Existence and uniqueness of solutions to dynamic models with occasionally binding constraints.

Tom D. Holden[1], *School of Economics, University of Surrey*

Abstract: We present the first necessary and sufficient conditions for there to be a unique perfect-foresight solution to an otherwise linear dynamic model with occasionally binding constraints, given a fixed terminal condition. We derive further conditions on the existence of a solution in such models. These results give determinacy conditions for models with occasionally binding constraints, much as Blanchard and Kahn (1980) did for linear models. In an application, we show that widely used New Keynesian models with endogenous states possess multiple perfect foresight equilibrium paths when there is a zero lower bound on nominal interest rates, even when agents believe that the central bank will eventually attain its long-run, positive inflation target. This illustrates that a credible long-run inflation target does not render the Taylor principle sufficient for determinacy in the presence of the zero lower bound. However, we show that price level targeting does restore determinacy in these situations.

Keywords: *occasionally binding constraints, zero lower bound, existence, uniqueness, price targeting, Taylor principle, linear complementarity problem*

JEL Classification: *C62, E3, E4, E5*

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The latest version of this paper may be downloaded from:
https://github.com/tholden/dynareOBC/raw/master/TheoryPaper.pdf

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[1] Contact address: School of Economics, FASS, University of Surrey, Guildford, GU2 7XH, England
Contact email: thomas.holden@gmail.com Web-site: http://www.tholden.org/
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1. Introduction

Since the financial crisis of 2007-2008, many central banks around the world have chosen to keep their nominal interest rate close to 0%. While in a few countries, rates on some assets have gone slightly negative, central banks are unable to push their target rate to the level a Taylor rule might suggest since agents always have the option of holding cash. In practice then, central banks face a zero lower bound (ZLB) on their policy rate, which limits their ability to provide stimulus in severe recessions. Furthermore, during the crisis, both households, firms and banks have hit their borrowing constraints, which has limited their ability to smooth out its effects. However, the theoretical results on determinacy which justify the Taylor principle do not apply to models with occasionally binding constraints (OBCs), such as the zero lower bound, or a borrowing constraint, meaning that the profession still lacks all of the necessary tools for understanding the behaviour of such models.

In this paper, we develop theoretical tools for understanding the behaviour of otherwise linear models with occasionally binding constraints.[2] Much as the seminal paper of Blanchard and Kahn (1980) provided necessary and sufficient conditions for the existence of a unique perfect foresight solution to a linear model that returns to a given steady-state, we will provide the first necessary and sufficient conditions for there to be a unique perfect foresight solution, returning to a given steady-state, in otherwise linear models with occasionally binding constraints. We further show that these conditions are not satisfied in standard New Keynesian (NK) models.

We will also provide both necessary conditions and sufficient conditions for there to exist any such solutions. When no solution always returning to the given steady-state exists, as we show to be the case in standard NK models, this implies that the model must converge to some alternative steady-state. We note that while in the fully linear case, rational expectations and perfect-foresight solutions coincide, in the otherwise linear case considered here, this will not be the case. However, since under quite mild assumptions, there are weakly more solutions under rational expectations than under perfect foresight (as proven in online appendix A), our results will imply lower bounds on the number of solutions under rational expectations.

As was observed by Benhabib, Schmitt-Grohé, and Uribe (2001a; 2001b), in the presence of OBCs, there are often multiple steady-states. For example, a model with a zero lower bound on nominal interest rates and Taylor rule monetary policy when away from the bound will have an additional "bad" deflationary steady-state in which nominal interest rates are zero. The presence of such multiple steady-states means that there can be sunspot equilibria which jump between the neighbourhoods of the two

[2] A companion paper (Holden 2016), develops computational tools for understanding the same thing.

steady-states. Furthermore, if agents put a positive probability on being in the neighbourhood of the "bad" steady-state in future, then since this "bad" steady-state is indeterminate, then by a backwards induction argument, there is indeterminacy now. The consequences of indeterminacy of these kinds have been explored by Schmitt-Grohé and Uribe (2012), Mertens and Ravn (2014) and Aruoba, Cuba-Borda, and Schorfheide (2014), amongst others. In all cases, the key to generating indeterminacy is that agents' beliefs about the point to which the economy would converge without future uncertainty are switching from one steady-state to the other.

However, the central banks of most major economies have announced (positive) inflation targets. Thus, convergence to a deflationary steady-state would represent a spectacular failure to hit the target. As argued by Christiano and Eichenbaum (2012), a central bank may rule out the deflationary equilibria in practice by switching to a money growth rule following severe deflation, along the lines of Christiano and Rostagno (2001). Furthermore, Richter and Throckmorton (2015) and Appendix B of Gavin et al. (2015) present evidence that the deflationary equilibrium is unstable[3] under rational expectations if shocks are large enough, making it much harder for agents to coordinate upon it. This suggests that at least in the long-run, agents ought to believe that we will return to the vicinity of the inflation target, and they ought to place zero probability on paths converging to deflation. Such beliefs also appear to be in line with the empirical evidence of Gürkaynak, Levin, and Swanson (2010). If agents' beliefs satisfy these restrictions, then the kinds of multiplicity studied by the authors cited in the previous paragraph are ruled out. It is an important question, then, whether there are still multiple equilibria even when all agents believe that in the long-run, the economy will return to a particular steady-state.

It is on such equilibria that we focus on in this paper, providing necessary and sufficient conditions for the existence of a unique perfect-foresight path, and also examining whether it is actually consistent with rationality for agents to believe that the economy will eventually return to the given steady state. In an application, we show that many otherwise linear New Keynesian models featuring endogenous state variables (e.g. price dispersion), do not possess a unique perfect-foresight path. This means that even when agents' long-run expectations are pinned down, there is still multiplicity of equilibria. Thus, the Taylor principle is not sufficient for determinacy in the presence of occasionally binding constrains. Indeed, we show that in these models, there are some initial states from which the economy has one return path that never hits the ZLB, and another that does hit it, so there may be multiplicity even when away from the bound. However, we show that under a price-targeting regime, there is a unique equilibrium path even when we impose the ZLB. Thus, if policy makers are

[3] In particular, they show that policy function iteration is not stable in the vicinity of the deflationary equilibria.

convinced by the arguments for the Taylor principle, then, given they face the zero lower bound, they ought to consider adopting a price level target.

Our paper also provides both necessary and sufficient conditions for the existence of any perfect-foresight solutions which return to the original ("good") steady-state. These results include both global (non-)existence results (i.e. ones independent of the initial state and/or shock realisation), and particular (non-)existence results (i.e. ones that are conditional on the current realisations). When no perfect-foresight equilibrium returning to the "good" steady-state exists, agents must switch their beliefs to the other ("bad") steady-state, where they will remain in the absence of any way for agents to coordinate back on the "good" steady-state.

We show that for standard New Keynesian models with endogenous state variables, there is a positive probability of ending up in a state of the world (i.e. with certain state variables and shock realisations) in which there is no perfect foresight path returning to the "good" steady-state. Furthermore, if we suppose that in the stochastic model, agents deal with uncertainty by integrating over the space of possible future shock sequences, as in the original stochastic extended path algorithm of Adjemian and Juillard (2013),[4] then such agents would always put positive probability on tending to the "bad" steady-state. Since the second steady-state is indeterminate in NK models, then this implies global indeterminacy by a backwards induction argument. Once again though, price level targeting would be sufficient to restore determinacy.

1.1. Further related literature

A growing literature has already looked at equilibrium non-existence or multiplicity in New Keynesian models subject to the ZLB, even aside from the literature started by Benhabib, Schmitt-Grohé, and Uribe (2001a; 2001b) and discussed previously. We will first discuss prior work on non-existence, before considering existing results on multiplicity. We conclude this section by arguing that our approach here is a conservative one that makes it as unlikely as possible that we should find multiplicity. That we still do find multiplicity in standard models is thus all the more surprising.

Several papers have found that New Keynesian models with a zero lower bound might have no solution at all if the variance of shocks is too high. Mendes (2011) derived analytic results on existence as a function of the variance of a demand shock, and Basu and Bundick (2015) showed the potential quantitative relevance of such results. Furthermore, conditions for the existence of an equilibrium in a simple NK model with discretionary monetary policy are derived in close form for a model with a two-state Markov shock by Nakata and Schmidt (2014). The conditions imply that

[4] Strictly, this is not fully rational, as it is equivalent to assuming that agents act as if the uncertainty in all future periods would be resolved next period. However, in practice this appears to be a close approximation to full rationality, as demonstrated by Holden (2016). The authors of the original stochastic path method now have a more complicated version that is fully consistent with rationality (Adjemian and Juillard 2016).

the economy must spend a small amount of time in the bad state for the equilibrium to exist, which again links existence to variance.

Our results will not be directly related to the variance of shocks, as we work under perfect foresight. Nonetheless, our theoretical results under perfect foresight may help explain some of the prior results in the stochastic case. We will show that whether or not a perfect foresight solution exists will depend on the perfect-foresight path taken by nominal interest rates in the absence of the bound. Throughout, we will assume that this path is arbitrary, as there is always some information about future shocks that could be revealed today to produce a given path. However, in a model with a small number of shocks, all of bounded support, and no information about future shocks, clearly not all paths are possible for nominal interest rates in the absence of the bound. The more shocks are added (e.g. news shocks), and the wider their support, the greater will be the support of the space of possible paths for nominal interest rates in the absence of the ZLB, and hence, the more likely will be non-existence of a solution for a positive measure of paths. This gives new intuition for the prior results.

There has also been some prior work by Richter and Throckmorton (2015) and Gavin et al. (2015; Appendix B) that has related a kind of eductive stability (the convergence of policy function iteration) to other properties of the model. Non-convergence of policy function iteration is suggestive of non-existence, though not definitive evidence. While the procedure of the cited authors has the advantage of working with the fully non-linear model under rational expectations, this limitation means that it cannot directly address the question of existence. By contrast, our results are theoretical and directly address existence. Thus, both procedures should be viewed as complementary; while ours definitively answers the question of existence in the slightly limited world of perfect foresight, otherwise linear models, the Richter and Throckmorton results give answers on stability in a richer setting.

