-\frac{10535}{5184}(x-16)(17-x)^3-\frac{11159}{3456}\%1-\frac{12407}{5184}(x-16)^3(17-x)\\
&\quad+\frac{6457}{20736}(x-16)^4)+6(y-6)^2(7-y)^2(-\frac{2375}{20736}(17-x)^4\\
&\quad-\frac{2375}{5184}(x-16)(17-x)^3-\frac{5303}{3456}\%1-\frac{11159}{5184}(x-16)^3(17-x)+\frac{793}{20736}(x-16)^4\\
&\quad)+4(y-6)^3(7-y)(\frac{1705}{20736}(17-x)^4+\frac{1705}{5184}(x-16)(17-x)^3-\frac{2375}{3456}\%1\\
&\quad-\frac{10535}{5184}(x-16)^3(17-x)-\frac{2039}{20736}(x-16)^4)+(y-6)^4(\frac{1705}{20736}(17-x)^4\\
&\quad+\frac{1705}{5184}(x-16)(17-x)^3-\frac{2375}{3456}\%1-\frac{10535}{5184}(x-16)^3(17-x)-\frac{2039}{20736}(x-16)^4\\
&\quad))(z-\frac{7}{2})^2(\frac{9}{2}-z)^2+4((7-y)^4(\frac{145}{20736}(17-x)^4+\frac{145}{5184}(x-16)(17-x)^3\\
&\quad+\frac{3553}{3456}\%1+\frac{10369}{5184}(x-16)^3(17-x)+\frac{41329}{20736}(x-16)^4)+4(y-6)(7-y)^3(\\
&\quad-\frac{10079}{20736}(17-x)^4-\frac{10079}{5184}(x-16)(17-x)^3-\frac{10127}{3456}\%1\\
&\quad-\frac{10223}{5184}(x-16)^3(17-x)+\frac{10369}{20736}(x-16)^4)+6(y-6)^2(7-y)^2(\\
&\quad-\frac{3071}{20736}(17-x)^4-\frac{3071}{5184}(x-16)(17-x)^3-\frac{5423}{3456}\%1-\frac{10127}{5184}(x-16)^3(17-x)\\
&\quad+\frac{3553}{20736}(x-16)^4)+4(y-6)^3(7-y)\%3+(y-6)^4\%3)(z-\frac{7}{2})^3(\frac{9}{2}-z)+(\\
&\quad(7-y)^4(\frac{17065}{20736}(17-x)^4+\frac{17065}{5184}(x-16)(17-x)^3+\frac{22201}{3456}\%1\\
&\quad+\frac{32473}{5184}(x-16)^3(17-x)+\frac{68617}{20736}(x-16)^4)+4(y-6)(7-y)^3(\frac{1657}{20736}(17-x)^4
\end{aligned}
$$

$$+\frac{1657}{5184}(x-16)(17-x)^3+\frac{3337}{3456}\%1+\frac{6697}{5184}(x-16)^3(17-x)+\frac{32473}{20736}(x-16)^4)$$
$$+6(y-6)^2(7-y)^2(\frac{5209}{20736}(17-x)^4+\frac{5209}{5184}(x-16)(17-x)^3+\frac{4585}{3456}\%1$$
$$+\frac{3337}{5184}(x-16)^3(17-x)+\frac{22201}{20736}(x-16)^4)+4(y-6)^3(7-y)\%2+(y-6)^4\%2$$
$$)(z-\frac{7}{2})^4$$
$$\%1:=(x-16)^2(17-x)^2$$
$$\%2:=\frac{6985}{20736}(17-x)^4+\frac{6985}{5184}(x-16)(17-x)^3+\frac{5209}{3456}\%1+\frac{1657}{5184}(x-16)^3(17-x)$$
$$+\frac{17065}{20736}(x-16)^4$$
$$\%3:=\frac{433}{20736}(17-x)^4+\frac{433}{5184}(x-16)(17-x)^3-\frac{3071}{3456}\%1-\frac{10079}{5184}(x-16)^3(17-x)$$
$$+\frac{145}{20736}(x-16)^4$$

5 Scaling of curves

In this section a curve similar to the one given in Example 4 of Section 3 is studied. The difference, though, is that the current one has rational coefficients, as opposed the floating–point ones used in Section 3.

