

Geopotential

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This document describes the computations that are performed by the method `GeneralSphericalHarmonicsAcceleration` of class `Geopotential` to determine the acceleration exerted by a non-spherical celestial on a point mass.

Let \mathbf{r} be the vector going from the centre of the celestial to the point mass. Let $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ be a (direct) base whose \mathbf{z} axis is along the axis of rotation of the celestial and whose \mathbf{x} axis points toward a reference point on the celestial. In this base \mathbf{r} has coordinates (x, y, z) which can be expressed in terms of the latitude β and the longitude λ :

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \beta \cos \lambda \\ r \cos \beta \sin \lambda \\ r \sin \beta \end{pmatrix}$$

where r is the norm of \mathbf{r} .

The gravitational potential exerted by the celestial has the form:

$$U(\mathbf{r}) = -\frac{\mu}{r} \left(1 + \sum_{n=1}^{\infty} \sum_{m=0}^n \left(\frac{R}{r} \right)^n P_{mn}(\sin \beta) (C_{mn} \cos m\lambda + S_{mn} \sin m\lambda) \right)$$

where P_{mn} is the associated Legendre function defined as:

$$P_{mn} = (1 - t^2)^{\frac{m}{2}} \frac{d^m P_n(t)}{dt^m}$$

The terms that are relevant for evaluating the acceleration exerted by the spherical harmonics on the point mass have the form:

$$V_{nm}(\mathbf{r}) = \frac{1}{r} \left(\frac{R}{r} \right)^n P_{mn}(\sin \beta) (C_{mn} \cos m\lambda + S_{mn} \sin m\lambda)$$

and the acceleration itself is proportional to $\nabla V_{nm}(\mathbf{r})$.

$V_{nm}(\mathbf{r})$ can be written as the product of a radial term, a latitudinal term and a longitudinal term, dependent of r , β , and λ , respectively:

$$V_{nm}(\mathbf{r}) = \mathfrak{R}(r) \mathfrak{B}(\beta) \mathfrak{L}(\lambda)$$

where:

$$\begin{cases} \mathfrak{R}(r) &= \frac{1}{r} \left(\frac{R}{r} \right)^n \\ \mathfrak{B}(\beta) &= P_{mn}(\sin \beta) \\ \mathfrak{L}(\lambda) &= C_{mn} \cos m\lambda + S_{mn} \sin m\lambda \end{cases}$$

The overall gradient can then be written as:

$$\nabla V_{nm}(\mathbf{r}) = \nabla \mathfrak{R}(r) \mathfrak{B}(\beta) \mathfrak{L}(\lambda) + \mathfrak{R}(r) \nabla \mathfrak{B}(\beta) \mathfrak{L}(\lambda) + \mathfrak{R}(r) \mathfrak{B}(\beta) \nabla \mathfrak{L}(\lambda)$$

To compute this expression, we first determine the gradient of various elements appearing in the potential. Starting with the \mathfrak{R} term, we have trivially:

$$\nabla r^2 = \nabla(x^2 + y^2 + z^2) = 2\mathbf{r}$$

from which we deduce:

$$\nabla r^n = \nabla((r^2)^{\frac{n}{2}}) = \frac{n}{2} (r^2)^{\frac{n}{2}-1} (2\mathbf{r}) = nr^{n-2} \mathbf{r}$$

Noting that $\nabla z = \mathbf{z}$ we can compute the following gradient needed for the \mathfrak{B} term:

$$\nabla \sin \beta = \nabla \frac{z}{r} = \frac{r\mathbf{z} - zr^{-1}\mathbf{r}}{r^2} = \frac{\mathbf{z}}{r} - \mathbf{r} \frac{z}{r^3}$$

which can be written, in coordinates:

$$\nabla \sin \beta = \frac{1}{r^3} \begin{pmatrix} -xz \\ -yz \\ x^2 + y^2 \end{pmatrix} = \frac{1}{r^3} \begin{pmatrix} -r^2 \sin \beta \cos \beta \cos \lambda \\ -r^2 \sin \beta \cos \beta \sin \lambda \\ r^2 \cos^2 \beta \end{pmatrix} = \frac{\cos \beta}{r} \begin{pmatrix} -\sin \beta \cos \lambda \\ -\sin \beta \sin \lambda \\ \cos \beta \end{pmatrix}$$

For the \mathfrak{L} term, we need to evaluate:

$$\begin{cases} \nabla \cos m\lambda &= -m \sin m\lambda \nabla \lambda \\ \nabla \sin m\lambda &= m \cos m\lambda \nabla \lambda \end{cases}$$

The angle λ is $\arctan \frac{y}{x}$ thus:

$$\nabla \lambda = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \nabla \left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{x\mathbf{y} - y\mathbf{x}}{x^2} = \frac{x\mathbf{y} - y\mathbf{x}}{x^2 + y^2}$$

which can be written, in coordinates:

$$\nabla \lambda = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \frac{1}{r^2 \cos^2 \beta} \begin{pmatrix} -r \cos \beta \sin \lambda \\ r \cos \beta \cos \lambda \\ 0 \end{pmatrix} = \frac{1}{r \cos \beta} \begin{pmatrix} -\sin \lambda \\ \cos \lambda \\ 0 \end{pmatrix}$$