Geopotential

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This document describes the computations that are performed by the method `GeneralSphericalHarmonicsAcceleration` of class `Geopotential` to determine the acceleration exerted by a non-spherical celestial on a point mass.

Notation

Let $\boldsymbol{r}$ be the vector going from the centre of the celestial to the point mass. Let $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ be a (direct) base whose $\hat{\boldsymbol{z}}$ axis is along the axis of rotation of the celestial and whose $\hat{\boldsymbol{x}}$ axis points toward a reference point on the celestial. In this base $\boldsymbol{r}$ has coordinates (x, y, z) which can be expressed in terms of the latitude $\beta \in \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the longitude $\lambda \in [0, 2\pi]$:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos\beta \cos\lambda \\ r \cos\beta \sin\lambda \\ r \sin\beta \end{pmatrix}$$

where r is the norm of $\boldsymbol{r}$. Note that $\cos\beta > 0$, which will come handy when simplifying expressions like $\sqrt{1 - \sin^2\beta}$.

Potential and acceleration

The gravitational potential due to the celestial has the form[PL10]:

$$U(\boldsymbol{r}) = -\frac{\mu}{r}\left(1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n \mathrm{P}_{nm}(\sin\beta)(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right)$$

where P_{nm} is the associated Legendre function defined as[Rie+16, appendix]:

$$\mathrm{P}_{nm} = (1 - t^2)^{\frac{m}{2}} \frac{\mathrm{d}^m \, \mathrm{P}_n(t)}{\mathrm{d}\, t^m}$$

The terms that are relevant for evaluating the acceleration exerted by the spherical harmonics on the point mass have the form:

$$V_{nm}(\boldsymbol{r}) = \frac{1}{r}\left(\frac{R}{r}\right)^n \mathrm{P}_{nm}(\sin\beta)(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)$$

and the acceleration itself is proportional to $\nabla V_{nm}(\boldsymbol{r})$.

$V_{nm}(\boldsymbol{r})$ can be written as the product of a radial term, a latitudinal term and a longitudinal term, dependent of r, β, and λ, respectively:

$$V_{nm}(\boldsymbol{r}) = \mathfrak{R}(r)\,\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda)$$

where:

$$\begin{cases} \mathfrak{R}(r) & = \frac{1}{r}\left(\frac{R}{r}\right)^n \\ \mathfrak{B}(\beta) & = \mathrm{P}_{nm}(\sin\beta) \\ \mathfrak{L}(\lambda) & = C_{nm}\cos m\lambda + S_{nm}\sin m\lambda \end{cases}$$

The overall gradient can then be written as:

$$\nabla V_{nm}(\boldsymbol{r}) = \nabla\mathfrak{R}(r)\,\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda) + \mathfrak{R}(r)\,\nabla\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda) + \mathfrak{R}(r)\,\mathfrak{B}(\beta)\,\nabla\mathfrak{L}(\lambda)$$

Lemmata

To compute the acceleration, we first determine the gradient of various elements appearing in the potential. The following lemma determines the gradient of the $\mathfrak{R}$ term:

Lemma.

$$\nabla r^n = n r^{n-2} \boldsymbol{r}.$$

Proof. We have trivially:

$$\nabla r = \nabla \sqrt{x^2 + y^2 + z^2} = \frac{\boldsymbol{r}}{r}$$

from which we deduce:

$$\nabla r^n = n r^{n-1} \nabla r = n r^{n-2} \boldsymbol{r} \qquad \square$$

The following lemma is useful for computing the gradient of the $\mathfrak{B}$ term:

Lemma.

$$\nabla \sin\beta = \frac{\cos\beta}{r} \begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix}.$$

