On an Article by Celledoni et al.

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This document provides clarifications, corrections, and accuracy improvements to the formulæ presented in [CFSZ08]. It follows the notation and conventions of that paper.

Preamble

We remind the reader of the derivation formulæ for the Jacobian elliptic functions ([OLBC10], section 22.13(i)):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{sn} u &= \operatorname{cn} u \operatorname{dn} u \\ \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{cn} u &= -\operatorname{sn} u \operatorname{dn} u \\ \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{dn} u &= -k^2 \operatorname{sn} u \operatorname{cn} u \end{cases}$$

The equations of motion

We start by writing equation (1) of [CFSZ08] in coordinates. The coordinates of m and I are defined by:

$$\boldsymbol{m} \coloneqq \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

and:

$$\boldsymbol{I} \coloneqq \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Euler's equation $\dot{m} = m \wedge (I^{-1} m)$ can be written in coordinates:

$$\dot{\boldsymbol{m}} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \wedge \begin{pmatrix} m_1/I_1 \\ m_2/I_2 \\ m_3/I_3 \end{pmatrix}$$

thus:

$$\begin{cases} \dot{m}_1 &= m_2 m_3 \left(1/I_3 - 1/I_2\right) \\ \dot{m}_2 &= m_3 m_1 \left(1/I_1 - 1/I_3\right) \\ \dot{m}_3 &= m_1 m_2 \left(1/I_2 - 1/I_1\right) \end{cases} \tag{1}$$

Solution of Euler's equation, case (i)

The case (i) of the solution of Euler's equation in section 2.2 of [CFSZ08] is:

$$\boldsymbol{m}_t = \begin{pmatrix} \sigma B_{13} \operatorname{dn}(\lambda t - \nu, k) \\ -B_{21} \operatorname{sn}(\lambda t - \nu, k) \\ B_{31} \operatorname{cn}(\lambda t - \nu, k) \end{pmatrix}$$

If we derive this expression with respect to t, inject in into (1), and eliminate the elliptic functions we obtain:

$$\begin{cases} -\sigma\lambda k^2 B_{13} & = -B_{21}B_{31}\big(1/I_3 - 1/I_2\big) \\ -\lambda B_{21} & = \sigma B_{13}B_{31}\big(1/I_1 - 1/I_3\big) \\ -\lambda B_{31} & = -\sigma B_{13}B_{21}\big(1/I_2 - 1/I_1\big) \end{cases} \tag{2}$$

The last equation of (2) yields the following value for λ:

$$\lambda = \sigma \frac{B_{13}B_{21}}{B_{31}} \frac{I_1 - I_2}{I_1 I_2} = \sigma \sqrt{\frac{I_1\Delta_3}{I_{13}} \frac{I_2\Delta_1}{I_{21}} \frac{I_{31}}{I_3\Delta_1}} \frac{I_1 - I_2}{I_1 I_2} = \sigma \sqrt{\frac{\Delta_3}{I_{21}I_1I_2I_3}} (I_1 - I_2) = -\sigma \sqrt{\frac{\Delta_3 I_{21}}{I_1I_2I_3}} = -\sigma\lambda_3$$

The sign change when moving $I_1 - I_2$ under the radical is necessary because $I_1 - I_2 < 0$.

It is straightforward to check that this value of λ also satisfies the other equations of (2). Note that it differs in sign from the one given by [CFSZ08]: the sign error is visible in that it does not yield the proper precession direction.

References

[CFSZ08] E. Celledoni, F. Fassò, N. Säfström, and A. Zanna. "The exact computation of the free rigid body motion and its use in splitting methods". In: *SIAM J. Scientific Computing* 30 (May 2008), pp. 2084–2112.

[OLBC10] F. Olver, D. Lozier, R. Boisvert, and C. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.