Existence, uniqueness and computation of solutions to DSGE models with occasionally binding constraints.

Tom Holden[1]*, School of Economics, University of Surrey*

Abstract: *In this paper we present the first necessary and sufficient conditions for the existence of a unique perfect-foresight solution to an otherwise linear model with occasionally binding constraints, where solution paths are constrained to return to the original steady-state. We also derive both necessary conditions and sufficient conditions for the existence of a solution to such a model. We show that widely used models such as those of Smets and Wouters (2003; 2007) possess multiple perfect foresight equilibria paths when augmented with a zero lower bound on nominal interest rates, and that in some states of the world they have no equilibrium at all. We go on to construct the first solution algorithm for these problems that is guaranteed to return a solution in finite time, if one exists. Moreover, in the presence of multiple solutions, the solution algorithm is able to always select the one that maximises an intuitive criterion. This solution algorithm is first extended to non-linear models, and then further extended to take into account future uncertainty, both in the short run, and in the longer run via a novel hybrid of local and global approaches. Our solution algorithm is shown to produce accurate simulations on a wide range of DSGE models with occasionally binding constraints.*

Keywords: *occasionally binding constraints, zero lower bound, DSGE, linear complementarity problem, LCP*
JEL Classification: *TODO*

Please make sure you have the latest version of this paper from:
https://raw.githubusercontent.com/tholden/DynareOBC/master/paper.pdf

[1] Contact address: School of Economics, FBEL, University of Surrey, Guildford, GU2 7XH, England
Contact email: thomas.holden@gmail.com Web-site: http://www.tholden.org/
The author would particularly like to thank Michael Paetz for joint work on an earlier algorithm for simulating models with occasionally binding constraints. Additionally, the author would like to thank the participants in numerous seminars at which versions of this paper were presented. Further special thanks are due to the following individuals for helpful discussions: Martin M. Andreasen, Charles Brendon, Michael Funke, William T. Gavin, Pablo Guerrón-Quintana, Matteo Iacoviello, Michel Juillard, Hong Lan, Paul Levine, Jesper Lindé, Albert Marcet, Antonio Mele, Alexander Meyer-Gohde, Matthias Paustian, Søren Ravn, Jonathan Swarbrick and Simon Wren-Lewis. Financial support provided to the author by the ESRC and the EC is also greatly appreciated. Furthermore, the author gratefully acknowledges the use of the University of Surrey FBEL cluster for the numerical computations performed in this paper.

1. Introduction

TODO

2. Occasionally binding constraints in otherwise linear models under perfect foresight

2.1. Theoretical results

2.1.1. Problem set-ups

Suppose that for $t \in \mathbb{N}^+$, (i.e. $t \in \mathbb{N}, t > 0$):

$$\left(\hat{A} + \hat{B} + \hat{C}\right)\hat{\mu} = \hat{A}\hat{x}_{t-1} + B\hat{x}_t + \hat{C}\mathbb{E}_t\hat{x}_{t+1} + \hat{D}\varepsilon_t,$$

where $\mathbb{E}_{t-1}\varepsilon_t = 0$ for all $t \in \mathbb{N}^+$, and suppose that $\hat{x}_0$ is given as an initial condition. Furthermore, suppose that $\varepsilon_t = 0$ for $t > 1$, as in an impulse response or perfect foresight simulation exercise. Additionally, we assume the existence of a the terminal condition of the form $\hat{x}_t \to \hat{\mu}$ as $t \to \infty$, coming, for example, from the model's transversality constraints.

For $t \in \mathbb{N}^+$, define:

$$x_t := \begin{bmatrix} \hat{x}_t \\ \varepsilon_{t+1} \end{bmatrix}, \qquad \mu := \begin{bmatrix} \hat{\mu} \\ 0 \end{bmatrix}, \qquad A := \begin{bmatrix} \hat{A} & \hat{D} \\ 0 & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} \hat{B} & 0 \\ 0 & I \end{bmatrix}, \qquad C := \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix}$$

then, for $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1},$$

and we have the extended initial condition $x_0 = \begin{bmatrix} \hat{x}_0 \\ \varepsilon_1 \end{bmatrix}$, and the extended terminal condition $x_t \to \mu$ as $t \to \infty$.

Expectations have disappeared since there is no uncertainty after period 0. Thus, the problem of solving the original model has the form as that given in:

Problem 1

Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1}.$$

We make the following assumption in all of the following:

Assumption 1

For any given $x_0 \in \mathbb{R}^n$, Problem 1 has a unique solution, which takes the form $x_t = (I - F)\mu + Fx_{t-1}$, for $t \in \mathbb{N}^+$, where $F = -(B + CF)^{-1}A$.

Sims's (2002) generalisation of the standard Blanchard-Kahn (1980) conditions is necessary and sufficient for this.

We are interested in models featuring occasionally binding constraints. For now, we will concentrate on models featuring a single zero lower bound type constraint in their first equation, which we treat as defining the first element of x_t. Generalising from this special case will be straightforward. First, let us define I as the $T \times T$ identity matrix, and let us write $x_{1,t}$, $I_{1,\cdot}$, $A_{1,\cdot}$, $B_{1,\cdot}$, $C_{1,\cdot}$ for the first row of x_t, I, A, B, C (respectively) and

$x_{-1,t}$, $I_{-1,\cdot}$, $A_{-1,\cdot}$, $B_{-1,\cdot}$, $C_{-1,\cdot}$ for the remainders. (Similar notation will be adopted for the rest of this paper.) Then we are interested in the solution to:

Problem 2

Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $T \in \mathbb{N}$ and $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$x_{1,t} = \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu)\},$$
$$(A_{-1,\cdot} + B_{-1,\cdot} + C_{-1,\cdot})\mu = A_{-1,\cdot}x_{t-1} + B_{-1,\cdot}x_t + C_{-1,\cdot}x_{t+1},$$

and such that $x_{1,t} > 0$ for $t > T$.

Note that in this problem we are implicitly ruling out any solutions which get permanently stuck at an alternative steady-state, by assuming that the terminal condition remains as before. Indeed, we are assuming that the bound is only relevant for some finite number of periods T. We continue to assume that there is no uncertainty after period 0, so, in this non-linear model, the path of the endogenous variables will not necessarily match up with the path of their expectation in a richer model in which there was uncertainty after period 0.

We will analyse this problem with the help of solutions to the following auxiliary problem:

Problem 3

Suppose that $T \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^T$ is given. Find $x_t \in \mathbb{R}^n, y_t \in \mathbb{R}^T$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$, $y_t \to 0$, as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1} + I_{\cdot,1}I_{1,\cdot}y_{t-1},$$
$$\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t} = y_{i+1,t-1},$$
$$y_{T,t} = 0.$$

This may be thought of as a version of Problem 1 with news shocks up to horizon T added to the first equation.[2] The value of $y_{t,0}$ gives the news shock that hits in period t, i.e. $I_{1,\cdot}y_{t-1} = y_{1,t-1} = y_{t,0}$ for $t \leq T$, and $I_{1,\cdot}y_{t-1} = y_{1,t-1} = 0$ for $t > T$.

2.1.2. Relationships between the problems

Together with Assumption 1, the fact that $I_{1,\cdot}y_{t-1} = 0$ for $t > T$ implies that $(x_{T+1} - \mu) = F(x_T - \mu)$, so with $t = T$, defining $s_{T+1} := 0$, $(x_{t+1} - \mu) = s_{t+1} + F(x_t - \mu)$. Proceeding now by backwards induction on t, note that:

$$0 = A(x_{t-1} - \mu) + B(x_t - \mu) + CF(x_t - \mu) + Cs_{t+1} + I_{\cdot,1}y_{t,0},$$

so:

$$(x_t - \mu) = -(B + CF)^{-1}[A(x_{t-1} - \mu) + Cs_{t+1} + I_{\cdot,1}y_{t,0}] = F(x_{t-1} - \mu) - (B + CF)^{-1}(Cs_{t+1} + I_{\cdot,1}y_{t,0}),$$

i.e., if we define: $s_t := -(B + CF)^{-1}(Cs_{t+1} + I_{\cdot,1}y_{t,0})$, then $(x_t - \mu) = s_t + F(x_{t-1} - \mu)$. By induction then, this holds for all $t \in \{1, \dots, T\}$.[3] Hence, we have proved the following lemma:

Lemma 1

There is a unique solution to Problem 3 that is linear in x_0 and y_0.

[2] The idea of imposing the zero lower bound by adding news shocks is also present in Holden (2010), Hebden et al. (2011), Holden & Paetz (2012) and Bodenstein et al. (2013).

[3] This representation of the solution to Problem 3 was inspired by that of Anderson (2015).

For future reference, let $x_t^{(3,k)}$ be the solution to Problem 3 when $x_0 = \mu$, $y_0 = I_{\cdot,k}$ (i.e. a vector which is all zeros apart from a 1 in position k). Then, by linearity, for arbitrary y_0 the solution to Problem 3 when $x_0 = \mu$ is given by:

$$x_t - \mu = \sum_{k=1}^{T} y_{k,0}\left(x_t^{(3,k)} - \mu\right).$$

Let $M \in \mathbb{R}^{T\times T}$ satisfy:

$$M_{t,k} = x_{1,t}^{(3,k)} - \mu_1, \qquad \forall t,k \in \{1,..,T\}, \tag{1}$$

i.e. M horizontally stacks the (column-vector) relative impulse responses to the news shocks. Then this result implies that for arbitrary y_0, the path of the first variable in the solution to Problem 3 when $x_0 = \mu$ is given by: $(x_{1,1\ldots T})' = \mu_1 + My_0$, where $x_{1,1\ldots T}$ is the row vector of the first T values of the first component of x_t. Furthermore, for both arbitrary x_0 and y_0, the path of the first variable in the solution to Problem 3 is given by: $(x_{1,1\ldots T})' = q + My_0$, where $q := \left(x_{1,1\ldots T}^{(1)}\right)'$ and $x_t^{(1)}$ is the unique solution to Problem 1, for the given x_0.[4] This ease in solving Problem 3 given y_0 will be crucial to the efficiency of our eventual solution algorithm for Problem 2.

Now let $x_t^{(2)}$ be a solution to Problem 2 given an arbitrary x_0. Since $x_t^{(2)} \to \mu$ as $t \to \infty$, there exists $T' \in \mathbb{N}$ such that for all $t > T'$, $x_{1,t}^{(2)} > 0$. We assume without loss of generality that $T' \leq T$. We seek to relate the solution to Problem 2 with the solution to Problem 3 for an appropriate choice of y_0.

First, for all $t \in \mathbb{N}^+$, let:

$$e_t := \begin{cases} -\left[I_{1,\cdot}\mu + A_{1,\cdot}\left(x_{t-1}^{(2)} - \mu\right) + \left(B_{1,\cdot} + I_{1,\cdot}\right)\left(x_t^{(2)} - \mu\right) + C_{1,\cdot}\left(x_{t+1}^{(2)} - \mu\right)\right] & \text{if } x_{1,t}^{(2)} = 0 \\ 0 & \text{if } x_{1,t}^{(2)} > 0 \end{cases}, \tag{2}$$

i.e. e_t is the shock that would need to hit the first equation for the positivity constraint on $x_{1,t}^{(2)}$ to be enforced. Note for future reference that by the definition of Problem 2, $e_t \geq 0$ and $x_{1,t}^{(2)} e_t = 0$, for all $t \in \mathbb{N}^+$. From this definition, we also have that for all $t \in \mathbb{N}^+$:

$$0 = A\left(x_{t-1}^{(2)} - \mu\right) + B\left(x_t^{(2)} - \mu\right) + C\left(x_{t+1}^{(2)} - \mu\right) + I_{\cdot,1}e_t.$$

Furthermore, if $t > T$, then $t > T'$, and hence $e_t = 0$. Hence, by Assumption 1, $\left(x_{T+1}^{(2)} - \mu\right) = F\left(x_T^{(2)} - \mu\right)$. Thus, much as before, with $t = T$, defining $r_{T+1} := 0$, $\left(x_{t+1}^{(2)} - \mu\right) = r_{t+1} + F\left(x_t^{(2)} - \mu\right)$. Consequently:

$$0 = A\left(x_{t-1}^{(2)} - \mu\right) + B\left(x_t^{(2)} - \mu\right) + CF\left(x_t^{(2)} - \mu\right) + Cr_{t+1} + I_{\cdot,1}e_t,$$

so:

$$\left(x_t^{(2)} - \mu\right) = F\left(x_{t-1}^{(2)} - \mu\right) - (B + CF)^{-1}(Cr_{t+1} + I_{\cdot,1}e_t),$$

i.e., if we define: $r_t := -(B + CF)^{-1}(Cr_{t+1} + I_{\cdot,1}e_t)$, then $\left(x_t^{(2)} - \mu\right) = r_t + F\left(x_{t-1}^{(2)} - \mu\right)$. As before, by induction this must hold for all $t \in \{1, \ldots, T\}$. By comparing the definitions of s_t and r_t, and the laws of motion of x_t under both problems, we then immediately have that if Problem 3 is started with $x_0 = x_0^{(2)}$ and $y_0 = e'_{1\ldots T}$, then $x_t^{(2)}$ solves Problem 3. Conversely, if $x_t^{(2)}$ solves Problem 3 for some y_0, then from the laws of motion of x_t under both problems it must be the case that $r_t = s_t$ for all $t \in \mathbb{N}$, and hence from the definitions of s_t and r_t, we have that $y_0 = e'_{1\ldots T}$. This has established the following result:

[4] This representation was also exploited by Holden (2010) and Holden & Paetz (2012).

Lemma 2

For any solution, $x_t^{(2)}$ to Problem 2:

1) With $e_{1\ldots T}$ as defined in equation (2), $e_{1\ldots T} \geq 0$, $x_{1,1\ldots T}^{(2)} \geq 0$ and $x_{1,1\ldots T}^{(2)} \circ e_{1\ldots T} = 0$, where $\circ$ denotes the Hadamard (entry-wise) product.
2) $x_t^{(2)}$ is also the unique solution to Problem 3 when started with $x_0 = x_0^{(2)}$ and with $y_0 = e'_{1\ldots T}$.
3) If $x_t^{(2)}$ solves Problem 3 when started with $x_0 = x_0^{(2)}$ and with some y_0, then $y_0 = e'_{1\ldots T}$.

However, to use the easy solution to Problem 3 to assist us in solving Problem 2 requires a slightly stronger result. Suppose that $y_0 \in \mathbb{R}^T$ is such that $y_0 \geq 0$, $x_{1,1\ldots T}^{(3)} \circ y_0' = 0$ and $x_{1,t}^{(3)} \geq 0$ for all $t \in \mathbb{N}$, where $x_t^{(3)}$ is the unique solution to Problem when started at x_0, y_0. We would like to prove that in this case $x_t^{(3)}$ must also be a solution to Problem 2. I.e., we must prove that for all $t \in \mathbb{N}^+$:

$$x_{1,t}^{(3)} = \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}^{(3)} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t^{(3)} - \mu) + C_{1,\cdot}(x_{t+1}^{(3)} - \mu)\},$$
$$(A_{-1,\cdot} + B_{-1,\cdot} + C_{-1,\cdot})\mu \neq A_{-1,\cdot}x_{t-1}^{(3)} + B_{-1,\cdot}x_t^{(3)} + C_{-1,\cdot}x_{t+1}^{(3)}.$$

By the definition of Problem 3, the latter equation must hold with equality, so there is nothing to prove there. Hence we just need to prove that for all $t \in \mathbb{N}^+$:

$$x_{1,t}^{(3)} = \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}^{(3)} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t^{(3)} - \mu) + C_{1,\cdot}(x_{t+1}^{(3)} - \mu)\}.$$

Let $t \in \mathbb{N}^+$. Now, if $x_{1,t}^{(3)} > 0$, then $y_{t,0} = 0$, by the complementary slackness type condition ($x_{1,1\ldots T}^{(3)} \circ y_0' = 0$). Thus, from the definition of Problem 3:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1},$$

so:

$$\begin{aligned} x_{1,t}^{(3)} &= I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}^{(3)} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t^{(3)} - \mu) + C_{1,\cdot}(x_{t+1}^{(3)} - \mu) \\ &= \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}^{(3)} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t^{(3)} - \mu) + C_{1,\cdot}(x_{t+1}^{(3)} - \mu)\}, \end{aligned}$$

as required. The only remaining case is that $x_{1,t}^{(3)} = 0$ (since $x_{1,t}^{(3)} \geq 0$ for all $t \in \mathbb{N}$, by assumption), which implies that:

$$\begin{aligned} x_{1,t}^{(3)} = 0 &= A_{1,\cdot}(x_{t-1} - \mu) + B_{1,\cdot}(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) + y_{t,0} \\ &= I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) + y_{t,0}, \end{aligned}$$

by the definition of Problem 3. Thus:

$$I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) = -y_{t,0} \leq 0,$$

where the inequality is an immediate consequence of another of our assumptions. Consequently:

$$x_{1,t}^{(3)} = \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}^{(3)} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t^{(3)} - \mu) + C_{1,\cdot}(x_{t+1}^{(3)} - \mu)\},$$

as required. Together with Lemma 1, Lemma 2, and our representation of the solution of Problem 3, this completes the proof of the following proposition:

Proposition 1

1) Let $x_t^{(3)}$ be the unique solution to Problem 3 when initialized with some x_0, y_0. Then $x_t^{(3)}$ is a solution to Problem 2 when initialized with x_0 if and only if $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$ and $x_{1,t}^{(3)} \geq 0$ for all $t \in \mathbb{N}$ with $t > T$.
2) Let $x_t^{(2)}$ be any solution to Problem 2 when initialized with x_0. Then there exists a $y_0 \in \mathbb{R}^T$ such that $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$, such that $x_t^{(2)}$ is the unique solution to Problem 3 when initialized with x_0, y_0.

2.1.3. The linear complementarity representation

Proposition 1 establishes that providing we initially choose T sufficiently high, to find a solution to Problem 2, it is sufficient to solve Problem 4 then perform some trivial linear calculations.

Problem 4 (Linear Complementarity Problem (LCP))

Suppose $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T\times T}$ are given. Find $y \in \mathbb{R}^T$ such that $y \geq 0$, $y \circ (q + My) = 0$ and $q + My \geq 0$. We call this the linear complementarity problem (q, M).

