Geopotential

Pascal Leroy (phl)

2018-09-16

This document describes the computations that are performed by the method `GeneralSphericalHarmonicsAcceleration` of class `Geopotential` to determine the acceleration exerted by a non-spherical celestial on a point mass.

Notation

Let $\boldsymbol{r}$ be the vector going from the centre of the celestial to the point mass. Let $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ be a (direct) base whose $\hat{\boldsymbol{z}}$ axis is along the axis of rotation of the celestial and whose $\hat{\boldsymbol{x}}$ axis points toward a reference point on the celestial. In this base $\boldsymbol{r}$ has coordinates (x, y, z) which can be expressed in terms of the latitude $\beta \in \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the longitude $\lambda \in [0, 2\pi]$:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\beta\cos\lambda \\ r\cos\beta\sin\lambda \\ r\sin\beta \end{pmatrix}$$

where r is the norm of $\boldsymbol{r}$. Note that $\cos\beta > 0$, which will come handy when simplifying expressions like $\sqrt{1 - \sin^2\beta}$.

Potential and acceleration

The gravitational potential due to the celestial has the form:

$$U(\boldsymbol{r}) = -\frac{\mu}{r}\left(1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n P_{mn}(\sin\beta)(C_{mn}\cos m\lambda + S_{mn}\sin m\lambda)\right)$$

where P_{mn} is the associated Legendre function defined as:

$$P_{mn} = (1 - t^2)^{\frac{m}{2}} \frac{\mathrm{d}^m P_n(t)}{\mathrm{d}t^m}$$

The terms that are relevant for evaluating the acceleration exerted by the spherical harmonics on the point mass have the form:

$$V_{nm}(\boldsymbol{r}) = \frac{1}{r}\left(\frac{R}{r}\right)^n P_{mn}(\sin\beta)(C_{mn}\cos m\lambda + S_{mn}\sin m\lambda)$$

and the acceleration itself is proportional to $\nabla V_{nm}(\boldsymbol{r})$.

$V_{nm}(\boldsymbol{r})$ can be written as the product of a radial term, a latitudinal term and a longitudinal term, dependent of r, β, and λ, respectively:

$$V_{nm}(\boldsymbol{r}) = \mathfrak{R}(r)\,\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda)$$

where:

$$\begin{cases} \mathfrak{R}(r) & = \frac{1}{r}\left(\frac{R}{r}\right)^n \\ \mathfrak{B}(\beta) & = P_{mn}(\sin\beta) \\ \mathfrak{L}(\lambda) & = C_{mn}\cos m\lambda + S_{mn}\sin m\lambda \end{cases}$$

The overall gradient can then be written as:

$$\nabla V_{nm}(\boldsymbol{r}) = \nabla\mathfrak{R}(r)\,\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda) + \mathfrak{R}(r)\,\nabla\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda) + \mathfrak{R}(r)\,\mathfrak{B}(\beta)\,\nabla\mathfrak{L}(\lambda)$$

Lemmata

To compute the acceleration, we first determine the gradient of various elements appearing in the potential. Starting with the $\mathfrak{R}$ term, we have trivially:

$$\nabla r^2 = \nabla\left(x^2 + y^2 + z^2\right) = 2\boldsymbol{r}$$

from which we deduce:

$$\nabla r^n = \nabla\left((r^2)^{\frac{n}{2}}\right) = \frac{n}{2}(r^2)^{\frac{n}{2}-1}(2\boldsymbol{r}) = nr^{n-2}\boldsymbol{r}$$

Noting that $\nabla z = \hat{\boldsymbol{z}}$ we can compute the following gradient needed for the $\mathfrak{B}$ term:

$$\nabla \sin\beta = \nabla\frac{z}{r} = \frac{r\hat{\boldsymbol{z}} - zr^{-1}\boldsymbol{r}}{r^2} = \frac{\hat{\boldsymbol{z}}}{r} - \boldsymbol{r}\frac{z}{r^3}$$

which can be written, in coordinates:

$$\nabla \sin\beta = \frac{1}{r^3}\begin{pmatrix} -xz \\ -yz \\ x^2 + y^2 \end{pmatrix} = \frac{1}{r^3}\begin{pmatrix} -r^2 \sin\beta\cos\beta\cos\lambda \\ -r^2\sin\beta\cos\beta\sin\lambda \\ r^2\cos^2\beta \end{pmatrix} = \frac{\cos\beta}{r}\begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix}$$

For the $\mathfrak{L}$ term, we need to evaluate:

$$\begin{cases} \nabla \cos m\lambda & = -m\sin m\lambda\,\nabla\lambda \\ \nabla\sin m\lambda & = m\cos m\lambda\,\nabla\lambda \end{cases}$$

The angle λ is $\arctan\frac{y}{x}$ thus:

$$\nabla\lambda = \frac{1}{1+\left(\frac{y}{x}\right)^2}\nabla\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}}{x^2} = \frac{x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}}{x^2+y^2}$$

which can be written, in coordinates:

$$\nabla\lambda = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \frac{1}{r^2\cos^2\beta}\begin{pmatrix} -r\cos\beta\sin\lambda \\ r\cos\beta\cos\lambda \\ 0 \end{pmatrix} = \frac{1}{r\cos\beta}\begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}$$

