Geopotential

Pascal Leroy (phl)

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This document describes the computations that are performed by the method `GeneralSphericalHarmonicsAcceleration` of class `Geopotential` to determine the acceleration exerted by a non-spherical celestial on a point mass.

Notation

Let $\boldsymbol{r}$ be the vector going from the centre of the celestial to the point mass. Let $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ be a (direct) base whose $\hat{\boldsymbol{z}}$ axis is along the axis of rotation of the celestial and whose $\hat{\boldsymbol{x}}$ axis points toward a reference point on the celestial. In this base $\boldsymbol{r}$ has coordinates (x, y, z) which can be expressed in terms of the latitude $\beta \in \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the longitude $\lambda \in [0, 2\pi]$:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos\beta \cos\lambda \\ r \cos\beta \sin\lambda \\ r \sin\beta \end{pmatrix}$$

where r is the norm of $\boldsymbol{r}$. Note that $\cos\beta > 0$, which will come handy when simplifying expressions like $\sqrt{1 - \sin^2\beta}$.

Potential and acceleration

The gravitational potential due to the celestial has the form[PL10]:

$$U(\boldsymbol{r}) = -\frac{\mu}{r}\left(1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n \mathrm{P}_{nm}(\sin\beta)\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)\right)$$

where P_{nm} is the associated Legendre function defined as[Rie+16, appendix]:

$$\mathrm{P}_{nm} = (1 - t^2)^{\frac{m}{2}} \frac{\mathrm{d}^m \,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}$$

The terms that are relevant for evaluating the acceleration exerted by the spherical harmonics on the point mass have the form:

$$V_{nm}(\boldsymbol{r}) = \frac{1}{r}\left(\frac{R}{r}\right)^n \mathrm{P}_{nm}(\sin\beta)\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)$$

and the acceleration itself is proportional to $V_{nm}(\boldsymbol{r})$.

$V_{nm}(\boldsymbol{r})$ can be written as the product of a radial term, a latitudinal term and a longitudinal term, dependent of r, β, and λ, respectively:

$$V_{nm}(\boldsymbol{r}) = \mathfrak{R}(r)\,\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda)$$

where:

$$\begin{cases} \mathfrak{R}(r) & = \frac{1}{r}\left(\frac{R}{r}\right)^n \\ \mathfrak{B}(\beta) & = \mathrm{P}_{nm}(\sin\beta) \\ \mathfrak{L}(\lambda) & = C_{nm}\cos m\lambda + S_{nm}\sin m\lambda \end{cases}$$

The overall gradient can then be written as:

$$V_{nm}(\boldsymbol{r}) = \mathfrak{R}(r)\,\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda) + \mathfrak{R}(r)\,\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda) + \mathfrak{R}(r)\,\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda)$$

Lemmata

To compute the acceleration, we first determine the gradient of various elements appearing in the potential. The following lemma determines the gradient of the $\mathfrak{R}$ term:

Lemma.

$$r^n = nr^{n-2}\boldsymbol{r}.$$

Proof. We have trivially:

$$r = \sqrt{x^2 + y^2 + z^2} = \frac{\boldsymbol{r}}{r}$$

from which we deduce:

$$r^n = nr^{n-1}r = nr^{n-2}\boldsymbol{r} \qquad \square$$

The following lemma is useful for computing the gradient of the $\mathfrak{B}$ term:

Lemma.

$$\sin\beta = \frac{\cos\beta}{r}\begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix}.$$

Proof. Noting that $z = \hat{\boldsymbol{z}}$ we have:

$$\sin\beta = \frac{z}{r} = \frac{r\hat{\boldsymbol{z}} - zr^{-1}\boldsymbol{r}}{r^2} = \frac{r^2\hat{\boldsymbol{z}} - z\boldsymbol{r}}{r^3}$$

which can be written, in coordinates:

$$\sin\beta = \frac{1}{r^3}\begin{pmatrix} -xz \\ -yz \\ x^2 + y^2 \end{pmatrix} = \frac{1}{r^3}\begin{pmatrix} -r^2\sin\beta\cos\beta\cos\lambda \\ -r^2\sin\beta\cos\beta\sin\lambda \\ r^2\cos^2\beta \end{pmatrix} = \frac{\cos\beta}{r}\begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix} \qquad \square$$

For the $\mathfrak{L}$ term, we will need to evaluate the quantities:

$$\begin{cases} \cos m\lambda & = -m\sin m\lambda\,\lambda \\ \sin m\lambda & = m\cos m\lambda\,\lambda \end{cases}$$

The following lemma helps with that computation:

Lemma.

$$\lambda = \frac{1}{r\cos\beta}\begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}.$$

Proof. The angle λ is $\arctan\frac{y}{x}$ thus:

$$\lambda = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\frac{x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}}{x^2} = \frac{x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}}{x^2 + y^2}$$

which can be written, in coordinates:

$$\lambda = \frac{1}{x^2 + y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \frac{1}{r^2\cos^2\beta}\begin{pmatrix} -r\cos\beta\sin\lambda \\ r\cos\beta\cos\lambda \\ 0 \end{pmatrix} = \frac{1}{r\cos\beta}\begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix} \qquad \square$$

