

Calculations for the second-order zonal harmonic

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Notations:

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, |\mathbf{z}| = 1, r = |\mathbf{r}|.$$

For oblateness along the z-axis, the potential is

$$\frac{J_2}{2r^5}(3z^2 - r^2).$$

For oblateness along \mathbf{z} ,

$$U(\mathbf{r}) = \frac{J_2}{2r^5}(3(\mathbf{r} \cdot \mathbf{z})^2 - r^2).$$

Differentiating,

$$\frac{dU}{d\mathbf{r}} = \frac{J_2}{2} \left(-\frac{5}{r^6} \frac{d\mathbf{r}}{d\mathbf{r}} (3(\mathbf{r} \cdot \mathbf{z})^2 - r^2) + \frac{1}{r^5} \left(3 \frac{d}{d\mathbf{r}} (\mathbf{r} \cdot \mathbf{z})^2 - 2r \frac{d\mathbf{r}}{d\mathbf{r}} \right) \right).$$

Recall that $\frac{d\mathbf{r}}{d\mathbf{r}} = \frac{\mathbf{r}}{r}$,

$$\begin{aligned} \frac{dU}{d\mathbf{r}} &= \frac{J_2}{2} \left(-\frac{5\mathbf{r}}{r^7} (3(\mathbf{r} \cdot \mathbf{z})^2 - r^2) + \frac{1}{r^5} \left(3 \frac{d}{d\mathbf{r}} (\mathbf{r} \cdot \mathbf{z})^2 - 2\mathbf{r} \right) \right) \\ &= \frac{J_2}{2} \left(-\frac{15\mathbf{r}}{r^7} (\mathbf{r} \cdot \mathbf{z})^2 + \frac{3\mathbf{r}}{r^5} + \frac{3}{r^5} \frac{d}{d\mathbf{r}} (\mathbf{r} \cdot \mathbf{z})^2 \right). \end{aligned}$$

With $\frac{d}{d\mathbf{r}} (\mathbf{r} \cdot \mathbf{z})^2 = 2(\mathbf{r} \cdot \mathbf{z}) \frac{d(\mathbf{r} \cdot \mathbf{z})}{d\mathbf{r}} = 2(\mathbf{r} \cdot \mathbf{z})\mathbf{z}$,

$$\begin{aligned} \frac{dU}{d\mathbf{r}} &= \frac{J_2}{2} \left(-\frac{15\mathbf{r}}{r^7} (\mathbf{r} \cdot \mathbf{z})^2 + \frac{3\mathbf{r}}{r^5} + \frac{6\mathbf{z}}{r^5} (\mathbf{r} \cdot \mathbf{z}) \right) \\ &= \frac{3J_2}{2r^5} \left(2\mathbf{z}(\mathbf{r} \cdot \mathbf{z}) + \mathbf{r} \left(1 - \frac{5(\mathbf{r} \cdot \mathbf{z})^2}{r^2} \right) \right). \end{aligned}$$

Note that this is invariant under $\mathbf{z} \mapsto -\mathbf{z}$ (but not under $\mathbf{r} \mapsto -\mathbf{r}$).