Another approach to establishing the existence of an equilibrium is to produce it to satisfactory accuracy, by solving the model in some way. Under perfect foresight, the procedure outlined in this paper's companion is a possibility (Holden 2016), and the method of Guerrieri and Iacoviello (2012) is a prominent alternative. Under rational expectations, policy function iteration methods have been used by Fernández-Villaverde et al. (2015) and Richter and Throckmorton (2015), amongst others. However, this approach cannot generally establish non-existence or prove uniqueness. As such it is of little use to the policy maker who wants policy guidance to ensure existence and/or uniqueness. Furthermore, if the problem is solved globally, one cannot in general rule out that there is not an area of non-existence outside of the grid on which the model was solved. Similarly, if the model is solved under perfect foresight for a given initial state, then the fact that a solution exists for that initial point gives no guarantees that a solution should exist for other initial points. Thus it is essential to produce more general results on global existence, as we do here.

Assorted papers have also looked at multiplicity of equilibria in models containing a zero lower bound. This of course started with Benhabib, Schmitt-Grohé, and Uribe (2001a; 2001b) as already discussed, but there are several other key threads to the literature that are related to our work. One branch that is only tangentially related and so will not be extensively discussed looks at Markov switching policy rules. Key papers include Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2010; 2011). While this literature has been able to derive tight theoretical results, the usually assumed exogeneity of the switching means that the model's shocks cannot drive the model into the ZLB state, limiting its application to the present context. Some determinacy results with endogenous switching were derived by Marx and Barthelemy (2013), but unfortunately they only apply to forward looking models that are sufficiently close to ones with exogenous switching, and there is no reason to think that a standard New Keynesian model with a ZLB should satisfy this property.

Perfect foresight equilibria of New Keynesian models with terminal conditions were also examined by Brendon, Paustian, and Yates (2013; 2016), henceforth abbreviated to BPY. In BPY (2013), the authors show analytically that in a very simple NK model, featuring a response to the growth rate in the Taylor rule, there are multiple perfect-foresight equilibria when all agents believe that with probability one, in one period's time, they will escape the bound and return to the neighbourhood of the "good" steady-state. Furthermore, in the aforementioned paper, and in BPY (2016), the authors show numerically that in some select other models, there are multiple perfect-foresight equilibria when the economy begins at the steady-state, and all agents believe that the economy will jump to the bound, remain there for some weakly positive number of periods, before leaving it endogenously, after which they believe they will never hit the bound again.

Relative to these authors, we will provide far more general theoretical results, and these will permit numerical analysis that is both more robust and less restrictive. This robustness and generality will prove crucial in showing multiplicity even in simple NK models, with entirely standard Taylor rules. For example, whereas BPY (2016) write that price-dispersion "does not have a strong enough impact on equilibrium allocations for the sort of propagation that we need", we will show that in fact the presence of price dispersion is sufficient for multiplicity. Likewise, whereas BPY (2013; 2016) find a much weaker role for multiplicity when the monetary rule does not include a response to the growth rate of output, our findings of multiplicity will not be at all dependent on such a response, implying very different policy prescriptions.

In other related work, Armenter (2016) shows that in a simple otherwise linear New Keynesian model with a ZLB, if the central bank pursues Markov (discretionary) policy subject to an objective targeting inflation, nominal GDP or the price level, then there are multiple equilibria quite generally. The existence of multiple discretionary equilibria in models of monetary policy without a ZLB is well established (see e.g. Albanesi, Chari, and Christiano (2003) and the other papers cited by Armenter (2016)), but Armenter shows that even in a model that would have a unique equilibrium in the absence of the ZLB, its introduction produces additional Markov equilibria. By contrast, we will find that adopting a Taylor rule including a term in the price level leads to there being a unique perfect-foresight path returning to the inflationary steady-state. This difference between our results and those of Armenter (2016) is driven both by our assumption of commitment to a rule, and by the fact that we rule out getting stuck in the neighbourhood of the deflationary steady-state by assumption.

In further related results, Braun, Körber, and Waki (2012) show that there may be multiple perfect-foresight solutions to a non-linear New-Keynesian model, converging to the non-deflationary steady-state. However, it turns out that the linearized version of their model has a unique equilibrium, even when the ZLB is imposed. Thus, the multiplicity we find is strictly in addition to the multiplicity found by those authors. While the theoretical and computational methods used by Braun, Körber, and Waki (2012) have the great advantage that they can cope with fully non-linear models, it appears that they cannot cope with endogenous state variables, which limits their applicability. By producing tools for analysing otherwise linear models including state variables, our tools and results provide a complement to those of Braun, Körber, and Waki (2012). Evidence of their continued relevance in a non-linear setting is provided by the fact that the multiplicity found in a simple linearized model by Brendon, Paustian, and Yates (2013) (and by us in this paper) is also found in the equivalent non-linear model by Brendon, Paustian, and Yates (2016).

Of course, ideally we would have liked to analyse models with other nonlinearities apart from the occasionally binding constraint(s). Linearization can artificially exclude equilibria (see e.g. Bodenstein (2010)), and, as just mentioned, Braun, Körber, and Waki (2012) show examples of linearization excluding equilibria in the presence of the ZLB. Nonetheless, we maintain that studying multiplicity in otherwise linear models is still an important exercise. Firstly, macroeconomists have long relied on existence and uniqueness results based on linearization of models without occasionally binding constraints, despite the fact that this may produce spurious uniqueness in some circumstances. Secondly, it is nearly impossible to find all perfect foresight solutions in general non-linear models, since this is equivalent to finding all of the solutions to a huge system of non-linear equations, when even finding all of the solutions to large systems of quadratic equations is computationally intractable. At least if we have the full set of solutions to the otherwise linear model, we may use homotopy continuation methods to map these solutions into solutions of the non-linear model. Furthermore, finding all solutions under uncertainty is at least as difficult in general, as the policy function is also defined by a large system of non-linear equations. Thirdly, Christiano and Eichenbaum (2012) argue that e-learnability considerations render the additional equilibria of Braun, Körber, and Waki (2012) as merely "mathematical curiosities", suggesting that the equilibria that exist in the linearized model are of independent interest, whatever one's view on this debate. Finally, our main results for New-Keynesian models will imply non-uniqueness, so concerns of spurious uniqueness under linearization will not be relevant in these cases.

From the preceding discussion, we see that our choice to focus on otherwise-linear models under perfect-foresight, with fixed terminal conditions, may bias our results in favour of uniqueness for four distinct reasons. Firstly, because there are potentially more solutions under rational expectations than under perfect-foresight; secondly, because there are potentially other solutions returning to alternate steady-states; thirdly, because the original fully non-linear model may possess yet more solutions; and fourthly because there may be further equilibria under discretionary policy. This means that our results on the multiplicity of solutions to New Keynesian models are all the more surprising, and that it is all the more likely that multiplicity of equilibria is an important factor in explaining actual economies' spells at the zero lower bound.

1.2. Intuition for the multiplicity mechanism

The key idea behind all of our proofs is that an OBC provides a source of endogenous news about the future. When a shock hits, driving the economy to the bound in some future periods, that tells us that in those future periods, the (lower) bounded variable will be higher that it would be otherwise.[5] Hence, any shock that causes the ZLB to be hit may be thought of as providing a source of endogenous news about future innovations to the monetary rule, of just the right magnitude needed to impose the ZLB. For example, if the magnitude of a productivity shock is such that in the absence of the ZLB, nominal interest rates would be negative a year after the original shock, then, in the presence of the ZLB, the shock is providing endogenous news that nominal interest rates will be higher than normal in one year's time.

Thinking in terms of endogenous news shocks also helps to provide intuition for the presence of multiple equilibria in these models. As a first step towards such intuition, consider a New Keynesian model in which learning about a future positive shock to nominal interest rates actually leads to lower rates in the period in which the shock

[5] The idea of imposing the zero lower bound by adding news shocks is also present in Holden (2010), Hebden et al. (2011), Holden and Paetz (2012) and Bodenstein et al. (2013). News shocks were introduced to the literature by Beaudry and Portier (2006).

arrives. This is true, for example, in the model of Smets and Wouters (2003) at the posterior mode, in part since that model features a strong response to the growth rate of output.[6] The mechanism there is as follows: since output growth must eventually be positive following the arrival of a contractionary shock, nominal interest rates will be relatively high after such a shock, compared to a world in which there was no response to growth rates. This pushes down future output and inflation, and hence depresses current output and inflation due to consumption smoothing and forward looking price setting. If the response to growth rates is large enough, then in the period of the shock, nominal interest rates will actually fall, in response to the contemporaneous fall in output and inflation.

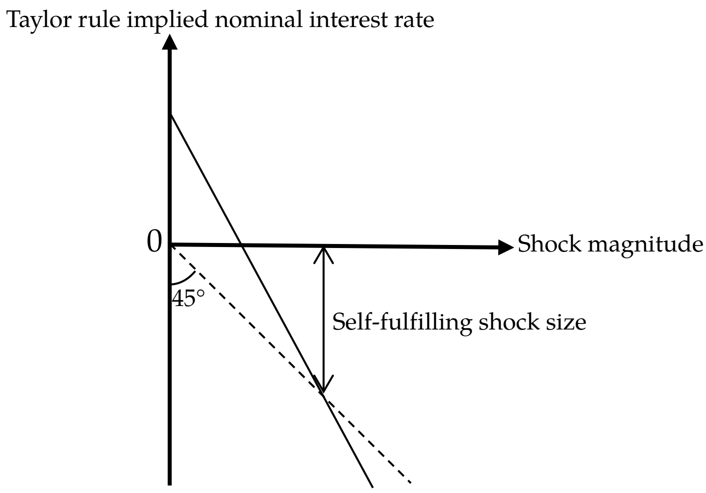


Figure 1: Self-fulfilling news shocks

Now, in such a model, there will be some magnitude of news shock to nominal interest rates today at which the news is of precisely the correct magnitude to bring the negative interest rates implied by the Taylor rule up to zero, in that period. A news shock of this magnitude thus becomes a self-fulfilling prophecy, as illustrated in Figure 1. If agents believed such a shock would occur, then Taylor rule implied interest rates would be negative in future, so such a shock would indeed "occur" in order to impose the ZLB.