The rational–coefficient curve is considered at first in the unit box. Then the curve is scaled to a gradually increasing area of interest.

In the first case the rational curve is considered in the box $[0,1]\times[0,1]$ and is given by:

$$p_{[0,1]\times[0,1]} = x^9-x^7\,y+3\,x^2\,y^6-y^3+y^5+y^4\,x-4\,y^4\,x^3$$

and its Bernstein form:

$$\begin{aligned}
bf_{[0,1]\times[0,1]} = {} & x^9(1-y)^6+6(-\frac{1}{6}x^7(1-x)^2-\frac{1}{3}x^8(1-x)+\frac{5}{6}x^9)\,y\,(1-y)^5\\
&+15(-\frac{1}{3}x^7(1-x)^2-\frac{2}{3}x^8(1-x)+\frac{2}{3}x^9)\,y^2(1-y)^4+20(-\frac{1}{20}(1-x)^9\\
&-\frac{9}{20}x(1-x)^8-\frac{9}{5}x^2(1-x)^7-\frac{21}{5}x^3(1-x)^6-\frac{63}{10}x^4(1-x)^5-\frac{63}{10}x^5(1-x)^4\\
&-\frac{21}{5}x^6(1-x)^3-\frac{23}{10}x^7(1-x)^2-\frac{29}{20}x^8(1-x)+\frac{9}{20}x^9)y^3(1-y)^3+15(\\
&-\frac{1}{5}(1-x)^9-\frac{26}{15}x(1-x)^8-\frac{20}{3}x^2(1-x)^7-\frac{76}{5}x^3(1-x)^6-\frac{346}{15}x^4(1-x)^5\\
&-\frac{368}{15}x^5(1-x)^4-\frac{92}{5}x^6(1-x)^3-10\,x^7(1-x)^2-\frac{21}{5}x^8(1-x)-\frac{1}{15}x^9)y^4\\
&(1-y)^2+6(-\frac{1}{3}(1-x)^9-\frac{8}{3}x(1-x)^8-\frac{28}{3}x^2(1-x)^7-20\,x^3(1-x)^6\\
&-\frac{94}{3}x^4(1-x)^5-\frac{116}{3}x^5(1-x)^4-36\,x^6(1-x)^3-\frac{47}{2}x^7(1-x)^2-10\,x^8(1-x)\\
&-\frac{7}{6}x^9)y^5(1-y)+(x(1-x)^8+11\,x^2(1-x)^7+45\,x^3(1-x)^6+95\,x^4(1-x)^5\\
&+115\,x^5(1-x)^4+81\,x^6(1-x)^3+30\,x^7(1-x)^2+3\,x^8(1-x))y^6
\end{aligned}$$

In the second case the rational curve is scaled to the box $[0,2]\times[0,2]$ and is given by:

$$p_{[0,2]\times[0,2]} = \frac{1}{512}x^9-\frac{1}{256}x^7\,y+\frac{3}{256}x^2\,y^6-\frac{1}{8}y^3+\frac{1}{32}y^5+\frac{1}{32}y^4\,x-\frac{1}{32}y^4\,x^3$$

and its Bernstein form in this box is:

$$
\begin{aligned}
bf_{[0,2]\times[0,2]} \;=\;& \frac{1}{512}\,x^9\,(1-\frac{1}{2}\,y)^6+3\,(-\frac{1}{768}\,x^7\,(1-\frac{1}{2}\,x)^2-\frac{1}{768}\,x^8\,(1-\frac{1}{2}\,x)+\frac{5}{3072}\,x^9)\,y\,(1-\frac{1}{2}\,y)^5 \\
&+\frac{15}{4}\,(-\frac{1}{384}\,x^7\,(1-\frac{1}{2}\,x)^2-\frac{1}{384}\,x^8\,(1-\frac{1}{2}\,x)+\frac{1}{768}\,x^9)\,y^2\,(1-\frac{1}{2}\,y)^4+\frac{5}{2}(\\
&-\frac{1}{20}\,(1-\frac{1}{2}\,x)^9-\frac{9}{40}\,x\,(1-\frac{1}{2}\,x)^8-\frac{9}{20}\,x^2\,(1-\frac{1}{2}\,x)^7-\frac{21}{40}\,x^3\,(1-\frac{1}{2}\,x)^6 \\
&-\frac{63}{160}\,x^4\,(1-\frac{1}{2}\,x)^5-\frac{63}{320}\,x^5\,(1-\frac{1}{2}\,x)^4-\frac{21}{320}\,x^6\,(1-\frac{1}{2}\,x)^3-\frac{23}{1280}\,x^7\,(1-\frac{1}{2}\,x)^2 \\
&-\frac{29}{5120}\,x^8\,(1-\frac{1}{2}\,x)+\frac{9}{10240}\,x^9)y^3\,(1-\frac{1}{2}\,y)^3+\frac{15}{16}(-\frac{1}{5}\,(1-\frac{1}{2}\,x)^9-\frac{13}{15}\,x\,(1-\frac{1}{2}\,x)^8 \\
&-\frac{5}{3}\,x^2\,(1-\frac{1}{2}\,x)^7-\frac{19}{10}\,x^3\,(1-\frac{1}{2}\,x)^6-\frac{173}{120}\,x^4\,(1-\frac{1}{2}\,x)^5-\frac{23}{30}\,x^5\,(1-\frac{1}{2}\,x)^4 \\
&-\frac{23}{80}\,x^6\,(1-\frac{1}{2}\,x)^3-\frac{5}{64}\,x^7\,(1-\frac{1}{2}\,x)^2-\frac{21}{1280}\,x^8\,(1-\frac{1}{2}\,x)-\frac{1}{7680}\,x^9)y^4\,(1-\frac{1}{2}\,y)^2 \\
&+\frac{3}{16}(-\frac{1}{3}\,(1-\frac{1}{2}\,x)^9-\frac{4}{3}\,x\,(1-\frac{1}{2}\,x)^8-\frac{7}{3}\,x^2\,(1-\frac{1}{2}\,x)^7-\frac{5}{2}\,x^3\,(1-\frac{1}{2}\,x)^6 \\
&-\frac{47}{24}\,x^4\,(1-\frac{1}{2}\,x)^5-\frac{29}{24}\,x^5\,(1-\frac{1}{2}\,x)^4-\frac{9}{16}\,x^6\,(1-\frac{1}{2}\,x)^3-\frac{47}{256}\,x^7\,(1-\frac{1}{2}\,x)^2 \\
&-\frac{5}{128}\,x^8\,(1-\frac{1}{2}\,x)-\frac{7}{3072}\,x^9)y^5\,(1-\frac{1}{2}\,y)+\frac{1}{64}(\frac{1}{2}\,x\,(1-\frac{1}{2}\,x)^8+\frac{11}{4}\,x^2\,(1-\frac{1}{2}\,x)^7 \\
&+\frac{45}{8}\,x^3\,(1-\frac{1}{2}\,x)^6+\frac{95}{16}\,x^4\,(1-\frac{1}{2}\,x)^5+\frac{115}{32}\,x^5\,(1-\frac{1}{2}\,x)^4+\frac{81}{64}\,x^6\,(1-\frac{1}{2}\,x)^3 \\
&+\frac{15}{64}\,x^7\,(1-\frac{1}{2}\,x)^2+\frac{3}{256}\,x^8\,(1-\frac{1}{2}\,x))y^6
\end{aligned}
$$

