# An Accessible Introduction to the Chaotic Nature of the Complex Exponential Map

Project 1 presentation

Jacob Rider

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Georgia State University

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# Intro

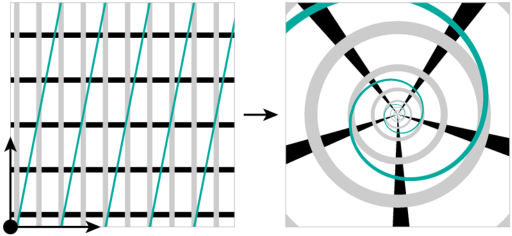
# The Complex Exponential Map

The complex exponential map is defined by the following equations:

$$f(z) = e^z \, ; z \in \mathbb{C} \Rightarrow z = x + iy \, ; x, y \in \mathbb{R}, \, i = \sqrt{-1}$$

$$f^n(z_0) = z_n = e^{z_{n-1}} \text{ and } f^{-n}(z_n) = z_0$$

$$e^z = e^{x+iy} = e^x \cdot (\cos(y) + i \cdot \sin(y))$$



# Main Results

*Deep Breath*

# Overview

## The following theorems describe the main results of the paper:

**Theorem 1.1 (Orbits of the complex exponential map).** *Each of the following sets is dense in the complex plane:*

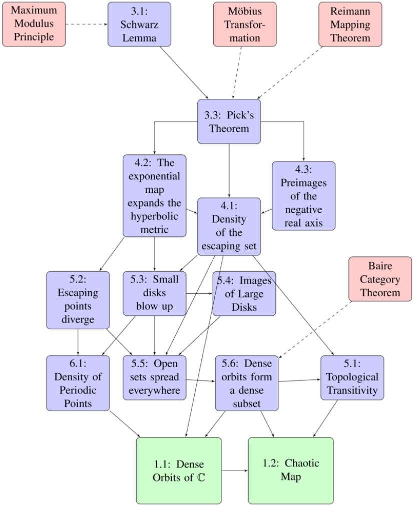
1. *the set of starting values $z_0$ whose orbit (defined by* (1.1)*) diverges to $\infty$;*
2. *the set of starting values $z_0$ whose orbit forms a dense subset of the plane;*
3. *the set of* periodic points*; i.e., starting values $z_0$ such that $z_{n+k} = z_n$ for some $k > 0$ and all $n \geq 0$.*

**Theorem 1.2 (The exponential map is chaotic).** *The complex exponential map $f : \mathbb{C} \to \mathbb{C};\ z \mapsto e^z$ is chaotic in the sense of Devaney.*

**Definition 2.1 (Devaney chaos).** Let $X \subset \mathbb{C}$ be infinite, and let $f : X \to X$ be continuous. We say that $f$ is *chaotic* (in the sense of Devaney) if the following two conditions are satisfied.

1. The set of periodic points of $f$ is *dense* in $X$.
2. The function $f$ is *topologically transitive*; that is, for all open sets $U, V \subset \mathbb{C}$ that intersect $X$, there is a point $z \in U \cap X$ and $n \geq 0$ such that $f^n(z) \in V$.

The following diagram shows the proof structure of the paper written by Shen and Rempe-Gillen (2015):



# Pick's Theorem: Introduction of the Hyperbolic Metric

**Theorem 3.3 (Pick's theorem).** *For every simply connected domain $U \subsetneq \mathbb{C}$, there exists a unique conformal metric $\rho_U(z)|\mathrm{d}z|$ on $U$, called the* hyperbolic metric, *such that the following hold:*

1. $\rho_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}$ *for all $z \in \mathbb{D}$;*
2. *if $f : U \to V$ is holomorphic, then $f$ does not increase hyperbolic distance; i.e.,*

$$\|\mathrm{D}f(z)\|_U^V := |f'(z)| \cdot \frac{\rho_V(f(z))}{\rho_U(z)} \le 1;$$

3. *for any $z \in U$ and any $f$ as above, we have $\|\mathrm{D}f(z)\|_U^V = 1$ if and only if $f$ is a conformal isomorphism between $U$ and $V$;*
4. *if $U \subsetneq V$, then $\rho_U(z) > \rho_V(z)$ for all $z \in U$.*

(a) The unit disk $\mathbb{D}$: Escher's "Circle Limit IV"

(b) The slit plane $\mathbb{C} \setminus (-\infty, 0]$

**Purpose:** establishes the concept of a hyperbolic metric.

**Main idea:** an alternative notion of distance based on a restricted domain (e.g. Arc length on a sphere).

**Notation:**

$||Df(\zeta_n)||_{U \to U} \Rightarrow$ Hyperbolic Distance

$\rho_U(z) \Rightarrow$ Hyperbolic Density

**Example:** $\rho_{\mathbb{C}\setminus[0,\infty)}(z) = \dfrac{1}{2|z|\sin(\arg(z)/2)}$

# Theorem 4.1: Density of the Escaping Set

**Theorem 4.1 (Density of the escaping set).** *The set $I(f)$, consisting of those points $z_0$ whose $f$-orbits converge to infinity, is a dense subset of the complex plane $\mathbb{C}$.*