A similar channel can produce multiplicity even in models without a response to growth rates in the Taylor rule, and even when news about positive shocks to the monetary rules always results in increases in interest rates in the period in which the shock arrives. The requirement is that there is some combination of future periods such that with appropriate positive shocks anticipated to arrive in each, interest rates are below steady-state in each of the given periods, allowing for a similar self-fulfilling prophecy. Informally, what is needed is that the impulse responses to positive news shocks to interest rates are sufficiently negative for a sufficiently high amount of time

[6] This is the mechanism stressed by Brendon, Paustian, and Yates (2013; 2016).

that a linear combination of them could be negative in every period in which a shock arrives. Since non-persistent shocks cannot have persistent effects without endogenous state variables, in practice endogenous state variables are likely to be necessary for multiplicity.[7]

To make this clearer, compare a standard Calvo (1983)-pricing New Keynesian model without capital or price indexation, but with positive trend inflation, to one with full indexation to steady-state inflation. More precisely, the models considered are that of Fernández-Villaverde et al. (2015) with the ZLB removed, and a version of the same model with indexation. Further details are given in section 3.3.

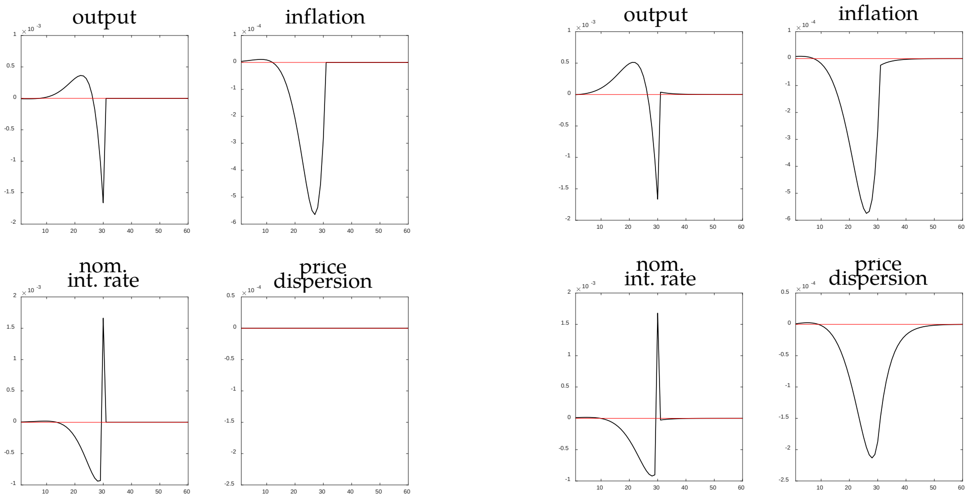


Figure 2: Impulse responses to a shock announced in period 1, but hitting in period 30, in basic New Keyenesian models with (left) and without (right) indexation to steady-state inflation.

All variables are in logarithms. In both cases, the model and parameters are taken from Fernández-Villaverde et al. (2015), the only change being the addition of complete price indexation to steady-state inflation for non-updating firms in the left hand plots.

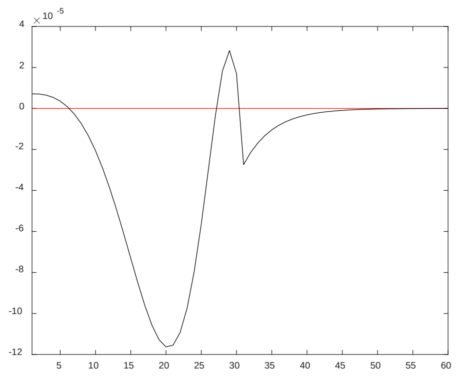


Figure 3: Difference between the impulse responses of nominal interest rates from the two models shown in Figure 2.

Negative values imply that nominal interest rates are lower in the model without indexation.

To a first order approximation, the model with full indexation never has any price dispersion, and thus has no endogenous state variables. In Figure 2 we plot the

[7] Esoteric examples of multiplicity without endogenous state variables may be constructed; one is given in online appendix B.

impulse responses of first order approximations to both models to a shock to nominal interest rates that is announced in period one but that does not hit until period thirty. For both models, the shape is similar, however, in the model without indexation, the presence of price dispersion reduces inflation both before and after the shock hits. This is because the predicted fall in inflation compresses the price distribution, reducing dispersion, and thus reducing the number of firms making very large adjustments. The fall in price dispersion also increases output, due to lower efficiency losses from miss-pricing. However, the effect on interest rates is dominated by the negative inflation effect, as the Taylor-rule coefficient on output cannot be too high if there is to be determinacy.[8] For reference, the difference between the IRFs of nominal interest rates in each model is plotted in Figure 3, which makes clear that interest rates are on average lower following the shock in the model without indexation.

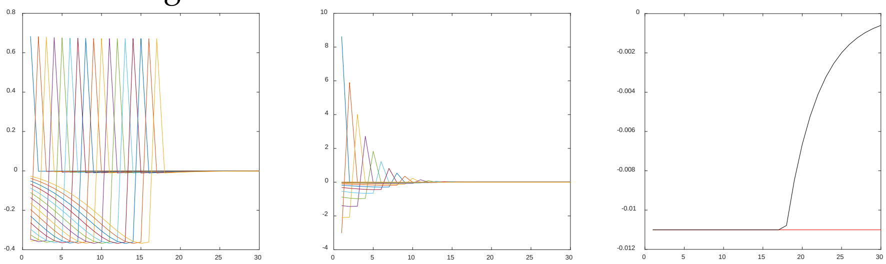


Figure 4: Construction of multiple equilibria in the basic NK model with price dispersion.
The left-hand plot shows the impulse responses to news shocks arriving zero to sixteen quarters after becoming known. The middle plot shows the same impulse responses scaled appropriately. The right-hand plot shows the sum of the scaled impulse responses shown in the central figure, where the red line gives the ZLB's location, relative to steady-state.

Remarkably, this small difference in the impulse responses between models is enough that the linearized model without indexation has multiple equilibria given a ZLB, but the linearized model with full indexation is determinate. We illustrate how multiplicity emerges in the model without indexation by showing, in Figure 4, the construction of an additional equilibrium which jumps to the ZLB for seventeen quarters.[9] If the economy is to be at the bound for seventeen quarters, then for those seventeen quarters, the nominal interest rate must be higher than it would be according to the Taylor rule, meaning that we need to consider seventeen endogenous news shocks, at horizons from zero to sixteen quarters into the future. The impulse responses to unit shocks of this kind are shown in the leftmost plot. Each impulse response has broadly the same shape as the one shown for nominal interest rates in the right of Figure 2. The central figure plots the same impulse responses again, but now each line is scaled by a constant so that their sum gives the line shown in black in

[8] One might think that the situation would be substantially different if the coefficient on output was high, so that the rise in output after the shock produced a rise in interest rates. However, as observed by Ascari and Ropele (2009), the determinacy region is much smaller in the presence of price dispersion than would be suggested by the standard Taylor criterion. Numerical experiments suggest that in all of the determinate region, interest rates are below steady-state following the shock.
[9] Seventeen quarters was the minimum span for which an equilibrium of this form could be found.

the rightmost plot. In this rightmost plot, the red line gives the ZLB's location, relative to steady-state, thus the combined impulse response spends seventeen quarters at the ZLB before returning to steady-state. Since there are only "news shocks" in the periods in which the economy is at the ZLB, this gives a perfect foresight rational expectations equilibrium which makes a self-fulfilling jump to the ZLB.

In richer New Keynesian models, both real and nominal rigidities help reduce the average value of the impulse response to a positive news shock to the monetary rule. Following the shock's arrival, they help ensure that the fall in output is persistent. Prior to the arrival, consumption smoothing (aided by internal habits) and capital or investment adjustment costs help produce a larger anticipatory recession. Given these mechanisms, we will see that multiplicity is the rule in NK models. Its absence in the basic model without price indexation is a knife-edge result, as evidenced by how small a change to impulse responses is sufficient to produce multiplicity.

1.3. Outline of the following

Our paper is structured as follows. In the following section, section 2, we present our key theoretical results on otherwise linear perfect foresight models. We then discuss the application of these results to New Keynesian models in section 3. Section 4 concludes. All files needed for the replication of this paper's numerical results are included in the "Examples" directory of the author's DynareOBC toolkit,[10] which implements an algorithm for simulating models with occasionally binding constraints that we discuss in a companion paper (Holden 2016), as well as checking the existence and uniqueness conditions that we will discuss here.

2. Theoretical results on occasionally binding constraints in otherwise linear models under perfect foresight

In this section, we present our main theoretical results on existence and uniqueness of perfect foresight solutions to models which are linear apart from an occasionally binding constraint. We start by defining the problem to be solved, and examining its relationship both to the problem without OBCs, and to a related problem with news shocks to the bounded variable. Using the news shock representation, we demonstrate that solving the model with OBCs is equivalent to solving a linear complementarity problem. We then discuss some theoretical background on these problems, before presenting the main existence and uniqueness results. Further results are contained in the appendix, with section 6.1 presenting additional results on the number and convexity of solutions, and section 6.2 giving further propositions relating our results to the properties of models solvable via dynamic programming. We conclude this section with a practical guide to checking the existence and uniqueness conditions.

[10] These files may be viewed online at https://github.com/tholden/dynareOBC/tree/master/Examples.

2.1. Problem set-ups

We start by describing the problem set-up without bounds. Suppose that for $t \in \mathbb{N}^+$, (i.e. $t \in \mathbb{N}$, $t > 0$), the first order conditions of some model may be represented as:

$$(\hat{A} + \hat{B} + \hat{C})\hat{\mu} = \hat{A}\hat{x}_{t-1} + \hat{B}\hat{x}_t + \hat{C}\mathbb{E}_t\hat{x}_{t+1} + \widehat{D}\varepsilon_t,$$

where $\hat{\mu} \in \mathbb{R}^{\hat{n}}$ and $\hat{x}_t \in \mathbb{R}^{\hat{n}}$, $\varepsilon_t \in \mathbb{R}^m$, $\mathbb{E}_{t-1}\varepsilon_t = 0$ for all $t \in \mathbb{N}^+$, and suppose that $\hat{x}_0$ is given as an initial condition. Throughout this paper, we will refer to first order conditions such as these as "the model", conflating them with the optimisation problem(s) which gave rise to them.