Finally, we will need the gradient of the associated Legendre polynomial, given by the following lemma:

Lemma.

$$\mathrm{P}'_{nm}(t) = (1-t^2)^{\frac{m}{2}} \frac{\mathrm{d}^{m+1}\,\mathrm{P}_n(t)}{\mathrm{d}\,t^{m+1}} - mt(1-t^2)^{\frac{m-2}{2}} \frac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}.$$

Proof. This follows trivially from the definition:

$$\mathrm{P}'_{nm}(t) = \frac{m}{2}(1-t^2)^{\frac{m}{2}-1}(-2t)\frac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m} + (1-t^2)^{\frac{m}{2}} \frac{\mathrm{d}^{m+1}\,\mathrm{P}_n(t)}{\mathrm{d}\,t^{m+1}} \qquad \square$$

Gradients

We can now compute the gradient of the three terms that make up $V_{nm}(\boldsymbol{r})$. First, the radial term:

$$\mathfrak{R}(r) = R^n r^{-(n+1)} = -(n+1)R^n r^{-(n+3)}\boldsymbol{r} = -(n+1)\frac{\mathfrak{R}(r)}{r^2}\boldsymbol{r}$$

For the latitudinal term, the chain rule yields:

$$\mathfrak{B}(\beta) = \mathrm{P}_{nm}\big(\sin\beta\big)\ \sin\beta = \mathrm{P}'_{nm}\big(\sin\beta\big)\frac{\cos\beta}{r}\begin{pmatrix}-\sin\beta\cos\lambda\\ -\sin\beta\sin\lambda\\ \cos\beta\end{pmatrix}$$

Substituting $\sin\beta$ for the argument of P_{nm} and its derivative we obtain:

$$\begin{cases}\mathrm{P}_{nm}\big(\sin\beta\big) = \cos^m\beta\ \left.\dfrac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}\right|_{t=\sin\beta}\\ \mathrm{P}'_{nm}\big(\sin\beta\big) = \cos^m\beta\ \left.\dfrac{\mathrm{d}^{m+1}\,\mathrm{P}_n(t)}{\mathrm{d}\,t^{m+1}}\right|_{t=\sin\beta} - m\sin\beta\big(\cos\beta\big)^{m-2}\ \left.\dfrac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}\right|_{t=\sin\beta}\end{cases}$$

and thus:

$$\begin{cases}\mathfrak{B}(\beta) = \cos^m\beta\ \left.\dfrac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}\right|_{t=\sin\beta}\\ \mathfrak{B}(\beta) = \dfrac{1}{r}\left(\big(\cos\beta\big)^{m+1}\ \left.\dfrac{\mathrm{d}^{m+1}\,\mathrm{P}_n(t)}{\mathrm{d}\,t^{m+1}}\right|_{t=\sin\beta} - m\sin\beta\big(\cos\beta\big)^{m-1}\ \left.\dfrac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}\right|_{t=\sin\beta}\right)\begin{pmatrix}-\sin\beta\cos\lambda\\ -\sin\beta\sin\lambda\\ \cos\beta\end{pmatrix}\end{cases}$$

For the longitudinal term we have:

$$\mathfrak{L}(\lambda) = \big(-mC_{nm}\sin m\lambda + mS_{nm}\cos m\lambda\big)\lambda = \frac{m}{r\cos\beta}\big(S_{nm}\cos m\lambda - C_{nm}\sin m\lambda\big)\begin{pmatrix}-\sin\lambda\\ \cos\lambda\\ 0\end{pmatrix}$$

Singularities

While the above formulæ are all we need to compute the gradient of the geopotential, they present a number of singularities that require some care for implementation purposes.

There is obviously a pole when $r = 0$. This one is not very interesting as we are never going to compute the acceleration at the centre of the celestial.

There are however removable singularities that arise when $\cos\beta$ appears at the denominator: any point on the axis of rotation of the celestial has $\cos\beta = 0$, but clearly the acceleration there is finite.

Consider $\mathfrak{B}(\beta)$. When $m = 0$ it includes a term in $(\cos\beta)^{-1}$. However, that term is multiplied by m, so it vanishes. Thus, for $m = 0$ we must use the special formula:

$$\mathfrak{B}(\beta)_{m=0} = \frac{\cos\beta}{r} \frac{\mathrm{d}\,\mathrm{P}_n(t)}{\mathrm{d}\,t}\bigg|_{t=\sin\beta} \begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix}$$

Similarly $\mathfrak{L}(\lambda)$ has $\cos\beta$ as its denominator. To eliminate this term we note that $\mathfrak{L}(\lambda)$ always occurs in a product involving $\mathfrak{B}(\beta)$. Let's write this product:

$$\mathfrak{B}(\beta)\mathfrak{L}(\lambda) = \frac{m(\cos\beta)^{m-1}}{r} \frac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}\bigg|_{t=\sin\beta} (S_{nm}\cos m\lambda - C_{nm}\sin m\lambda) \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}$$

