Conditional Probability and Bayes Theorem Assignment

June 1, 2024

Conditional Probability

Question 1

In a survey among few people, 60% read Hindi newspaper, 40% read English newspaper and 20% read both. If a person is chosen at random and if he already reads English newspaper find the probability that he also reads Hindi newspaper.

Solution

Given:

$$P(H) = 0.6, \quad P(E) = 0.4, \quad P(H \cap E) = 0.2$$

We need to find $P(H|E)$.

Using the definition of conditional probability:

$$P(H|E) = \frac{P(H \cap E)}{P(E)} = \frac{0.2}{0.4} = 0.5$$

So, the probability that a person reads Hindi newspaper given that they read English newspaper is 0.5.

Question 2

You are given a set of cards numbered from 1 to 15. You choose two cards at random such that the sum of the numbers on the cards is even. What is the probability that both the cards you chose have odd numbers?

Solution

Number of odd numbered cards: 8 (1, 3, 5, 7, 9, 11, 13, 15)

Number of even numbered cards: 7 (2, 4, 6, 8, 10, 12, 14)

Ways to choose 2 odd cards out of 8:

$$\binom{8}{2} = \frac{8 \times 7}{2} = 28$$

Ways to choose 2 even cards out of 7:

$$\binom{7}{2} = \frac{7 \times 6}{2} = 21$$

Total favorable outcomes:

$$28 + 21 = 49$$

Probability that both chosen cards have odd numbers given their sum is even:

$$P(\text{both cards are odd} \mid \text{sum is even}) = \frac{28}{49} = \frac{4}{7}$$

So, the probability that both chosen cards have odd numbers is $\frac{4}{7}$.

Question 3

Let E and F be events of an experiment such that $P(E) = \frac{3}{10}, P(F) = \frac{1}{2}, P(F|E) = \frac{2}{5}$. Find:

1. $P(E \cap F)$
2. $P(E|F)$
3. $P(E \cup F)$

Solution

i. $P(E \cap F)$:

Using the definition of conditional probability:

$$P(F|E) = \frac{P(E \cap F)}{P(E)}$$

$$\frac{2}{5} = \frac{P(E \cap F)}{\frac{3}{10}}$$

$$P(E \cap F) = \frac{2}{5} \times \frac{3}{10} = \frac{6}{50} = \frac{3}{25}$$

ii. $P(E|F)$:

Using the definition of conditional probability:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$P(E|F) = \frac{\frac{3}{25}}{\frac{1}{2}} = \frac{3}{25} \times \frac{2}{1} = \frac{6}{25}$$

iii. $P(E \cup F)$:
Using the formula for the union of two events:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$P(E \cup F) = \frac{3}{10} + \frac{1}{2} - \frac{3}{25}$$

$$P(E \cup F) = \frac{15}{50} + \frac{25}{50} - \frac{6}{50} = \frac{34}{50} = \frac{17}{25}$$

Bayes Theorem

Question 4

Three persons A, B and C have applied for a job in a private company. The chance of their selections is in the ratio 1 : 2 : 4. The probabilities that A, B and C can introduce changes to improve the profits of the company are 0.8, 0.5 and 0.3, respectively. If the change does not take place, find the probability that it is due to the appointment of C.

Solution

Selection probabilities ratio: 1:2:4.

$$P(A) = \frac{1}{7}, \quad P(B) = \frac{2}{7}, \quad P(C) = \frac{4}{7}$$

Probabilities of introducing changes:

$$P(I_A) = 0.8, \quad P(I_B) = 0.5, \quad P(I_C) = 0.3$$

Find the probability that changes are not introduced:

$$P(I^c) = 1 - P(I)$$

where $P(I)$ is the probability that changes are introduced.

