

Physics of Complex Systems

https://github.com/Grufoony/Physics_Unibo

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Chapter 1

Introduction to Complex Systems

Complex systems are interdisciplinary: you have to communicate your results to experts from different science field. You need to provide an easy answer to a complex problem, no matter how difficult it's to get this answer.

1.1 Emergence

One of the main ideas in complex systems is emergence. Emergence means that the structure of the particles is simple and they are not so important like interactions between them.

An example of this is the Central limit theorem, which comes from mathematics.

If you have a random variable x_k , with average value $\langle x_k \rangle = 0$ and finite variance σ^2 , then the central limit theorem says that, if the variables are independent (and in physics this is usually a fair assumption) for every value of k , then the normalized sum

$$z_n = \frac{1}{\sqrt{N}} \sum_{k=0}^N x_k$$

then the distribution of this variable z_n is known, and it is gaussian, for a big enough value of N .

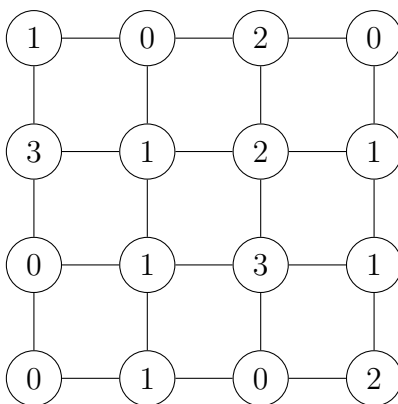
$$\rho(z_n) \sim \exp\left(-\frac{z_n^2}{2\sigma^2}\right)$$

Despite the fact that we, as humans, need a cause-effect relationship to describe a phenomenon, nature loves independent events, like DNA mutations. The gaussian describes the fluctuations of a system at equilibrium, not the complexity. You cannot extract work from fluctuations at equilibrium, otherwise you violate thermodynamics (and that's no good).

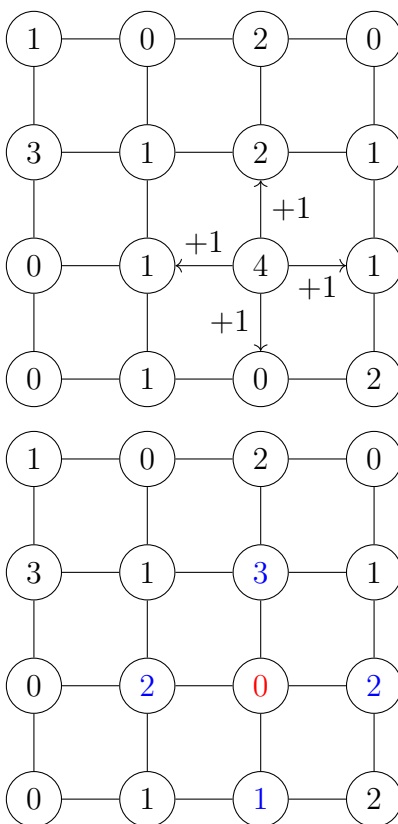
The gaussian function is not a physical function, because it implies non zero probabilities to events which are impossible. For example, if we take a particle in a basin of attraction of a potential, the non zero probability given by the gaussian fluctuations

allows the particle to jump out of the pit. But this violates the second law of thermodynamics.

A way to defy the gaussian properties is to allow a system to have memory, so to remove the independence of the variable.



One example of this is the sand pile model. You have a lattice, and each point is connected to its four neighbors. At this point one particle is put in a randomly chosen point. Each node can have four possible states, 0, 1, 2, 3, that is four possible numbers of particles. If a node reaches 4 particles, the 4 particles are distributed to the 4 neighbouring nodes.



For the nodes in the border, what happens is that 2 of the particles are redistributed and the other 2 are released in the environment, so they are lost.

So this system has memory, and this means that the states of all the nodes are not independent. This memory turns the gaussian distribution into a power law

$$p(n) \propto n^{-\alpha}$$

with $\alpha > 1$.

The decay of the power law is much much slower than that of the gaussian, which means that the probability to have events in the extremes is significantly higher with the power law distribution. Memory is related to power laws, but viceversa is not guaranteed. We can also have power laws in physics, for example in the Ising model. In physics, power laws typically represent a phase transition for a dynamic system in a non-equilibrium state.

In the sand pile model, we notice a self-organized criticality: the system naturally goes into a critical state and does a phase transition.

Another example of a natural power law is the Kleiber law which relates the metabolic rate (amount of energy you need to survive) to the mass.

$$E(m) \propto m^{\frac{4}{3}} \quad (1.1)$$

We can also observe that heartbeat rate decreases with mass and lifetime increases.

1.2 The broken stick model

Let's consider the portion $[0, 1]$ of the axis, which represents a segment of length 1. Suppose that we extract randomly a point x_1 on that segment, and we cut it in correspondence of that point, thus obtaining the portion $[0, x_1]$ of the axis. If we do it many times, we obtain a system with memory, because of course the length of the segment at a certain iteration depends on all the previous iterations.

For this system one expects to have a power law distribution, because if we restale the variable x , the distribution must not change.

$$p(x) \sim \frac{1}{x^a}$$

After N iterations one tries to construct the probability distribution of the cuts (fallo e poi riscrivi).

