Geopotential

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This document describes the computations that are performed by the method `GeneralSphericalHarmonicsAcceleration` of class `Geopotential` to the determine the acceleration exerted by a non-spherical celestial on a point mass.

Notation

Let $\boldsymbol{r}$ be the vector going from the centre of the celestial to the point mass. Let $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ be a (direct) base whose $\boldsymbol{z}$ axis is along the axis of rotation of the celestial and whose $\boldsymbol{x}$ axis points toward a reference point on the celestial. In this base $\boldsymbol{r}$ has coordinates (x, y, z) which can be expressed in terms of the latitude β and the longitude λ:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos\beta \cos\lambda \\ r \cos\beta \sin\lambda \\ r \sin\beta \end{pmatrix}$$

where r is the norm of $\boldsymbol{r}$.

Potential and acceleration

The gravitational potential exerted by the celestial has the form:

$$U(\boldsymbol{r}) = -\frac{\mu}{r}\left(1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n P_{mn}(\sin\beta)(C_{mn}\cos m\lambda + S_{mn}\sin m\lambda)\right)$$

where P_{mn} is the associated Legendre function defined as:

$$P_{mn} = (1 - t^2)^{\frac{m}{2}}\frac{\mathrm{d}^m P_n(t)}{\mathrm{d}\, t^m}$$

The terms that are relevant for evaluating the acceleration exerted by the spherical harmonics on the point mass have the form:

$$V_{nm}(\boldsymbol{r}) = \frac{1}{r}\left(\frac{R}{r}\right)^n P_{mn}(\sin\beta)(C_{mn}\cos m\lambda + S_{mn}\sin m\lambda)$$

and the acceleration itself is proportional to $\nabla V_{nm}(\boldsymbol{r})$.

$V_{nm}(\boldsymbol{r})$ can be written as the product of a radial term, a latitudinal term and a longitudinal term, dependent of r, β, and λ, respectively:

$$V_{nm}(\boldsymbol{r}) = \Re(r)\,\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda)$$

where:

$$\begin{cases} \Re(r) & = \dfrac{1}{r}\left(\dfrac{R}{r}\right)^n \\ \mathfrak{B}(\beta) & = P_{mn}(\sin\beta) \\ \mathfrak{L}(\lambda) & = C_{mn}\cos m\lambda + S_{mn}\sin m\lambda \end{cases}$$

The overall gradient can then be written as:

$$\nabla V_{nm}(\boldsymbol{r}) = \nabla\Re(r)\,\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda) + \Re(r)\,\nabla\mathfrak{B}(\beta)\,\mathfrak{L}(\lambda) + \Re(r)\,\mathfrak{B}(\beta)\,\nabla\mathfrak{L}(\lambda)$$

Lemmata

To compute the acceleration, we first determine the gradient of various elements appearing in the potential. Starting with the $\Re$ term, we have trivially:

$$\nabla r^2 = \nabla\left(x^2 + y^2 + z^2\right) = 2\boldsymbol{r}$$

from which we deduce:

$$\nabla r^n = \nabla\left((r^2)^{\frac{n}{2}}\right) = \frac{n}{2}(r^2)^{\frac{n}{2}-1}(2\boldsymbol{r}) = nr^{n-2}\boldsymbol{r}$$

Noting that $\nabla z = \boldsymbol{z}$ we can compute the following gradient needed for the $\mathfrak{B}$ term:

$$\nabla \sin\beta = \nabla\frac{z}{r} = \frac{r\boldsymbol{z} - zr^{-1}\boldsymbol{r}}{r^2} = \frac{\boldsymbol{z}}{r} - \boldsymbol{r}\frac{z}{r^3}$$

which can be written, in coordinates:

$$\nabla \sin\beta = \frac{1}{r^3}\begin{pmatrix} -xz \\ -yz \\ x^2 + y^2 \end{pmatrix} = \frac{1}{r^3}\begin{pmatrix} -r^2 \sin\beta \cos\beta \cos\lambda \\ -r^2 \sin\beta \cos\beta \sin\lambda \\ r^2 \cos^2\beta \end{pmatrix} = \frac{\cos\beta}{r}\begin{pmatrix} -\sin\beta \cos\lambda \\ -\sin\beta \sin\lambda \\ \cos\beta \end{pmatrix}$$

For the $\mathfrak{L}$ term, we need to evaluate:

$$\begin{cases} \nabla \cos m\lambda &= -m \sin m\lambda \, \nabla\lambda \\ \nabla \sin m\lambda &= m \cos m\lambda \, \nabla\lambda \end{cases}$$

The angle λ is $\arctan\frac{y}{x}$ thus:

$$\nabla\lambda = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\nabla\left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\frac{x\boldsymbol{y} - y\boldsymbol{x}}{x^2} = \frac{x\boldsymbol{y} - y\boldsymbol{x}}{x^2 + y^2}$$

which can be written, in coordinates:

$$\nabla\lambda = \frac{1}{x^2 + y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \frac{1}{r^2 \cos^2\beta}\begin{pmatrix} -r \cos\beta \sin\lambda \\ r \cos\beta \cos\lambda \\ 0 \end{pmatrix} = \frac{1}{r \cos\beta}\begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}$$