Furthermore, suppose that $\varepsilon_t = 0$ for $t > 1$, as in an impulse response or perfect foresight simulation exercise. Additionally, we assume the existence of a terminal condition of the form $\hat{x}_t \to \hat{\mu}$ as $t \to \infty$, coming, for example, from the source model's transversality constraints.

For $t \in \mathbb{N}^+$, define $x_t := \begin{bmatrix} \hat{x}_t \\ \varepsilon_{t+1} \end{bmatrix}$, $\mu := \begin{bmatrix} \hat{\mu} \\ 0 \end{bmatrix}$, $A := \begin{bmatrix} \hat{A} & \widehat{D} \\ 0 & 0 \end{bmatrix}$, $B := \begin{bmatrix} \hat{B} & 0 \\ 0 & I \end{bmatrix}$, $C := \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix}$, then, for $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1}, \tag{1}$$

and we have the extended initial condition $x_0 = \begin{bmatrix} \hat{x}_0 \\ \varepsilon_1 \end{bmatrix}$, and the extended terminal condition $x_t \to \mu$ as $t \to \infty$. Expectations have disappeared since there is no uncertainty after period 0. Thus, the problem of solving the original model has the same form as that given in:

Problem 1 Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$, equation (1) holds.

We make the following assumption in all of the following:

Assumption 1 For any given $x_0 \in \mathbb{R}^n$, Problem 1 has a unique solution, which takes the form $x_t = (I - F)\mu + Fx_{t-1}$, for $t \in \mathbb{N}^+$, where $F = -(B + CF)^{-1}A$, and where all of the eigenvalues of F are weakly inside the unit circle.

Sims's (2002) generalisation of the standard Blanchard-Kahn (1980) conditions is necessary and sufficient for this. Further, to avoid dealing specially with the knife-edge case of exact unit eigenvalues (even if they are constrained to the part of the model that is solved forward), in the following we rule it out with the subsequent assumption, which is, in any case, a necessary condition for perturbation to produce a consistent approximation to a source non-linear model, and which is also necessary for the linear model to have a unique steady-state:

Assumption 2 $\det(A + B + C) \neq 0$.

The combination of Assumption 1 and Assumption 2 imply that all of the eigenvalues of F are strictly inside the unit circle.

We are interested in models featuring occasionally binding constraints. We will concentrate on models featuring a single zero lower bound type constraint in their first equation, which we treat as defining the first element of x_t. Generalising from this special case to models with one or more fully general bounds is straightforward, and is discussed in the companion paper (Holden 2016). All of the results below go through in the more general case with minimal effort.

First, let us write $x_{1,t}$, $I_{1,\cdot}$, $A_{1,\cdot}$, $B_{1,\cdot}$, $C_{1,\cdot}$ for the first row of x_t, I, A, B, C (respectively) and $x_{-1,t}$, $I_{-1,\cdot}$, $A_{-1,\cdot}$, $B_{-1,\cdot}$, $C_{-1,\cdot}$ for the remainders. Likewise, we write $I_{\cdot,1}$ for the first column of I, and so on. Then, we are interested in the solution to:

***Problem* 2** Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $T \in \mathbb{N}$ and $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$x_{1,t} = \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu)\},$$

$$(A_{-1,\cdot} + B_{-1,\cdot} + C_{-1,\cdot})\mu = A_{-1,\cdot}x_{t-1} + B_{-1,\cdot}x_t + C_{-1,\cdot}x_{t+1},$$

and such that $x_{1,t} > 0$ for $t > T$.

Note that in this problem we are implicitly ruling out any solutions which get permanently stuck at an alternative steady-state, by assuming that the terminal condition remains as before. In the monetary policy context, this amounts to assuming that the central banks' inflation target is credible. Since $x_{1,t} \to \mu_1$ as $t \to \infty$, it is without loss of generality to assume the existence of a $T \in \mathbb{N}$ such that $x_{1,t} > 0$ for $t > T$, but this T will play an important role in the below, so we introduce it now. We continue to assume that there is no uncertainty after period 0, so, in this non-linear model, the path of the endogenous variables will not necessarily match up with the path of their expectation in a richer model in which there was uncertainty.

In many models, the occasionally binding constraint comes from the KKT conditions of an optimisation problem, which take the form $y \geq 0$, $\lambda \geq 0$ and $y\lambda = 0$. These may be converted into the max/min form since they are equivalent to the single equation $0 = \min\{y, \lambda\}$, which holds if and only if $y = \max\{0, y - \lambda\}$, which is in the form of that of Problem 2. Additionally, in the appendix, section 6.2, we give a more natural, alternative procedure for converting KKT conditions into a problem in the form of that Problem 2. The intuition is that one can use the model's equations to find the value the (lower) constrained variable would take were there no constraint and were the Lagrange multiplier on the constraint equal to zero today. This gives a "shadow" value of the constrained variable, and the actual value it takes will be the maximum of the bound and this shadow value.

We will analyse Problem 2 with the help of solutions to the auxiliary problem:

Problem 3 Suppose that $T \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^T$ is given. Find $x_t \in \mathbb{R}^n, y_t \in \mathbb{R}^T$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$, $y_t \to 0$, as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1} + I_{\cdot,1}y_{1,t-1},$$
$$y_{T,t} = 0, \qquad \forall i \in \{1, \dots, T-1\}, \qquad y_{i,t} = y_{i+1,t-1}.$$

This may be thought of as a version of Problem 1 with news shocks up to horizon T added to the first equation. The value of $y_{t,0}$ gives the news shock that hits in period t, i.e. $y_{1,t-1} = y_{t,0}$ for $t \le T$, and $y_{1,t-1} = 0$ for $t > T$.

2.2. Relationships between the problems

Since $y_{1,t-1} = 0$ for $t > T$, and using Assumption 1, $(x_{T+1} - \mu) = F(x_T - \mu)$, so with $t = T$, defining $s_{T+1} \coloneqq 0$, $(x_{t+1} - \mu) = s_{t+1} + F(x_t - \mu)$. Proceeding now by backwards induction on t, note that $0 = A(x_{t-1} - \mu) + B(x_t - \mu) + CF(x_t - \mu) + Cs_{t+1} + I_{\cdot,1}y_{t,0}$, so:

$$\begin{aligned}(x_t - \mu) &= -(B + CF)^{-1}[A(x_{t-1} - \mu) + Cs_{t+1} + I_{\cdot,1}y_{t,0}] \\ &= F(x_{t-1} - \mu) - (B + CF)^{-1}(Cs_{t+1} + I_{\cdot,1}y_{t,0}),\end{aligned}$$

i.e., if we define: $s_t \coloneqq -(B + CF)^{-1}(Cs_{t+1} + I_{\cdot,1}y_{t,0})$, then $(x_t - \mu) = s_t + F(x_{t-1} - \mu)$. By induction then, this holds for all $t \in \{1, \dots, T\}$.[11] Hence, we have proved the following lemma:

Lemma 1 There is a unique solution to Problem 3 that is linear in x_0 and y_0.

For future reference, let $x_t^{(3,k)}$ be the solution to Problem 3 when $x_0 = \mu$, $y_0 = I_{\cdot,k}$ (i.e. a vector which is all zeros apart from a 1 in position k). Then, by linearity, for arbitrary y_0 the solution to Problem 3 when $x_0 = \mu$ is given by:

$$x_t - \mu = \sum_{k=1}^{T} y_{k,0}\left(x_t^{(3,k)} - \mu\right).$$

Let $M \in \mathbb{R}^{T \times T}$ satisfy:

$$M_{t,k} = x_{1,t}^{(3,k)} - \mu_1, \qquad \forall t, k \in \{1, .., T\}, \tag{2}$$

i.e. M horizontally stacks the (column-vector) relative impulse responses to the news shocks. Then this result implies that for arbitrary y_0, the path of the first variable in the solution to Problem 3 when $x_0 = \mu$ is given by: $(x_{1,1:T})' = \mu_1 + My_0$, where $x_{1,1:T}$ is the row vector of the first T values of the first component of x_t. Furthermore, for both arbitrary x_0 and y_0, the path of the first variable in the solution to Problem 3 is given by: $(x_{1,1:T})' = q + My_0$, where $q \coloneqq \left(x_{1,1:T}^{(1)}\right)'$ and $x_t^{(1)}$ is the unique solution to Problem 1, for the given x_0.[12]

[11] This representation of the solution to Problem 3 was inspired by that of Anderson (2015).
[12] This representation was also exploited by Holden (2010) and Holden and Paetz (2012).