Since we have:

$$\begin{aligned} \langle x_k \rangle &= \frac{\langle x_{k-1} \rangle}{2} \\ \rho_N(2x) &= \frac{1}{2} \rho_{N-1}(x) \end{aligned}$$

Which means that, as N goes to infinity we get

$$\rho(2x) = \frac{1}{2}\rho(x)$$

$$\rho(x) = \frac{1}{x}$$

Let's take again a segment, but try to implement the system without the memory. So we have N independent variables x_k uniformly distributed. For each variable we define it's distance from the previous one, Δx .

The probability to find Δx is the probability of not finding x in any segment, so

$$p(\Delta x) \approx \left(1 - \frac{\Delta x}{L}\right)$$

We then take a new variable $y = N\Delta x$ and we obtain the probability distribution

$$p = \exp(-y/L)$$

as N goes to infinity, which of course is an exponential law.

Suppose that the whole stick is a state, and we want to distribute the population inside of it. If we divide the stick uniformly in portions and diivde the populations in this group, we obtain an exponential law, as we have just seen.

Another way, which contains memory, consists of creating a city and letting it grow, and only then introducing a second one, and repeating the process until all the space is occupied.

Let's take a sphere of radius L , containing nodes of size l , with a numero of dimensions equal to D .

$$\left(\frac{L}{l}\right)^D = N$$

For a node we define its energy flow φ . How much energy do you need to insert in the system?

If we have to distribute something to all the nodes, the most basic way to do it is to connect one on one all the nodes, as to form a long chain.

If the flow out of the source S is ϕ , the flow after the first node is $\phi - \varphi$, and so on, and we expect that the flow arriving to the final node will be φ .

So the total energy is

$$E_T \approx l \sum_{k=0}^{N-1} (\phi - k\varphi) = l\varphi \sum_{k=0}^{N-1} k$$

$$E_T \propto l\varphi N^2$$

Another possible connection of the nodes is the folloing: This is what is called a tree structure. The main problem with such a structure is that the path between the nodes is always the same, but it is impossible to design a city in such a way.

The number of nodes is 2^m , where m is the number of levels, so one has that $2^{m+1} = N$. The relation for the flow in each level is

$$2\phi_k + \varphi = \phi_{k-1}$$

where we assume again that the flow in the final level is φ .

ϕ_k then is

$$\phi_k = (2^{m+1-k} - 1)\varphi$$

Then:

$$E_T = d\varphi \sum_{k=1}^m 2^k (2^{m+1-k} - 1) \approx d\varphi m 2^{m+1}$$

$$E_T = d\phi N \log_2(N)$$

Which is much better then the previous alternative, although it is impossible to put to practice.

So we need to find a solution that is in the middle of these two.

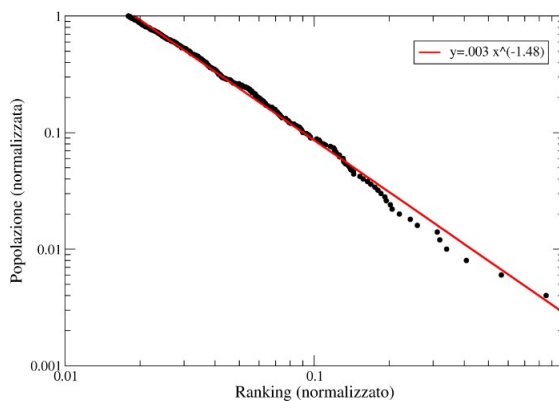
Chapter 2

Rankings

A ranking is an ordered subset of measures of the same quantity, between which exists an order relation. Rankings allow us to obtain complex informations with the use of very little data, since the information is contained in the order of the values.

An example of the use of rankings can be found in web pages, where the search engine shows to the user the results of the search, and the results are ordered depending on their relevance, so to allow them to find the desired information as quickly as possible. In this example, the quantity is the relevance of a result to the search.

Another example is the population of cities. The graph below shows the population of the 300 biggest cities in Europe. The graph is in log-log scale, and since it shows a straight line, it means that the distribution is a power law distribution.



This can be explained considering the fact that, historically, the population used to migrate towards the biggest cities of the country, because there were more job opportunities. Thus one can say that big cities have a preferential attachment.

Let x be a random variable and $\{x_1, x_2, \dots, x_N\}$ a sample of experimental observations of the variable, ordered in such a way that $x_1 \geq x_2 \geq \dots \geq x_N$.

Then, the ranking distribution is created by assigning to the position j in the order the corresponding value in the sample, which is given by the map $x_j = f(j)$. Also, usually the elements are normalized like $y_j = x_j/x_1$.

By definition, j/N is the frequency of the event $/x \mid x \geq x_j/$, that is, the probability that the variable x has a value larger than x_j for larger values of j .

The cumulative distribution is defined as $F(x) = P\{u \mid u \leq x_j\}$, that is, the probability that the value u is smaller than x . Of course, this probability is equal to the integral of the distribution in the range $[-\infty, x]$.

The relation between the frequency j/N and the cumulative distribution is given by:

$$F(x_j) = 1 - \frac{j}{N} = 1 - \frac{J(x_j)}{N}$$

where $J(x_j)$ is the inverse of the ranking, so it's the function that returns the position of a certain value of x in the sample's order.

Since the cumulative distribution is the integral of the probability distribution, the latter can be obtained by differentiating the first:

$$p(x) = \frac{dF}{dx} \longrightarrow p(x) = -\frac{1}{N} \frac{dJ}{dx}$$