Finally, we will also need the derivative of the associated Legendre function:

$$P'_{nm}(t) = \frac{m}{2}(1 - t^2)^{\frac{m}{2}-1}(-2t)\frac{\mathrm{d}^m P_n(t)}{\mathrm{d}t^m} + (1 - t^2)^{\frac{m}{2}}\frac{\mathrm{d}^{m+1} P_n(t)}{\mathrm{d}t^{m+1}}$$

$$= (1 - t^2)^{\frac{m}{2}-1}\left((1 - t^2)\frac{\mathrm{d}^{m+1} P_n(t)}{\mathrm{d}t^{m+1}} - mt\frac{\mathrm{d}^m P_n(t)}{\mathrm{d}t^m}\right)$$

Gradients

We can now compute the gradient of the three terms that make up $V_{mn}(\boldsymbol{r})$. First, the radial term:

$$\nabla\Re(r) = R^n\nabla r^{-(n+1)} = -(n + 1)R^n r^{-(n+3)}\boldsymbol{r} = -(n + 1)\frac{\Re(r)}{r^2}\boldsymbol{r}$$

For the latitudinal term, the chain rule yields:

$$\nabla\mathfrak{B}(\beta) = P'_{nm}(\sin\beta)\,\nabla\sin\beta = P'_{nm}(\sin\beta)\frac{\cos\beta}{r}\begin{pmatrix} -\sin\beta \cos\lambda \\ -\sin\beta \sin\lambda \\ \cos\beta \end{pmatrix}$$

Substituting $\sin\beta$ for the argument of P_{nm} and its derivative we obtain:

$$\begin{cases} P_{nm}(\sin\beta) = \cos^m\beta\,\dfrac{\mathrm{d}^m P_n(\sin\beta)}{\mathrm{d}t^m} \\ P'_{nm}(\sin\beta) = \cos^m\beta\,\dfrac{\mathrm{d}^{m+1} P_n(\sin\beta)}{\mathrm{d}t^{m+1}} - m\sin\beta(\cos\beta)^{m-2}\dfrac{\mathrm{d}^m P_n(\sin\beta)}{\mathrm{d}t^m} \end{cases}$$

and thus:

$$
\begin{cases}
\mathfrak{B}(\beta) = \cos^m \beta \, \dfrac{\mathrm{d}^m P_n(\sin\beta)}{\mathrm{d}\,t^m} \\[2ex]
\nabla\mathfrak{B}(\beta) = \dfrac{1}{r}\left((\cos\beta)^{m+1} \dfrac{\mathrm{d}^{m+1} P_n(\sin\beta)}{\mathrm{d}\,t^{m+1}} - m\sin\beta(\cos\beta)^{m-1} \dfrac{\mathrm{d}^m P_n(\sin\beta)}{\mathrm{d}\,t^m}\right)\begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix}
\end{cases}
$$

For the longitudinal term we have:

$$
\nabla\mathfrak{L}(\lambda) = (-mC_{nm}\sin m\lambda + mS_{nm}\cos m\lambda)\nabla\lambda = \frac{m}{r\cos\beta}(S_{nm}\cos m\lambda - C_{nm}\sin m\lambda)\begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}
$$

Singularities

While the above formulae are all we need to compute the gradient of the geopotential, they present a number of singularities that required some care for implementation purposes.

There is obviously an essential singularity when $r = 0$. This one is not very interesting as we are never going to compute the acceleration at the centre of the celestial.

There are however non-essential singularities that arise when $\cos\beta$ appears at the denominator: any point on the axis of rotation of the celestial has $\cos\beta = 0$, but clearly the acceleration there is finite.

Consider $\nabla\mathfrak{B}(\beta)$. When $m = 0$ it includes a term in $(\cos\beta)^{-1}$. However, that term is multiplied by m, so it vanishes. Thus, for $m = 0$ we must use the special formula:

$$
\nabla\mathfrak{B}(\beta)_{m=0} = \frac{1}{r}\cos\beta\,\frac{\mathrm{d}\,P_n(\sin\beta)}{\mathrm{d}\,t}\begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix}
$$

Similarly $\nabla\mathfrak{L}(\lambda)$ has $\cos\beta$ as its denominator. To eliminate this term we note that $\nabla\mathfrak{L}(\lambda)$ always occurs in a product involving $\mathfrak{B}(\beta)$. Let's write this product:

$$
\mathfrak{B}(\beta)\nabla\mathfrak{L}(\lambda) = \frac{m(\cos\beta)^{m-1}}{r}\frac{\mathrm{d}^m P_m(\sin\beta)}{\mathrm{d}\,t^m}(S_{nm}\cos m\lambda - C_{nm}\sin m\lambda)\begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}
$$

When $m = 0$ this expression has a term in $(\cos\beta)^{-1}$ so we need to special-case it:

$$
\mathfrak{B}(\beta)\nabla\mathfrak{L}(\lambda)_{m=0} = 0
$$