On an Article by Celledoni et al.

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This document provides clarifications, corrections, and accuracy improvements to the formulæ presented in [CFSZ08]. It follows the notation and conventions of that paper.

Preamble

We remind the reader of the derivation formulæ for the Jacobian elliptic functions ([OLBC10], section 22.13(i)):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{sn} u &= \operatorname{cn} u \operatorname{dn} u \\ \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{cn} u &= -\operatorname{sn} u \operatorname{dn} u \\ \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{dn} u &= -k^2 \operatorname{sn} u \operatorname{cn} u \end{cases}$$

and for the hyperbolic functions ([OLBC10], section 4.34):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{th} u &= \operatorname{sech}^2 u \\ \dfrac{\mathrm{d}}{\mathrm{d}\,u}\operatorname{sech} u &= -\operatorname{sech} u \operatorname{th} u \end{cases}$$

The equations of motion

We start by writing equation (1) of [CFSZ08] in coordinates. The coordinates of m and I are defined by:

$$\boldsymbol{m} \coloneqq \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

and:

$$\boldsymbol{I} \coloneqq \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Euler's equation $\dot{m} = m \wedge (I^{-1}m)$ can be written in coordinates:

$$\dot{\boldsymbol{m}} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \wedge \begin{pmatrix} m_1/I_1 \\ m_2/I_2 \\ m_3/I_3 \end{pmatrix}$$

thus:

$$\begin{cases} \dot{m}_1 &= m_2 m_3 \big(1/I_3 - 1/I_2\big) \\ \dot{m}_2 &= m_3 m_1 \big(1/I_1 - 1/I_3\big) \\ \dot{m}_3 &= m_1 m_2 \big(1/I_2 - 1/I_1\big) \end{cases} \tag{1}$$

Solution of Euler's equation, case (i)

The case (i) of the solution of Euler's equation in section 2.2 of [CFSZ08] is:

$$\boldsymbol{m}_t = \begin{pmatrix} \sigma B_{13}\,\mathrm{dn}(\lambda t-\nu,k) \\ -B_{21}\,\mathrm{sn}(\lambda t-\nu,k) \\ B_{31}\,\mathrm{cn}(\lambda t-\nu,k) \end{pmatrix}$$

If we derive this expression with respect to t, inject in into (1), and eliminate the elliptic functions we obtain:

$$\begin{cases} -\sigma\lambda k^2 B_{13} & = -B_{21}B_{31}\big(1/I_3 - 1/I_2\big) \\ -\lambda B_{21} & = \sigma B_{13}B_{31}\big(1/I_1 - 1/I_3\big) \\ -\lambda B_{31} & = -\sigma B_{13}B_{21}\big(1/I_2 - 1/I_1\big) \end{cases} \tag{2}$$

The last equation of (2) yields the following value for λ:

$$\lambda = \sigma\frac{B_{13}B_{21}}{B_{31}}\frac{I_1-I_2}{I_1I_2} = \sigma\sqrt{\frac{I_1\Delta_3}{I_{13}}\frac{I_2\Delta_1}{I_{21}}\frac{I_{31}}{I_3\Delta_1}}\frac{I_1-I_2}{I_1I_2} = \sigma\sqrt{\frac{\Delta_3}{I_{21}I_1I_2I_3}}(I_1-I_2) = -\sigma\sqrt{\frac{\Delta_3 I_{21}}{I_1I_2I_3}} = -\sigma\lambda_3$$

The sign change when moving $I_1 - I_2$ under the radical is necessary because $I_1 - I_2 < 0$.

It is straightforward to check that this value of λ also satisfies the other equations of (2). Note that it differs in sign from the one given by [CFSZ08]: the sign error is visible in that it does not yield the proper precession direction.

Solution of Euler's equation, case (ii)

The case (ii) of the solution of Euler's equation in section 2.2 of [CFSZ08] is:

$$\boldsymbol{m}_t = \begin{pmatrix} B_{13}\,\mathrm{cn}(\lambda t-\nu,k^{-1}) \\ -B_{23}\,\mathrm{sn}(\lambda t-\nu,k^{-1}) \\ \sigma B_{31}\,\mathrm{dn}(\lambda t-\nu,k^{-1}) \end{pmatrix}$$

Just as we did above, we derive this expression with respect to t, inject in into (1), and eliminate the elliptic functions:

$$\begin{cases} -\lambda B_{13} & = -\sigma B_{23}B_{31}\big(1/I_3 - 1/I_2\big) \\ -\lambda B_{23} & = \sigma B_{13}B_{31}\big(1/I_1 - 1/I_3\big) \\ -\sigma\lambda k^{-2} & = -B_{13}B_{21}\big(1/I_2 - 1/I_1\big) \end{cases} \tag{3}$$

The first equation of (3) yields the following value for λ:

$$\lambda = \sigma\frac{B_{23}B_{31}}{B_{13}}\frac{I_2-I_3}{I_2I_3} = \sigma\sqrt{\frac{I_2\Delta_3}{I_{23}}\frac{I_3\Delta_1}{I_{31}}\frac{I_{13}}{I_1\Delta_3}}\frac{I_2-I_3}{I_2I_3} = \sigma\sqrt{\frac{\Delta_1}{I_{23}I_1I_2I_3}}(I_2-I_3) = -\sigma\sqrt{\frac{\Delta_1 I_{23}}{I_1I_2I_3}} = -\sigma\lambda_1$$

Again, note the change of sign due to the fact that $I_2 - I_3 < 0$. And again, the same value of λ can be shown to satisfy the other equations of (3).

