

Geopotential

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2018-09-16

This document describes the computations that are performed by the method `GeneralSphericalHarmonicsAcceleration` of class `Geopotential` to determine the acceleration exerted by a non-spherical celestial on a point mass.

Notation

Let \mathbf{r} be the vector going from the centre of the celestial to the point mass. Let $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ be a (direct) base whose $\hat{\mathbf{z}}$ axis is along the axis of rotation of the celestial and whose $\hat{\mathbf{x}}$ axis points toward a reference point on the celestial. In this base \mathbf{r} has coordinates (x, y, z) which can be expressed in terms of the latitude $\beta \in \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the longitude $\lambda \in [0, 2\pi]$:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \beta \cos \lambda \\ r \cos \beta \sin \lambda \\ r \sin \beta \end{pmatrix}$$

where r is the norm of \mathbf{r} . Note that $\cos \beta > 0$, which will come handy when simplifying expressions like $\sqrt{1 - \sin^2 \beta}$.

Potential and acceleration

The gravitational potential due to the celestial has the form[PL10]:

$$U(\mathbf{r}) = -\frac{\mu}{r} \left(1 + \sum_{n=1}^{\infty} \sum_{m=0}^n \left(\frac{R}{r} \right)^n P_{nm}(\sin \beta) (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \right)$$

where P_{nm} is the associated Legendre function defined as[Rie+16, appendix]:

$$P_{nm} = (1 - t^2)^{\frac{m}{2}} \frac{d^m P_n(t)}{dt^m}$$

The terms that are relevant for evaluating the acceleration exerted by the spherical harmonics on the point mass have the form:

$$V_{nm}(\mathbf{r}) = \frac{1}{r} \left(\frac{R}{r} \right)^n P_{nm}(\sin \beta) (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)$$

and the acceleration itself is proportional to $\nabla V_{nm}(\mathbf{r})$.

$V_{nm}(\mathbf{r})$ can be written as the product of a radial term, a latitudinal term and a longitudinal term, dependent of r , β , and λ , respectively:

$$V_{nm}(\mathbf{r}) = \mathfrak{R}(r) \mathfrak{B}(\beta) \mathfrak{L}(\lambda)$$

where:

$$\begin{cases} \mathfrak{R}(r) &= \frac{1}{r} \left(\frac{R}{r} \right)^n \\ \mathfrak{B}(\beta) &= P_{nm}(\sin \beta) \\ \mathfrak{L}(\lambda) &= C_{nm} \cos m\lambda + S_{nm} \sin m\lambda \end{cases}$$

The overall gradient can then be written as:

$$\nabla V_{nm}(\mathbf{r}) = \nabla \mathfrak{R}(r) \mathfrak{B}(\beta) \mathfrak{L}(\lambda) + \mathfrak{R}(r) \nabla \mathfrak{B}(\beta) \mathfrak{L}(\lambda) + \mathfrak{R}(r) \mathfrak{B}(\beta) \nabla \mathfrak{L}(\lambda)$$

Lemmata

To compute the acceleration, we first determine the gradient of various elements appearing in the potential. Starting with the \mathfrak{R} term, we have trivially:

$$\nabla r^2 = \nabla(x^2 + y^2 + z^2) = 2\mathbf{r}$$

from which we deduce:

$$\nabla r^n = \nabla((r^2)^{\frac{n}{2}}) = \frac{n}{2}(r^2)^{\frac{n}{2}-1}(2\mathbf{r}) = nr^{n-2}\mathbf{r}$$

Noting that $\nabla z = \hat{\mathbf{z}}$ we can compute the following gradient needed for the \mathfrak{B} term:

$$\nabla \sin \beta = \nabla \frac{z}{r} = \frac{r\hat{\mathbf{z}} - zr^{-1}\mathbf{r}}{r^2} = \frac{\hat{\mathbf{z}}}{r} - \mathbf{r} \frac{z}{r^3}$$

which can be written, in coordinates:

$$\nabla \sin \beta = \frac{1}{r^3} \begin{pmatrix} -xz \\ -yz \\ x^2 + y^2 \end{pmatrix} = \frac{1}{r^3} \begin{pmatrix} -r^2 \sin \beta \cos \beta \cos \lambda \\ -r^2 \sin \beta \cos \beta \sin \lambda \\ r^2 \cos^2 \beta \end{pmatrix} = \frac{\cos \beta}{r} \begin{pmatrix} -\sin \beta \cos \lambda \\ -\sin \beta \sin \lambda \\ \cos \beta \end{pmatrix}$$

For the \mathfrak{L} term, we need to evaluate:

$$\begin{cases} \nabla \cos m\lambda &= -m \sin m\lambda \nabla \lambda \\ \nabla \sin m\lambda &= m \cos m\lambda \nabla \lambda \end{cases}$$

The angle λ is $\arctan \frac{y}{x}$ thus:

$$\nabla \lambda = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \nabla \left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{x\hat{\mathbf{y}} - y\hat{\mathbf{x}}}{x^2} = \frac{x\hat{\mathbf{y}} - y\hat{\mathbf{x}}}{x^2 + y^2}$$

which can be written, in coordinates:

$$\nabla \lambda = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \frac{1}{r^2 \cos^2 \beta} \begin{pmatrix} -r \cos \beta \sin \lambda \\ r \cos \beta \cos \lambda \\ 0 \end{pmatrix} = \frac{1}{r \cos \beta} \begin{pmatrix} -\sin \lambda \\ \cos \lambda \\ 0 \end{pmatrix}$$