Now let $x_t^{(2)}$ be a solution to Problem 2 given an arbitrary x_0. Since $x_t^{(2)} \to \mu$ as $t \to \infty$, there exists $T' \in \mathbb{N}$ such that for all $t > T'$, $x_{1,t}^{(2)} > 0$. We assume without loss of generality that $T' \le T$. We seek to relate the solution to Problem 2 with the solution to Problem 3 for an appropriate choice of y_0. First, for all $t \in \mathbb{N}^+$, let:

$$e_t := \begin{cases} -\left[I_{1,\cdot}\mu + A_{1,\cdot}\left(x_{t-1}^{(2)} - \mu\right) + (B_{1,\cdot} + I_{1,\cdot})\left(x_t^{(2)} - \mu\right) + C_{1,\cdot}\left(x_{t+1}^{(2)} - \mu\right)\right] & \text{if } x_{1,t}^{(2)} = 0 \\ 0 & \text{if } x_{1,t}^{(2)} > 0 \end{cases}, \quad (3)$$

i.e. e_t is the shock that would need to hit the first equation for the positivity constraint on $x_{1,t}^{(2)}$ to be enforced. Note for future reference that by the definition of Problem 2, $e_t \ge 0$ and $x_{1,t}^{(2)} e_t = 0$, for all $t \in \mathbb{N}^+$. From this definition, we also have that for all $t \in \mathbb{N}^+$, $0 = A\left(x_{t-1}^{(2)} - \mu\right) + B\left(x_t^{(2)} - \mu\right) + C\left(x_{t+1}^{(2)} - \mu\right) + I_{\cdot,1} e_t$. Furthermore, if $t > T$, then $t > T'$, and hence $e_t = 0$. Hence, by Assumption 1, $\left(x_{T+1}^{(2)} - \mu\right) = F\left(x_T^{(2)} - \mu\right)$. Thus, much as before, with $t = T$, defining $\tilde{s}_{T+1} := 0$, $\left(x_{t+1}^{(2)} - \mu\right) = \tilde{s}_{t+1} + F\left(x_t^{(2)} - \mu\right)$. Consequently, $0 = A\left(x_{t-1}^{(2)} - \mu\right) + B\left(x_t^{(2)} - \mu\right) + CF\left(x_t^{(2)} - \mu\right) + C\tilde{s}_{t+1} + I_{\cdot,1} e_t$, so $\left(x_t^{(2)} - \mu\right) = F\left(x_{t-1}^{(2)} - \mu\right) - (B + CF)^{-1}\left(C\tilde{s}_{t+1} + I_{\cdot,1} e_t\right)$, i.e., if we define: $\tilde{s}_t := -(B + CF)^{-1}\left(C\tilde{s}_{t+1} + I_{\cdot,1} e_t\right)$, then $\left(x_t^{(2)} - \mu\right) = \tilde{s}_t + F\left(x_{t-1}^{(2)} - \mu\right)$. As before, by induction this must hold for all $t \in \{1, \dots, T\}$. By comparing the definitions of s_t and $\tilde{s}_t$, and the laws of motion of x_t under both problems, we then immediately have that if Problem 3 is started with $x_0 = x_0^{(2)}$ and $y_0 = e_{1:T}'$, then $x_t^{(2)}$ solves Problem 3. Conversely, if $x_t^{(2)}$ solves Problem 3 for some y_0, then from the laws of motion of x_t under both problems it must be the case that $\tilde{s}_t = s_t$ for all $t \in \mathbb{N}$, and hence from the definitions of s_t and $\tilde{s}_t$, we have that $y_0 = e_{1:T}'$. This has established the following result:

***Lemma* 2** For any solution, $x_t^{(2)}$ to Problem 2:

1) With $e_{1:T}$ as defined in equation (3), $e_{1:T} \ge 0$, $x_{1,1:T}^{(2)} \ge 0$ and $x_{1,1:T}^{(2)} \circ e_{1:T} = 0$, where $\circ$ denotes the Hadamard (entry-wise) product.
2) $x_t^{(2)}$ is also the unique solution to Problem 3 with $x_0 = x_0^{(2)}$ and $y_0 = e_{1:T}'$.
3) If $x_t^{(2)}$ solves Problem 3 with $x_0 = x_0^{(2)}$ and with some y_0, then $y_0 = e_{1:T}'$.

However, to use the easy solution to Problem 3 to assist us in solving Problem 2 requires a slightly stronger result. Suppose that $y_0 \in \mathbb{R}^T$ is such that $y_0 \ge 0$, $x_{1,1:T}^{(3)} \circ y_0' = 0$ and $x_{1,t}^{(3)} \ge 0$ for all $t \in \mathbb{N}$, where $x_t^{(3)}$ is the unique solution to Problem 3 when started at x_0, y_0. We would like to prove that in this case $x_t^{(3)}$ must also be a solution to Problem 2. I.e., we must prove that for all $t \in \mathbb{N}^+$:

$$x_{1,t}^{(3)} = \max\left\{0, I_{1,\cdot}\mu + A_{1,\cdot}\left(x_{t-1}^{(3)} - \mu\right) + (B_{1,\cdot} + I_{1,\cdot})\left(x_t^{(3)} - \mu\right) + C_{1,\cdot}\left(x_{t+1}^{(3)} - \mu\right)\right\}, \quad (4)$$

$$(A_{-1,\cdot} + B_{-1,\cdot} + C_{-1,\cdot})\mu = A_{-1,\cdot} x_{t-1}^{(3)} + B_{-1,\cdot} x_t^{(3)} + C_{-1,\cdot} x_{t+1}^{(3)}.$$

By the definition of Problem 3, the latter equation must hold with equality, so there is nothing to prove there. Hence we just need to prove that equation (4) holds for all $t \in \mathbb{N}^+$. So let $t \in \mathbb{N}^+$. Now, if $x_{1,t}^{(3)} > 0$, then $y_{t,0} = 0$, by the complementary slackness type condition ($x_{1,1:T}^{(3)} \circ y_0' = 0$). Thus, from the definition of Problem 3:

$$x_{1,t}^{(3)} = I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}^{(3)} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t^{(3)} - \mu) + C_{1,\cdot}(x_{t+1}^{(3)} - \mu)$$
$$= \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1}^{(3)} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t^{(3)} - \mu) + C_{1,\cdot}(x_{t+1}^{(3)} - \mu)\},$$

as required. The only remaining case is that $x_{1,t}^{(3)} = 0$ (since $x_{1,t}^{(3)} \geq 0$ for all $t \in \mathbb{N}$, by assumption), which implies that:

$$x_{1,t}^{(3)} = 0 = A_{1,\cdot}(x_{t-1} - \mu) + B_{1,\cdot}(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) + y_{t,0}$$
$$= I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) + y_{t,0},$$

by the definition of Problem 3. Thus:

$$I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + (B_{1,\cdot} + I_{1,\cdot})(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu) = -y_{t,0} \leq 0,$$

where the inequality is an immediate consequence of another of our assumptions. Consequently, equation (4) holds in this case too. Together with Lemma 1, Lemma 2, and our representation of the solution of Problem 3, this completes the proof of the following proposition:

Proposition 1 The following hold:

1) Let $x_t^{(3)}$ be the unique solution to Problem 3 when initialized with some x_0, y_0. Then $x_t^{(3)}$ is a solution to Problem 2 when initialized with x_0 if and only if $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$ and $x_{1,t}^{(3)} \geq 0$ for all $t \in \mathbb{N}$ with $t > T$.
2) Let $x_t^{(2)}$ be any solution to Problem 2 when initialized with x_0. Then there exists a $y_0 \in \mathbb{R}^T$ such that $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$, such that $x_t^{(2)}$ is the unique solution to Problem 3 when initialized with x_0, y_0.

2.3. The linear complementarity representation

Proposition 1 establishes that providing we initially choose T sufficiently high, to find a solution to Problem 2, it is sufficient to solve the following problem instead:

Problem 4 Suppose $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given. Find $y \in \mathbb{R}^T$ such that $y \geq 0$, $y \circ (q + My) = 0$ and $q + My \geq 0$. We call this the **linear complementarity problem (LCP)** (q, M). (Cottle 2009)

These problems have been extensively studied, and so we can import results on the properties of LCPs to derive results on the properties of models with OBCs.

All of the results in the mathematical literature rest on properties of the matrix M, thus we would like to establish if the structure of our particular M implies it has any special properties. Unfortunately, we prove the following result in this paper's companion paper (Holden 2016), which implies that M has no general properties:

Proposition 2 For any matrix $\mathcal{M} \in \mathbb{R}^{T \times T}$, there exists a model in the form of Problem 2 with a number of state variables given by a quadratic in T, such that $M = \mathcal{M}$ for that model, where M is defined as in equation (2), and such that for all $\mathcal{q} \in \mathbb{R}^T$, there exists an initial state for which $q = \mathcal{q}$, where q is the path of the bounded variable when constraints are ignored. (Holden 2016)

We now introduce some definitions of matrix properties that are necessary for the statement of our key existence and uniqueness results. The ultimate properties of the solutions to the OBC model are determined by which of these matrix properties M possesses. In each case, we give the definitions in a constructive form which makes clear both how the property might be verified computationally, and the links between definitions. These are not necessarily in the form which is standard in the original literature, however. For both the original definitions, and the proofs of equivalence between the ones below and the originals, see Cottle, Pang, and Stone (2009a) and Xu (1993) (for the characterisation of sufficient matrices).

Definition 1 (Principal sub-matrix, Principal minor) For a matrix $M \in \mathbb{R}^{T\times T}$, the **principal sub-matrices** of M are the matrices:

$$\left\{ \left[M_{i,j}\right]_{i,j=k_1,\dots,k_S} \middle| S, k_1, \dots, k_S \in \{1, \dots, T\}, k_1 < k_2 < \dots < k_S \right\},$$

i.e. the **principal sub-matrices** of M are formed by deleting the same rows and columns. The **principal minors** of M are the collection of values:

$$\left\{ \det\left(\left[M_{i,j}\right]_{i,j=k_1,\dots,k_S} \right) \middle| S, k_1, \dots, k_S \in \{1, \dots, T\}, k_1 < k_2 < \dots < k_S \right\},$$

i.e. the **principal minors** of M are the determinants of M's principal sub-matrices.

Definition 2 ($P_{(0)}$-matrix) A matrix $M \in \mathbb{R}^{T\times T}$ is called a **P-matrix (P_0-matrix)** if the principal minors of M are all strictly (weakly) positive. *Note: for symmetric M, M is a $P_{(0)}$-matrix if and only if it is positive (semi-)definite.*

Definition 3 (General positive (semi-)definite) A matrix $M \in \mathbb{R}^{T\times T}$ is called **general positive (semi-)definite** if $M + M'$ is a P-matrix (P_0-matrix). *Note: if M is symmetric, then, M is general positive (semi-)definite if and only if it is positive (semi-)definite.*

For intuition on the relevance of these properties, recall that the definition of a linear complementarity problem (Problem 4) contained the complementary slackness type condition, $y \circ (q + My) = 0$. Equivalently then, $0 = y'(q + My) = y'q + y'My$. Now, if there is no multiplicity, $y'q$ is likely to be negative as the bound usually binds when q (the path in the absence of the bound) is negative. Thus, for the equation to be satisfied, $y'My = \frac{1}{2}y'(M + M')y$ should be positive, which certainly holds when M is general positive definite (so $M + M'$ is positive definite). More generally, y will usually have many zeros, since y is zero whenever the model is away from the bound. The remaining non-zero elements of y select a principal sub-matrix of M, which will be a P-matrix if M is a P-matrix. Since being a P-matrix is an alternative generalisation of positive-definiteness to non-symmetric matrices, this turns out to be sufficient for there to be a solution to the original equation.