Using the law of total probability:

$$P(I) = P(I|A)P(A) + P(I|B)P(B) + P(I|C)P(C)$$

$$P(I) = (0.8 \times \frac{1}{7}) + (0.5 \times \frac{2}{7}) + (0.3 \times \frac{4}{7}) = \frac{0.8}{7} + \frac{1}{7} + \frac{1.2}{7} = \frac{3}{7}$$

Therefore:

$$P(I^c) = 1 - \frac{3}{7} = \frac{4}{7}$$

Find the probability that changes are not introduced due to C's appointment, $P(C|I^c)$:

$$P(C|I^c) = \frac{P(I^c|C)P(C)}{P(I^c)}$$

We need $P(I^c|C)$:

$$P(I^c|C) = 1 - P(I|C) = 1 - 0.3 = 0.7$$

Substituting the values:

$$P(C|I^c) = \frac{0.7 \times \frac{4}{7}}{\frac{4}{7}} = 0.7$$

So, the probability that changes are not introduced due to the appointment of C is 0.7.

Question 5

A new virus test has been developed. The test's accuracy is as follows: If a person is infected, the test correctly identifies it with probability $P(T|I) = 0.98$. If a person is not infected, the test incorrectly identifies them as infected with probability $P(T|N) = 0.03$. The overall prevalence of the virus in the population is 1%. If a person tests positive, what is the probability that they are actually infected?

Solution

Define the events:

$$I : \text{Person is infected}$$
$$N : \text{Person is not infected}$$
$$T : \text{Test is positive}$$

Given probabilities:

$$P(T|I) = 0.98, \quad P(T|N) = 0.03, \quad P(I) = 0.01, \quad P(N) = 0.99$$

Using Bayes' Theorem:

$$P(I|T) = \frac{P(T|I)P(I)}{P(T)}$$

First, calculate $P(T)$ using the law of total probability:

$$P(T) = P(T|I)P(I) + P(T|N)P(N)$$

$$P(T) = (0.98 \times 0.01) + (0.03 \times 0.99) = 0.0098 + 0.0297 = 0.0395$$

Now, using Bayes' Theorem:

$$P(I|T) = \frac{0.98 \times 0.01}{0.0395} = \frac{0.0098}{0.0395} \approx 0.2481$$

So, the probability that a person is actually infected given that they have tested positive is approximately 0.2481 or 24.81%.

Question 6

There are three identical cards except that both the sides of the first card are coloured red, both sides of the second card are coloured blue and for the third card one side is coloured red and the other side is blue. One card is randomly selected from these three cards and put down and the visible side of the card is red. What is the probability that the other side is blue?

Solution

Define the events:

$$A : \text{The chosen card is the one with one red side and one blue side}$$

$$R : \text{The visible side of the chosen card is red}$$

Identify the possible outcomes:

- Card 1 (Red/Red): both sides are red.
- Card 2 (Blue/Blue): both sides are blue.
- Card 3 (Red/Blue): one side is red, the other side is blue.

Each card has an equal probability of being chosen:

$$P(\text{Card 1}) = P(\text{Card 2}) = P(\text{Card 3}) = \frac{1}{3}$$

Calculate the probability of seeing a red side given each card:

$$P(R|\text{Card 1}) = 1, \quad P(R|\text{Card 2}) = 0, \quad P(R|\text{Card 3}) = \frac{1}{2}$$

Using the law of total probability:

$$P(R) = P(R|\text{Card 1})P(\text{Card 1}) + P(R|\text{Card 2})P(\text{Card 2}) + P(R|\text{Card 3})P(\text{Card 3})$$

$$P(R) = (1 \times \frac{1}{3}) + (0 \times \frac{1}{3}) + (\frac{1}{2} \times \frac{1}{3}) = \frac{1}{3} + 0 + \frac{1}{6} = \frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

Apply Bayes' Theorem:

$$P(A|R) = \frac{P(R|A)P(A)}{P(R)}$$

Here, A corresponds to choosing Card 3. Thus:

$$P(R|A) = \frac{1}{2}, \quad P(A) = \frac{1}{3}, \quad P(R) = \frac{1}{2}$$

Substituting the values:

$$P(A|R) = \left(\frac{1}{2}\right) \times \left(\frac{1}{3}\right) \div \left(\frac{1}{2}\right) = \frac{1}{6} \div \frac{1}{2} = \frac{1}{6} \times 2 = \frac{1}{3}$$

So, the probability that the other side of the card is blue given that the visible side is red is $\frac{1}{3}$.