When $m = 0$ this expression has a term in $(\cos\beta)^{-1}$ so we need to special-case it:

$$\mathfrak{B}(\beta)\mathfrak{L}(\lambda)_{m=0} = 0$$

Implementation considerations

A naïve implementation of the preceding formulæ yields poor performance because of the computation of trigonometric functions, powers and polynomials. It is advantageous to use simple geometric identities and recurrence formulæ to avoid these computations, at the expense of a small amount of storage. First, we note that:

$$x^2 + y^2 = r^2\cos^2\beta$$

and therefore we obtain the trigonometric functions for β:

$$\begin{cases} \cos\beta = \dfrac{\sqrt{x^2+y^2}}{r} \\ \sin\beta = \dfrac{z}{r} \end{cases}$$

and for λ:

$$\begin{cases} \cos\lambda = \dfrac{x}{\sqrt{x^2+y^2}} \\ \sin\lambda = \dfrac{y}{\sqrt{x^2+y^2}} \end{cases}$$

In the case where $\sqrt{x^2+y^2}$ is 0, any value will do for λ, so we can for instance pick $\cos\lambda = 1$ and $\sin\lambda = 0$.

The value of $\cos^m\beta$ may be computed by recurrence in a way that entails $O(log_2 m)$ rounding errors. If m is even, we compute:

$$\begin{cases} h = \dfrac{m}{2} \\ \cos^m\beta = (\cos^h\beta)^2 \end{cases}$$

and if m is odd:

$$\begin{cases} h_1 = \left\lfloor \dfrac{m}{2} \right\rfloor \\ h_2 = m - h_1 \\ \cos^m\beta = (\cos^{h_1}\beta)(\cos^{h_2}\beta) \end{cases}$$

The value of $\cos m\lambda$ and $\sin m\lambda$ may similarly be computed using elementary trigonometric identities with $O(log_2 m)$ rounding errors. If m is even, we compute:

$$\begin{cases} h = \dfrac{m}{2} \\ \cos m\lambda = \cos^2 h\lambda - \sin^2 h\lambda \\ \sin m\lambda = 2\sin h\lambda\cos h\lambda \end{cases}$$

and if m is odd:

$$\begin{cases} h_1 = \left\lfloor \dfrac{m}{2} \right\rfloor \\ h_2 = m - h_1 \\ \sin m\lambda = \sin h_1\lambda \cos h_2\lambda + \cos h_1\lambda \sin h_2\lambda \\ \cos m\lambda = \cos h_1\lambda \cos h_2\lambda - \sin h_1\lambda \sin h_2\lambda \end{cases}$$

For the Legendre polynomials, we are looking for a recurrence relationship among terms of the form $\frac{\mathrm{d}^m \mathrm{P}_n(t)}{\mathrm{d}\, t^m}$. First note that these terms are 0 when $m > n$. We start with the recurrence defining the polynomials themselves:

$$n\,\mathrm{P}_n(t) = (2n-1)t\,\mathrm{P}_{n-1}(t) - (n-1)\,\mathrm{P}_{n-2}(t)$$

and we derive m times:

$$n\frac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m} = (2n-1)\frac{\mathrm{d}^m\,(t\,\mathrm{P}_{n-1}(t))}{\mathrm{d}\,t^m} - (n-1)\frac{\mathrm{d}^m\,\mathrm{P}_{n-2}(t)}{\mathrm{d}\,t^m}$$

It's easy to verify, by recurrence, that:

$$\frac{\mathrm{d}^m\,(t\,\mathrm{P}_n(t))}{\mathrm{d}\,t^m} = t\frac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m} + m\frac{\mathrm{d}^{m-1}\,\mathrm{P}_n(t)}{\mathrm{d}\,t^{m-1}}$$

so we end up with:

$$n\frac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m} = (2n-1)\left(t\frac{\mathrm{d}^m\,\mathrm{P}_{n-1}(t)}{\mathrm{d}\,t^m} + m\frac{\mathrm{d}^{m-1}\,\mathrm{P}_{n-1}(t)}{\mathrm{d}\,t^{m-1}}\right) - (n-1)\frac{\mathrm{d}^m\,\mathrm{P}_{n-2}(t)}{\mathrm{d}\,t^m}$$

This recurrence starts with $\mathrm{P}_0(t) = 1$, $\mathrm{P}_1(t) = t$ and $\frac{\mathrm{d}\,\mathrm{P}_1(t)}{\mathrm{d}\,t} = 1$. It is similar to [Wes17, eqn. (12)] and makes it possible to efficiently evaluate $\left.\frac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}\right|_{t=\sin\beta}$ for increasing values of n and m.

References

[PL10] G. Petit and B. Luzum. *IERS Conventions (2010)*. IERS Technical Note 36. International Earth Rotation and Reference Systems Service Convention Centre, 2010. eprint: `http://www.iers.org/IERS/EN/Publications/TechnicalNotes/tn36.html`.

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