We now return to further definitions.

Definition 4 (Sufficient matrices) Let $M \in \mathbb{R}^{T\times T}$. M is called **column sufficient** if M is a P$_0$-matrix, and for each principal sub-matrix $W := \left[M_{i,j}\right]_{i,j=k_1,\dots,k_S}$ of M, with zero determinant, and for each proper principal sub-matrix $\left[W_{i,j}\right]_{i,j=l_1,\dots,l_R}$ of W ($R < S$), with zero determinant, the columns of $\left[W_{i,j}\right]_{\substack{i=1,\dots,S\\ j=l_1,\dots,l_R}}$ do not form a basis for the column space of W.[13] M is called **row sufficient** if M' is column sufficient. M is called **sufficient** if it is column sufficient and row sufficient.

Definition 5 (S($_0$)-matrix) A matrix $M \in \mathbb{R}^{T\times T}$ is called an **S-matrix** (**S$_0$-matrix)** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \gg 0$ ($My \geq 0$).[14]

Definition 6 ((Strictly) Semi-monotone) A matrix $M \in \mathbb{R}^{T\times T}$ is called (**strictly) semi-monotone** if each of its principal sub-matrices is an **S$_0$-matrix** (**S-matrix)**.

Definition 7 ((Strictly) Copositive) A matrix $M \in \mathbb{R}^{T\times T}$ is called (**strictly) copositive** if $M + M'$ is (strictly) semi-monotone.[15]

Cottle, Pang, and Stone (2009a) note the following relationships between these classes (amongst others):

Lemma 3 The following hold:

1) All general positive semi-definite matrices are copositive and sufficient.
2) P$_0$ includes skew-symmetric matrices, general positive semi-definite matrices, sufficient matrices and P-matrices.
3) All P$_0$-matrices, and all copositive matrices are semi-monotone, and all P-matrices, and all strictly copositive matrices are strictly semi-monotone (and hence also S-matrices).

Additionally, from considering the 1×1 principal sub-matrices of M, we have the following restrictions on the diagonal of M:

Lemma 4 All general positive semi-definite, semi-monotone, sufficient, P$_0$ and copositive matrices have non-negative diagonals, and all general positive definite, strictly semi-monotone, P and strictly copositive matrices have positive diagonals.

For many macroeconomic models, this simple condition is sufficient to rule out membership of these matrix classes, as medium-scale DSGE models[16] with a ZLB frequently have negative elements on the diagonal of their M matrix, when T is large

[13] This may be checked via the singular value decomposition.

[14] These condition may be rewritten as $\sup\{\varsigma \in \mathbb{R} | \exists y \geq 0 \text{ s.t. } \forall t \in \{1,\dots,T\}, (My)_t \geq \varsigma \wedge y_t \leq 1\} > 0$, and $\sup\left\{\sum_{t=1}^{T} y_t \,\middle|\, y \geq 0, My \geq 0 \wedge \forall t \in \{1,\dots,T\}, y_t \leq 1\right\} > 0$, respectively. As linear-programming problems, these may be verified in time polynomial in T using the methods described in e.g. Roos, Terlaky, and Vial (2006). Alternatively, by Ville's theorem of the alternative (Cottle, Pang, and Stone 2009b), M is not an S$_0$-matrix if and only if $-M'$ is an S-matrix.

[15] Väliaho (1986) contains an alternative characterisation which avoids solving any linear programming problems.

[16] This applies, for example, to the Smets and Wouters (2003) model, as we will show in section 3.5.

enough. Thus, following the intuition of Figure 1, such models will satisfy the conditions to have multiple equilibria, though they will not be the only such models.

A common "intuition" is that in models without state variables, M must be both a P matrix, and an S matrix. In fact, this is not true. Indeed, there are even purely static models for which M is not in either of these classes. For example, in online appendix B, we construct a purely static model for which $M_{1:\infty,1:\infty} = -I_{\infty\times\infty}$, which is neither a P-matrix, nor an S-matrix, for any T.

2.4. Existence results

We start by considering necessary or sufficient conditions for the existence of a solution to a model with occasionally binding constraints. Ideally, we would like the solution to exist for any possible path the bounded variable might have taken in the future were there no OBC, i.e. for any possible q. To see this, note that under a perfect foresight exercise we are ignoring the fact that shocks might hit the economy in future. More properly, we ought to take future uncertainty into account. One way to do this would be to follow the original stochastic extended path approach of Adjemian and Juillard (2013) by drawing lots of samples of future shocks for periods $1, \dots, S$, and averaging over these draws. [17] However, in a linear model with shocks with unbounded support, providing at least one shock has an impact on a given variable, the distribution of future paths of that variable has positive support over the entirety of $\mathbb{R}^S$. Thus, ideally we would like M to be such that for any q, the linear complementarity problem (q, M) has a solution.

Definition 8 (Feasible LCP) Suppose $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T\times T}$ are given. The LCP (q, M) is called **feasible** if there exists $y \in \mathbb{R}^T$ such that $y \geq 0$ and $q + My \geq 0$.

By construction, if an LCP (q, M) has a solution, then it is feasible, i.e. being feasible is a necessary condition for existence. Checking feasibility is straightforward for any particular (q, M), since to find a feasible solution we just need to solve a standard linear programming problem, which is possible in an amount of time that is polynomial in T.

Note that if the LCP (q, M) is not feasible, then for any $\hat{q} \leq q$, if $y \geq 0$, then $\hat{q} + My \leq q + My < 0$ since (q, M) is not feasible, so the LCP $(\hat{q}, M)$ is also not feasible. Consequently, if there are any q for which the LCP is non-feasible, then there is a positive measure of such q. Thus, in a model in which q is uncertain, if there are some q for which the model has no solution satisfying the terminal condition, even with arbitrarily large T, then the model will have no solution satisfying the terminal condition with positive probability. Hence it is not consistent with rationality for

[17] See footnote 4 for caveats to this procedure.

agents to believe that our terminal condition is satisfied with certainty, so they would have to place some positive probability on getting stuck in an alternative steady-state.

The following proposition gives an easily verified necessary condition for the global existence of a solution to the model with occasionally binding constraints, given some fixed horizon T:

Proposition 3 The LCP (q, M) is feasible for all $q \in \mathbb{R}^T$ if and only if M is an S-matrix. (Cottle, Pang, and Stone 2009a) [18]

Of course, it may be the case that the M matrix is only an S-matrix when T is very large, so we must be careful in using this condition to imply non-existence of a solution. Furthermore, it may be the case that although there exists some $y \in \mathbb{R}^T$ with $y \geq 0$ such that $M_{1:T,1:T}y \gg 0$ (indexing the M matrix by its size for clarity), for any such y, $\inf_{t \in \mathbb{N}^+} M_{t,1:T}y < 0$, so for some $q \in \mathbb{R}^{\mathbb{N}^+}$, the infinite LCP $(q, M_{1:\infty,1:\infty})$ is not feasible under the additional restriction that $y_t = 0$ for $t > T$. Strictly, it is this infinite LCP which we ought to be solving, subject to the additional constraint that y has only finitely many non-zero elements, as implied by our terminal condition.

From Proposition 3, we immediately have the following result on feasibility of the infinite problem:

Corollary 1 The infinite LCP $(q, M_{1:\infty,1:\infty})$ is feasible if and only if:

$$\varsigma := \sup_{\substack{y \in [0,1]^{\mathbb{N}^+} \\ \exists T \in \mathbb{N} \text{ s.t. } \forall t > T, y_t = 0}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty}y > 0.$$

Consequently, if $\varsigma > 0$ then for every $q \in \mathbb{R}^{\mathbb{N}^+}$, for sufficiently large T, the finite problem $(q_{1:T}, M_{1:T,1:T})$ will be feasible, which is a sufficient condition for solvability. In order to evaluate this limit, we first need to derive constructive bounds on the M matrix for large T. We do this in the online appendix C, where we prove that the rows and columns of M are converging to 0 (with constructive bounds), and that the k^{th} diagonal of the M matrix is converging to the value $d_{1,k}$, to be defined (again with constructive bounds), where diagonals are indexed such that the principal diagonal is index 0, and indices increase as one moves up and to the right in the M matrix. To explain the origins of $d_{1,k}$ we note the following lemma proved in online appendix C:

Lemma 5 The (time-reversed) difference equation $A\hat{d}_{k+1} + B\hat{d}_k + C\hat{d}_{k-1} = 0$ for all $k \in \mathbb{N}^+$ has a unique solution satisfying the terminal condition $\hat{d}_k \to 0$ as $k \to \infty$, given by $\hat{d}_k = H\hat{d}_{k-1}$, for all $k \in \mathbb{N}^+$, for some H with eigenvalues in the unit circle.

---

[18] Most of the results on LCPs in both this and the following section are restatements of (assorted) results contained in Cottle, Pang, and Stone (2009a) and Väliaho (1986) (for the characterisation of "copositive-plus" matrices), and the reader is referred to those works for proofs and further references.

Then, we define $d_0 := -(AH + B + CF)^{-1} I_{\cdot,1}$, $d_k = H d_{k-1}$, for all $k \in \mathbb{N}^+$, and $d_{-t} = F d_{-(t-1)}$, for all $t \in \mathbb{N}^+$, so d_k follows the time reversed difference equation for positive indices, and the original difference equation for negative indices. This is opposite to what one might perhaps expect since time is increasing as one descends the rows of M, but diagonal indices are decreasing as one descends in M.

Using the resulting bounds on M, we can construct upper and lower bounds on ς, which are described in the following propositions, also proven in online appendix C:

Proposition 4 There exists $\underline{\varsigma}_T, \overline{\varsigma}_T \geq 0$, defined in the online appendix C, computable in time polynomial in T, such that $\underline{\varsigma}_T \leq \varsigma \leq \overline{\varsigma}_T$, and $|\underline{\varsigma}_T - \overline{\varsigma}_T| \to 0$ as $T \to \infty$.