In the last case the rational curve is scaled to the box $[0,50]\times[0,50]$ and is given by:

$$
\begin{aligned}
p_{[0,50]\times[0,50]} \;=\;& \frac{1}{1953125000000000}\,x^9-\frac{1}{39062500000000}\,x^7\,y+\frac{3}{39062500000000}\,x^2\,y^6-\frac{1}{125000}\,y^3 \\
&+\frac{1}{312500000}\,y^5+\frac{1}{312500000}\,y^4\,x-\frac{1}{195312500000}\,y^4\,x^3
\end{aligned}
$$

and its Bernstein form:

$$
\begin{aligned}
bf_{[0,50]\times[0,50]} \;=\;& \frac{1}{1953125000000000}\,x^9\,(1-\frac{1}{50}\,y)^6+\frac{3}{25}(-\frac{1}{4687500000000}\,x^7\,(1-\frac{1}{50}\,x)^2 \\
&-\frac{1}{117187500000000}\,x^8\,(1-\frac{1}{50}\,x)+\frac{1}{2343750000000000}\,x^9)y\,(1-\frac{1}{50}\,y)^5+\frac{3}{500} \\
&(-\frac{1}{2343750000000}\,x^7\,(1-\frac{1}{50}\,x)^2-\frac{1}{58593750000000}\,x^8\,(1-\frac{1}{50}\,x) \\
&+\frac{1}{2929687500000000}\,x^9)y^2\,(1-\frac{1}{50}\,y)^4+\frac{1}{6250}(-\frac{1}{20}\,(1-\frac{1}{50}\,x)^9 \\
&-\frac{9}{1000}\,x\,(1-\frac{1}{50}\,x)^8-\frac{9}{12500}\,x^2\,(1-\frac{1}{50}\,x)^7-\frac{21}{625000}\,x^3\,(1-\frac{1}{50}\,x)^6 \\
&-\frac{63}{62500000}\,x^4\,(1-\frac{1}{50}\,x)^5-\frac{63}{3125000000}\,x^5\,(1-\frac{1}{50}\,x)^4 \\
&-\frac{21}{78125000000}\,x^6\,(1-\frac{1}{50}\,x)^3-\frac{23}{7812500000000}\,x^7\,(1-\frac{1}{50}\,x)^2 \\
&-\frac{29}{781250000000000}\,x^8\,(1-\frac{1}{50}\,x)+\frac{9}{39062500000000000}\,x^9)y^3\,(1-\frac{1}{50}\,y)^3 \\
&+\frac{3}{1250000}(-\frac{1}{5}\,(1-\frac{1}{50}\,x)^9-\frac{13}{375}\,x\,(1-\frac{1}{50}\,x)^8-\frac{1}{375}\,x^2\,(1-\frac{1}{50}\,x)^7 \\
&-\frac{19}{156250}\,x^3\,(1-\frac{1}{50}\,x)^6-\frac{173}{46875000}\,x^4\,(1-\frac{1}{50}\,x)^5-\frac{23}{292968750}\,x^5\,(1-\frac{1}{50}\,x)^4 \\
&-\frac{23}{19531250000}\,x^6\,(1-\frac{1}{50}\,x)^3-\frac{1}{78125000000}\,x^7\,(1-\frac{1}{50}\,x)^2
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{21}{195312500000000}\,x^8\,(1-\frac{1}{50}\,x) - \frac{1}{29296875000000000}\,x^9)y^4\,(1-\frac{1}{50}\,y)^2 \\
&+ \frac{3}{156250000}(-\frac{1}{3}\,(1-\frac{1}{50}\,x)^9 - \frac{4}{75}\,x\,(1-\frac{1}{50}\,x)^8 - \frac{7}{1875}\,x^2\,(1-\frac{1}{50}\,x)^7 \\
&- \frac{1}{6250}\,x^3\,(1-\frac{1}{50}\,x)^6 - \frac{47}{9375000}\,x^4\,(1-\frac{1}{50}\,x)^5 - \frac{29}{234375000}\,x^5\,(1-\frac{1}{50}\,x)^4 \\
&- \frac{9}{3906250000}\,x^6\,(1-\frac{1}{50}\,x)^3 - \frac{47}{1562500000000}\,x^7\,(1-\frac{1}{50}\,x)^2 \\
&- \frac{1}{3906250000000}\,x^8\,(1-\frac{1}{50}\,x) - \frac{7}{117187500000000000}\,x^9)y^5\,(1-\frac{1}{50}\,y) \\
&+ \frac{1}{15625000000}(\frac{1}{50}\,x\,(1-\frac{1}{50}\,x)^8 + \frac{11}{2500}\,x^2\,(1-\frac{1}{50}\,x)^7 + \frac{9}{25000}\,x^3\,(1-\frac{1}{50}\,x)^6 \\
&+ \frac{19}{1250000}\,x^4\,(1-\frac{1}{50}\,x)^5 + \frac{23}{62500000}\,x^5\,(1-\frac{1}{50}\,x)^4 + \frac{81}{15625000000}\,x^6\,(1-\frac{1}{50}\,x)^3 \\
&+ \frac{3}{78125000000}\,x^7\,(1-\frac{1}{50}\,x)^2 + \frac{3}{39062500000000}\,x^8\,(1-\frac{1}{50}\,x))y^6
\end{aligned}
$$