Solution of Euler's equation, case (iii)

The case (iii) of the solution of Euler's equation in section 2.2 of [CFSZ08] is clearly incorrect as it implies that m_1 and m_3 always have the same sign, whereas it is straightforward to choose initial conditions where they do not. Instead, we introduce an extra parameter $\sigma'' = \pm 1$ and posit a solution of the form:

$$\boldsymbol{m}_t = \begin{pmatrix} \sigma' B_{13}\,\mathrm{sech}(\lambda t-\nu) \\ \mathrm{th}(\lambda t-\nu) \\ \sigma'' B_{31}\,\mathrm{sech}(\lambda t-\nu) \end{pmatrix}$$

Deriving this expression and injecting it into (1) yields:

$$\begin{cases} -\sigma'\lambda B_{13} & = \sigma'' B_{31}\big(1/I_3 - 1/I_2\big) \\ \lambda & = \sigma'\sigma'' B_{13}B_{31}\big(1/I_1 - 1/I_3\big) \\ -\sigma''\lambda B_{31} & = \sigma' B_{13}\big(1/I_2 - 1/I_1\big) \end{cases} \tag{4}$$

The second equation of (4) gives the following value for λ:

$$\lambda = \sigma'\sigma'' B_{13}B_{31}\frac{I_3 - I_1}{I_1 I_3} = \sigma'\sigma''\sqrt{\frac{I_1\Delta_3}{I_{13}}\frac{I_3\Delta_1}{I_{31}}}\frac{I_3 - I_1}{I_1 I_3} = \sigma'\sigma''\sqrt{\frac{\Delta_1\Delta_3}{I_1 I_3}}$$

In this case it is a bit less obvious that the other equations yield the same value of λ. We detail the derivation for the first equation, using the fact that ${\sigma'}^2 = 1$:

$$\lambda = -\sigma'\sigma''\frac{B_{31}}{B_{13}}\frac{I_2 - I_3}{I_2 I_3} = -\sigma'\sigma''\sqrt{\frac{I_3\Delta_1}{I_{31}}\frac{I_{13}}{I_1\Delta_3}}\frac{I_2 - I_3}{I_2 I_3} = -\sigma'\sigma''\sqrt{\frac{\Delta_1}{I_1 I_3\Delta_3}}\frac{I_2 - I_3}{I_2} = \sigma'\sigma''\sqrt{\frac{\Delta_1}{I_1 I_3\Delta_3}}\left(\frac{I_3}{I_2} - 1\right)$$

Now note that in case (iii) we have $2TI_2 = 1$ thus $1/I_2 = 2T$. λ can be rewritten as:

$$\lambda = \sigma'\sigma''\sqrt{\frac{\Delta_1}{I_1 I_3\Delta_3}}(2TI_3 - 1) = \sigma'\sigma''\sqrt{\frac{\Delta_1\Delta_3}{I_1 I_3}}$$

where we have used the fact that $2TI_3 - 1 = 2T(I_3 - I_2) > 0$.

It is easy to see that the radical is the common value of λ_1 and λ_3, so σ' and σ'' are free parameters and:

$$\lambda = \sigma'\sigma''\lambda_1 = \sigma'\sigma''\lambda_3$$

Phase and initial value

The phase ν and the free parameters σ, σ' and σ'' are determined from the initial value $\boldsymbol{m}_0$ by setting $t = 0$.

Case (i)

We have:

$$\boldsymbol{m}_0 = \begin{pmatrix} \sigma B_{13}\,\mathrm{dn}(-\nu, k) \\ -B_{21}\,\mathrm{sn}(-\nu, k) \\ B_{31}\,\mathrm{cn}(-\nu, k) \end{pmatrix}$$

First, we set σ to be the sign of m_{01}. Then, forming the quotient of the last two coordinates we find:

$$\frac{m_{02}}{m_{03}} = \frac{B_{21}}{B_{31}}\tan(\mathrm{am}(\nu, k))$$

thus:

$$\tan^{-1}\left(\frac{m_{02}}{m_{03}}\frac{B_{31}}{B_{21}}\right) = \mathrm{am}(\nu, k)$$

and finally we obtain ν as:

$$\nu = F\left(\tan^{-1}\left(\frac{m_{02}}{m_{03}}\frac{B_{31}}{B_{21}}\right), k\right)$$

Case (ii)