*[Paraphrased proof]:* "$\mathbb{R} \subset I(f)$, so if $\exists n \in \mathbb{N}$ s/t $f^n(z) \in \mathbb{R}$ then $z \in I(f)$. It is necessary to define the slit plane, $U := \mathbb{C} \setminus [0, \infty)$. The proof assumes (by contradiction) that $\exists$ a small disk, $D$ s/t $D \cap I(f) = \emptyset$. Since there are no escaping points in D, this implies that $f^n(D)$ has a finite accumulation point (the smallest point of a sequence such that every point after it is contained in an arbitrarily small disk around the finite accumulation point). Lemma 4.2 shows that all finite accumulation points $(w_n)$ must lie on the interval $[0, \infty)$. Let $z_0$ be the smallest accumulation point of $(w_n)$ s/t $w_{n_k} \to z_0$. Recall: since $w_{n_k} = e^{w_{n_{k-1}}} \Rightarrow log|w_{n_k}| = Re(w_{n_{k-1}})$. A contradiction is reached since: $log|w_{n_k}| = Re(w_{n_{k-1}}) \to log|z_0| < z_0$. Therefore $z_0$ is not the smallest accumulation point of $(w_n)$ and thus $D \cap I(f) \neq \emptyset$." ■

I.e. Escaping points (points that diverge to $\infty$ over the exponential map) are scattered densely over the complex plane.

# Lemma 4.2: Expansion of the Hyperbolic Metric

**Lemma 4.2 (The exponential map expands the hyperbolic metric).** *The complex exponential map $f$ locally expands the hyperbolic metric of $U := \mathbb{C} \setminus [0, \infty)$. That is, $\|\mathrm{D}f(\zeta)\|_U^U > 1$ for all $\zeta \in f^{-1}(U)$.*

*Moreover, suppose that $(\zeta_n)_{n\geq 0}$ is a sequence with $\zeta_n \in f^{-1}(U)$ for all $n$ and $\|\mathrm{D}f(\zeta_n)\|_U^U \to 1$ as $n \to \infty$. Then $\min\big(|\zeta_n|, \operatorname{Arg}(\zeta_n)\big) \to 0$ as $n \to \infty$.*

*[Paraphrased proof]:* "Let the polar form of a complex variable be written as: $\zeta = re^{i\theta}$ s/t $0 < \theta < 2\pi$, then $\omega = f(\zeta) = e^{re^{i\theta}} = e^{r(\cos(\theta)+i\cdot\sin(\theta))}$. Using the formula for hyperbolic derivative:

$$\underset{U\to U}{||Df(\zeta)||} = |f'(\zeta)| \cdot \frac{\rho_U(f(\zeta))}{\rho_U(\zeta)}, \text{ where } \rho_U(f(\zeta)) = \frac{1}{2|\zeta|\sin(\frac{arg(\zeta)}{2})} \Rightarrow$$

$$\underset{U\to U}{||Df(\zeta)||} = |f'(\zeta)| \cdot \frac{2|\zeta|\cdot\sin(.5\cdot arg(\zeta))}{2|e^{\zeta}|\cdot\sin(.5\cdot arg(e^{\zeta}))} = \frac{r\cdot\sin(.5\cdot\theta)}{\sin(.5\cdot arg(e^{\zeta}))} \underset{(1)}{=} \frac{r\cdot\sin(.5\cdot\theta)}{\sin(.5\cdot r\cdot\sin(\theta))} \Rightarrow$$

$$\frac{r\cdot\sin(.5\cdot\theta)}{\sin(.5\cdot r\cdot\sin(\theta))} \underset{(2)}{\geq} \frac{r\cdot\sin(.5\cdot\theta)}{.5r\cdot\sin|\theta|} = \frac{2\cdot\sin(.5\cdot\theta)}{|\sin(\theta)|} \underset{(3)}{=} \frac{1}{\cos(.5\cdot\theta)} > 1$$

(1) $\theta \in (0, 2\pi) \Rightarrow arg(e^{\zeta}) \equiv r\cdot\sin(\theta)(mod 2\pi) \Rightarrow |\sin(.5\cdot arg(e^{\zeta}))| = |\sin(.5r\cdot\sin(\theta)|$

(2) $\forall x \in \mathbb{R}, |\sin(x)| \leq |x|$

(3) $\sin(2x) = 2\cdot\sin(x)\cdot\cos(x)$

$\therefore \forall\zeta_n$ s/t $\underset{U\to U}{||Df(\zeta_n)||} \to 1$, $|cos(\frac{\theta_n}{2}| \to 1 \Rightarrow \theta_n \to 0$. This shows that $\underset{U\to U}{||Df(\zeta_n)||} \to 1 \Leftrightarrow \zeta_n \to \mathbb{R}$" ■

I.e. The set of finite accumulation points of the inverse map is a subset of the positive real numbers.