These examples show that scaling does not affect the shape of the curve: when a curve is scaled to a different area of interest, the coefficients of both the power and the Bernstein form are adjusted so that the curve fits the new box.

6 Conclusions

In general the Bernstein–form polynomials generated by the methods introduced in Section 2 have more non–zero terms than their corresponding power forms.

The power base $(1, x, x^2, x^3, ...)$ is the same for all curves, but the power–form coefficients differ from curve to curve. However, when a curve is translated to a new area of interest, its power form changes radically, in that its new coefficients are completely different.

Quite the opposite happens to the Bernstein form. A Bernstein form can only be calculated with respect to a region of interest. The Bernstein basis corresponding to that region has a finite number of terms. When a given Bernstein–form polynomial is translated in a new area of interest, only the Bernstein basis changes, but its coefficients stay the same. However, the Bernstein coefficients do change if the new area of interest is clipped so that it contains only part of the surface.

Because of the advantages listed above (see Section 1.2), the implicit Bernstein form of a polynomial could be an alternative to the classic power–form surface representation in CSG. This alternative requires all the surfaces to be rewritten in their Bernstein forms and all the methods to be applied to Bernstein representations. This involves the manipulation (e.g. arithmetic) of multivariate Bernstein–form polynomials.

7 Further work

To handle or to display curves and surfaces in computer aided design it is necessary to locate these objects in the modelling space. The location can be carried out by classifying whole regions of space at the same time. This can be achieved, for example, by using interval arithmetic.

Given the axially–aligned box $[\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] \times [\underline{z}, \overline{z}]$, the three variables of the surface expression x, y and z are replaced by the three interval coordinates $[\underline{x}, \overline{x}]$, $[\underline{y}, \overline{y}]$ and $[\underline{z}, \overline{z}]$ respectively. This substitution produces an interval expression. By applying interval arithmetic rules, evaluating the surface expression results in an interval.

If this interval is all negative then the box contains solid; if the interval is all positive then the box contains air; if the interval straddles zero the box is assumed to contain surface, solid and air (and the method classifies it as an *unknown box*).

A study of how interval arithmetic is influenced by the polynomial representation is necessary. In our paper "Experiments on multivariate power– and Bernstein–form polynomials using interval arithmetic" (submitted for a Special Issue of the Computational Geometry Journal) the behaviour of the interval arithmetic applied to multivariate power– and Bernstein–form polynomials will be studied.

Our set–theoretic geometric modeller sVLIS [1] uses interval arithmetic on implicit power–form polynomials. Of course, it would also be possible to use the Bernstein form of polynomials instead. This modification would automatically improve the accuracy of the location methods, and therefore the accuracy of the geometric representation itself.

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