In the mathematical and optimisation literatures, Problem 4 is called a linear complementarity problem or LCP (Cottle 2009). These problems have been extensively studied, and so we can import results on the properties of LCPs to derive results on the properties of solutions to models with occasionally binding constraints.

All of the results in the mathematical literature rest on properties of the matrix M, thus we must first establish if the structure of our particular M implies it has any special properties. Unfortunately it seems that M has no general properties. We show this by constructing a model for each matrix in $\mathbb{R}^{T\times T}$, such that the M matrix the model produces is precisely the matrix we started with.

Let $\mathcal{M} \in \mathbb{R}^{T\times T}$. Consider a model with the following equations:

$$\begin{aligned}
a_t &= \max\{0, b_t\},\\
a_t &= 1 + \sum_{j=1}^{T}\sum_{k=1}^{T} \mathcal{M}_{j,k}\left(c_{j-1,k-1,t} - c_{j,k,t}\right),\\
c_{0,0,t} &= a_t - b_t,\\
c_{0,k,t} &= \mathbb{E}_t c_{0,k-1,t+1}, \qquad \forall k \in \{1,\dots,T\},\\
c_{j,k,t} &= c_{j-1,k,t-1}, \qquad \forall j \in \{1,\dots,T\}, k \in \{0,\dots,T\},
\end{aligned}$$

with steady-state $a_\cdot = b_\cdot = 1$, $c_{j,k,\cdot} = 0$ for all $j, k, \in \{0,\dots,T\}$. Defining:

$$x_t := [a_t \quad b_t \quad (\operatorname{vec} c_{\cdot,\cdot,t})']'$$

and dropping expectations, this model is then in the form of Problem 2.

Now consider the model's Problem 3 type equivalent, in which for $t \in \mathbb{N}^+$:

$$a_t = \begin{cases} b_t + y_{t,0} & \text{if } t \leq T \\ b_t & \text{if } t > T' \end{cases}$$

where $y_{\cdot,\cdot}$ is defined as in Problem 3.

Thus, if $c_{j,k,0} = 0$ for all $j, k \in \{0,\dots,T\}$, then for all $t \in \mathbb{N}^+$, $j, k \in \{0,\dots,T\}$:

$$c_{0,k,t} = \begin{cases} y_{t+k,0} & \text{if } t+k \leq T \\ 0 & \text{if } t+k > 0' \end{cases}$$

$$c_{j,k,t} = \begin{cases} c_{0,k,t-j} & \text{if } t-j > 0 \\ 0 & \text{if } t-j \leq 0 \end{cases} = \begin{cases} y_{t+k-j,0} & \text{if } t-j > 0, t+k-j \leq T \\ 0 & \text{otherwise} \end{cases}.$$

Hence, for all $t \in \mathbb{N}^+$, $j, k \in \{1,\dots,T\}$:

$$c_{j-1,k-1,t} - c_{j,k,t} = \begin{cases} y_{t+k-j,0} & \text{if } t-j = 0, t+k-j \leq T \\ 0 & \text{otherwise} \end{cases} = \begin{cases} y_{k,0} & \text{if } t = j \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, for all $t \in \{1,\dots,T\}$:

$$a_t - 1 = \sum_{k=1}^{T} \mathcal{M}_{t,k} y_{k,0}.$$

Consequently, if $y_{k,0} = I_{\cdot,l}$ for some $l \in \{1,\dots,T\}$, then $a_t - 1 = \mathcal{M}_{t,l}$, i.e. the relative impulse response to a news-shock at horizon l is the l^{th} column of $\mathcal{M}$. This establishes the following proposition:

Proposition 2

For any matrix $\mathcal{M} \in \mathbb{R}^{T\times T}$, there exists a model in the form of Problem 2 with a number of state variables given by a quadratic in T, such that $M = \mathcal{M}$ for that model, where M is defined as in equation (1).

We now introduce some definitions of matrix properties that are necessary for the statement of our key existence and uniqueness results. The ultimate properties of the occasionally binding constraint model are determined by which of these matrix properties M possesses. In each case, we give the definitions in a constructive form which makes clear both how the property might be verified computationally, and the links between definitions. These are not necessarily in the form which is standard in the original literature, however. For both the original definitions, and the proofs of equivalence between the ones below and the originals, see Cottle, Pang, and Stone (2009) and Xu (1993) (for the characterisation of sufficient matrices).

Definition 1 (Principal sub-matrix, Principal minor)

For a matrix $M \in \mathbb{R}^{T\times T}$, the **principal sub-matrices** of M are the matrices:

$$\left\{\left[M_{i,j}\right]_{i,j=k_1,\dots,k_S}\middle| S, k_1, \dots, k_S \in \{1,\dots,T\}, k_1 < k_2 < \cdots < k_S\right\},$$

i.e. the **principal sub-matrices** of M are formed by deleting the same rows and columns.

The **principal minors** of M are the collection of values:

$$\left\{\det\left(\left[M_{i,j}\right]_{i,j=k_1,\dots,k_S}\right)\middle| S, k_1, \dots, k_S \in \{1,\dots,T\}, k_1 < k_2 < \cdots < k_S\right\},$$

i.e. the **principal minors** of M are the determinants of the principal sub-matrices of M.

Definition 2 ((Non-)Degenerate matrix)

A matrix $M \in \mathbb{R}^{T\times T}$ is called a **non-degenerate matrix** if the principal minors of M are all non-zero. M is called a **degenerate matrix** if it is not a non-degenerate matrix.

Definition 3 (P_0-matrix)

A matrix $M \in \mathbb{R}^{T\times T}$ is called a **P-matrix** if the principal minors of M are all strictly positive. M is called a **P_0-matrix** if the principal minors of M are all non-negative. *Note: for symmetric M, M is a P-matrix if and only if all of its eigenvalues are strictly positive, and M is a P_0-matrix if and only if all of its eigenvalues are non-negative.*

Definition 4 (General positive (semi-)definite)

A matrix $M \in \mathbb{R}^{T\times T}$ is called **general positive definite** if $M + M'$ is a P-matrix. If $M + M'$ is a P_0-matrix, then M is called **general positive semi-definite**. *Note: that we do not require that M is symmetric in either case, but that if M is symmetric, then, M is general positive-definite if and only if it is a P-matrix and M is a general positive semi-definite if and only if it is a P_0-matrix, so in this case the definitions coincide with the standard ones.*

Definition 5 (S_0-matrix)

A matrix $M \in \mathbb{R}^{T\times T}$ is called an **S-matrix** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \gg 0$. M is called an **S_0-matrix** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \geq 0$.[5]

[5] The former condition may be rewritten as $\sup\{\alpha \in \mathbb{R} | \exists y \geq 0 \text{ s.t. } \forall t \in \{1,\dots,T\}, (My)_t \geq \alpha \wedge y_t \leq 1\} > 0$, and the latter may be rewritten as $\sup\left\{\sum_{t=1}^{T} y_t \middle| y \in \mathbb{R}^T, My \geq 0 \wedge \forall t \in \{1,\dots,T\}, y_t \leq 1\right\} > 0$. As linear-programming problems, these may be verified in time polynomial in the size of the matrix using the methods described in e.g. Roos, Terlaky, and Vial (2006).

Definition 6 ((Strictly) Semi-monotone)

A matrix $M \in \mathbb{R}^{T\times T}$ is called **strictly semi-monotone** if each of its principal sub-matrices is an **S-matrix**. M is called **semi-monotone** if each of its principal sub-matrices is an **S_0-matrix**.

Definition 7 ((Strictly) Copositive)

A matrix $M \in \mathbb{R}^{T\times T}$ is called **strictly copositive** if $M + M'$ is strictly semi-monotone. If $M + M'$ is semi-monotone then M is called **copositive**.[6]

Definition 8 (Sufficient matrices)

Let $M \in \mathbb{R}^{T\times T}$. M is called **column sufficient** if M is a P_0-matrix, and for each principal sub-matrix $W := \left[M_{i,j}\right]_{i,j=k_1,\ldots,k_S}$ of M, with zero determinant, and for each proper principal sub-matrix $\left[W_{i,j}\right]_{i,j=l_1,\ldots,l_R}$ of W ($R < S$), with zero determinant, the columns of $\left[W_{i,j}\right]_{\substack{i=1,\ldots,S\\ j=l_1,\ldots,l_R}}$ do not form a basis for the column space of W.[7]

M is called **row sufficient** if M' is column sufficient. M is called **sufficient** if it is column sufficient and row sufficient.

Cottle, Pang, and Stone (2009) note the following relationships between these classes (amongst others):

1. All general positive semi-definite matrices are copositive and sufficient.
2. P_0 includes skew-symmetric matrices, general positive semi-definite matrices, sufficient matrices and P-matrices.
3. All P_0-matrices, and all copositive matrices are semi-monotone, and all P-matrices, and all strictly copositive matrices are strictly semi-monotone.

Additionally, from considering the 1×1 principal sub-matrices of M, we have the following restrictions on the diagonal of M:

4. All general positive semi-definite, semi-monotone, sufficient, P_0 and copositive matrices have non-negative diagonals, and all general positive definite, strictly semi-monotone, P and strictly copositive matrices have strictly positive diagonals.

For many models encountered in macroeconomics, this simple condition is already sufficient to rule out membership of these matrix classes, as medium-scale DSGE models (including, for example, Smets and Wouters (2003), as we will show in section 2.2.3) frequently have negative elements on the diagonal of the M matrix coming from the zero lower bound on nominal interest rates, when T is large enough. Unfortunately, for all of these matrix classes except the classes of general positive (semi-)definite matrices, and $S(_0)$-matrices, no algorithm which runs in an amount of time that is polynomial in T is known, thus verifying class membership may not be feasible with large T. However, disproving class membership only requires finding one principal sub-matrix which fails to have the required property, and for this, starting with the 1×1 principal sub-matrices (e.g. the diagonal), then considering the 2×2 ones (etc.) is often a good strategy. The facts that all of the eigenvalues of a $T \times T$ P-matrix have complex arguments in the interval $\left(-\pi + \frac{\pi}{T}, \pi - \frac{\pi}{T}\right)$, and all of the eigenvalues of a $T \times T$ P_0-matrix have complex arguments in the interval $\left[-\pi + \frac{\pi}{T}, \pi - \frac{\pi}{T}\right]$ (Fang 1989) may also assist in ruling out these matrix classes.

[6] Väliaho (1986) contains an alternative characterisation which ought to be faster to check, since it avoids solving any linear programming problems.
[7] Again, this may be checked via the singular value decomposition.

2.1.4. Existence results

We start by considering necessary or sufficient conditions for the existence of a solution to a model with occasionally binding constraints. Ideally, we would like the solution to exist for any possible path the bounded variable might have taken in the future were there no occasionally binding constraint, i.e. for any possible q. To see this, note that under a perfect foresight exercise we are ignoring the fact that shocks might hit the economy in future. More properly, we ought to integrate over future uncertainty, as in the stochastic extended path approach of Adjemian and Juillard (2013). A crude way to do this would just be to draw lots of samples of future shocks for periods $1, \dots, S$, and average over these draws. But in a linear model with shocks with unbounded support, providing at least one shock has an impact on a given variable, then the distribution of future paths of that variable has positive support over the entirety of $\mathbb{R}^S$. Thus, ideally we would like M to be such that for any q, the linear complementarity problem (q, M) has a solution.

Definition 9 (Feasible LCP)

Suppose $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given. The LCP corresponding to M and q is called **feasible** if there exists $y \in \mathbb{R}^T$ such that $y \geq 0$ and $q + My \geq 0$.

By construction, if an LCP (q, M) has a solution, then it is feasible, i.e. being feasible is a necessary condition for existence. Checking feasibility is straightforward for any particular (q, M), since to find a feasible solution we just need to solve a standard linear programming problem, which is possible in an amount of time that is polynomial in T.

Proposition 3

The LCP (q, M) is feasible for all $q \in \mathbb{R}^T$ if and only if M is an S-matrix. (Cottle, Pang, and Stone 2009)[8]

This gives an easily verified necessary condition for the global existence of a solution to the model with occasionally binding constraints. We now turn to sufficient conditions.

Proposition 4

The LCP (q, M) is solvable if it is feasible and, either:

1. M is row-sufficient, or,
2. M is copositive and for all non-singular principal sub-matrices $W := \left[M_{i,j}\right]_{i,j=k_1,\dots,k_S}$ of M, all non-negative columns of W^{-1} possess a non-zero diagonal element.

(Cottle, Pang, and Stone 2009; Väliaho 1986)

If either condition 1 or condition 2 of Proposition 4 is satisfied, then to check existence for any particular q, we only need to solve a linear programming problem to see if a solution exists for a particular q. As this may be substantially faster than solving the LCP, this may be helpful in practice.

[8] All of the results on LCPs in both this and the following section are restatements of (assorted) results contained in Cottle, Pang, and Stone (2009) and Väliaho (1986) (for the characterisation of "copositive-plus" matrices), and the reader is referred to those works for proofs and further references. Cottle, Pang, and Stone (2009) also contains many additional necessary or sufficient conditions for the various properties of interest.

Proposition 5

The LCP (q, M) is solvable for all $q \in \mathbb{R}^T$, if at least one of the following conditions holds:

1. M is an S-matrix, and either condition 1 or condition 2 of Proposition 4 are satisfied.
2. M is copositive and non-degenerate.
3. M is a P-matrix, a strictly copositive matrix or a strictly semi-monotone matrix.

(Cottle, Pang, and Stone 2009)

If condition 1, 2 or 3 of Proposition 5 is satisfied, then we know that the LCP will always have a solution. Therefore, for any path of the bounded variable in the absence of the bound, we will be able to solve the model even with the bound. Since it is unclear what happens in a New-Keynesian model when there is no solution that imposes the zero lower bound on nominal interest rates, monetary policy makers should always choose a policy rule that produces a model that satisfies one of these three conditions, if they can.

Ideally, we might have liked conditions for the existence of a solution that are both necessary and sufficient, but unfortunately at present no such conditions exist in full generality. However, in the special case of M matrices with nonnegative entries, we have the following result:

Proposition 6

If M is a matrix with nonnegative entries, then the LCP (q, M) is solvable for all $q \in \mathbb{R}^T$, if and only if M has a strictly positive diagonal. (Cottle, Pang, and Stone 2009)

2.1.5. Uniqueness results

While no fully general necessary and sufficient conditions have been derived for existence, such conditions have been derived for the existence of a unique solution, in particular:

Proposition 7

The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$, if and only if M is a P-matrix. (Samelson, Thrall, and Wesler 1958; Cottle, Pang, and Stone 2009)

This proposition is the equivalent for models with occasionally binding constraints of the key proposition of Blanchard and Kahn (1980). By testing whether our matrix M is a P-matrix we can immediately determine if the model possesses a unique solution in any state of the world, and for any sequence of future shocks. In our experience, this condition is frequently satisfied in efficient models, such as models of irreversible investment, but is not generally satisfied in medium-scale New-Keynesian models with a zero lower bound on nominal interest rates.

Since some classes of models almost never possess a unique solution when at the zero lower bound, we might reasonably require a lesser condition, namely that at least when the solution to the model without a bound is a solution to the model with the bound (i.e. when the solution to Problem 1 is a solution to Problem 2), then it ought to be the unique solution. This is equivalent to requiring that when q is non-negative, the LCP (q, M) has a unique solution. Conditions for this are given in the following propositions:

Proposition 8

The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$ with $q \gg 0$ if and only if M is semi-monotone. (Cottle, Pang, and Stone 2009)

Proposition 9

The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$ with $q \geq 0$ if and only if M is strictly semi-monotone. (Cottle, Pang, and Stone 2009)

Hence, by verifying that M is (strictly) semi-monotone, we can reassure ourselves that merely introducing the bound will not change the solution away from the bound. When this condition is violated, even when the economy is a long way from the bound, there may be solutions which jump to the bound.

2.1.6. Properties of the solution set

Where there are multiple solutions, we might like to be able to select one via some objective function. This is particularly tractable when either the number of solutions is finite, or the solution set is convex. Conditions for this are given in the following propositions:

Proposition 10

The LCP (q, M) has a finite (possibly zero) number of solutions for all $q \in \mathbb{R}^T$ if and only if M is non-degenerate. (Cottle, Pang, and Stone 2009)

Proposition 11

The LCP (q, M) has a convex (possibly empty) set of solutions for all $q \in \mathbb{R}^T$ if and only if M is column sufficient. (Cottle, Pang, and Stone 2009)

2.1.7. Generalisations to richer otherwise linear models

It is straightforward to generalise these results for Problem 2 to less restrictive otherwise linear models with occasionally binding constraints.

Firstly, if the constraint is on a variable other than $x_{1,t}$, or in another equation than the first, then it is immediately clear that all of the results must go through as before (just by relabelling and rearranging). Furthermore, if the constraint takes the form of $z_{1,t} = \max\{z_{2,t}, z_{3,t}\}$, where $z_{1,t}$, $z_{2,t}$ and $z_{3,t}$ are linear expressions in the contemporaneous values, lags and leads of x_t, then, assuming without loss of generality that $z_{3,\cdot} > z_{2,\cdot}$ in steady-state, we have that:

$$z_{1,t} - z_{2,t} = \max\{0, z_{3,t} - z_{2,t}\}.$$

Hence, adding a new auxiliary variable $x_{n+1,t}$, with the associated equation:

$$x_{n+1,t} = z_{1,t} - z_{2,t},$$

and replacing the constrained equation with:

$$x_{n+1,t} = \max\{0, z_{3,t} - z_{2,t}\},$$

we have a new equation in the form covered by our original results. Moreover, if rather than a max we have a min, we just use the fact that if $z_{1,t} = \min\{z_{2,t}, z_{3,t}\}$, then $-z_{1,t} = \max\{-z_{2,t}, -z_{3,t}\}$, which is in the form covered by the generalisation just established.