Finally, we will also need the derivative of the associated Legendre function:

$$\begin{aligned} P'_{nm}(t) &= \frac{m}{2}(1-t^2)^{\frac{m}{2}-1}(-2t)\frac{\mathrm{d}^m\,P_n(t)}{\mathrm{d}\,t^m} + (1-t^2)^{\frac{m}{2}}\frac{\mathrm{d}^{m+1}\,P_n(t)}{\mathrm{d}\,t^{m+1}} \\ &= (1-t^2)^{\frac{m}{2}}\frac{\mathrm{d}^{m+1}\,P_n(t)}{\mathrm{d}\,t^{m+1}} - mt(1-t^2)^{\frac{m-2}{2}}\frac{\mathrm{d}^m\,P_n(t)}{\mathrm{d}\,t^m} \end{aligned}$$

Gradients

We can now compute the gradient of the three terms that make up $V_{mn}(\boldsymbol{r})$. First, the radial term:

$$\nabla\mathfrak{R}(r) = R^n\nabla r^{-(n+1)} = -(n+1)R^n r^{-(n+3)}\boldsymbol{r} = -(n+1)\frac{\mathfrak{R}(r)}{r^2}$$

For the latitudinal term, the chain rule yields:

$$\nabla\mathfrak{B}(\beta) = P'_{nm}(\sin\beta)\,\nabla\sin\beta = P'_{nm}(\sin\beta)\frac{\cos\beta}{r}\begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix}$$

Substituting $\sin\beta$ for the argument of P_{nm} and its derivative we obtain:

$$\begin{cases} P_{nm}(\sin\beta) = \cos^m\beta\left.\frac{\mathrm{d}^m\,P_n(t)}{\mathrm{d}\,t^m}\right|_{t=\sin\beta} \\ P'_{nm}(\sin\beta) = \cos^m\beta\left.\frac{\mathrm{d}^{m+1}\,P_n(t)}{\mathrm{d}\,t^{m+1}}\right|_{t=\sin\beta} - m\sin\beta\left(\cos\beta\right)^{m-2}\left.\frac{\mathrm{d}^m\,P_n(t)}{\mathrm{d}\,t^m}\right|_{t=\sin\beta} \end{cases}$$

and thus:

$$\begin{cases} \mathfrak{B}(\beta) = \cos^m \beta \dfrac{\mathrm{d}^m\, P_n(t)}{\mathrm{d}\, t^m}\Bigg|_{t=\sin\beta} \\ \nabla\mathfrak{B}(\beta) = \dfrac{1}{r}\left(\left(\cos\beta\right)^{m+1} \dfrac{\mathrm{d}^{m+1}\, P_n(t)}{\mathrm{d}\, t^{m+1}}\Bigg|_{t=\sin\beta} - m\sin\beta \left(\cos\beta\right)^{m-1} \dfrac{\mathrm{d}^m\, P_n(t)}{\mathrm{d}\, t^m}\Bigg|_{t=\sin\beta} \right) \begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix} \end{cases}$$

For the longitudinal term we have:

$$\nabla\mathfrak{L}(\lambda) = (-mC_{nm}\sin m\lambda + mS_{nm}\cos m\lambda)\nabla\lambda = \frac{m}{r\cos\beta}(S_{nm}\cos m\lambda - C_{nm}\sin m\lambda) \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}$$

Singularities

While the above formulæ are all we need to compute the gradient of the geopotential, they present a number of singularities that require some care for implementation purposes.

There is obviously an essential singularity when $r = 0$. This one is not very interesting as we are never going to compute the acceleration at the centre of the celestial.

There are however non-essential singularities that arise when $\cos\beta$ appears at the denominator: any point on the axis of rotation of the celestial has $\cos\beta = 0$, but clearly the acceleration there is finite.

Consider $\nabla\mathfrak{B}(\beta)$. When $m = 0$ it includes a term in $\left(\cos\beta\right)^{-1}$. However, that term is multiplied by m, so it vanishes. Thus, for $m = 0$ we must use the special formula:

$$\nabla\mathfrak{B}(\beta)_{m=0} = \frac{\cos\beta}{r}\frac{\mathrm{d}\, P_n(t)}{\mathrm{d}\, t}\Bigg|_{t=\sin\beta} \begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix}$$

Similarly $\nabla\mathfrak{L}(\lambda)$ has $\cos\beta$ as its denominator. To eliminate this term we note that $\nabla\mathfrak{L}(\lambda)$ always occurs in a product involving $\mathfrak{B}(\beta)$. Let's write this product:

$$\mathfrak{B}(\beta)\nabla\mathfrak{L}(\lambda) = \frac{m\left(\cos\beta\right)^{m-1}}{r}\frac{\mathrm{d}^m\, P_m(t)}{\mathrm{d}\, t^m}\Bigg|_{t=\sin\beta}(S_{nm}\cos m\lambda - C_{nm}\sin m\lambda) \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}$$

When $m = 0$ this expression has a term in $\left(\cos\beta\right)^{-1}$ so we need to special-case it:

$$\mathfrak{B}(\beta)\nabla\mathfrak{L}(\lambda)_{m=0} = 0$$