# Section 5: Theorems and Informal Translations

**Theorem 5.1 (Topological transitivity).** *If $U$, $V$ are nonempty and open, then there exists $n \geq 0$ such that $f^n(U) \bigcap V \neq \emptyset$.*

$\Rightarrow$ 5.1: Formal definition of topological transitivity.

**Observation 5.2 (Expansion along escaping orbits).** *Let $z_0 \in I(f)$, and define $z_n := f^n(z_0)$ for $n \geq 1$. Then $\operatorname{Re} z_n \to \infty$ and $|(f^n)'(z_0)| \to \infty$ as $n \to \infty$.*

$\Rightarrow$ 5.2: Points in the escaping set diverge to $\infty$

**Proposition 5.3 (Small disks blow up).** *Let $z_0 \in I(f)$. For $n \geq 1$, set $z_n := f^n(z_0)$ and consider the disk $D_n$ of radius $2\pi$ centered at $z_n$; i.e. $D_n := D_{2\pi}(z_n)$. Then there are $n_0 \in \mathbb{N}$ and a sequence $(\phi_n)_{n \geq n_0}$ of holomorphic maps $\phi_n \colon D_n \to \mathbb{C}$ with the following properties:*

1. $\phi_n(z_n) = z_0$,
2. $f^n(\phi_n(z)) = z$ *for all* $z \in D_n$,
3. $\sup_{z \in D_n} |\phi_n'(z)| \to 0$ *as* $n \to \infty$, *and*
4. $\operatorname{diam}(\phi(D_n)) \to 0$ *as* $n \to \infty$.

$\Rightarrow$ 5.3: Defines the inverse map, $\phi_n$, so any disk of radius $2\pi$ shrinks under this map.

**Observation 5.4 (Images of large disks).** *Let $K$ be any nonempty compact subset of the punctured plane $\mathbb{C} \setminus \{0\}$. Then there is $\rho > 0$ with the following property. Suppose that $D$ is a disk of radius $2\pi$, centered at a point $\zeta$ having real part at least $\rho$. Then $K \subset f^2(D)$.*

$\Rightarrow$ 5.4: The twice-iterated image of any disk of radius $2\pi$ centered at $\rho$ covers any bounded set.

**Corollary 5.5 (Open sets spread everywhere).** *Let $K \subset \mathbb{C} \setminus \{0\}$ be compact, and let $U \subset \mathbb{C}$ be open and nonempty. Then there is some $N \in \mathbb{N}$ such that $K \subset f^n(U)$ for all $n \geq N$.*

$\Rightarrow$ 5.5: The image of any arbitrary disk covers any bounded set of the plane.

# Conclusion

# 5.6: Dense Orbits

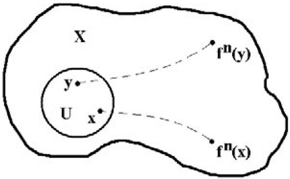
**Corollary 5.6 (Dense orbits).** *The set $\mathcal{T}$ of all points $z_0 \in \mathbb{C}$ with dense orbits under the exponential map is uncountable and dense in $\mathbb{C}$.*

*[Paraphrased proof]:* "The proof relies on the Baire category theorem which states that every countable intersection of open and dense subsets of $\mathbb{C}$ is dense and uncountable. Let $U$ be some arbitrary open set. Then the inverse orbit of $U$ is defined as: $\mathbb{O}^-(U) := \bigcup_{n \geq 0} f^{-n}(U)$. Theorem 5.1 implies that $\mathbb{O}^-(U)$ is a dense open subset of $\mathbb{C}$. Let $\mathbb{D}_{\mathbb{Q}}$ define the set of all open disks with rational centers and radii. Thus an element, $z$, has a dense orbit iff $\exists n \in \mathbb{N}$ s/t $f^n(z) \in \mathbb{D}_{\mathbb{Q}}$. So the set of all points with dense orbits can be defined as:

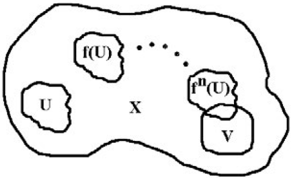
$$\tau = \bigcap_{U \in \mathbb{D}_{\mathbb{Q}}} \mathbb{O}^-(U)$$

Since the set, $\mathbb{Q}$ is countable $\Rightarrow$ $\mathbb{D}_{\mathbb{Q}}$ is countable $\Rightarrow$ $\tau$ is a countable intersection of open and dense sets $\Rightarrow$ $\tau$ is uncountable and dense in $\mathbb{C}$." ■

# Conclusion: Topological Transitivity and Chaos




(a) Sensitive dependence




(b) topological transitivity

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# References

## Paper Sources


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## Image sources


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Questions?