We may also readily deal with multiple occasionally binding constraints, following the representation used in Holden & Paetz (2012). Suppose there are c constrained variables in the model. For $a \in \{1, \dots, c\}$, let $q^{(a)}$ be the path of the a^{th} constrained variable in the absence of all constraints. For $a, b \in \{1, \dots, c\}$, let $M^{(a,b)}$ be the

matrix created by horizontally stacking the column vector relative impulse responses of the a^{th} constrained variable to magnitude 1 news shocks at horizon $0,\dots,T-1$ to the equation defining the b^{th} constrained variables. For example, if $c=1$ so there is a single constraint, then we would have that $M^{(1,1)}=M$ as defined in equation (1). Finally, let:

$$q \coloneqq \begin{bmatrix} q^{(1)} \\ \vdots \\ q^{(c)} \end{bmatrix}, \qquad M \coloneqq \begin{bmatrix} M^{(1,1)} & \cdots & M^{(1,c)} \\ \vdots & \ddots & \vdots \\ M^{(c,1)} & \cdots & M^{(c,c)} \end{bmatrix},$$

and let y be a solution to the LCP (q,M). Then the vertically stacked paths of the constrained variables in a solution which satisfies these constraints is given by $q+My$, and again any solution satisfying the constraints corresponds to a solution to the LCP. Thus, in the multiple constraint case, all of our previous results go through, with this redefined q vector and M matrix.

2.2. Select examples

Brendon, Paustian, and Yates (2015) consider multiple equilibria in a simple New Keynesian model with an output growth rate term in the Taylor rule. They show that with sufficiently large reaction to the growth rate, there can be multiple equilibria today, even when the policy rule used to form tomorrow's expectations is held fixed. This is equivalent to the existence of multiple equilibria even when $T=1$. In the first sub-section here, we give an alternative analytic proof of this using our results, and discuss the generalisation to higher T. We refer the reader to the original paper for an elegant discussion of the intuition behind these results.

Brendon, Paustian, and Yates (2015) go on to show numerical results from the model with persistence in the shadow nominal interest rate (i.e. the rate which would obtain were it not for the zero lower bound). In the second sub-section here we numerically examine this case with larger T, illustrating how multiple equilibria tend to become easier to support when T is large. We conclude by examining the M matrices generated by the posterior-modes of the Smets and Wouters (2003) and Smets and Wouters (2007) models, illustrating that problems of multiplicity are not confined to carefully constructed theoretical examples.

2.2.1. The simple Brendon, Paustian, and Yates (2015) model

The equations of the simple Brendon, Paustian, and Yates (2015) model are as follows:

$$\begin{aligned} x_{i,t} &= \max\{0, 1-\beta+\alpha_{\Delta y}(x_{y,t}-x_{y,t-1})+\alpha_{\pi}x_{\pi,t}\}, \\ x_{y,t} &= \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}(x_{i,t}+\beta-1-\mathbb{E}_t x_{\pi,t+1}), \\ x_{\pi,t} &= \beta\mathbb{E}_t x_{\pi,t+1}+\gamma x_{y,t}, \end{aligned}$$

where $x_{i,t}$ is the nominal interest rate, $x_{y,t}$ is the deviation of output from steady-state, $x_{\pi,t}$ is the deviation of inflation from steady-state, and $\beta\in(0,1)$, $\gamma,\sigma,\alpha_{\Delta y}\in(0,\infty)$, $\alpha_{\pi}\in(1,\infty)$ are parameters. Defining $x_t=[x_{i,t} \quad x_{y,t} \quad x_{\pi,t}]'$, this is in the form of Problem 2, with:

$$A \coloneqq \begin{bmatrix} 0 & -\alpha_{\Delta y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B \coloneqq \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_{\pi} \\ -\frac{1}{\sigma} & -1 & 0 \\ 0 & \gamma & -1 \end{bmatrix}, \qquad C \coloneqq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & \beta \end{bmatrix}.$$

Let $f := F_{2,2}$, where F is as in Assumption 1. Then:

$$F = \begin{bmatrix} 0 & \alpha_{\Delta y}(f-1) + \alpha_\pi \dfrac{\gamma f}{1-\beta f} & 0 \\ 0 & f & 0 \\ 0 & \dfrac{\gamma f}{1-\beta f} & 0 \end{bmatrix}.$$

Hence:

$$f = f^2 - \frac{1}{\sigma}\left(\alpha_{\Delta y}(f-1) + \alpha_\pi \frac{\gamma f}{1-\beta f} - \frac{\gamma f^2}{1-\beta f}\right),$$

i.e.:

$$\beta\sigma f^3 - \Big((\alpha_{\Delta y}+\sigma)\beta + \gamma + \sigma\Big)f^2 + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big)f - \alpha_{\Delta y} = 0. \tag{3}$$

When $f \leq 0$, the left hand side is negative, and when $f = 1$, the left hand side equals $(\alpha_\pi - 1)\gamma > 0$ (by assumption on α_π), hence equation (3) has either one or three solutions in $(0,1)$, and no solutions in $(-\infty, 0]$. We wish to prove there is a unique solution in $(-1,1)$. First note that when $\alpha_\pi = 1$, the discriminant of the polynomial is:

$$\Big((1-\beta)(\alpha_{\Delta y} - \sigma) - \gamma\Big)^2 \Big(\big(\beta\alpha_{\Delta y}\big)^2 + 2\beta(\gamma - \sigma)\alpha_{\Delta y} + (\gamma+\sigma)^2\Big).$$

The first multiplicand is positive. The second is minimised when $\sigma = \beta\alpha_{\Delta y} - \gamma$, at the value $4\beta\gamma\alpha_{\Delta y} > 0$, hence this multiplicand is positive too. Consequently, at least for small α_π, there are three real solutions for f, so there may be multiple solutions in $(0,1)$. In the appendix, section 7.1, we prove that there is a unique solution to equation (3) in $(-1,1)$, establishing that Assumption 1 holds.

Now, when $T = 1$, M is equal to the top left element of the matrix $-(B + CF)^{-1}$, i.e.:

$$M = \frac{\beta\sigma f^2 - \big((1+\beta)\sigma + \gamma\big)f + \sigma}{\beta\sigma f^2 - \Big((1+\beta)\sigma + \gamma + \beta\alpha_{\Delta y}\Big)f + \sigma + \alpha_{\Delta y} + \gamma\alpha_\pi}.$$

Now, multiplying the denominator by f gives:

$$\begin{aligned} &\beta\sigma f^3 - \Big((1+\beta)\sigma + \gamma + \beta\alpha_{\Delta y}\Big)f^2 + (\sigma + \alpha_{\Delta y} + \gamma\alpha_\pi)f \\ &\qquad = \Big[\beta\sigma f^3 - \Big((\alpha_{\Delta y}+\sigma)\beta + \gamma + \sigma\Big)f^2 + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big)f - \alpha_{\Delta y}\Big] - \big[\beta\alpha_{\Delta y} f - \alpha_{\Delta y}\big] \\ &\qquad = (1-\beta f)\alpha_{\Delta y} > 0, \end{aligned}$$

by equation (3). Hence, the sign of M is that of $\beta\sigma f^2 - \big((1+\beta)\sigma + \gamma\big)f + \sigma$. I.e., M is negative if and only if:

$$\frac{\big((1+\beta)\sigma + \gamma\big) - \sqrt{\big((1+\beta)\sigma+\gamma\big)^2 - 4\beta\sigma^2}}{2\beta\sigma} < f < \frac{\big((1+\beta)\sigma + \gamma\big) + \sqrt{\big((1+\beta)\sigma+\gamma\big)^2 - 4\beta\sigma^2}}{2\beta\sigma}.$$

The upper limit is greater than 1, so only the lower is relevant. To translate this bound on f into a bound on $\alpha_{\Delta y}$, we first need to establish that f is monotonic in $\alpha_{\Delta y}$. Totally differentiating equation (3) gives:

$$\Big[3\beta\sigma f^2 - 2\Big((\alpha_{\Delta y}+\sigma)\beta + \gamma + \sigma\Big)f + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big)\Big]\frac{df}{d\alpha_{\Delta y}} = (1-\beta f)(1-f) > 0.$$

Thus, the sign of $\frac{df}{d\alpha_{\Delta y}}$ is equal to that of:

$$3\beta\sigma f^2 - 2\Big((\alpha_{\Delta y}+\sigma)\beta + \gamma + \sigma\Big)f + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big).$$

Note, however, that this expression is just the derivative of the left hand side of equation (3) with respect to f.

To establish the sign of $\frac{df}{d\alpha_{\Delta y}}$, we consider two cases. First, suppose that equation (3) has three real solutions. Then, the unique solution to equation (3) in (0,1) is its lowest solution. Hence, this solution must be below the first local maximum of the left hand side of equation (3). Consequently, at the $f \in (0,1)$, which solves equation (3), $3\beta\sigma f^2 - 2\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f + \Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big) > 0$. Alternatively, suppose that equation (3) has a unique real solution. Then the left hand side of this equation cannot change sign in between its local maximum and its local minimum (if it has any). Thus, at the $f \in (0,1)$ at which it changes sign, we must have that $3\beta\sigma f^2 - 2\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f + \Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big) > 0$. Therefore, in either case $\frac{df}{d\alpha_{\Delta y}} > 0$, meaning that f is monotonic increasing in $\alpha_{\Delta y}$.

Consequently, to find the critical $(f, \alpha_{\Delta y})$ at which M changes sign, it is sufficient to find the lowest solution with respect to both f and $\alpha_{\Delta y}$ of the pair of equations:

$$\beta\sigma f^2 - \big((1+\beta)\sigma+\gamma\big)f + \sigma = 0,$$
$$\beta\sigma f^3 - \Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f^2 + \Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big)f - \alpha_{\Delta y} = 0.$$

The former implies that:

$$\beta\sigma f^3 - \big((1+\beta)\sigma+\gamma\big)f^2 + \sigma f = 0,$$

so, by the latter:

$$\alpha_{\Delta y}\beta f^2 - \Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi\Big)f + \alpha_{\Delta y} = 0.$$

If $\alpha_{\Delta y} = \sigma\alpha_\pi$, then this equation holds if and only if:

$$\sigma\beta f^2 - \big((1+\beta)\sigma+\gamma\big)f + \sigma = 0.$$

Therefore, the critical $(f, \alpha_{\Delta y})$ at which M changes sign are given by:

$$\alpha_{\Delta y} = \sigma\alpha_\pi,$$
$$f = \frac{\big((1+\beta)\sigma+\gamma\big) - \sqrt{\big((1+\beta)\sigma+\gamma\big)^2 - 4\beta\sigma^2}}{2\beta\sigma}.$$

Thus, M is negative if and only if $\alpha_{\Delta y} > \sigma\alpha_\pi$, and M is zero if and only if $\alpha_{\Delta y} = \sigma\alpha_\pi$.

For a 1×1 matrix, checking the conditions from section 2.1.3 is trivial. In particular, we have that if $\alpha_{\Delta y} < \sigma\alpha_\pi$, then M is a general positive definite, strictly semi-monotone, strictly co-positive, sufficient, P, S matrix, if $\alpha_{\Delta y} \leq \sigma\alpha_\pi$, then M is a general positive semi-definite, semi-monotone, co-positive, sufficient, P_0, S_0 matrix, if $\alpha_{\Delta y} = \sigma\alpha_\pi$, then M is degenerate, and if $\alpha_{\Delta y} \neq \sigma\alpha_\pi$, then M is non-degenerate. Hence, when $T = 1$, if $\alpha_{\Delta y} < \sigma\alpha_\pi$ then the model has a unique solution for all q, if $\alpha_{\Delta y} \leq \sigma\alpha_\pi$ then the model has a convex set of solutions for all q, a unique solution whenever $q > 0$, and at least one solution when $q = 0$. When $\alpha_{\Delta y} > \sigma\alpha_\pi$, then M is negative, and so for any positive q, there exists $y > 0$ such that $q + My = 0$, so the model has multiple solutions in this case. I.e. there are solutions which jump to the bound, even when the nominal interest rate would always be positive were there no bound at all.

When $T > 1$, the previous results imply that if $\alpha_{\Delta y} > \sigma\alpha_\pi$, then M is neither P_0, general positive semi-definite, semi-monotone, co-positive, nor sufficient, since the top-left 1×1 principal sub-matrix of M is the same as it is when $T = 1$. Thus, if anything, when $T > 1$, the parameter region in which there are multiple solutions (when away from the bound or at it) is larger. However, numerical experiments suggest that this parameter region in fact remains the same as T increases, which is unsurprising given the weak persistence

of this model. Thus, if we want more interesting results with higher T, we need to consider a model with a stronger persistence mechanism.

2.2.2. The Brendon, Paustian, and Yates (2015) model with shadow interest rate persistence

If we introduce persistence in the shadow interest rate to the previous model, we obtain the following equations:

$$x_{i,t} = \max\{0, x_{d,t}\},$$
$$x_{d,t} = (1-\rho)\left(1-\beta+\alpha_{\Delta y}\left(x_{y,t}-x_{y,t-1}\right)+\alpha_{\pi}x_{\pi,t}\right)+\rho x_{d,t-1},$$
$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}\left(x_{i,t}+\beta-1-\mathbb{E}_t x_{\pi,t+1}\right),$$
$$x_{\pi,t} = \beta\mathbb{E}_t x_{\pi,t+1} + \gamma x_{y,t},$$

where parameters and variables are as before, with the addition of $x_{d,t}$, the shadow nominal interest rate, and ρ, its persistence. Defining $x_t = [x_{i,t} \quad x_{d,t} \quad x_{y,t} \quad x_{p,t}]'$, this is again in the form of Problem 2, with:

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho & -(1-\rho)\alpha_{\Delta y} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & (1-\rho)\alpha_{\Delta y} & (1-\rho)\alpha_{\pi} \\ -\frac{1}{\sigma} & 0 & -1 & 0 \\ 0 & 0 & \gamma & -1 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & 0 & \beta \end{bmatrix}.$$

In all of the following, we set $\sigma = 1$, $\beta = 0.99$, $\gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85}(2+\sigma)$, $\rho = 0.5$, following Brendon, Paustian, and Yates (2015). For each considered $\alpha_{\Delta y}$, α_{π}, we verify numerically that Assumption 1 is indeed satisfied.

In Figure 1, we plot the regions in $(\alpha_{\Delta y}, \alpha_{\pi})$ space in which M is a P-matrix (P_0-matrix) when $T = 2$ or $T = 4$. For this model, these correspond to the regions in which M is strictly semi-monotone (semi-monotone). As may be seen, in the smaller T case, the P-matrix region is much larger. This relationship appears to continue to hold for both larger and smaller T, with the equivalent $T = 1$ plot being almost entirely shaded, and the large T plot apparently tending to the equivalent plot from the model without monetary policy persistence. Intuitively, the persistence in the shadow nominal interest rate dampens the immediate response of nominal interest rates to inflation and output growth, making it harder to induce a zero lower bound episode over short-horizons.

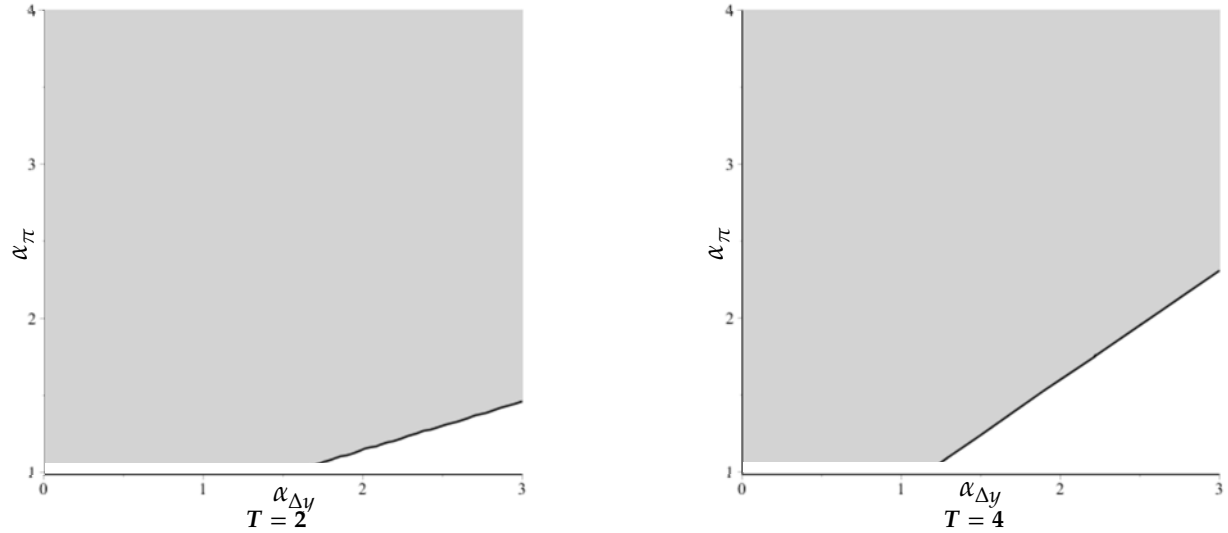


Figure 1: Regions in which M is a P-matrix (shaded grey) or a P_0-matrix (shaded grey, plus the black line), when $T = 2$ (left) or $T = 4$ (right).

2.2.3. The Smets and Wouters (2003) and Smets and Wouters (2007) models

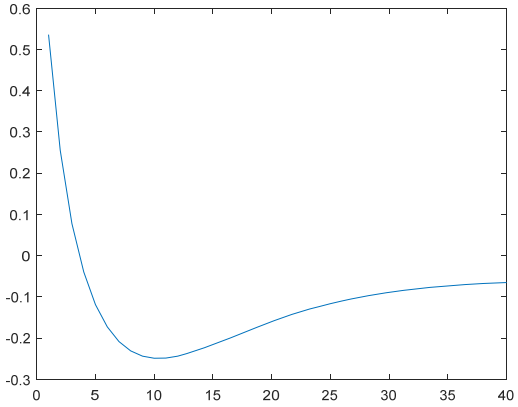
Smets and Wouters (2003) and Smets and Wouters (2007) are the canonical medium-scale linear DSGE models, featuring assorted shocks, habits, price and wage indexation, capital (with adjustment costs), (costly) variable utilisation and quite general monetary policy reaction functions. The former model is estimated on Euro area data, while the latter is estimated on US data. The latter model also contains trend growth (permitting its estimation on non-detrended data), and a slightly more general aggregator across industries. However, overall, they are really quite similar models, and any differences in their behaviour chiefly stems from differences in the estimated parameters. Since both models are incredibly well known in the literature, we omit their equations here, referring the reader to the original papers for further details.

