

Solutions for Advanced Math II Week II Slides

Jie Young

January 4, 2026

1 Lecture Slide 1: De Moivre's Theorem

Example 1: Solve $[5(\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4}))]^3$ and $5(\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4}))^3$

Given De Moivre's Theorem, $Z^n = r^n[\cos(n\theta) + i \sin(\theta)]$

$$[5(\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4}))]^3 = 125(\cos(\frac{3\pi}{4}) + i \sin(3\frac{\pi}{4}))$$

Where,

$$5(\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4})) = 5(\cos(\frac{3\pi}{4}) + i \sin(3\frac{\pi}{4}))$$

Example 2: Simplify $[3(\cos(\frac{2\pi}{3}) + i \sin(\frac{2\pi}{3}))]^4$

$$[3(\cos(\frac{2\pi}{3}) + i \sin(\frac{2\pi}{3}))]^4 = 81(\cos(\frac{8\pi}{3}) + i \sin(\frac{8\pi}{3}))$$

$\frac{8\pi}{3}$ can be simplified by deducting 2π

$$Z^n = 81(\cos(\frac{2\pi}{3}) + i \sin(\frac{2\pi}{3}))$$

Example 3: Simplify $\frac{1}{[2(\cos(\frac{\pi}{6}) + i \sin(\frac{\pi}{6}))]^4}$

By Converting to Exponential form, $Z^n = r^n e^{i\theta}$

$$\frac{1}{[2(\cos(\frac{\pi}{6}) + i \sin(\frac{\pi}{6}))]^4} = 2^{-4} e^{-i4\frac{\pi}{6}}$$

$$\frac{1}{[2(\cos(\frac{\pi}{6}) + i \sin(\frac{\pi}{6}))]^4} = \frac{1}{16} e^{-\frac{2\pi}{3}i}$$

Example 4: Express $(1 + i)^9$ in cartesian form

Converting to Exponential Form

$$r = \sqrt{1^2 + 1^2} \qquad \arg(z) = \tanh\left(\frac{1}{1}\right)$$

$$= \sqrt{2} \qquad \qquad \qquad = \frac{\pi}{4}$$

$$Z = \sqrt{2}^9 e^{\frac{9\pi}{4}i}$$

$$Z = 16\sqrt{2}[\cos(\frac{9\pi}{4}) + i \sin(\frac{9\pi}{4})]$$

$$Z = 16 + 16i$$

Example 5: Express $(-1 - \sqrt{3}i)^{-5}$ in cartesian form

Converting to Exponential Form

$$r = \sqrt{(-1)^2 + (-\sqrt{3})^2} \qquad \arg(z) = \tanh(\sqrt{3})$$

$$= 2 \qquad \qquad \qquad = \frac{\pi}{3}$$

$$\arg(z) \text{ is in Quadrant III, therefore } \arg(z) = \frac{-2\pi}{3}$$

$$Z = 2^{-5} e^{\frac{10\pi}{3}i}$$

$$Z = \frac{1}{32}[\cos(\frac{10\pi}{3}) + i \sin(\frac{10\pi}{3})]$$

$$Z = -\frac{1}{64} - \frac{\sqrt{3}}{64}i$$

Example 6: Simplify $\frac{\cos(4\theta) + i \sin(4\theta)}{\cos(3\theta) + i \sin(3\theta)}$ into Polar Form

$$\frac{\cos(4\theta) + i \sin(4\theta)}{\cos(3\theta) + i \sin(3\theta)} = \frac{e^{4i\theta}}{e^{3i\theta}} \quad Z = e^{i\theta} \quad Z = \cos(\theta) + i \sin(\theta)$$

Example 7: Find $\arg[(\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4}))^3(\cos(\frac{\pi}{8}) + i \sin(\frac{\pi}{8}))^2]$

$$\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4})^3(\cos(\frac{\pi}{8}) + i \sin(\frac{\pi}{8}))^2 = e^{\frac{3\pi}{4}i} + e^{\frac{\pi}{4}i} = e^{\pi i}$$

$$\arg(z) = \pi$$

1.1 General Form of Complex Numbers under De Moivre's Theorem

1.1.1 Cartesian Form

$$Z^n = (a + bi)^n$$

1.1.2 Polar Form

$$Z^n = r^n[\cos(n\theta) + i \sin(\theta)]$$

1.1.3 Exponential Form

$$Z^n = r^n e^{in\theta}$$

2 Lecture Slide 2: Functions of Complex Numbers

2.1 Ways of Expansions

2.1.1 Binomial Expansions

Pascal Triangle

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & 1 & & 1 & \\ & & 1 & & 2 & & 1 \\ & 1 & & 3 & & 3 & & 1 \\ & & 1 & & 4 & & 6 & & 4 & & 1 \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\ & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \end{array}$$