This condition gives a simple test for feasibility with sufficiently large T. It also provides a test giving strong numerical evidence of non-feasibility, since if $\overline{\varsigma}_T = 0 +$ numerical error, then $\varsigma = 0$ is likely.

We now turn to sufficient conditions for existence of a solution for some finite T.

Proposition 5 The LCP (q, M) is solvable if it is feasible and, either:

1. M is row-sufficient, or,
2. M is copositive and for all non-singular principal sub-matrices W of M, all non-negative columns of W^{-1} possess a non-zero diagonal element.

(Cottle, Pang, and Stone 2009a; Väliaho 1986)

If either condition 1 or condition 2 of Proposition 5 is satisfied, then to check existence for any particular q, we only need to solve a linear programming problem to see if a solution exists for a particular q. As this may be substantially faster than solving the LCP, this may be helpful in practice.

Proposition 6 The LCP (q, M) is solvable for all $q \in \mathbb{R}^T$, if at least one of the following conditions holds:

1. M is an S-matrix, and either condition 1 or condition 2 of Proposition 5 are satisfied.
2. M is copositive with non-zero principal minors.
3. M is a P-matrix, a strictly copositive matrix or a strictly semi-monotone matrix.

(Cottle, Pang, and Stone 2009a)

If condition 1, 2 or 3 of Proposition 6 is satisfied, then we know that the LCP will always have a solution. Therefore, for any path of the bounded variable in the absence of the bound, we will also be able to solve the model when the bound is imposed. Monetary policy makers should always choose a policy rule that produces a model that satisfies one of these three conditions, if they can, since otherwise there is a positive probability that only solutions converging to the "bad" steady-state will exist for some values of state variables and shock realisations.

Ideally, we might have liked conditions for the existence of a solution that are both necessary and sufficient, but unfortunately at present no such conditions exist in full generality. However, in the special case of M matrices with nonnegative entries, we have the following result:

Proposition 7 If M is a matrix with nonnegative entries, then the LCP (q, M) is solvable for all $q \in \mathbb{R}^T$ if and only if M has a strictly positive diagonal.
(Cottle, Pang, and Stone 2009a)

2.5. Uniqueness results

While no fully general necessary and sufficient conditions have been derived for existence, such conditions are available for uniqueness, in particular:

Proposition 8 The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$, if and only if M is a P-matrix. If M is not a P-matrix, then the LCP (q, M) has multiple solutions for some q. (Samelson, Thrall, and Wesler 1958; Cottle, Pang, and Stone 2009a)

This proposition is the equivalent for models with OBCs of the key proposition of Blanchard and Kahn (1980). By testing whether our matrix M is a P-matrix we can immediately determine if the model possesses a unique solution no matter what the initial state is, and no matter what shocks (if any) are predicted to hit the model in future, for a fixed T. In our experience, this condition is satisfied in efficient models, such as models of irreversible investment, as one would expect, but is not generally satisfied in medium-scale NK models with a ZLB on nominal interest rates. Given that if M is a P-matrix, so too are all its principal sub-matrices, if we see that M is not a P-matrix for some T, then we know that with larger T it would also not be a P-matrix. Thus, if for some T, M is not a P-matrix, then we know that the model does not have a unique solution, even for arbitrarily large T. Alternatively, we can prove that with large T some M is not a P-matrix by using the analytic formula for the limit of its diagonal given in the previous section, i.e. $d_{0,1} = -I_{1,\cdot}(AH + B + CF)^{-1}I_{\cdot,1}$. If this value is negative, then we know that with sufficiently large T, M will not be a P-matrix.

Since some classes of models almost never possess a unique solution when at the zero lower bound, we might reasonably require a lesser condition, namely that at least when the solution to the model without a bound is a solution to the model with the bound, then it ought to be the unique solution. This is equivalent to requiring that when q is non-negative, the LCP (q, M) has a unique solution. Conditions for this are given in the following proposition:

Proposition 9 The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$ with $q \gg 0$ $(q \geq 0)$ if and only if M is (strictly) semi-monotone. (Cottle, Pang, and Stone 2009a)

Hence, by verifying that M is (strictly) semi-monotone, we can reassure ourselves that merely introducing the bound will not change the solution away from the bound. When this condition is violated, even when the economy is a long way from the bound, there may be solutions which jump to the bound. Again, since principal sub-matrices of (strictly) semi-monotone are (strictly) semi-monotone, a failure of (strict) semi-monotonicity for some T implies a failure for all larger T. Furthermore, if $d_{0,1} < 0$ then again for sufficiently large T, M cannot be semi-monotone.

Where there are multiple solutions, we might like to be able to select one via some objective function. This is particularly tractable when either the number of solutions is finite, or the solution set is convex. Conditions for this are given in the appendix, section 6.1.

2.6. Checking the existence and uniqueness conditions in practice

This section has presented a large number of results on existence and uniqueness, but the practical details of what one should test and in what order may still be somewhat unclear. Luckily, a lot of the decisions are automated by the author's DynareOBC toolkit, but we present a suggested testing procedure here in any case. This also serves to give an overview of our results and their limitations.

For checking feasibility and existence, the most powerful result is the combination of Corollary 1 and Proposition 4. If the lower bound from Proposition 4 is positive, then Corollary 1 guarantees that for all sufficiently high T, the LCP is always feasible. If further conditions are satisfied for a given T, (see Proposition 5 and Proposition 6) then this guarantees existence for that particular T. However, since the additional conditions are sufficient and not necessary, in practice it may not be worth checking them, since we have never encountered a problem without a solution that was nonetheless feasible. Finding a T for which Proposition 4 produces a positive lower bound on ς requires a bit of trial and error. In general, T will need to be big enough that the asymptotic approximation is accurate, which in turn usually requires T to be bigger than the time it takes for the model's dynamics to die out. However, if T is too large, numerical inaccuracies can dominate.

For checking non-existence, Corollary 1 and Proposition 4 can still be useful, though in this case they do not provide definitive proof of non-feasibility, due to numerical inaccuracies. For a particular T, we may test if M is not an S-matrix in time polynomial in T by solving a simple linear programming problem. If M is not an S-matrix, then by Proposition 3, there are some q for which there is no solution which finally escapes the bound after at most T periods. With T larger than the time it takes for the model's dynamics to die out, this provides further evidence of non-existence for arbitrarily large T. In any case, given that only having a solution that stays at the bound for 250 years is arguably as bad as having no solution at all, for medium scale models, we suggest to just check if M is an S-matrix with $T = 1000$.

For checking uniqueness vs multiplicity, it is important to remember that while we can prove uniqueness for a given finite T by proving that the M matrix is a P-matrix, once we have found one T for which M is not a P-matrix (so there are multiple solutions, by Proposition 8), we know the same is true for all higher T. Additionally, it is much easier to prove a matrix is not a P-matrix than to prove that it is, as the former just requires us to find one principal sub-matrix with negative determinant, while the latter requires us to check all such sub-matrices. As a result, in practice, checking that M is a P-matrix for large T may not be computationally feasible, though finding a counter-example usually is.

If we wish to prove multiplicity by finding a principal sub-matrix with negative determinant, it is often sensible to begin by checking the contiguous principal sub-matrices.[19] These correspond to a single spell at the ZLB which is natural given that impulse responses in DSGE models tend to be single peaked. This is so reliable a diagnostic (and so fast) that DynareOBC reports it automatically for all models. Additionally, DynareOBC always checks the sign of $d_{0,1}$, and a condition on the arguments of the M matrix's eigenvalues,[20] which can also quickly help rule out being a P-matrix. If none of these conditions are informative, then it is sensible to start by checking all the 2×2 principal sub-matrices, then the 3×3 ones, and so on. With T around the half-life of the model's dynamics, one of these conditions will usually produce a counter-example reasonably quickly. A similar search strategy can be used to rule out semi-monotonicity, implying multiplicity even when away from the bound, by Proposition 9.

3. Applications to New Keynesian models

Brendon, Paustian, and Yates (2013) (henceforth: BPY) consider multiple equilibria in a simple New Keynesian model with an output growth rate term in the Taylor rule. They show that with sufficiently large reaction to the growth rate, there can be multiple equilibria today, even when the policy rule used to form tomorrow's expectations is held fixed. This is equivalent to the existence of multiple equilibria even when $T = 1$. In the first subsection here, we give an alternative analytic proof of this using our results, and discuss the generalisation to higher T. We go on to consider a variant of the BPY model with price targeting, and show that it produces determinacy.

However, we do not want to give the impression that multiplicity and non-existence are only caused by the central bank responding to the growth rate, or that they are

[19] Some care must be taken though as checking the signs of determinants of large matrices is numerically unreliable.

[20] All of the eigenvalues of a $T \times T$ P-matrix have complex arguments in the interval $\left(-\pi + \frac{\pi}{T}, \pi - \frac{\pi}{T}\right)$, and all of the eigenvalues of a $T \times T$ P_0-matrix have complex arguments in the interval $\left[-\pi + \frac{\pi}{T}, \pi - \frac{\pi}{T}\right]$ (Fang 1989).

only a problem in carefully constructed theoretical examples. In subsection 3.4, we show that a standard NK model with positive steady-state inflation and a ZLB possesses multiple equilibria in some states, and no solutions in others, even with an entirely standard Taylor rule. We also show that here too price level targeting is sufficient to restore determinacy. Next, we show that these conclusions also carry through to the posterior-modes of the Smets and Wouters (2003; 2007) models. We conclude this section by discussing the economic significance of multiplicity, arguing that self-fulfilling recessions could well explain some aspects of recent economic outcomes in the US, Europe and Japan.