Starting from:

$$\boldsymbol{m}_0 = \begin{pmatrix} B_{13}\,\mathrm{cn}(-\nu, k^{-1}) \\ -B_{23}\,\mathrm{sn}(-\nu, k^{-1}) \\ \sigma B_{31}\,\mathrm{dn}(-\nu, k^{-1}) \end{pmatrix}$$

we set σ to be the sign of m_{03} and form the quotient of the first two coordinates. We obtain:

$$\frac{m_{02}}{m_{01}} = \frac{B_{23}}{B_{13}}\tan(\mathrm{am}(\nu, k^{-1}))$$

and for ν:

$$\nu = F\left(\tan^{-1}\left(\frac{m_{02}}{m_{01}}\frac{B_{13}}{B_{23}}\right), k^{-1}\right)$$

Case (iii)

The initial value $\boldsymbol{m}_0$ is:

$$\boldsymbol{m}_0 = \begin{pmatrix} \sigma' B_{13} \operatorname{sech}(-\nu) \\ \operatorname{th}(-\nu) \\ \sigma'' B_{31} \operatorname{sech}(-\nu) \end{pmatrix}$$

σ' and σ'' are set to be the signs of m_{01} and m_{03}, respectively. The second coordinate immediately gives:

$$\nu = -\operatorname{th}^{-1}(m_{02})$$

For this formula to be homogeneous we need to restore the total angular momentum G thus:

$$\nu = -\operatorname{th}^{-1}\left(\frac{m_{02}}{G}\right)$$

Implementation considerations

Some of the formulæ given by [CFSZ08] do not lend themselves to an easy implementation or lead to numerical inaccuracies. We describe in this section the modifications we make to their formulæ in our implementation. We also restore dimensionful formulæ as needed.

First, it is simpler and more efficient to avoid absolute values, so we define:

$$I_{jh} \coloneqq I_j - I_h$$

This is the same quantity as in [CFSZ08] when $j \geq h$ but it has the opposite sign (it is negative) when $j < h$.

Next we notice that the computation of Δ_j may entail cancellations, so we go back to the definition of $|\boldsymbol{m}|$ and of the kinetic energy:

$$\begin{cases} G^2 & = m_1^2 + m_2^2 + m_3^2 \\ 2T & = \dfrac{m_1^2}{I_1} + \dfrac{m_2^2}{I_2} + \dfrac{m_3^2}{I_3} \end{cases}$$

and we define Δ_j without absolute values:

$$\Delta_j \coloneqq G^2 - 2TI_j$$

When, for instance, $j = 2$ this yields:

$$\begin{aligned} \Delta_2 &= m_1^2\left(1 - \frac{I_2}{I_1}\right) + m_3^2\left(1 - \frac{I_2}{I_3}\right) \\ &= m_1^2 \frac{I_{12}}{I_1} + m_3^2 \frac{I_{32}}{I_3} \end{aligned}$$

and similarly:

$$\begin{cases} \Delta_1 & = m_2^2 \dfrac{I_{21}}{I_2} + m_3^2 \dfrac{I_{31}}{I_3} \\ \Delta_3 & = m_1^2 \dfrac{I_{13}}{I_1} + m_2^2 \dfrac{I_{23}}{I_2} \end{cases}$$

It is easy to see that Δ_1 and Δ_3 are the sums of terms of the same sign, so they can be computed without cancellations and $\Delta_1 \geq 0$ and $\Delta_3 \leq 0$. Δ_2 can have either sign, which correspond exactly to cases (i) ($\Delta_2 < 0$), (ii) ($\Delta_2 > 0$) and (iii) ($\Delta_2 = 0$).

Finally, for the computation of the elliptic functions and integrals [CFSZ08] gives the value of k but we need the value of $m_c = 1 - m$ (see [OLBC10], section 19.1.2 for an overview of the notation). In case (i) we have:

$$m_c = 1 - k^2 = 1 + \frac{\Delta_1 I_{32}}{\Delta_3 I_{21}}$$

where we have used $\Delta_3 \leq 0$. This can be rewritten as follows:

$$m_c = \frac{\Delta_3 I_{21} + \Delta_1 I_{32}}{\Delta_3 I_{21}} = \frac{(G^2 - 2TI_3)(I_2 - I_1) + (G^2 - 2TI_1)(I_3 - I_2)}{\Delta_3 I_{21}}$$
$$= \frac{G^2(I_3 - I_1) + 2TI_2(I_1 - I_3)}{\Delta_3 I_{21}} = \frac{\Delta_2 I_{31}}{\Delta_3 I_{21}}$$

Similarly, in case (ii):

$$m_c = 1 - k^{-2} = 1 + \frac{\Delta_3 I_{21}}{\Delta_1 I_{32}} = \frac{\Delta_1 I_{32} + \Delta_3 I_{21}}{\Delta_1 I_{32}} = \frac{\Delta_2 I_{31}}{\Delta_1 I_{32}}$$

Note that in both cases we have $m_c \geq 0$.

References

[CFSZ08] E. Celledoni, F. Fassò, N. Säfström, and A. Zanna. "The exact computation of the free rigid body motion and its use in splitting methods". In: *SIAM J. Scientific Computing* 30 (May 2008), pp. 2084–2112.

[OLBC10] F. Olver, D. Lozier, R. Boisvert, and C. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.