To assess the likelihood of multiple equilibria at or away from the zero lower bound, we augment each model with a zero lower bound on nominal interest rates, and evaluate the properties of each model's M matrix with large T, at the estimated posterior-modes from the original papers. Note that we do not introduce an auxiliary for shadow nominal interest rates, so the monetary rules take the form of:

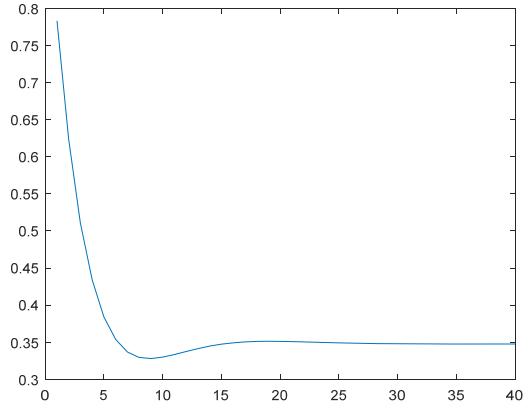
$$r_t = \max\{0, (1-\rho)(\cdots) + \rho r_{t-1} + \cdots\}$$

in both cases.

As discussed in section 2.1.3, if the diagonal of the M matrix ever goes negative, then the M matrix cannot be general positive semi-definite, semi-monotone, sufficient, P_0 or copositive, and hence the model will sometimes have multiple solutions even when away from the zero lower bound (i.e. for some strictly positive q). In Figure 2, we plot the diagonal of the M matrix for each model in turn, i.e. the impact on nominal interest rates in period t of news in period 1 that a positive, magnitude one shock will hit nominal interest rates in period t. Immediately, we see that while in the US model, these impacts remain positive at all horizons, in the Euro area model, these impacts turn negative after just a few periods, and remain so at least up to period 40. Therefore, in the zero lower bound augmented Smets and Wouters (2003) model, there is not always a unique equilibrium. Furthermore, there are sequences of predicted future shocks (with positive density) for which the model without the zero lower bound would always feature positive interest rates, but for which the model with the zero lower bound could hit zero.



The Smets and Wouters (2003) model



The Smets and Wouters (2007) model

Figure 2: The diagonals of the M matrices for the Smets and Wouters (2003) and Smets and Wouters (2007) models

It remains for us to assess whether M is a P_0-matrix or (strictly) semi-monotone for the Smets and Wouters (2007) model. Numerical calculations reveal that for $T < 9$, M is a P-matrix, and hence is strictly semi-monotone. However, with $T \geq 9$, M contains a 6×6 principal sub-matrix (with indices 1,2,4,6,7,9) with

negative determinant, which is neither an S nor an S_0-matrix. Thus, for $T \geq 9$, M is not a $P_{(0)}$-matrix or (strictly) semi-monotone, and hence this model also has multiple equilibria, even when away from the bound. Given that the US has been at the zero lower bound for over eight years, that T ought to be greater than eight quarters seems pretty uncontroversial. Hence, in both the Euro area and the US, we ought to take seriously the possibility that the existence of the zero lower bound produces non-uniqueness. Furthermore, it turns out that for neither model is M an S-matrix, and thus for both models there are some q for which no solution exists. Quite what happens in these economies in such cases is unclear.

2.3. Algorithms for the otherwise linear case

2.3.1. On the difficulty of the problem

We start with a note of caution. If no properties of the matrix M are known a priori, then Problem 4 is provably a computationally difficult problem, more formally, it may be shown to be "strongly-NP complete" (Chung 1989), and this remains true even if M is restricted to be a P_0 matrix (Kojima et al. 1991). This means that even if the inputs q and M have descriptions which are of a polynomial length in T, then if we could solve Problem 4 in an amount of time that was polynomial in T, then we could also solve in polynomial time any problem for which the solution could be verified in polynomial time. In the language of computer science, this would mean that "P=NP", something almost all computer scientists believe to be false. The strength of computer scientists conviction that this is not true is best exemplified by the fact that were P=NP, all commonly used forms of cryptography (such as those used to secure internet banking), would be defeated.

Since there is a bijection between solutions for Problem 4 and solutions for Problem 2, this means that while forming expectations in linear models without occasionally binding constraints is computationally easy (polynomial algorithms exist for it), in models with occasionally binding constraints, forming expectations may be incredibly difficult. It also means that we should be sceptical of claims of computational efficiency from other algorithms for solving models with occasionally binding constraints. A proof that such algorithms actually ran in time polynomial either in T or in the number of state variables of the model, on all models, would again function as a proof that "P=NP", since we showed in Proposition 2 that there is a model corresponding to any $T \times T$ M matrix, featuring polynomial in T state variables. Thus, for example, global methods will never escape the curse of dimensionality in general models with occasionally binding constraints, even using methods explicitly designed to do this such as that of Judd, Maliar, and Maliar (2012). In fact, even proving the finiteness of algorithms for solving these problems is non-trivial (see e.g. Csizmadia and Illés 2006), and, for example, there is no reason to believe that the iterations in Guerrieri and Iacoviello (2015) will converge in finite time on all models.

For some special classes though, it has been shown that the problem is solvable in polynomial time in T. In particular, if M is general positive semi-definite, then this is possible (Kojima, Mizuno, and Yoshise 1989). However, it appears that M is general positive semi-definite in only very few macroeconomic models. Furthermore, if either condition 1 or condition 2 of Proposition 4 is known to be satisfied (e.g. M is row sufficient), then we can find out if a solution exists in polynomial time, by solving the feasibility problem.

Moreover, a polynomial time algorithm exists (Illés, Nagy, and Terlaky 2010) which will give a certificate that one of the following is true, for a given q and real number $\bar{\kappa} \geq 0$:

- For any $\kappa \leq \bar{\kappa}$, M is not a member of the matrix class $P^*(\kappa)$, defined in the paper. (*Note that for $\kappa_1 < \kappa_2$, $P^*(\kappa_1) \subseteq P^*(\kappa_2)$, and that the class of sufficient matrices is the union of the classes of $P^*(\kappa)$ matrices for all $\kappa \geq 0$.)*
- The LCP (q, M) has no solution.
- The LCP (q, M) has the solution y.

Thus for "most" sufficient matrices we can find a solution (or a certificate that there is none), in polynomial time. It has been conjectured that in fact this holds for all sufficient matrices (Fukuda 2015).

Unfortunately, no algorithm is known for finding out if M is sufficient in polynomial time. Indeed, it has also been shown (Coxson 1994; Tseng 2000) that it is "co-NP complete" to test any of the following:

- M is non-degenerate.
- M is a P-matrix.
- M is a P_0-matrix.
- M is semi-monotone.
- M is strictly semi-monotone.
- M is column sufficient.
- M is row sufficient.

This means that were a polynomial time (in T) algorithm available for these things then we would have a proof that P=NP. Likewise, were a polynomial time (in T) algorithm available for testing if a particular model, e.g.:

- always had a unique solution,
- always had a unique solution when away from the bound,
- always had a finite number of solutions,
- always had a convex set of solutions,

then that algorithm would also serve as a proof that P=NP, since we showed in Proposition 2 that a model could be constructed which produces any given M matrix. Since the model we constructed to match a $T \times T$ M matrix featured polynomial in M state variables, this also means that an algorithm which answered these questions which ran in an amount of time polynomial in the number of state variables would also serve as a proof that P=NP.

2.3.2. The mixed integer linear programming representation

Given that there is no reason to believe that there is a polynomial time algorithm to solve the LCPs we encounter, it is important that we choose an algorithm, which, although it may not complete in polynomial time in the worst case, is nonetheless as computationally efficient as possible, particularly on average. One way to do this is to reduce the problem of finding an LCP to the solution of a problem for which highly efficient algorithms and computational libraries are available. One such problem is mixed integer linear programming (MILP), for which algorithms are included in most major optimisation suites (e.g. CPLEX, Gurobi, XPress MP, MOKEK, etc.). Conveniently, we can reduce the LCP problem to the MILP one in a way that gives not only a solution when one exists, but also a definite answer on whether or not there is a solution. This is an improvement over more naïve approaches, such as those of Holden (2010), Holden & Paetz (2012)

or Guerrieri and Iacoviello (2015), for which a failure of convergence may just mean that the optimiser got stuck at some local minimum.

To motivate the LCP representation, suppose that y solves the LCP (q, M). Then $y \geq 0$, $0 \leq q + My$ and if $y_j > 0$ then $(q + My)_j = 0$. Now let $\tilde{\omega} > 0$ be an arbitrary constant, let

$$\alpha := \min\{\|y\|_\infty^{-1}, \tilde{\omega}\|q + My\|_\infty^{-1}\} > 0$$

(where $\|\cdot\|_\infty$ is the usual sup norm), let $\hat{y} := \alpha y$, let $1_{T\times 1}$ be a $T \times 1$ vector of ones, and let $z \in \{0,1\}^T$ be such that for all $j \in \{1, \dots, T\}$, $z_j = 1$ if and only if $y_j > 0$. Then $0 \leq \hat{y} \leq 1_{T\times 1}$ and:

$$0 \leq \alpha q + M\hat{y} \leq \alpha\|q + My\|_\infty \leq \tilde{\omega} 1_{T\times 1}.$$

Now, if $z_j = 0$ for some $j \in \{1, \dots, T\}$, then $y_j = 0$. Hence, in fact, $0 \leq \hat{y} \leq z$. Likewise, if $z_j = 1$, then $y_j > 0$, so since y solves the LCP, $0 = \alpha(q + My)_j = (\alpha q + M\hat{y})_j$. Hence, similarly:

$$0 \leq \alpha q + M\hat{y} \leq \tilde{\omega}(1_{T\times 1} - z).$$

Moreover, for any $\tilde{\alpha} > \alpha$, we claim that there is no $z \in \{0,1\}^T$ such that $0 \leq \tilde{\alpha} y \leq z$ and $0 \leq \tilde{\alpha} q + M(\tilde{\alpha} y) \leq \tilde{\omega}(1_{T\times 1} - z)$. To see this, suppose for a contradiction that there were. Then $\tilde{\alpha}\|y\|_\infty \leq 1$, so $\alpha < \tilde{\alpha} \leq \|y\|_\infty^{-1}$. Hence, $\alpha = \tilde{\omega}\|q + My\|_\infty^{-1}$. But, by assumption $\tilde{\alpha}\|q + My\|_\infty \leq \tilde{\omega}$, hence $\tilde{\omega}\|q + My\|_\infty^{-1} = \alpha < \tilde{\alpha} \leq \tilde{\omega}\|q + My\|_\infty^{-1}$, which gives the required contradiction. Therefore, α, $\hat{y}$, z are feasible for the following MILP problem (though they may not necessarily be the solution):

Problem 5 (MILP representation of LCP)

Suppose $\tilde{\omega} > 0$, $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T\times T}$ are given. Find $\alpha \in \mathbb{R}$, $\hat{y} \in \mathbb{R}^T$, $z \in \{0,1\}^T$ to maximise α subject to the following constraints: $\alpha \geq 0$, $0 \leq \hat{y} \leq z$, $0 \leq \alpha q + M\hat{y} \leq \tilde{\omega}(1_{T\times 1} - z)$.

A version of this representation with $\tilde{\omega} = 1$ was first given by Pardalos and Rosen (1988), and its properties in that special case were proven by Rosen (1990).

We now establish that solutions of the MILP representation are solutions of the LCP. Suppose that α, $\hat{y}$, z solve Problem 5 (MILP representation of LCP). If $\alpha = 0$, then there is no $\alpha > 0$ such that $0 \leq \hat{y} \leq z$, $0 \leq \alpha q + M\hat{y} \leq \tilde{\omega}(1_{T\times 1} - z)$. Now, we showed above that if the LCP (q, M) had a solution, then there would be an $\alpha > 0$, $\hat{y}$ and z which were feasible for Problem 5 (MILP representation of LCP), hence, this $\alpha > 0$ provides a lower bound on the solution to Problem 5 (MILP representation of LCP). Thus, if $\alpha = 0$, the LCP cannot have a solution. Alternatively, suppose that $\alpha > 0$. Then if for some $j \in \{1, \dots, T\}$, $z_j = 1$, then $0 = (\alpha q + M\hat{y})_j$, and if for some $j \in \{1, \dots, T\}$, $z_j = 0$, then $\hat{y}_j = 0$. Thus, $\hat{y} \circ (\alpha q + M\hat{y}) = 0$. Finally, define $y := \frac{\hat{y}}{\alpha} \geq 0$, hence $\hat{y} = \alpha y$, $0 \leq q + My$ and $y \circ (q + My) = 0$, i.e. y solves the LCP (q, M). This completes the proof of the following result:

Proposition 12

Suppose $\tilde{\omega} > 0$, $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T\times T}$ are given. If y solves Problem 4 (Linear Complementarity Problem (LCP)), then the solution to Problem 5 (MILP representation of LCP) has $\alpha \geq \min\{\|y\|_\infty^{-1}, \tilde{\omega}\|q + My\|_\infty^{-1}\}$. If the solution to Problem 4 (Linear Complementarity Problem (LCP)) is unique, then this last inequality holds with equality, and $\hat{y} = \alpha y$, $z = \begin{cases} 1 & \text{if } y_j > 0 \\ 0 & \text{if } y_j = 0 \end{cases}$ in the solution to Problem 5 (MILP representation of LCP). Conversely, if α, $\hat{y}$, z solve Problem 5 (MILP representation of LCP), then if $\alpha = 0$, Problem 4 (Linear Complementarity Problem (LCP)) has no solution, and if $\alpha > 0$, then $y := \frac{\hat{y}}{\alpha}$ solves Problem 4 (Linear Complementarity Problem (LCP)).

This result establishes that we can use the MILP representation both to find out if the LCP problem has a solution, and to find a solution when one exists. Furthermore, by varying $\tilde{\omega}$ we can determine which solution is returned, when there are multiple. In the limit as $\tilde{\omega} \to 0$, the MILP solver will return the solution which minimises $\|q + My\|_\infty$, and in the limit as $\tilde{\omega} \to \infty$, the MILP solver will return the solution to the LCP which minimises $\|y\|_\infty$. The former objective would ensure that the returned solution does not generate large oscillations in the path of the constrained variable, and the latter would ensure that the violations of the bound are minimised. Intermediate values of $\tilde{\omega}$ result in a solution being returned that features balanced concern for these two extremes. In practice, we suggest choosing $\tilde{\omega} = \omega\|q\|_\infty$ where $\omega > 0$ is another constant, to ensure that the solution returned scales appropriately with q.

3. Algorithms for non-linear models, without perfect foresight

Up to now, we have solely been concerned with the perfect foresight solution of models which were linear apart from the occasionally binding constraint. In this section, we will apply these insights to the solution of general non-linear models, allowing for future uncertainty, i.e. we attempt to solve the following general problem:

Problem 6

Suppose that $x_0 \in \mathbb{R}^n$ is given and that $f: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^m \to \mathbb{R}^n$, $g, h: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^m \to \mathbb{R}^c$ are given continuously $d \in \mathbb{N}^+$ times differentiable functions. Find $x_t \in \mathbb{R}^n$ and $v_t \in \mathbb{R}^c$ for $t \in \mathbb{N}^+$ such that for all $t \in \mathbb{N}^+$:

$$0 = \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t),$$
$$v_t = \mathbb{E}_t \max\{h(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t), g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\}$$

where $\varepsilon_t \sim \text{NIID}(0, \Sigma)$, where the max operator acts elementwise on vectors, and where the information set is such that for all $t \in \mathbb{N}^+$, $\mathbb{E}_{t-1}\varepsilon_t = 0$ and $\mathbb{E}_t\varepsilon_t = \varepsilon_t$.

We construct our algorithm in three steps. Firstly, we present an algorithm which accounts for the effects of uncertainty that would be present even without the bound, but which treats hitting the bound as a probability zero event. Secondly, we extend the aforementioned algorithm to capture the risk of hitting the bound in the near future. Finally, we introduce a hybrid global/local algorithm which fully captures the risk of hitting the bound at all horizons. All of the algorithms discussed here are implemented in the author's open source "DynareOBC" toolkit, which is available from http://github.org/tholden/dynareOBC, and which extends Dynare (Adjemian et al. 2011) with the ability to deal with occasionally binding constraints.

3.1. Dealing with non-linearity other than the bounds

3.1.1. Setup and assumptions

Given a non-linear, but $d \in \mathbb{N}$ times continuously differentiable model, the standard practice in macroeconomics is to take a perturbation approximation to the model around its deterministic steady-state. Given that high order perturbation approximations are often unstable, the use of a "pruned" approximation (Kim et al. 2008) is usually advisable. We will proceed along similar lines, taking a perturbation approximation to the model ignoring the bound, and then imposing the bound on the approximated model. The advantage for our purposes of the pruned approximation is that the result is linear in an augmented

state space (including e.g. the products of all pairs of state variables), which will assist us transferring results from the linear case to the non-linear one.

We start by making a further assumption that is necessary for us to be able to construct a perturbation approximation to the model without the bound.

Assumption 2

In the setup of Problem 6, there exists $\mu_x \in \mathbb{R}^n$ and $\mu_v \in \mathbb{R}^c$ such that:

$$0 = f(\mu_x, \mu_x, \mu_x, \mu_v, 0),$$
$$\mu_v = \max\{h(\mu_x, \mu_x, \mu_x, \mu_v, 0), g(\mu_x, \mu_x, \mu_x, \mu_v, 0)\},$$

and such that for all $a \in \{1, \dots, c\}$:

$$\big(h(\mu_x, \mu_x, \mu_x, \mu_v, 0)\big)_a \neq \big(g(\mu_x, \mu_x, \mu_x, \mu_v, 0)\big)_a.$$

This is necessary because if any of the constraints just bind in steady-state, then the equation defining the corresponding element of μ_v is not differentiable at $(\mu_x, \mu_x, \mu_x, \mu_v, 0)$, preventing us from taking a perturbation approximation to the model without the bound.