Proof. Noting that $\nabla z = \hat{\boldsymbol{z}}$ we have:

$$\nabla \sin\beta = \nabla \frac{z}{r} = \frac{r\hat{\boldsymbol{z}} - z r^{-1} \boldsymbol{r}}{r^2} = \frac{r^2 \hat{\boldsymbol{z}} - z\boldsymbol{r}}{r^3}$$

which can be written, in coordinates:

$$\nabla \sin\beta = \frac{1}{r^3} \begin{pmatrix} -xz \\ -yz \\ x^2 + y^2 \end{pmatrix} = \frac{1}{r^3} \begin{pmatrix} -r^2 \sin\beta\cos\beta\cos\lambda \\ -r^2 \sin\beta\cos\beta\sin\lambda \\ r^2\cos^2\beta \end{pmatrix} = \frac{\cos\beta}{r} \begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix} \qquad \square$$

For the $\mathfrak{L}$ term we will use the following lemma:

Lemma.

$$\nabla\lambda = \frac{1}{r\cos\beta} \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}.$$

Proof. For the $\mathfrak{L}$ term, we need to evaluate:

$$\begin{cases} \nabla \cos m\lambda & = -m \sin m\lambda \, \nabla\lambda \\ \nabla \sin m\lambda & = m \cos m\lambda \, \nabla\lambda \end{cases}$$

The angle λ is arctan $\frac{y}{x}$ thus:

$$\nabla\lambda = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \nabla\left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}}{x^2} = \frac{x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}}{x^2 + y^2}$$

which can be written, in coordinates:

$$\nabla\lambda = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \frac{1}{r^2\cos^2\beta} \begin{pmatrix} -r\cos\beta\sin\lambda \\ r\cos\beta\cos\lambda \\ 0 \end{pmatrix} = \frac{1}{r\cos\beta} \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix} \qquad \square$$

Finally, we will need the gradient of the associated Legendre polynomial, given by the following lemma:

Lemma.

$$\mathrm{P}'_{nm}(t) = (1-t^2)^{\frac{m}{2}}\frac{\mathrm{d}^{m+1}\,\mathrm{P}_n(t)}{\mathrm{d}\,t^{m+1}} - mt(1-t^2)^{\frac{m-2}{2}}\frac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}.$$

Proof. This follows trivially from the definition:

$$\mathrm{P}'_{nm}(t) = \frac{m}{2}(1-t^2)^{\frac{m}{2}-1}(-2t)\frac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m} + (1-t^2)^{\frac{m}{2}}\frac{\mathrm{d}^{m+1}\,\mathrm{P}_n(t)}{\mathrm{d}\,t^{m+1}} \qquad \square$$

Gradients

We can now compute the gradient of the three terms that make up $V_{nm}(\boldsymbol{r})$. First, the radial term:

$$\nabla\mathfrak{R}(r) = R^n\nabla r^{-(n+1)} = -(n+1)R^n r^{-(n+3)}\boldsymbol{r} = -(n+1)\frac{\mathfrak{R}(r)}{r^2}\boldsymbol{r}$$

For the latitudinal term, the chain rule yields:

$$\nabla\mathfrak{B}(\beta) = \mathrm{P}_{nm}(\sin\beta)\,\nabla\sin\beta = \mathrm{P}'_{nm}(\sin\beta)\frac{\cos\beta}{r}\begin{pmatrix}-\sin\beta\cos\lambda\\ -\sin\beta\sin\lambda\\ \cos\beta\end{pmatrix}$$

Substituting $\sin\beta$ for the argument of P_{nm} and its derivative we obtain:

$$\begin{cases}\mathrm{P}_{nm}(\sin\beta) = \cos^m\beta\left.\dfrac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}\right|_{t=\sin\beta}\\ \mathrm{P}'_{nm}(\sin\beta) = \cos^m\beta\left.\dfrac{\mathrm{d}^{m+1}\,\mathrm{P}_n(t)}{\mathrm{d}\,t^{m+1}}\right|_{t=\sin\beta} - m\sin\beta(\cos\beta)^{m-2}\left.\dfrac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}\right|_{t=\sin\beta}\end{cases}$$

and thus:

$$\begin{cases}\mathfrak{B}(\beta) = \cos^m\beta\left.\dfrac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}\right|_{t=\sin\beta}\\ \nabla\mathfrak{B}(\beta) = \dfrac{1}{r}\left((\cos\beta)^{m+1}\left.\dfrac{\mathrm{d}^{m+1}\,\mathrm{P}_n(t)}{\mathrm{d}\,t^{m+1}}\right|_{t=\sin\beta} - m\sin\beta(\cos\beta)^{m-1}\left.\dfrac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}\right|_{t=\sin\beta}\right)\begin{pmatrix}-\sin\beta\cos\lambda\\ -\sin\beta\sin\lambda\\ \cos\beta\end{pmatrix}\end{cases}$$

For the longitudinal term we have:

$$\nabla\mathfrak{L}(\lambda) = (-mC_{nm}\sin m\lambda + mS_{nm}\cos m\lambda)\nabla\lambda = \frac{m}{r\cos\beta}(S_{nm}\cos m\lambda - C_{nm}\sin m\lambda)\begin{pmatrix}-\sin\lambda\\ \cos\lambda\\ 0\end{pmatrix}$$

Singularities

While the above formulæ are all we need to compute the gradient of the geopotential, they present a number of singularities that require some care for implementation purposes.

There is obviously a pole when $r = 0$. This one is not very interesting as we are never going to compute the acceleration at the centre of the celestial.

There are however removable singularities that arise when $\cos\beta$ appears at the denominator: any point on the axis of rotation of the celestial has $\cos\beta = 0$, but clearly the acceleration there is finite.

Consider $\nabla\mathfrak{B}(\beta)$. When $m = 0$ it includes a term in $\left(\cos\beta\right)^{-1}$. However, that term is multiplied by m, so it vanishes. Thus, for $m = 0$ we must use the special formula:

$$\nabla\mathfrak{B}(\beta)_{m=0} = \frac{\cos\beta}{r}\frac{\mathrm{d}\,\mathrm{P}_n(t)}{\mathrm{d}\,t}\bigg|_{t=\sin\beta}\begin{pmatrix}-\sin\beta\cos\lambda\\ -\sin\beta\sin\lambda\\ \cos\beta\end{pmatrix}$$

Similarly $\nabla\mathfrak{L}(\lambda)$ has $\cos\beta$ as its denominator. To eliminate this term we note that $\nabla\mathfrak{L}(\lambda)$ always occurs in a product involving $\mathfrak{B}(\beta)$. Let's write this product:

$$\mathfrak{B}(\beta)\nabla\mathfrak{L}(\lambda) = \frac{m\left(\cos\beta\right)^{m-1}}{r}\frac{\mathrm{d}^m\,\mathrm{P}_n(t)}{\mathrm{d}\,t^m}\bigg|_{t=\sin\beta}\left(S_{nm}\cos m\lambda - C_{nm}\sin m\lambda\right)\begin{pmatrix}-\sin\lambda\\ \cos\lambda\\ 0\end{pmatrix}$$

When $m = 0$ this expression has a term in $\left(\cos\beta\right)^{-1}$ so we need to special-case it:

$$\mathfrak{B}(\beta)\nabla\mathfrak{L}(\lambda)_{m=0} = 0$$

References

[PL10] G. Petit and B. Luzum. *IERS Conventions (2010)*. IERS Technical Note 36. International Earth Rotation and Reference Systems Service Convention Centre, 2010. eprint: `http://www.iers.org/IERS/EN/Publications/TechnicalNotes/tn36.html`.

[Rie+16] J. Ries, S. Bettadpur, R. Eanes, Z. Kang, U. Ko, C. McCullough, P. Nagel, N. Pie, S. Poole, H. Save, and B. Tapley. *The Combination Global Gravity Model GGM05C*. Technical Memorandum CSR-TM-16-01. Center for Space Research at the University of Texas at Austin, Jan. 2016. eprint: `ftp://ftp.csr.utexas.edu/pub/grace/GGM05/README_GGM05C.pdf`.