Finally, we will also need the derivative of the associated Legendre function:

$$\begin{aligned} P'_{nm}(t) &= \frac{m}{2}(1-t^2)^{\frac{m}{2}-1}(-2t) \frac{d^m P_n(t)}{dt^m} + (1-t^2)^{\frac{m}{2}} \frac{d^{m+1} P_n(t)}{dt^{m+1}} \\ &= (1-t^2)^{\frac{m}{2}} \frac{d^{m+1} P_n(t)}{dt^{m+1}} - mt(1-t^2)^{\frac{m-2}{2}} \frac{d^m P_n(t)}{dt^m} \end{aligned}$$

Gradients

We can now compute the gradient of the three terms that make up $V_{nm}(\mathbf{r})$. First, the radial term:

$$\nabla \mathfrak{R}(r) = R^n \nabla r^{-(n+1)} = -(n+1)R^n r^{-(n+3)}\mathbf{r} = -(n+1) \frac{\mathfrak{R}(r)}{r^2}$$

For the latitudinal term, the chain rule yields:

$$\nabla \mathfrak{B}(\beta) = P_{nm}(\sin \beta) \nabla \sin \beta = P'_{nm}(\sin \beta) \frac{\cos \beta}{r} \begin{pmatrix} -\sin \beta \cos \lambda \\ -\sin \beta \sin \lambda \\ \cos \beta \end{pmatrix}$$

Substituting $\sin \beta$ for the argument of P_{nm} and its derivative we obtain:

$$\begin{cases} P_{nm}(\sin \beta) = \cos^m \beta \frac{d^m P_n(t)}{dt^m} \Big|_{t=\sin \beta} \\ P'_{nm}(\sin \beta) = \cos^m \beta \frac{d^{m+1} P_n(t)}{dt^{m+1}} \Big|_{t=\sin \beta} - m \sin \beta (\cos \beta)^{m-2} \frac{d^m P_n(t)}{dt^m} \Big|_{t=\sin \beta} \end{cases}$$

and thus:

$$\begin{cases} \mathfrak{B}(\beta) = \cos^m \beta \frac{d^m P_n(t)}{dt^m} \Big|_{t=\sin \beta} \\ \nabla \mathfrak{B}(\beta) = \frac{1}{r} \left((\cos \beta)^{m+1} \frac{d^{m+1} P_n(t)}{dt^{m+1}} \Big|_{t=\sin \beta} - m \sin \beta (\cos \beta)^{m-1} \frac{d^m P_n(t)}{dt^m} \Big|_{t=\sin \beta} \right) \begin{pmatrix} -\sin \beta \cos \lambda \\ -\sin \beta \sin \lambda \\ \cos \beta \end{pmatrix} \end{cases}$$

For the longitudinal term we have:

$$\nabla \mathfrak{L}(\lambda) = (-m C_{nm} \sin m\lambda + m S_{nm} \cos m\lambda) \nabla \lambda = \frac{m}{r \cos \beta} (S_{nm} \cos m\lambda - C_{nm} \sin m\lambda) \begin{pmatrix} -\sin \lambda \\ \cos \lambda \\ 0 \end{pmatrix}$$

Singularities

While the above formulæ are all we need to compute the gradient of the geopotential, they present a number of singularities that require some care for implementation purposes.

There is obviously an essential singularity when $r = 0$. This one is not very interesting as we are never going to compute the acceleration at the centre of the celestial.

There are however non-essential singularities that arise when $\cos \beta$ appears at the denominator: any point on the axis of rotation of the celestial has $\cos \beta = 0$, but clearly the acceleration there is finite.

Consider $\nabla \mathfrak{B}(\beta)$. When $m = 0$ it includes a term in $(\cos \beta)^{-1}$. However, that term is multiplied by m , so it vanishes. Thus, for $m = 0$ we must use the special formula:

$$\nabla \mathfrak{B}(\beta)_{m=0} = \frac{\cos \beta}{r} \frac{d P_n(t)}{dt} \Big|_{t=\sin \beta} \begin{pmatrix} -\sin \beta \cos \lambda \\ -\sin \beta \sin \lambda \\ \cos \beta \end{pmatrix}$$

Similarly $\nabla \mathfrak{L}(\lambda)$ has $\cos \beta$ as its denominator. To eliminate this term we note that $\nabla \mathfrak{L}(\lambda)$ always occurs in a product involving $\mathfrak{B}(\beta)$. Let's write this product:

$$\mathfrak{B}(\beta) \nabla \mathfrak{L}(\lambda) = \frac{m (\cos \beta)^{m-1}}{r} \frac{d^m P_n(t)}{dt^m} \Big|_{t=\sin \beta} (S_{nm} \cos m\lambda - C_{nm} \sin m\lambda) \begin{pmatrix} -\sin \lambda \\ \cos \lambda \\ 0 \end{pmatrix}$$

When $m = 0$ this expression has a term in $(\cos \beta)^{-1}$ so we need to special-case it:

$$\mathfrak{B}(\beta) \nabla \mathfrak{L}(\lambda)_{m=0} = 0$$

References

- [PL10] G. Petit and B. Luzum. *IERS Conventions (2010)*. IERS Technical Note 36. International Earth Rotation and Reference Systems Service Convention Centre, 2010. eprint: <http://www.iers.org/IERS/EN/Publications/TechnicalNotes/tn36.html>.
- [Rie+16] J. Ries, S. Bettadpur, R. Eanes, Z. Kang, U. Ko, C. McCullough, P. Nagel, N. Pie, S. Poole, H. Save, and B. Tapley. *The Combination Global Gravity Model GGM05C*. Technical Memorandum CSR-TM-16-01. Center for Space Research at the University of Texas at Austin, Jan. 2016. eprint: ftp://ftp.csr.utexas.edu/pub/grace/GGM05/README_GGM05C.pdf.