Henceforth, we suppose without loss of generality that $h(\mu_x, \mu_x, \mu_x, \mu_v, 0) \ll g(\mu_x, \mu_x, \mu_x, \mu_v, 0)$. We claim that we may further assume without loss of generality that $h(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t) \equiv 0$. First, note that, we can rewrite the equation defining v_t as:

$$v_t = h(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t) + \mathbb{E}_t \max\{0, g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t) - h(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\}.$$

Then if we define:

$$\hat{x}_t := \begin{bmatrix} x_t \\ v_t \end{bmatrix}, \forall t \in \mathbb{N},$$
$$\hat{g}(\hat{x}_{t-1}, \hat{x}_t, \hat{x}_{t+1}, \hat{v}_t, \varepsilon_t) := g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t) - h(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t),$$
$$\hat{f}(\hat{x}_{t-1}, \hat{x}_t, \hat{x}_{t+1}, \hat{v}_t, \varepsilon_t) := f(x_{t-1}, x_t, x_{t+1}, g_1(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t) + \hat{v}_t, \varepsilon_t),$$

then for all $t \in \mathbb{N}^+$:

$$0 = \mathbb{E}_t \hat{f}(\hat{x}_{t-1}, \hat{x}_t, \hat{x}_{t+1}, \hat{v}_t, \varepsilon_t),$$
$$\hat{v}_t = \mathbb{E}_t \max\{0, \hat{g}(\hat{x}_{t-1}, \hat{x}_t, \hat{x}_{t+1}, \hat{v}_t, \varepsilon_t)\},$$

which is again in the form of Problem 6. Thus, without loss of generality, we can indeed assume that $h(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t) \equiv 0$, meaning that we work with the system:

$$0 = \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t),$$
$$v_t = \mathbb{E}_t \max\{0, g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\},$$

where $g(\mu_x, \mu_x, \mu_x, \mu_v, 0) \gg 0$.

3.1.2. First order approximations

Now, suppose that we believe that a first order approximation gives adequate accuracy away from the bound. This system is locally $d \geq 1$ times differentiable in a neighbourhood of $(\mu_x, \mu_x, \mu_x, \mu_v, 0)$, so we can certainly take a first order approximation around this point. Doing this gives the following approximation for the equation for v_t:

$$v_t = \mu_v + g_1(x_{t-1} - \mu_x) + g_2(x_t - \mu_x) + g_3 \mathbb{E}_t(x_{t+1} - \mu_x) + g_4(v_t - \mu_v) + g_5 \varepsilon_t,$$

where $g_1, \dots, g_5$ are the matrices of partial derivatives of g with respect to its first to fifth arguments, respectively. This approximation obviously completely ignores the bound. Thus, we propose to increase its accuracy by imposing the bound on the linearized equations, i.e. by instead working with the equation:

$$v_t = \max\{0, \mu_v + g_1(x_{t-1} - \mu_x) + g_2(x_t - \mu_x) + g_3 \mathbb{E}_t(x_{t+1} - \mu_x) + g_4(v_t - \mu_v) + g_5 \varepsilon_t\}.$$

This gives a system of equations in nearly the same form as that for which we developed a solution algorithm in section 2.3, the only difference being the presence of expectations operators and uncertainty. We deal with these following the extended path approach of Fair and Taylor (1983). I.e., if we are currently in period t of a simulation, we assume that the agents in the model believe that for all $s > t$, $\varepsilon_s = 0$. Thus, in each period of a simulation run, we merely have to solve a perfect foresight problem of the form of Problem 2, using the methods of section 2.3. We then advance one period, draw new shocks, and repeat the process.

We can also use a slightly modified form of the representation of Problem 3 to track the endogenous "news" that is coming from the bound, following Holden and Paetz (2012). In particular, we are effectively replacing the bounded equations with equations of the form:

$$v_{a,t} = \mathbb{E}_t\big(g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\big)_a + I_{1,\cdot} y_t^{(a)},$$

for all $a \in \{1, \dots, c\}$, where, for all $a \in \{1, \dots, c\}$:

$$\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t}^{(a)} = y_{i+1,t-1}^{(a)} + \eta_{i,t}^{(a)},$$
$$y_{T,t}^{(a)} = \eta_{T,t}^{(a)},$$

implying that $y_{1,t}^{(a)} = \sum_{i=1}^{T} \eta_{i,t-i}^{(a)}$, where $\eta_{i,t}^{(a)}$ contains the news about the likelihood of the a^{th} bound binding in period $t+i$. Whereas the ys found by the LCP solver will always be positive, the implied $\eta_{i,t}^{(a)}$ need not necessarily be positive, as shocks may hit today which result in the economy moving away from the bound.

3.1.3. Higher order approximations

While the applicability of the results of section 2 to linearized models was probably immediately clear, applying these methods to models solved with a higher order approximation is slightly more subtle. Recall that in linear models, we started by introducing news shocks to the bounded equation(s) and stacking the impulse responses to these news shocks into the M matrix. In order to use these impulse responses to tell us about the path of the bounded variable, we exploited the fact that the impulse response to a linear combination of shocks is the same linear combination of the individual impulse responses. It was this linearity that gave the $q + My$ representation of the path of the bounded variable.

Now, consider what would happen in a pruned or non-pruned second order approximation following a similar linear combination of shocks. Under such an approximation, it is no longer true in general that the impulse response to a linear combination of shocks is a linear combination of the impulse responses, since the second order approximation captures interactions between the shocks (amongst other things). However, if the partial derivative of f and g with respect to each of the shocks being combined is zero, then the shocks only have second or higher order effects, hence, any interaction between them would be a fourth order effect or higher, and so would not be captured contemporaneously by the second order approximation. The period after the shocks hit, though, linearity would again be broken if a non-pruned second order approximation had been taken, since the slope of the response of the states to their lags vary with the states' levels. This is not true under a pruned perturbation approximation though, since under such an approximation, the solution takes the form:

$$x_t^{(1)} = \alpha x_{t-1}^{(1)} + \beta_0 \varepsilon_t,$$
$$x_t^{(2)} = \alpha x_{t-1}^{(2)} + \frac{1}{2}\beta_{22}\big(x_{t-1}^{(1)} \otimes x_{t-1}^{(1)}\big) + \beta_{20}\big(x_{t-1}^{(1)} \otimes \varepsilon_t\big) + \frac{1}{2}\beta_{00}(\varepsilon_t \otimes \varepsilon_t),$$
$$x_t = \mu_x + x^{(0)} + x_t^{(1)} + x_t^{(2)},$$

where $x^{(0)} \in \mathbb{R}^n$ is a constant, $x_t^{(1)}$ is the first order component of the approximation, $x_t^{(2)}$ is the second order component of the approximation, and where β_0 has zero columns corresponding to each shock with respect to which the partial derivatives of f and g is zero (Kim et al. 2008). Thus, $x_t^{(1)}$ does not respond to any shocks with respect to which the partial derivatives of f and g is zero, and hence $x_t^{(2)}$ and x_t are linear in such shocks.

In light of this discussion, in order to preserve the $q + My$ representation, we just need to define M as stacking the impulse responses of the bounded equation(s) to news shocks which hit the bounded equation(s) raised to the power of two, rather than in levels. This generalises to higher order pruned perturbation approximations as one would expect. Hence, in a d^{th} order pruned perturbation approximation, we replace the bounded equations with equations of the form:

$$v_{a,t} = \mathbb{E}_t\big(g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\big)_a + I_{1,\cdot} y_t^{(a)},$$

for all $a \in \{1, \dots, c\}$, where, now, for all $a \in \{1, \dots, c\}$:

$$\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t}^{(a)} = y_{i+1,t-1}^{(a)} + \kappa\big(\eta_{i,t}^{(a)}\big)^d,$$
$$y_{T,t}^{(a)} = \big(\eta_{T,t}^{(a)}\big)^d,$$

where κ is a very small constant in order to ensure that the presence of the additional shocks does not have an unwanted risk effect elsewhere in the model. We obviously scale the impulse responses which make up M by $\frac{1}{\kappa}$ to correct for this. In practice, we do not actually need to augment the model we approximate (in, e.g. Dynare) with all of these additional equations. This is because in the limit as $\kappa \to 0$, the generated impulse responses tend to the impulse responses used to construct the M matrix at first order (since the contemporaneous response must be the same, and the subsequent response is given by $x_t - x^{(0)} = \alpha\big(x_{t-1} - x^{(0)}\big)$ in both cases).

Hence, in order to impose the bound in non-linear models solved by perturbation, we can proceed much as we did at first order. At each time step, we first evaluate the expected path of the bounded variable(s), and stack the results in q. Thanks to the augmented state space representation of pruned perturbation solutions, this is possible without any Monte-Carlo simulation, as we show in section 7.2 in the appendices. Then, we use the M matrix derived from the first order approximation, and calculated as in section 2.1.2 in the LCP (q, M). Finally, we use the solution to this LCP to calculate the required offsets to each variable this period, again based on the first order approximation to the model. Since we are not actually augmenting the model's state space, even for high degree approximations to the model, imposing the bound will not slow down simulation much more than it does at order one.

3.2. Integrating over future uncertainty

The downside to the approach discussed in the previous section is that the news shocks that hit the bounded equation(s) will not be conditionally mean zero, i.e. $\mathbb{E}_{t-1}\eta_{i,t}^{(a)} \neq 0$. For example, in a model with a zero lower bound on nominal interest rates and persistence in productivity growth rates, the lower is the state of productivity's growth rate, the higher the chances of hitting the bound today, thus $\mathbb{E}_{t-1}\eta_{i,t}^{(a)}$ will be decreasing in this state. This failure of rationality with respect to expectations at the bound stems from the fact that we are still treating the bound in a pseudo-perfect foresight manner. In each period, agents act as if they believed that no future news shocks would ever hit the bounded equation. Due to the strict convexity of the $x \mapsto \max\{0, x\}$ mapping and Jensen's inequality, this manifests itself as a systematic downward bias in expectations of v_t.

To rectify this bias, we need to integrate over future uncertainty to calculate the expectation of the cumulated news shocks (the elements of y). We do this following the stochastic extended path approach of Adjemian and Juillard (2013). However, in our context this will be much easier than in the general fully non-linear context of Adjemian and Juillard (2013). In particular, in the basic algorithm of that paper, to integrate over S periods of future uncertainty, in a model with m shocks, they have to solve the perfect foresight model p^{mS} times, for some constant $p \geq 2$. While they are able to reduce this somewhat through the removal of low weighted quadrature nodes to produce a sparse tree of shocks, the resulting distribution of paths will significantly under-estimate the true variance of the model, and they still have to solve the perfect foresight model many more times when the number of shocks is high.

In our context, we will be able to do much better. In particular, we will be able to attain comparable accuracy with the evaluation of only a polynomial in S number of solutions of the perfect foresight problem, regardless of the number of shocks in the model. Furthermore, since solving an LCP is much easier than solving a general fully non-linear perfect foresight problem, each of these solutions is orders of magnitude faster for us. The key to our invariance to the number of shocks in the model is the fact that in the absence of any bounds, we are able to write down a closed form expression for the conditional covariance of the bounded variables, thanks to the properties of pruned perturbation solutions. To be slightly more specific, suppose that $w_{t,s}$ is the value the bounded variables would take at s if the constraints did not apply from period t onwards. Then, we will be able to calculate $\text{cov}_t\left(w_{t,t+i}, w_{t,t+j}\right)$, for $t, i, j \in \mathbb{N}$, without any numerical integration. The derivation of this is contained in section 7.3 in the appendices.

For clarity of presentation, let us assume that there is a single bounded variable. As ever, the generalisation to multiple bounded variables will be straightforward. Now, ideally we would like to integrate over infinitely many periods of future uncertainty, but clearly this is not practical in reality. Instead, Adjemian and Juillard (2013) advocate integrating over $S \in \mathbb{N}^+$ periods of future uncertainty, and then ignoring uncertainty from period $S+1$ onwards. By introducing a "discontinuity" in time of this sort, we would risk getting spurious movement in the expected path of variables around S periods into the future. Indeed, this occurred in some early numerical experiments that took this approach. Instead then, we apply a smooth windowing function to the covariance. In particular, if we define Ω_t by:

$$\Omega_{t,i,j} = \frac{1}{4}\left(1 + \cos\left(\pi\frac{i-1}{S}\right)\right)\left(1 + \cos\left(\pi\frac{j-1}{S}\right)\right)\text{cov}_t\left(w_{t,t+i}, w_{t,t+j}\right),$$

then for the purposes of integration, we make the approximation that:

$$[w_{t,t+1} \quad \cdots \quad w_{t,t+S}]' \sim \text{N}(\mathbb{E}_t[w_{t,t+1} \quad \cdots \quad w_{t,t+S}]', \Omega_t).$$

Of course the cosine window is ad hoc, but so too is the step-function window used by Adjemian and Juillard (2013). The legitimacy of both come from the fact that as $S \to \infty$, the error in this approximation would go to zero were $[w_{t,t+1} \quad \cdots \quad w_{t,t+S}]'$ really normally distributed. One further argument in favour of our cosine window is that it is widely used in signal processing due to its low distortion in the frequency domain (see e.g. Harris 1978). In this literature, it is termed the Hann or Hanning window. We also note that for most DSGE models, the additional error coming from the normal approximation will be minimal, since it is exact at first order, and higher order approximations are usually dominated by their first order terms.

Given this normal approximation, integration is then relatively straightforward. We first take the Schur decomposition of Ω_t, giving $\Omega_t = UDU'$, where U is an orthogonal matrix and $D \geq 0$ is diagonal, with the elements sorted in decreasing order. To reduce the integration dimension without overly affecting accuracy,

we set any elements of D which are less than 1% of the maximum element of D to zero, as these components are unlikely to have a big impact. Indeed, when S is very large, it may be advisable to set all but the $\tilde{S}$ largest elements of D to zero, which means the cost of integration will scale in $\tilde{S}$ not S. After these steps we that that $D = \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix}$, where $\dim D_{11} = \hat{S} \times \hat{S}$ for some $\hat{S} \leq S$. Conformably partitioning U as $U = [U_{\cdot 1} \quad U'_{\cdot 1}]$,and defining $\Lambda := U_{\cdot 1}\sqrt{D_{11}}$, we then have that $\Lambda\Lambda' \approx \Omega_t$. Then if $\zeta \sim \mathrm{N}(0, I_{\hat{S}})$, then $(\mathbb{E}_t[w_{t,t+1} \quad \cdots \quad w_{t,t+S}]') + \Lambda\zeta$ has approximately the same distribution as $[w_{t,t+1} \quad \cdots \quad w_{t,t+S}]'$. We have thus transformed the problem of integrating over the distribution of $[w_{t,t+1} \quad \cdots \quad w_{t,t+S}]'$ to that of integrating over the $\hat{S}$ independent standard normals which make up ζ. We do this using either quasi-Monte Carlo methods, the sparse nested Gaussian cubature rules of Genz and Keister (1996), or the equal weight degree 3 monomial cubature rule with $2\hat{S} + 1$ nodes[9]. Of course, ideally we would like to break the domain of integration into pieces on which the integrand was continuous, but this is not computationally practical for even moderately large $\hat{S}$.[10]

The latter rule exactly integrates all degree 3 monomials in the components of ν. While a third order approximation to the cumulated news shocks, y, as a function of ζ may do a poor job at capturing this highly non-linear (and even non-differentiable) mapping, in practice the approximation to the integral is often surprisingly accurate. This is in a large part due to the robustness of the integration rule which stems from its equal, positive weights. All known higher degree integration rules that do not use more than polynomial in $\hat{S}$ nodes also feature negative weights on at least some nodes (Cools 2003), which means that their result is not guaranteed to lie within the convex hull of the source evaluations, and, in this case in which we are integrating a positive function (y), it further means the result can have the wrong sign.

The Genz and Keister (1996) rules allow one to choose the maximum degree of monomial that should be integrated exactly, up to a maximum order of 51. The number of points used is $\mathrm{O}(\hat{S}^K)$, where $2K + 1$ is the degree of monomial that is integrated exactly. When $K > 0$ and $\hat{S} > 1$, the rule features negative weights on at least one node, which means it is susceptible to the problems mentioned above. However, it has a few points in its favour. Firstly, by using negative weights, the rule is able to ensure that the maximum over the absolute vectors of integration points is independent of $\hat{S}$. This contrasts with the aforementioned rule in which the higher is $\hat{S}$, the further into the tails of the distribution one has to evaluate the integrand. Given the extreme non-linearity of the integrand, evaluating far into the tails can lead the equal weighted integration rule to produce a heavily upwards biased estimate of the integral. Secondly, by using a higher degree rule, we can generally obtain a better approximation to the integrand, despite its non-differentiability. Finally, the Genz and Keister (1996) rules are nested, which means that we can use an adaptive integration degree without wasting evaluations, continuing to increase the degree until approximate convergence. In practice, the results of these rules often repeatedly flip from biased down to biased up as the degree increases, due to the discontinuities. To lessen this, we take a statistical approach to smoothing across degrees, as detailed in section 7.4 in the appendices. We show there that with a high enough maximum degree, this

[9] While there is a degree 3 monomial cubature rule with only $2\hat{S}$ nodes, since we are evaluating the zero point anyway, it is costless to include it in the cubature, and doing so will generally reduce the error of the rule. As well as the zero point, the rule we use evaluates at $\pm\frac{1}{2}\sqrt{2 + 4\hat{S}}$ with respect to each coordinate.

[10] For example, with $\hat{S} = 1$, there are at least as many discontinuities as there are non-zero elements in Λ. While we could in theory get the full set of discontinuities at arbitrary dimension using a parametric linear complementarity problem solver such as the algorithm of Jones and Morrari (2006) which works providing that M is general positive semi-definite, this is computationally intractable for $\hat{S}$ or T bigger than (about) ten, and integrating over all of these regions separately is computationally intractable even for much smaller $\hat{S}$.

smoothed method still exactly integrates monomials of a given degree, albeit less efficiently than the unsmoothed one. Its practical performance on the discontinuous functions we need to integrate appears much improved over the base Genz and Keister (1996) rules though.

The final integration method we consider is quasi Monte Carlo, generating points from a Sobol sequence (Sobol 1967). Given that the functions we are integrating are absolutely continuous (as they are piecewise polynomial, with a finite number of manifolds of non-differentiability), quasi Monte Carlo with $2^{1+l}-1$ draws will produce an error that decays as $\mathrm{O}\left(\frac{l^{\hat{S}}}{2^l}\right)$. With the Sobol sequence, the choice of $2^{1+l}-1$ integration points for some $l \in \mathbb{N}$ also ensures that the points are exactly mean zero, hopefully lessening overall bias. However, on functions that are well approximated by a polynomial, quasi Monte Carlo will generally require far more evaluations of the integrand for a similar accuracy than the Genz and Keister (1996) rules would. Which dominates in practice will depend on the precise integrand, which in turn will depend on the model and its current state. At times where the bound is either highly likely to bind or highly likely not to bind, whatever future shocks hit, it is likely that our smoothed variant of the Genz and Keister (1996) rules will dominate, however, at times when the bound is only binding with moderate probability, quasi Monte Carlo's "dumb" approach may give it better performance.