Binomial Theorem The formula for $(a + b)^n$ is given by:

$$\sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

Where,

$$\binom{n}{k} = {}^n C_r = \frac{n!}{r!(n-r)!}$$

2.1.2 De Moivre's Theorem

The powers of sine and cosine functions can be expressed as Complex Numbers in Exponential Form. As shown in table below:

$e^{ix} = \cos(x) + i \sin(x)$	$e^{ix} - e^{-ix} = i \sin(x)$
$e^{ix} + e^{-ix} = 2 \cos(x)$	$e^{-ix} = \cos(x) - i \sin(x)$
$e^{-ix} + e^{ix} = 2 \cos(x)$	$e^{ix} - e^{-ix} = 2i \sin(x)$

Table 1: Table of Function of Complex Numbers

2.2 Solutions for slides II

Example 1: Expand $(a + b)^3$, $(a + b)^5$

$$\begin{aligned}(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\(a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

Example 2: Solve $\cos^3(x)$

$$e^{ix} + e^{-ix} = 2 \cos(x)$$

$$\cos^3(x) = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3$$

$$\cos^3(x) = \frac{1}{8}(e^{i3x} + 3e^{ix} + 3e^{-ix} + e^{-i3x})$$

$$\cos^3(x) = \frac{1}{8}(e^{i3x} + e^{-i3x} + 3e^{ix} + 3e^{-ix})$$

$$\cos^3(x) = \frac{1}{8}(2 \cos(3x) + 3(2 \cos(x)))$$

$$\cos^3(x) = \frac{1}{4} \cos(3x) + \frac{3}{4} \cos(x)$$

Example 3: Solve $\sin^3(x)$

$$e^{ix} - e^{-ix} = 2i \sin(x)$$

$$\cos^3(x) = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3$$

$$\cos^3(x) = -\frac{1}{8}(e^{i3x} - 3e^{ix} + 3e^{-ix} + e^{-i3x})$$

$$\cos^3(x) = -\frac{1}{8}(e^{i3x} - e^{-i3x} + 3e^{ix} - 3e^{-ix})$$

$$\cos^3(x) = -\frac{1}{8}(2i \sin(3x) + 3(2i \sin(x)))$$

$$\cos^3(x) = -\frac{1}{4}i \sin(3x) + \frac{3}{4}i \sin(x)$$

Example 4: Express $\sin^3(x)$ as the sum of powers of $\sin(x)$ only, and express $\cos^3(x)$ as the sum of power of $\cos(x)$ only

By De Moivre's Theorem:

$$(\cos(x) + i \sin(x))^3 = \cos(3x) + i \sin(3x)$$

By Binomial Expansion:

$$(\cos(x) + i \sin(x))^3 = \cos^3(x) + 3i \cos^2(x) \sin(x) - 3 \cos(x) \sin^2(x) - i \sin^3(x)$$

$$\cos(3x) + i \sin(3x) = \cos^3(x) - 3 \cos(x) \sin^2(x) + i[3 \cos^2(x) \sin(x) - \sin^3(x)]$$

For $\cos(3x)$:

$$\cos(3x) = \cos^3(x) - 3 \cos(x) \sin^2(x)$$

$$\cos(3x) = \cos^3(x) - 3 \cos(x)(1 - \cos^2(x))$$

$$\cos(3x) = 4 \cos^3(x) - 3 \cos(x)$$

For $\sin(3x)$:

$$\sin(3x) = 3 \cos^2(x) \sin(x) - \sin^3(x)$$

$$\sin(3x) = 3(1 - \sin^2(x)) \sin(x) - \sin^3(x)$$

$$\sin(3x) = 3 \sin(x) - 4 \sin^3(x)$$