Whichever approach to integration is taken, we end with an approximation to the expected value of the "y" vector of cumulated news shocks needed to impose the bound. This y will imply a set of news shocks that hit today, just as it does when we ignore future uncertainty. We can thus proceed with the simulation exactly as we do in the case without integrating over future uncertainty.

3.3. Hybrid local/global approach

Ideally, we would like to integrate over infinitely many periods of future uncertainty each period, but clearly this is infeasible. Thus, an alternative approach is needed. Recall, that the source of the inaccuracy of the base algorithm (without integrating over future uncertainty), was that the resulting news shocks were partially predictable by the model's state variables, and by the other shocks in the model. This suggests that if we could somehow tweak the bounded equation so that the needed news shocks were orthogonal to the states and the shock, then we would have a much more accurate simulation.

This is the approach we pursue here. In particular, given original equations with bounds of the form:

$$v_t = \mathbb{E}_t \max\{0, g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\},$$

where $\mathrm{g}(\mu_x, \mu_x, \mu_x, \mu_v, 0) \gg 0$, we replace them with equations of the form:

$$v_{a,t} = \mathbb{E}_t\big(g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\big)_a + I_{1,\cdot} y_t^{(a)} + \theta'_{\cdot,a} \hat{s}_t,$$
$$\tilde{v}_{a,t} = \mathbb{E}_t\big(g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\big)_a + I_{1,\cdot} y_t^{(a)},$$

for all $a \in \{1, \dots, c\}$, where $\theta = [\theta_{\cdot,1} \quad \cdots \quad \theta_{\cdot,c}]$ is a matrix of unknown parameters,

$$\hat{s}_t := \begin{bmatrix} 1 \\ x_{t-1} \\ \varepsilon_t \end{bmatrix}^{\otimes d},$$ [11]

where d is again the degree of perturbation approximation being taken, and $\cdot^{\otimes d}$ denotes Kronecker exponentiation, and where, just as in section 3.1, for all $a \in \{1, \dots, c\}$:

[11] In practice, we just use the state variables from x_{t-1}.

$$\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t}^{(a)} = y_{i+1,t-1}^{(a)} + \eta_{i,t}^{(a)},$$
$$y_{T,t}^{(a)} = \eta_{T,t}^{(a)}.$$

In the first step of the algorithm, we set $\eta_{i,t}^{(a)} = 0$ for all i, t, a and attempt to solve for θ such that:

$$0 = \mathbb{E}\delta_t(\max\{0, \tilde{v}_t\} - v_t)' = \mathbb{E}\delta_t(\max\{0, \tilde{v}_t\} - \tilde{v}_t - \theta'\delta_t)' = \mathbb{E}\delta_t(\max\{-\tilde{v}_t, 0\} - \theta'\delta_t)'.$$

In other words, we try to find a θ at which the error in the approximation of $\max\{0, \tilde{v}_t\}$ by v_t is orthogonal to the states and shocks of the model. This gives an optimal polynomial approximation to the bounded variables, and has a direct impact on the policy functions for other variables, capturing, for example, the precautionary motive to avoid the bound. This procedure is similar in spirit to the parameterised expectations algorithm of den Haan and Marcet (1990).

Now, at a θ that solved the previous equations, we would have that:

$$\mathbb{E}\delta_t \max\{-\tilde{v}_t, 0\}' = \mathbb{E}\delta_t\delta_t'\theta,$$

so:

$$\theta = (\mathbb{E}\delta_t\delta_t')^{-1}\mathbb{E}\delta_t \max\{-\tilde{v}_t, 0\}'.$$

Now, thanks to the existence of an augmented state-space representation for pruned perturbation solutions (as discussed in section 7.2 in the appendices) we can evaluate $\mathbb{E}\delta_t$, $\mathbb{E}\tilde{v}_t$, $\mathbb{E}\delta_t\delta_t'$, $\mathbb{E}\delta_t\tilde{v}_t'$ without any Monte Carlo simulation, or numerical integration. Furthermore, note that if:

$$\begin{bmatrix} p \\ q \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} \mu_p \\ \mu_q \end{bmatrix}, \begin{bmatrix} \sigma_{pp} & \sigma_{pq} \\ \sigma_{qp} & \sigma_{qq} \end{bmatrix}\right),$$

then, by explicit integration:

$$\mathbb{E}p \max\{-q, 0\} = \mu_p \exp\left(-\frac{\mu_q^2}{2\sigma_{qq}}\right)\sqrt{\frac{\sigma_{qq}}{2\pi}} - \left(\mu_p\mu_q + \sigma_{pq}\right)\left(1 - \Phi\left(\frac{\mu_q}{\sqrt{\sigma_{qq}}}\right)\right),$$

where Φ is the cumulative distribution function of a standard normal. Thus, if we approximate the joint distribution of δ_t and v_t by a Gaussian, then we can also calculate $\mathbb{E}\delta_t \max\{-\tilde{v}_t, 0\}'$ and hence θ too, all without any Monte Carlo simulation, or numerical integration. Since DSGE models are usually dominated by their first order terms, in practice, not much accuracy is lost by this joint Gaussian approximation, so this is the approach we take here.

The existence of a formula for θ suggests a fixed point algorithm. We first guess θ, then we solve the model by perturbation given that value for θ . We then calculate a new value for θ using $\theta = (\mathbb{E}\delta_t\delta_t')^{-1}\mathbb{E}\delta_t \max\{-\tilde{v}_t, 0\}'$. In practice, this fixed point iteration is non-convergent for many frequently encountered models, so we instead just solve for the fixed point using a non-linear equation solver, such as "fmincon" in MATLAB. Given that each evaluation only requires us to solve the model by perturbation and then to make some relatively straightforward numerical calculations, this system of non-linear equations can be solved very quickly even when the dimensions of θ is large. As such, although this is a global algorithm since we are integrating over the entire state space, it is orders of magnitude faster than standard global algorithms which require either numerical integration at each point on a grid, or long simulation runs.

Once we have found θ, simulation is a simple extension of the simulation algorithm involving integrating over future uncertainty. Rather than constructing a single M matrix, we now have one coming from the responses of $\tilde{v}$, which we continue to call M, and one coming from the responses of v, which we now call $\overline{M}$. As ever, at each simulation period, we need to solve for the y which makes expectations close to rational.

Conceptually, we do this by finding a y which brings $q + \overline{M}y$ as close as possible to our best estimate of rational expectations.

First, note that v_t is not likely to be a particularly accurate approximation to $\max\{0, \tilde{v}_t\}$, since polynomials cannot well approximate discontinuities. Thus, it is more appropriate as a model of long-run expectations, rather than as a model of the near term. However, by integrating over future uncertainty (with LCPs using the M matrix, not the $\overline{M}$ one), as in section 3.2, we can get a reasonable estimates of the path of the constrained variables. Let us write $\mathcal{E}_{s,t}$ for this approximation to the expected value of v_{t+s} from the "true" model with occasionally binding constraints, when we set $S := T$. Then we define a scaled deviation from $\mathcal{E}_{s,t}$, which we call $e_{s,t}$, by:

$$\mathbb{E}_t v_{t+s} = \frac{1}{2}\left(1 + \cos\left(\pi \frac{\max\{0, s-1\}}{T}\right)\right)\mathcal{E}_{s,t} + \left[1 - \frac{1}{2}\left(1 + \cos\left(\pi \frac{\max\{0, s-1\}}{T}\right)\right)\right][\mathbb{E}_t v_{t+s} + e_{s,t}],$$

which treats $\mathbb{E}_t v_{t+s}$ as a linear interpolation between $\mathcal{E}_{s,t}$ and $\mathbb{E}_t v_{t+s} + e_{s,t}$ with weights equal to the windowing function applied to the covariance matrix when integrating over future uncertainty, so as to capture the fact that departures from $\mathcal{E}_{s,t}$ are more acceptable at long horizons where it is less accurate. This seems a natural choice, given our windowing function. It implies:

$$0 = \frac{1}{2}\left(1 + \cos\left(\pi \frac{\max\{0, s-1\}}{T}\right)\right)[\mathcal{E}_{s,t} - \mathbb{E}_t v_{t+s}] + \left[1 - \frac{1}{2}\left(1 + \cos\left(\pi \frac{\max\{0, s-1\}}{T}\right)\right)\right] e_{s,t},$$

so for $s = 0$ and $s = 1$ we have $\mathcal{E}_{s,t} = \mathbb{E}_t v_{t+s}$.

We then choose $e_t := \text{vec}[e_{0,t} \quad \cdots \quad e_{T-1,t}] \in \mathbb{R}^{cT \times 1}$ and $y_t := \text{vec}\left[y_t^{(1)} \quad \cdots \quad y_t^{(c)}\right] \in \mathbb{R}^{Tc \times 1}$ to minimise:

$$\omega e_t' e_t + (y_t - \mathbb{E}_{t-1} y_t)'(y_t - \mathbb{E}_{t-1} y_t),$$

where ω is an arbitrary constant, subject to the constraints:

$$\mathbb{E}_t v_{t+s} \geq 0, \qquad \forall s \in \{0, \dots, T-1\},$$

$$\begin{bmatrix} y_{s+1,t}^{(1)} \\ \vdots \\ y_{s+1,t}^{(c)} \end{bmatrix} \geq \mathbb{E}_t \tilde{v}_{t+s} - \mathbb{E}_t v_{t+s}, \qquad \forall s \in \{0, \dots, T-1\},$$

where $\mathbb{E}_t v_{t+s}$, $\mathbb{E}_t \tilde{v}_{t+s}$, are functions of y_t, for $s \in \{0, \dots, T-1\}$, since:

$$\text{vec}[[\mathbb{E}_t v_t \quad \cdots \quad \mathbb{E}_t v_{t+T-1}]'] = q + \overline{M} y_t,$$

and:

$$\text{vec}[[\mathbb{E}_t \tilde{v}_t \quad \cdots \quad \mathbb{E}_t \tilde{v}_{t+T-1}]'] = \tilde{q} + M y_t,$$

for some $q, \tilde{q} \in \mathbb{R}^{Tc \times 1}$. By construction, this is a concave quadratic programming problem, and so may be solved in time polynomial in cT using the ellipsoid method (Kozlov, Tarasov, and Khachiyan 1980). The first constraint on the problem ensures that expectations of the bounded variable are positive, which they must be as the bound is at zero by assumption. The second constraint ensures that at each horizon s, $\mathbb{E}_t v_{t+s} \geq \mathbb{E}_t g(x_{t+s-1}, x_{t+s}, x_{t+s+1}, v_{t+s}, \varepsilon_{t+s})$, which should hold since $\max\{0, g(x_{t+s-1}, x_{t+s}, x_{t+s+1}, v_{t+s}, \varepsilon_{t+s})\} \geq g(x_{t+s-1}, x_{t+s}, x_{t+s+1}, v_{t+s}, \varepsilon_{t+s})$. Finally, the objective trades off the magnitude of the news shocks, with the magnitude of the scaled deviations. With our suggested setting of $\omega := 1$, a deviation between $\mathcal{E}_{s,t}$ and $\mathbb{E}_t v_{t+s}$ at a horizon of $s = 1 + \frac{T}{2}$ is given equal weighting to the news shocks, while deviations at earlier horizons are given greater weightings, and deviations at later horizons are given lesser ones. This seems a reasonable compromise as with $q \gg 0$, $y_t = \mathbb{E}_{t-1} y_t$ is usually feasible, so we can afford to place a substantial weight on the constraint without distorting y_t too much at long horizons. As with any such decision, the ultimate test is its practical performance, which we discuss in section 4, after first discussing the implementation of these algorithms in the next section.

3.4. The DynareOBC toolkit

Code implementing all of the algorithms discussed here is contained in the author's "DynareOBC" toolkit which is a suite of MATLAB files designed to augment the abilities of Dynare (Adjemian et al. 2011). The toolkit may be downloaded from http://github.org/tholden/dynareOBC, and this site also contains complete documentation for its assorted options. To use it, one merely has to include a "max", "min" or "abs" in the "MOD" file describing the DSGE model to be simulated, and then to invoke DynareOBC with the MATLAB command "dynareOBC ModFileName.MOD".

Internally, DynareOBC uses the "YALMIP" (Löfberg 2004) MATLAB toolkit as an interface to a wide variety of open source and commercial mixed integer linear programming solvers. The distribution of DynareOBC comes with a variety of open source solvers, so DynareOBC is certainly not dependent on any particular commercial packages (other than MATLAB itself). DynareOBC also attempts to obtain a parametric solution to the LCP (q, M) for q which only violate the bound in at most the first few periods, using the MPT toolkit (Herceg et al. 2013), which in turn uses an algorithm due to Jones and Morrari (2006). The resulting parametric solution takes the form of a compiled MEX function, which, when passed a q, returns the y that solves the LCP. This reduces the number of times the LCP needs to be solved in inner loops, increasing performance. Furthermore, DynareOBC includes efficient code for testing whether M is a P-matrix, based on an algorithm of Tsatsomeros and Li (2000), and can also test if M is an S-matrix or (strictly) semi-monotone. Additionally, DynareOBC contains code for facilitating the calculation of Jin and Judd (2002) style accuracy checks. Thus, DynareOBC functions as an easy to use, one stop shop for all queries one might have of a model with occasionally binding constraints.

4. Performance of our algorithm

TODO

5. Conclusion

TODO Suggestion that policy makers may wish to choose policy rules to ensure that solving the LCP problem is possible in polynomial time, and that the LCP is at least always feasible.

6. References

Adjemian, Stéphane, Houtan Bastani, Fréderic Karamé, Michel Juillard, Junior Maih, Ferhat Mihoubi, George Perendia, Johannes Pfeifer, Marco Ratto, and Sébastien Villemot. 2011. Dynare: Reference Manual Version 4. CEPREMAP.

Adjemian, Stéphane, and Michel Juillard. 2013. "Stochastic Extended Path Approach."

Anderson, Gary. 2015. "A Solution Strategy for Occasionally Binding Constraints." Unpublished.

Blanchard, Olivier Jean, and Charles M. Kahn. 1980. "The Solution of Linear Difference Models under Rational Expectations." Econometrica 48 (5): 1305–1311.

Bodenstein, Martin, Luca Guerrieri, and Christopher J. Gust. 2013. "Oil Shocks and the Zero Bound on Nominal Interest Rates." Journal of International Money and Finance 32 (February): 941–967.

Brendon, Charles, Matthias Paustian, and Tony Yates. 2015. "Self-Fulfilling Recessions at the Zero Lower Bound." Unpublished.

Chung, S.J. 1989. “NP-Completeness of the Linear Complementarity Problem.” Journal of Optimization Theory and Applications 60 (3) (March 1): 393–399.

Cools, Ronald. 2003. “An Encyclopaedia of Cubature Formulas.” Oberwolfach Special Issue 19 (3) (June): 445–453.

Cottle, Richard W. 2009. “Linear Complementarity Problem.” In Encyclopedia of Optimization, edited by Christodoulos A. Floudas and Panos M. Pardalos, 1873–1878. Springer US.

Cottle, Richard W., Jong-Shi Pang, and Richard E. Stone. 2009. “3. Existence and Multiplicity.” In The Linear Complementarity Problem, 137–223. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics.

Coxson, Gregory E. 1994. “The P-Matrix Problem Is Co-NP-Complete.” Mathematical Programming 64 (1-3) (March 1): 173–178.

Csizmadia, Zsolt, and Tibor Illés. 2006. “New Criss-Cross Type Algorithms for Linear Complementarity Problems with Sufficient Matrices.” Optimization Methods and Software 21 (2) (April 1): 247–266.

Fair, Ray C, and John B Taylor. 1983. “Solution and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models.” Econometrica 51 (4) (July): 1169–85.

Fang, Li. 1989. “On the Spectra of P- and P0-Matrices.” Linear Algebra and Its Applications 119 (July): 1–25.

Fukuda, Komei. 2015. “Toward a Strongly Polynomial Algorithm for the Linear Complementarity Problems with Sufficient Matrices.” Unpublished (September 22).

Genz, Alan, and Bradley D. Keister. 1996. “Fully Symmetric Interpolatory Rules for Multiple Integrals over Infinite Regions with Gaussian Weight.” Journal of Computational and Applied Mathematics 71 (2): 299–309.

Guerrieri, Luca, and Matteo Iacoviello. 2015. “OccBin: A Toolkit for Solving Dynamic Models with Occasionally Binding Constraints Easily.” Journal of Monetary Economics 70 (March): 22–38.

den Haan, Wouter J, and Albert Marcet. 1990. “Solving the Stochastic Growth Model by Parameterizing Expectations.” Journal of Business & Economic Statistics 8 (1) (January): 31–34.

Harris, F.J. 1978. “On the Use of Windows for Harmonic Analysis with the Discrete Fourier Transform.” Proceedings of the IEEE 66 (1) (January): 51–83.

Hebden, James, Jesper Lindé, and Lars Svensson. 2011. “Optimal Monetary Policy in the Hybrid New Keynesian Model under the Zero Lower Bound.” Unpublished.

Herceg, Martin, Michal Kvasnica, Colin N. Jones, and Manfred Morari. 2013. “Multi-Parametric Toolbox 3.0.” In Control Conference (ECC), 2013 European, 502–510. IEEE.

Holden, Tom. 2010. Products, Patents and Productivity Persistence: A DSGE Model of Endogenous Growth. Economics Series Working Paper. University of Oxford, Department of Economics.

Holden, Tom, and Michael Paetz. 2012. “Efficient Simulation of DSGE Models with Inequality Constraints.” School of Economics Discussion Papers.

Illés, Tibor, Marianna Nagy, and Tamás Terlaky. 2010. “A Polynomial Path-Following Interior Point Algorithm for General Linear Complementarity Problems.” Journal of Global Optimization 47 (3) (July 1): 329–342.

Jin, Hehui, and Kenneth L. Judd. 2002. “Perturbation Methods for General Dynamic Stochastic Models.”

Jones, C.N., and M. Morrari. 2006. “Multiparametric Linear Complementarity Problems.” In Decision and Control, 2006 45th IEEE Conference on, 5687–5692.

Judd, Kenneth L., Lilia Maliar, and Serguei Maliar. 2012. Merging Simulation and Projection Approaches to Solve High-Dimensional Problems. National Bureau of Economic Research, Inc.

Kim, Sunghyun Henry, Jinill Kim, Ernst Schaumburg, and Christopher A. Sims. 2008. "Calculating and Using Second Order Accurate Solutions of Discrete Time Dynamic Equilibrium Models." Journal of Economic Dynamics and Control 32 (11): 3397–3414.

Kojima, M., N. Megiddo, T. Noma, and A. Yoshise. 1991. A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems. Springer.

Kojima, M., Shinji Mizuno, and Akiko Yoshise. 1989. "A Polynomial-Time Algorithm for a Class of Linear Complementarity Problems." Mathematical Programming 44 (1-3) (May 1): 1–26.

Kozlov, Mikhail K., Sergei P. Tarasov, and Leonid G. Khachiyan. 1980. "The Polynomial Solvability of Convex Quadratic Programming." USSR Computational Mathematics and Mathematical Physics 20 (5): 223–228.

Lan, Hong, and Alexander Meyer-Gohde. 2013a. "Solving DSGE Models with a Nonlinear Moving Average." Journal of Economic Dynamics and Control 37 (12). Journal of Economic Dynamics and Control: 2643–2667.

———. 2013b. Pruning in Perturbation DSGE Models - Guidance from Nonlinear Moving Average Approximations. SFB 649 Discussion Papers. Sonderforschungsbereich 649, Humboldt University, Berlin, Germany.

Löfberg, J. 2004. "YALMIP : A Toolbox for Modeling and Optimization in MATLAB." In Proceedings of the CACSD Conference. Taipei, Taiwan.

Magnus, Jan R., and H. Neudecker. 1979. "The Commutation Matrix: Some Properties and Applications." Ann. Statist. (2) (March): 381–394.

Pardalos, P. M., and J. B. Rosen. 1988. "Global Optimization Approach to the Linear Complementarity Problem." SIAM J. Sci. Stat. Comput. 9 (2): 341–354.

Roos, C., T. Terlaky, and J. P. Vial. 2006. Interior Point Methods for Linear Optimization. Springer.

Rosen, J. B. 1990. "Minimum Norm Solution to the Linear Complementarity Problem." Functional Analysis, Optimization and Economics: 208–216.

Samelson, Hans, Robert M. Thrall, and Oscar Wesler. 1958. "A Partition Theorem for Euclidean N-Space." Proceedings of the American Mathematical Society 9 (5): 805–807.

Sims, ChristopherA. 2002. "Solving Linear Rational Expectations Models." Computational Economics 20 (1-2) (October 1): 1–20.

Smets, Frank, and Rafael Wouters. 2003. "An Estimated Dynamic Stochastic General Equilibrium Model of the Euro Area." Journal of the European Economic Association 1 (5): 1123–1175.

———. 2007. "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach." American Economic Review 97 (3) (June): 586–606.

Sobol, Ilya M. 1967. "On the Distribution of Points in a Cube and the Approximate Evaluation of Integrals." USSR Computational Mathematics and Mathematical Physics (7): 86–112.

Tsatsomeros, Michael J., and Lei Li. 2000. "A Recursive Test for P-Matrices." BIT Numerical Mathematics 40 (2) (June 1): 410–414.

Tseng, Paul. 2000. "Co-NP-Completeness of Some Matrix Classification Problems." Mathematical Programming 88 (1) (June 1): 183–192.

Väliaho, Hannu. 1986. "Criteria for Copositive Matrices." Linear Algebra and Its Applications 81 (September 1): 19–34.

Xu, Song. 1993. "Notes on Sufficient Matrices." Linear Algebra and Its Applications 191 (September 15): 1–13.

7. Appendices

7.1. Proof of uniqueness of the solution to example 2.2.1 without the bound

Suppose for a contradiction that there were at least three solutions to equation (3) in (0,1) (double counting repeated roots), even for arbitrary large $\beta \in (0,1)$. Let $f_1, f_2, f_3 \in (0,1)$ be the three roots. Then, by Vieta's formulas:

$$3 > f_1 + f_2 + f_3 = \frac{(\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma}{\beta\sigma},$$
$$3 > f_1 f_2 + f_1 f_3 + f_2 f_3 = \frac{(1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma}{\beta\sigma},$$
$$1 > f_1 f_2 f_3 = \frac{\alpha_{\Delta y}}{\beta\sigma},$$

so:

$$(2\beta - 1)\sigma > \beta\alpha_{\Delta y} + \gamma > \gamma > 0$$
$$\beta > \frac{1}{2}, \qquad (2\beta - 1)\sigma > \gamma,$$
$$\beta\sigma > \beta\alpha_{\Delta y} + \gamma + \sigma(1-\beta),$$
$$2\beta\sigma > (1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma(1-\beta),$$
$$\beta\sigma > \alpha_{\Delta y}.$$

Also, the first derivative of equation (3) must be strictly positive at $f = 1$, so:

$$(1-\beta)(\alpha_{\Delta y} - \sigma) + (\alpha_\pi - 2)\gamma > 0.$$

Combining all of these inequalities gives the bounds:

$$0 < \alpha_{\Delta y} < 2\sigma - \frac{\gamma + \sigma}{\beta},$$
$$2 + \frac{(1-\beta)(\sigma - \alpha_{\Delta y})}{\gamma} < \alpha_\pi < \frac{(3\beta - 1)\sigma - (1+\beta)\alpha_{\Delta y}}{\gamma}.$$

Furthermore, if there are multiple solutions to equation (3), then the discriminant of its first derivative must be weakly positive, i.e.:

$$\Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)^2 - 3\beta\sigma\Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big) \geq 0.$$

Therefore, we have the following bounds on α_π:

$$2 + \frac{(1-\beta)(\sigma - \alpha_{\Delta y})}{\gamma} < \alpha_\pi \leq \frac{\Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)^2 - 3\beta\sigma\Big((1+\beta)\alpha_{\Delta y} + \sigma\Big)}{3\beta\sigma\gamma}$$

since,

$$\frac{(3\beta - 1)\sigma - (1+\beta)\alpha_{\Delta y}}{\gamma} - \frac{\Big((\alpha_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)^2 - 3\beta\sigma\Big((1+\beta)\alpha_{\Delta y} + \sigma\Big)}{3\beta\sigma\gamma}$$
$$= \frac{\Big((2\sigma - \alpha_{\Delta y})\beta - \gamma - \sigma\Big)\Big((4\sigma + \alpha_{\Delta y})\beta + \gamma + \sigma\Big)}{3\beta\gamma\sigma} > 0$$

as $\alpha_{\Delta y} < 2\sigma - \frac{\gamma+\sigma}{\beta}$. Consequently, there exists $\lambda, \mu, \kappa \in [0,1]$ such that:

$$\alpha_\pi = (1-\lambda)\left[2 + \frac{(1-\beta)(\sigma - \alpha_{\Delta y})}{\gamma}\right] + \lambda\left[\frac{\left((\alpha_{\Delta y}+\sigma)\beta + \gamma + \sigma\right)^2 - 3\beta\sigma\left((1+\beta)\alpha_{\Delta y} + \sigma\right)}{3\beta\sigma\gamma}\right],$$

$$\alpha_{\Delta y} = (1-\mu)[0] + \mu\left[2\sigma - \frac{\gamma+\sigma}{\beta}\right],$$

$$\gamma = (1-\kappa)[0] + \kappa[(2\beta-1)\sigma]$$

These simultaneous equations have unique solutions for α_π, $\alpha_{\Delta y}$ and γ in terms of λ, μ and κ. Substituting these solutions into the discriminant of equation (3) gives a polynomial in $\lambda, \mu, \kappa, \beta, \sigma$. As such, an exact global maximum of the discriminant may be found subject to the constraints $\lambda, \mu, \kappa \in [0,1]$, $\beta \in \left[\frac{1}{2}, 1\right]$, $\sigma \in [0, \infty)$, by using an exact compact polynomial optimisation solver, such as that in the Maple computer algebra package. Doing this gives a maximum of 0 when $\beta \in \left\{\frac{1}{2}, 1\right\}$, $\kappa = 1$ and $\sigma = 0$. But of course, we actually require that $\beta \in \left(\frac{1}{2}, 1\right)$, $\kappa < 1$, $\sigma > 0$. Thus, by continuity, the discriminant is strictly negative over the entire possible domain. This gives the required contradiction to our assumption of three roots to the polynomial.

7.2. The augmented state-space representation of a pruned perturbation solution

We seek to convert the model into the form:

$$z_t = o + P\tilde{z}_{t-1} + Q\xi_t,$$
$$x_t = u + Vz_t,$$

where $\mathbb{E}_{t-1}\xi_t = 0$, and where throughout, $\tilde{}$s over variables denote the subset of state variables. We proceed by taking each order of approximation in turn. We assume that the original model has l state variables. Of the assorted algorithms available for pruning, it appears that Lan and Meyer-Gohde's (2013a) algorithm is the most accurate (Lan and Meyer-Gohde 2013b), and so both the discussion below, and the implementation in DynareOBC is based on this approach, however, everything we say in the below would also go through with alternative pruning algorithms.

7.2.1. Order 1

At order 1:

$$x_t^{(1)} = \alpha\tilde{x}_{t-1}^{(1)} + \beta_0\varepsilon_t,$$
$$x_t = \mu_x + x_t^{(1)},$$

so if we define:

$$z_t := x_t^{(1)}, \qquad \tilde{z}_t := \tilde{x}_t^{(1)}, \qquad o := 0, \qquad P := \alpha, \qquad Q := \beta_0, \qquad \xi_t := \varepsilon_t, \qquad u := \mu_x, \qquad V := I_n,$$

then we are done.

7.2.2. Order 2

At order 2:

$$x_t^{(1)} = \alpha\tilde{x}_{t-1}^{(1)} + \beta_0\varepsilon_t,$$
$$x_t^{(2)} = \alpha\tilde{x}_{t-1}^{(2)} + \frac{1}{2}\beta_{22}\left(\tilde{x}_{t-1}^{(1)} \otimes \tilde{x}_{t-1}^{(1)}\right) + \beta_{20}\left(\tilde{x}_{t-1}^{(1)} \otimes \varepsilon_t\right) + \frac{1}{2}\beta_{00}(\varepsilon_t \otimes \varepsilon_t),$$
$$x_t = \mu_x + x^{(0)} + x_t^{(1)} + x_t^{(2)},$$

for some constant $x^{(0)}$.

Now, note that:

$$\tilde{x}_t^{(1)} \otimes \tilde{x}_t^{(1)} = \left(\tilde{\alpha}\tilde{x}_{t-1}^{(1)} + \tilde{\beta}_0\varepsilon_t\right) \otimes \left(\tilde{\alpha}\tilde{x}_{t-1}^{(1)} + \tilde{\beta}_0\varepsilon_t\right)$$

$$
\begin{aligned}
&= \tilde{\alpha}\tilde{x}_{t-1}^{(1)} \otimes \tilde{\alpha}\tilde{x}_{t-1}^{(1)} + \tilde{\alpha}\tilde{x}_{t-1}^{(1)} \otimes \tilde{\beta}_0\varepsilon_t + \tilde{\beta}_0\varepsilon_t \otimes \tilde{\alpha}\tilde{x}_{t-1}^{(1)} + \tilde{\beta}_0\varepsilon_t \otimes \tilde{\beta}_0\varepsilon_t \\
&= (\tilde{\alpha} \otimes \tilde{\alpha})\big(\tilde{x}_{t-1}^{(1)} \otimes \tilde{x}_{t-1}^{(1)}\big) + \big(\tilde{\alpha} \otimes \tilde{\beta}_0\big)\big(\tilde{x}_{t-1}^{(1)} \otimes \varepsilon_t\big) + \big(\tilde{\beta}_0 \otimes \tilde{\alpha}\big)\big(\varepsilon_t \otimes \tilde{x}_{t-1}^{(1)}\big) + \big(\tilde{\beta}_0 \otimes \tilde{\beta}_0\big)(\varepsilon_t \otimes \varepsilon_t) \\
&= (\tilde{\alpha} \otimes \tilde{\alpha})\big(\tilde{x}_{t-1}^{(1)} \otimes \tilde{x}_{t-1}^{(1)}\big) + \Big(\big(\tilde{\alpha} \otimes \tilde{\beta}_0\big) + \big(\tilde{\beta}_0 \otimes \tilde{\alpha}\big)K_{m,l}\Big)\big(\tilde{x}_{t-1}^{(1)} \otimes \varepsilon_t\big) + \big(\tilde{\beta}_0 \otimes \tilde{\beta}_0\big)(\varepsilon_t \otimes \varepsilon_t) \\
&= (\tilde{\alpha} \otimes \tilde{\alpha})\big(\tilde{x}_{t-1}^{(1)} \otimes \tilde{x}_{t-1}^{(1)}\big) + \Big(\big(\tilde{\alpha} \otimes \tilde{\beta}_0\big) + K_{l,l}\big(\tilde{\alpha} \otimes \tilde{\beta}_0\big)K_{l,m}K_{m,l}\Big)\big(\tilde{x}_{t-1}^{(1)} \otimes \varepsilon_t\big) + \big(\tilde{\beta}_0 \otimes \tilde{\beta}_0\big)(\varepsilon_t \otimes \varepsilon_t) \\
&= (\tilde{\alpha} \otimes \tilde{\alpha})\big(\tilde{x}_{t-1}^{(1)} \otimes \tilde{x}_{t-1}^{(1)}\big) + \big(I_{l^2} + K_{l,l}\big)\big(\tilde{\alpha} \otimes \tilde{\beta}_0\big)\big(\tilde{x}_{t-1}^{(1)} \otimes \varepsilon_t\big) + \big(\tilde{\beta}_0 \otimes \tilde{\beta}_0\big)(\varepsilon_t \otimes \varepsilon_t),
\end{aligned}
$$

where $K_{l,m} \in \mathbb{R}^{lm \times lm}$ is the commutation matrix for $l \times m$ matrices, i.e. it is the unique matrix such that for all $D \in \mathbb{R}^{l \times m}$, $K_{l,m} \operatorname{vec} D = \operatorname{vec} D'$ (Magnus and Neudecker 1979). Thus, if we define:

$$
z_t := \begin{bmatrix} x_t^{(1)} \\ x_t^{(2)} \\ \tilde{x}_t^{(1)} \otimes \tilde{x}_t^{(1)} \end{bmatrix}, \qquad \tilde{z}_t := \begin{bmatrix} \tilde{x}_t^{(1)} \\ \tilde{x}_t^{(2)} \\ \tilde{x}_t^{(1)} \otimes \tilde{x}_t^{(1)} \end{bmatrix},
$$

$$
P := \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & \frac{1}{2}\beta_{22} \\ 0 & 0 & \tilde{\alpha} \otimes \tilde{\alpha} \end{bmatrix}, \qquad Q := \begin{bmatrix} \beta_0 & 0 & 0 \\ 0 & \frac{1}{2}\beta_{00} & \beta_{20} \\ 0 & \tilde{\beta}_0 \otimes \tilde{\beta}_0 & \big(I_{l^2} + K_{l,l}\big)\big(\tilde{\alpha} \otimes \tilde{\beta}_0\big) \end{bmatrix},
$$

$$
\xi_t := \begin{bmatrix} \varepsilon_t \\ \varepsilon_t \otimes \varepsilon_t - \operatorname{vec}\Sigma \\ \tilde{x}_{t-1}^{(1)} \otimes \varepsilon_t \end{bmatrix}, \qquad o := \begin{bmatrix} 0 \\ \frac{1}{2}\beta_{00} \operatorname{vec}\Sigma \\ \big(\tilde{\beta}_0 \otimes \tilde{\beta}_0\big) \operatorname{vec}\Sigma \end{bmatrix},
$$

$$
u := \mu_x + x^{(0)}, \qquad V := [I \quad I \quad 0],
$$

then we are done.

7.2.3. Order 3

At order 3:

$$
\begin{aligned}
x_t^{(1)} &= \alpha\tilde{x}_{t-1}^{(1)} + \beta_0\varepsilon_t, \\
x_t^{(2)} &= \alpha\tilde{x}_{t-1}^{(2)} + \frac{1}{2}\beta_{22}\big(\tilde{x}_{t-1}^{(1)} \otimes \tilde{x}_{t-1}^{(1)}\big) + \beta_{20}\big(\tilde{x}_{t-1}^{(1)} \otimes \varepsilon_t\big) + \frac{1}{2}\beta_{00}(\varepsilon_t \otimes \varepsilon_t), \\
x_t^{(1,\sigma^2)} &= \alpha\tilde{x}_{t-1}^{(1,\sigma^2)} + \frac{1}{2}\beta_{\sigma^2,0}\varepsilon_t + \frac{1}{2}\beta_{\sigma^2,1}\tilde{x}_{t-1}^{(1)}, \\
x_t^{(3)} &= \alpha\tilde{x}_{t-1}^{(3)} + \frac{1}{6}\beta_{333,1}\big(\tilde{x}_{t-1}^{(1)} \otimes \tilde{x}_{t-1}^{(1)} \otimes \tilde{x}_{t-1}^{(1)}\big) + \frac{1}{6}\beta_{000}(\varepsilon_t \otimes \varepsilon_t \otimes \varepsilon_t) + \frac{1}{2}\beta_{330,1}\big(\tilde{x}_{t-1}^{(1)} \otimes \tilde{x}_{t-1}^{(1)} \otimes \varepsilon_t\big) \\
&\quad + \frac{1}{2}\beta_{300}\big(\tilde{x}_{t-1}^{(1)} \otimes \varepsilon_t \otimes \varepsilon_t\big) + \beta_{22}\big(\tilde{x}_{t-1}^{(2)} \otimes \tilde{x}_{t-1}^{(1)}\big) + \beta_{20}\big(\tilde{x}_{t-1}^{(2)} \otimes \varepsilon_t\big), \\
x_t &= \mu_x + x^{(0)} + x_t^{(1)} + x_t^{(2)} + x_t^{(1,\sigma^2)} + x_t^{(3)},
\end{aligned}
$$

for the same constant $x^{(0)}$ as at order 2, providing the shocks have zero skewness (e.g. they are normally distributed). By similar calculations to those at second order, we then have that if we define:

$$
z_t^3 := \mathbb{E}_s \begin{bmatrix} x_t^{(1)} \\ x_t^{(2)} \\ \tilde{x}_t^{(1)} \otimes \tilde{x}_t^{(1)} \\ x_t^{(1,\sigma^2)} \\ x_t^{(3)} \\ \tilde{x}_t^{(2)} \otimes \tilde{x}_t^{(1)} \\ \tilde{x}_t^{(1)} \otimes \tilde{x}_t^{(1)} \otimes \tilde{x}_t^{(1)} \end{bmatrix}, \qquad z_t^3 := \mathbb{E}_s \begin{bmatrix} \tilde{x}_t^{(1)} \\ \tilde{x}_t^{(2)} \\ \tilde{x}_t^{(1)} \otimes \tilde{x}_t^{(1)} \\ \tilde{x}_t^{(1,\sigma^2)} \\ \tilde{x}_t^{(3)} \\ \tilde{x}_t^{(2)} \otimes \tilde{x}_t^{(1)} \\ \tilde{x}_t^{(1)} \otimes \tilde{x}_t^{(1)} \otimes \tilde{x}_t^{(1)} \end{bmatrix},
$$

$$P := \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & \frac{1}{2}\beta_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\alpha}\otimes\tilde{\alpha} & 0 & 0 & 0 & 0 \\ \frac{1}{2}\beta_{\sigma^2,1} & 0 & 0 & \alpha & 0 & 0 & 0 \\ \frac{1}{2}\beta_{300}(I_l\otimes \operatorname{vec}\Sigma) & 0 & 0 & 0 & \alpha & \beta_{22} & \frac{1}{6}\beta_{333,1} \\ \left(\tilde{\beta}_{20}\otimes\tilde{\beta}_0+\frac{1}{2}K_{l,l}(\tilde{\alpha}\otimes\tilde{\beta}_{00})\right)(I_l\otimes\operatorname{vec}\Sigma) & 0 & 0 & 0 & 0 & \tilde{\alpha}\otimes\tilde{\alpha} & \frac{1}{2}\tilde{\beta}_{22}\otimes\tilde{\alpha} \\ \left((I_{l^2}+K_{l,l})\otimes I_l + K_{l^2,l}\right)(\tilde{\alpha}\otimes\tilde{\beta}_0\otimes\tilde{\beta}_0)(I_l\otimes\operatorname{vec}\Sigma) & 0 & 0 & 0 & 0 & 0 & \tilde{\alpha}\otimes\tilde{\alpha}\otimes\tilde{\alpha} \end{bmatrix},$$

$$Q_{11} := \begin{bmatrix} \beta_0 & 0 & 0 \\ 0 & \frac{1}{2}\beta_{00} & \beta_{20} \\ 0 & \tilde{\beta}_0\otimes\tilde{\beta}_0 & (I_{l^2}+K_{l,l})(\tilde{\alpha}\otimes\tilde{\beta}_0) \\ \frac{1}{2}\beta_{\sigma^2,0} & 0 & 0 \end{bmatrix},$$

$$Q_{22} := \begin{bmatrix} \beta_{20} & \frac{1}{2}\beta_{330,1} & \frac{1}{2}\beta_{300} & \frac{1}{6}\beta_{000} \\ \tilde{\alpha}\otimes\tilde{\beta}_0 & \frac{1}{2}\tilde{\beta}_{22}\otimes\tilde{\beta}_0+(\tilde{\beta}_{20}\otimes\tilde{\alpha})(I_l\otimes K_{m,l}) & \tilde{\beta}_{20}\otimes\tilde{\beta}_0+\frac{1}{2}K_{l,l}(\tilde{\alpha}\otimes\tilde{\beta}_{00}) & \frac{1}{2}\tilde{\beta}_{00}\otimes\tilde{\beta}_0 \\ 0 & (I_l\otimes(I_{l^2}+K_{l,l})+K_{l,l^2})(\tilde{\alpha}\otimes\tilde{\alpha}\otimes\tilde{\beta}_0) & \left((I_{l^2}+K_{l,l})\otimes I_l+K_{l^2,l}\right)(\tilde{\alpha}\otimes\tilde{\beta}_0\otimes\tilde{\beta}_0) & \tilde{\beta}_0\otimes\tilde{\beta}_0\otimes\tilde{\beta}_0 \end{bmatrix},$$

$$Q := \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix},$$

$$\xi_t := \begin{bmatrix} \varepsilon_t \\ \varepsilon_t\otimes\varepsilon_t - \operatorname{vec}\Sigma \\ \tilde{x}^{(1)}_{t-1}\otimes\varepsilon_t \\ \tilde{x}^{(2)}_{t-1}\otimes\varepsilon_t \\ \tilde{x}^{(1)}_{t-1}\otimes\tilde{x}^{(1)}_{t-1}\otimes\varepsilon_t \\ \tilde{x}^{(1)}_{t-1}\otimes(\varepsilon_t\otimes\varepsilon_t-\operatorname{vec}\Sigma) \\ \varepsilon_t\otimes\varepsilon_t\otimes\varepsilon_t \end{bmatrix}, \qquad o := \begin{bmatrix} 0 \\ \frac{1}{2}\beta_{00}\operatorname{vec}\Sigma \\ (\tilde{\beta}_0\otimes\tilde{\beta}_0)\operatorname{vec}\Sigma \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$u := \mu_x + x^{(0)}, \qquad V := [I \quad I \quad 0 \quad I \quad I \quad 0 \quad 0],$$

then again we are done.

7.3. The conditional covariance of future variables under a pruned perturbation solution

First, suppose that:

$$z_t = o + Pz_{t-1} + Q\xi_t$$

where $\mathbb{E}_t\xi_{t+k} = 0$ for $k > 0$. Then:

$$\mathbb{E}_t z_{t+k} = \sum_{j=0}^{k-1} P^j o + P^k z_t,$$

so:

$$z_{t+k} - \mathbb{E}_t z_{t+k} = \sum_{j=1}^{k} P^{k-j} Q\xi_{t+j}.$$

Consequently:

$$\operatorname{cov}_t(z_{t+a}, z_{t+b}) = \sum_{i=1}^{a}\sum_{j=1}^{b} P^{a-i} Q\left(\mathbb{E}_t \xi_{t+i} \xi'_{t+j}\right) Q' P'^{b-j}$$

If $\mathbb{E}_t \xi_{t+i} \xi'_{t+j} = 0$ for $i \neq j$, then this simplifies to:

$$\operatorname{cov}_t(z_{t+a}, z_{t+b}) = P^{a-\min\{a,b\}} \left[\sum_{i=1}^{\min\{a,b\}} P^{\min\{a,b\}-i} Q(\mathbb{E}_t \xi_{t+i} \xi'_{t+i}) Q' P'^{\min\{a,b\}-i} \right] P'^{b-\min\{a,b\}}.$$

Now, in the previous section of these appendices (7.2), we showed that at order 1, 2 and 3 the pruned perturbation solutions may be represented in the form:

$$z_t = o + P\tilde{z}_{t-1} + Q\xi_t$$

where $\mathbb{E}_t \xi_{t+k} = 0$ for $k > 0$. It is trivial to add zero columns to P so that we instead have:

$$z_t = o + P z_{t-1} + Q\xi_t,$$

thus, we just need to evaluate $\mathbb{E}_t\left(\xi_{t+i} \xi'_{t+j}\right)$ in order to have a closed form expression for $\operatorname{cov}_t(z_{t+a}, z_{t+b})$, then from this and the fact that $x_t = u + V z_t$, we would have that:

$$\operatorname{cov}_t(x_{t+a}, x_{t+b}) = \operatorname{cov}_t(u + V z_{t+a}, u + V z_{t+b}) = V \operatorname{cov}_t(z_{t+a}, z_{t+b})\, V'.$$

We now proceed to evaluate $\mathbb{E}_t \xi_{t+i} \xi'_{t+j}$ for orders 1 and 2. We skip the third order case as a second order approximation to the conditional covariance is normally sufficient for reasonable accuracy, and as the third order conditional covariance is very slow to calculate.

7.3.1. Order 1

At order 1, $\xi_t := \varepsilon_t$, thus:

$$\mathbb{E}_t \xi_{t+i} \xi'_{t+j} = \begin{cases} \Sigma & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

7.3.2. Order 2

At order 2:

$$\xi_t := \begin{bmatrix} \varepsilon_t \\ \varepsilon_t \otimes \varepsilon_t - \operatorname{vec}\Sigma \\ \tilde{x}^{(1)}_{t-1} \otimes \varepsilon_t \end{bmatrix},$$

thus

$$\mathbb{E}_t \xi_{t+i} \xi'_{t+j} = 0 \text{ if } i \neq j,$$

and by theorem 4.3 of Magnus and Neudecker (1979):

$$\begin{aligned} \mathbb{E}_s \xi_t \xi'_t &= \mathbb{E}_s \begin{bmatrix} 1 \otimes \varepsilon_t \\ \varepsilon_t \otimes \varepsilon_t - \operatorname{vec}\Sigma \\ \tilde{x}^{(1)}_{t-1} \otimes \varepsilon_t \end{bmatrix} \begin{bmatrix} 1 \otimes \varepsilon'_t & \varepsilon'_t \otimes \varepsilon'_t - (\operatorname{vec}\Sigma)' & \tilde{x}^{1\prime}_{t-1} \otimes \varepsilon'_t \end{bmatrix} \\ &= \begin{bmatrix} \Sigma & 0 & \mathbb{E}_s \tilde{x}^{1\prime}_{t-1} \otimes \Sigma \\ 0 & (I_{m^2} + K_{m,m})(\Sigma \otimes \Sigma) & 0 \\ \mathbb{E}_s \tilde{x}^{(1)}_{t-1} \otimes \Sigma & 0 & \mathbb{E}_s\left(\tilde{x}^{(1)}_{t-1} \tilde{x}^{1\prime}_{t-1}\right) \otimes \Sigma \end{bmatrix}. \end{aligned}$$

7.4. Statistical smoothing of cubature results

Let $\mathcal{Y}_{\cdot,d} \in \mathbb{R}^T$ be the value of the integral calculated using a cubature rule that is exact for monomials up to degree d. Let $\mathcal{Y} \coloneqq [\mathcal{Y}_{\cdot,1} \quad \cdots \quad \mathcal{Y}_{\cdot,D}]$, for some $D \in \mathbb{N}^+$. Let μ be the true value of this integral. We seek to model the distribution of $\mathcal{Y}$. In particular, we suppose that $\operatorname{vec}\mathcal{Y}$ is jointly normally distributed, and that:

1) for all $d \in \{1,\dots,D\}$: $\mathbb{E}\mathcal{Y}_{\cdot,d} = \mu$,
2) for all $t \in \{1,\dots,T\}$, $d \in \{1,\dots,D\}$: $\operatorname{var}\mathcal{Y}_{t,d} = \frac{\sigma_t^2 \kappa^{2(d-1)}}{(1-\rho^2)(1-\phi^2)}$,
3) for all $s,t \in \{1,\dots,T\}$, $c,d \in \{1,\dots,D\}$: $\frac{\operatorname{cov}(\mathcal{Y}_{s,c},\mathcal{Y}_{t,d})}{\sqrt{\operatorname{var}\mathcal{Y}_{s,c}}\sqrt{\operatorname{var}\mathcal{Y}_{t,d}}} = \phi^{|s-t|}\rho^{|c-d|}$,

where $\rho, \phi \in (-1,1)$ and $\kappa \in (0,1)$. Here: 1) captures a belief that there is no systematic bias in our cubature rules over the space of functions we might like to integrate; 2) captures the fact that there is no reason to believe different coordinates of the integrand should be estimated equally precisely, through the σ_t^2 term; 2) also captures our belief that integrals evaluated at higher degrees should be more accurate, through the κ^{2d} term; finally, 3) captures our beliefs that errors are correlated both across nearby integration degrees, and across nearby elements of the integrand.

Given these assumptions, straightforward calculation reveals that:

$$\operatorname{vec}\mathcal{Y} \sim \mathrm{N}\Big(1_{D\times 1} \otimes \mu, \big((\operatorname{diag}\mathrm{K})\mathrm{P}(\operatorname{diag}\mathrm{K})\big) \otimes \big((\operatorname{diag}\sigma)\Phi(\operatorname{diag}\sigma)\big)\Big),$$

where:

$$\mathrm{K} \coloneqq \begin{bmatrix} 1 \\ \kappa \\ \vdots \\ \kappa^{D-1} \end{bmatrix}, \qquad \mathrm{P} \coloneqq \frac{1}{1-\rho^2}\begin{bmatrix} 1 & \rho & \cdots & \rho^{D-1} \\ \rho & 1 & \cdots & \rho^{D-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{D-1} & \rho^{D-2} & \cdots & 1 \end{bmatrix}, \qquad \Phi \coloneqq \frac{1}{1-\phi^2}\begin{bmatrix} 1 & \phi & \cdots & \phi^{T-1} \\ \phi & 1 & \cdots & \phi^{T-2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{T-1} & \phi^{T-2} & \cdots & 1 \end{bmatrix}.$$

Hence, given an observation of $\mathcal{Y}$ it is straightforward to find the μ, κ, ρ, ϕ which maximise the likelihood of that observation. We then take the maximum likelihood estimate of μ as our new estimate of the true value of the integral.

Alternatively, we may wish to impose a prior over κ, as when D is small, there is little information about this parameter, so the optimisation tends to push it towards zero. (For example, when $D = 2$, in the limit as $\kappa \to 0$, the likelihood goes to ∞.) In DynareOBC, we allow the user to specify the parameters of the following density:

$$f(\kappa) = \frac{1-(1-\kappa^\alpha)^\beta}{1-\frac{\Gamma(1+\beta)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(1+\beta+\frac{1}{\alpha}\right)}}$$

to use as a prior for κ. We then maximise the posterior density, implicitly placing a flat prior on all other parameters. This appears to improve accuracy for large as well as small D, by ensuring that κ is not driven too small, too quickly.

We end with some theoretical results to demonstrate that this integration method still exactly integrates polynomials, albeit with additional evaluations. First, note that two times the negative log likelihood is given by:

$$\begin{aligned} &\log\left|\big((\operatorname{diag}\mathrm{K})\mathrm{P}(\operatorname{diag}\mathrm{K})\big) \otimes \big((\operatorname{diag}\sigma)\Phi(\operatorname{diag}\sigma)\big)\right| \\ &\quad + \left(\operatorname{vec}\mathcal{Y} - 1_{D\times 1} \otimes \mu\right)'\left[\big((\operatorname{diag}\mathrm{K})\mathrm{P}(\operatorname{diag}\mathrm{K})\big) \otimes \big((\operatorname{diag}\sigma)\Phi(\operatorname{diag}\sigma)\big)\right]^{-1}\left(\operatorname{vec}\mathcal{Y} - 1_{D\times 1} \otimes \mu\right) \\ &\quad + TD\log(2\pi) \end{aligned}$$

$$= T[D(D-1)\log\kappa - \log(1-\rho) - \log(1+\rho)] + D[2(1'_{T\times 1}\log\sigma) - \log(1-\phi) - \log(1+\phi)]$$
$$+ \operatorname{vec}(\mathcal{Y} - \mu 1'_{D\times 1})' [((\operatorname{diag} K)^{-1} P^{-1} (\operatorname{diag} K)^{-1})$$
$$\otimes ((\operatorname{diag}\sigma)^{-1}\Phi^{-1}(\operatorname{diag}\sigma)^{-1})]\operatorname{vec}(\mathcal{Y} - \mu 1'_{D\times 1}) + TD\log(2\pi).$$

Now, suppose that for some $E \in \{0,\dots,D\}$, $\mathcal{Y}_{\cdot,E+1} = \mathcal{Y}_{\cdot,E+2} = \cdots = \mathcal{Y}_{\cdot,D}$, which is true for example if the integrand is a polynomial which the degree $E+1$ rule integrates exactly. Furthermore, suppose that $\rho = \phi = 0$, and $\mu = \mathcal{Y}_{\cdot,D}$. Then, at these parameters, two times the negative log likelihood is given by:

$$TD(D-1)\log\kappa + 2D\sum_{t=1}^{T}\log\sigma_t + \sum_{t=1}^{T}\sum_{d=1}^{E}\frac{(\mathcal{Y}_{t,d} - \mathcal{Y}_{t,D})^2}{\sigma_t^2\kappa^{2(d-1)}} + TD\log(2\pi).$$

The σ minimising this expression satisfies the following for all $t \in \{1,\dots,T\}$:

$$\sigma_t^2 = \frac{1}{D}\sum_{d=1}^{E}\frac{(\mathcal{Y}_{t,d} - \mathcal{Y}_{t,D})^2}{\kappa^{2(d-1)}}.$$

With this σ, twice the negative log likelihood equals:

$$TD(D-1)\log\kappa + D\sum_{t=1}^{T}\log\left[\frac{1}{D}\sum_{d=1}^{E}\frac{(\mathcal{Y}_{t,d} - \mathcal{Y}_{t,D})^2}{\kappa^{2(d-1)}}\right] + TD(1+\log(2\pi))$$
$$\leq TD(D-1)\log\kappa + TD\log\left[\kappa^{-2(E-1)}\frac{E}{D}\max\left\{(\mathcal{Y}_{t,d} - \mathcal{Y}_{t,D})^2 \middle| t \in \{1,\dots,T\}, d \in \{1,\dots,E\}\right\}\right] + TD(1+\log(2\pi))$$
$$= TD[(D-1) - 2(E-1)]\log\kappa + TD\log\left[\frac{E}{D}\max\left\{(\mathcal{Y}_{t,d} - \mathcal{Y}_{t,D})^2 \middle| t \in \{1,\dots,T\}, d \in \{1,\dots,E\}\right\}\right]$$
$$+ TD(1+\log(2\pi)).$$

Providing $D \geq 2E$, as $\kappa \to 0$, this last expression tends to $-\infty$, so the log likelihood must tend to ∞. Therefore, when $D \geq 2E$, $\kappa = \rho = \phi = 0$, $\mu = \mathcal{Y}_{\cdot,D}$ maximises the log-likelihood, implying that the statistical smoothing procedure will return the exact integral of the integrand when used without a prior on κ. When there is a prior on κ, the same result will hold for some sufficiently high D, providing that the prior has support on $(0,1)$ and providing that its log density is greater than $\hat{\alpha}\log\kappa$ in a neighbourhood of 0, for some $\hat{\alpha} \geq 0$. The prior given above satisfies this condition with $\hat{\alpha} = \alpha$, so it will integrate exactly in the setting above providing $D > 2E - 1 + \frac{2\alpha}{TD}$.