3.1. The simple Brendon, Paustian, and Yates (2013) (BPY) model

The equations of the simple Brendon, Paustian, and Yates (2013) model are as follows:

$$x_{i,t} = \max\{0,1-\beta+\alpha_{\Delta y}(x_{y,t}-x_{y,t-1})+\alpha_\pi x_{\pi,t}\},$$
$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}(x_{i,t}+\beta-1-\mathbb{E}_t x_{\pi,t+1}), \qquad x_{\pi,t} = \beta \mathbb{E}_t x_{\pi,t+1} + \gamma x_{y,t},$$

where $x_{i,t}$ is the nominal interest rate, $x_{y,t}$ is the deviation of output from steady-state, $x_{\pi,t}$ is the deviation of inflation from steady-state, and $\beta \in (0,1)$, $\gamma,\sigma,\alpha_{\Delta y} \in (0,\infty)$, $\alpha_\pi \in (1,\infty)$ are parameters. In online appendix D, we prove the following:

Proposition 10 The BPY model is in the form of Problem 2, and satisfies Assumptions 1 and 2. With $T=1$, $M<0$ ($M=0$) if and only if $\alpha_{\Delta y} > \sigma\alpha_\pi$ ($\alpha_{\Delta y} = \sigma\alpha_\pi$).

For a 1×1 matrix, checking the conditions from section 2.3 is trivial. In particular, we have that if $\alpha_{\Delta y} < \sigma\alpha_\pi$, M is a general positive definite, strictly semi-monotone, strictly co-positive, sufficient, P, S matrix; if $\alpha_{\Delta y} \le \sigma\alpha_\pi$, M is a general positive semi-definite, semi-monotone, co-positive, sufficient, P_0, S_0 matrix. Hence, when $T=1$, if $\alpha_{\Delta y} < \sigma\alpha_\pi$, the model has a unique solution for all q; if $\alpha_{\Delta y} \le \sigma\alpha_\pi$, the model has a unique solution whenever $q>0$, and at least one solution when $q=0$. When $\alpha_{\Delta y} > \sigma\alpha_\pi$, M is negative, and so for any positive q, there exists $y>0$ such that $q+My=0$, so the model has multiple solutions. I.e. there are solutions that jump to the bound, even when the nominal interest rate would be positive were there no bound at all.

We illustrate this by adding a shock to the Euler equation, and showing impulse responses for alternative solutions. In particular, we replace the Euler equation with:

$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}(x_{i,t}+\beta-1-\mathbb{E}_t x_{\pi,t+1}-(0.01)\varepsilon_t),$$

and take the parameterisation $\sigma=1$, $\beta=0.99$, $\gamma=\frac{(1-0.85)(1-\beta(0.85))}{0.85}(2+\sigma)$, following BPY, and we additionally set $\alpha_\pi=1.5$ and $\alpha_{\Delta y}=1.6$, to ensure we are in the region with multiple solutions. In Figure 5, we show two alternative solutions to the impulse response to a magnitude 1 shock to ε_t. The solid line in the left plot gives the solution which minimises $\|y\|_\infty$. This solution never hits the bound, and is moderately expansionary. The solid line in the right plot gives the solution which minimises $\|q + My\|_\infty$. (The dotted line in the right plot repeats the left plot, for comparison.) This solution stays at the bound for two periods, and is strongly contractionary, with a magnitude around 100 times larger than the other solution.

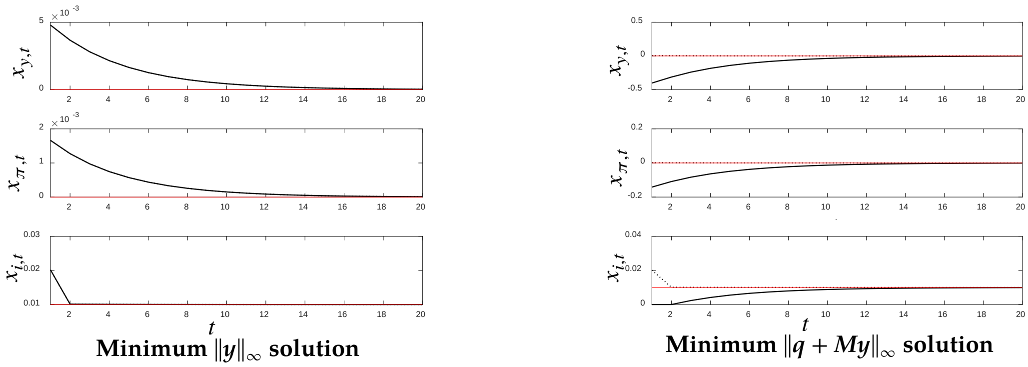


Figure 5: Alternative solutions following a magnitude 1 impulse to ε_t

When $T > 1$, the previous results imply that if $\alpha_{\Delta y} > \sigma\alpha_\pi$, then M is neither P_0, general positive semi-definite, semi-monotone, co-positive, nor sufficient, by Lemma 4, since the top-left 1×1 principal sub-matrix of M is the same as when $T = 1$. Thus, if anything, when $T > 1$, the parameter region in which there are multiple solutions (when away from the bound or at it) is larger. However, numerical experiments suggest that this parameter region in fact remains the same as T increases, which is unsurprising given the weak persistence of this model. Thus, if we want more interesting results with higher T, we need to consider a model with a stronger persistence mechanism. One obvious possibility is to consider models with either persistence in the interest rate, or persistence in the "shadow" rate that would hold were it not for the ZLB. However, perhaps unsurprisingly, in online appendix E we find that persistence in the shadow interest rate does not change the determinacy region providing T is large enough. Identical results can be shown for persistence in the actual interest rate.

3.2. The BPY model with price targeting

One way to introduce persistence to shadow interest rates is to set:

$$x_{d,t} = (1-\rho)\left(1 - \beta + \frac{\alpha_{\Delta y}}{1-\rho}\left(x_{y,t} - x_{y,t-1}\right) + \frac{\alpha_\pi}{1-\rho} x_{\pi,t}\right) + \rho x_{d,t-1}$$
$$= (1-\rho)(1-\beta) + \left(\alpha_{\Delta y}\left(x_{y,t} - x_{y,t-1}\right) + \alpha_\pi x_{\pi,t}\right) + \rho x_{d,t-1},$$

where $x_{i,t} = \max\{0, x_{d,t}\}$. If the second bracketed term was multiplied by $(1-\rho)$, then this would be entirely standard, however as written here, we have that in the limit as $\rho \to 1$, this tends to:

$$x_{d,t} = 1 - \beta + \alpha_{\Delta y} x_{y,t} + \alpha_\pi x_{p,t}$$

where $x_{p,t}$ is the price level, so $x_{\pi,t} = x_{p,t} - x_{p,t-1}$. This is a level targeting rule, with nominal GDP targeting as a special case with $\alpha_{\Delta y} = \alpha_\pi$. Note that the omission of the $(1-\rho)$ coefficient on $\alpha_{\Delta y}$ and α_π is akin to having a "true" response to output growth of $\frac{\alpha_{\Delta y}}{1-\rho}$ and a "true" response to inflation of $\frac{\alpha_\pi}{1-\rho}$, so in the limit as $\rho \to 1$, we effectively have an infinitely strong response to these quantities. It turns out that this is sufficient to produce determinacy for all $\alpha_{\Delta y}, \alpha_\pi \in (0,\infty)$.

In particular, given the model:

$$x_{i,t} = \max\{0, 1-\beta + \alpha_{\Delta y} x_{y,t} + \alpha_\pi x_{p,t}\},$$
$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}\left(x_{i,t} + \beta - 1 - \mathbb{E}_t x_{p,t+1} + x_{p,t}\right),$$
$$x_{p,t} - x_{p,t-1} = \beta \mathbb{E}_t x_{p,t+1} - \beta x_{p,t} + \gamma x_{y,t},$$

we prove in online appendix F that the following proposition holds:

Proposition 11 The BPY model with price targeting is in the form of Problem 2, and satisfies Assumptions 1 and 2. With $T = 1$, $M > 0$ for all $\alpha_\pi \in (0,\infty)$, $\alpha_{\Delta y} \in [0,\infty)$.

Furthermore, with $\sigma = 1$, $\beta = 0.99$, $\gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85}(2+\sigma)$, as before, and $\alpha_{\Delta y} = 1$, $\alpha_\pi = 1$, if we check our lower bound on ς with $T = 20$, we find that $\varsigma > 0.042$. Hence, this model is always feasible for any sufficiently large T. Given that $d_0 > 0$ for this model, and that for $T = 20$, M is a P-matrix, this is strongly suggestive of the existence of a unique solution for any q and for arbitrarily large T.

3.3. The linearized Fernández-Villaverde et al. (2015) model

The discussion of the BPY (2013) model might lead one to believe that multiplicity and non-existence is solely a consequence of overly aggressive monetary responses to output growth, and overly weak monetary responses to inflation. However, it turns out that in basic New Keynesian models with positive inflation in steady-state, and hence price dispersion, even without any monetary response to output growth, and even with extremely aggressive monetary responses to inflation, there are still multiple equilibria in some states of the world (i.e. for some q), and no solutions in others. Price level targeting is again sufficient to fix these problems though.

We show these results in the Fernández-Villaverde et al. (2015) model, which is a basic non-linear New Keynesian model without capital or price indexation of non-resetting firms, but featuring (non-valued) government spending and steady-state inflation (and hence price-dispersion). We refer the reader to the original paper for the model's equations. After substitutions, the model has four non-linear equations which are functions of gross inflation, labour supply, price dispersion and an auxiliary variable introduced from the firms' price-setting first order condition. Of these variables, only price dispersion enters with a lag. We linearize the model around its steady-state, and then reintroduce the "max" operator which linearization removed from the Taylor rule. [21] All parameters are set to the values given in Fernández-Villaverde et al. (2015). There is no term featuring output growth in the Taylor rule, so any multiplicity or non-existence in this model cannot be a consequence of the mechanism highlighted by BPY (2013).

For this model, numerical calculations reveal that with $T \le 14$, M is a P-matrix. However, with $T \ge 15$, M is not a P matrix, and thus there are certainly some states of the world (some q) in which the model has multiple solutions. One example was given in Figure 4. Furthermore, with $T = 1000$, our upper bound on ς from Proposition 4 implies that $\varsigma \le 0 + \text{numerical error}$, suggesting that M is not an S-matrix for arbitrarily large T, by Corollary 1. If this is correct, then even for arbitrarily large T, there are some q for which no solution exists.

However, if we replace inflation in the monetary rule with the price level relative to its linear trend, which evolves according to:

$$x_{p,t} = x_{p,t-1} + x_{\pi,t} - x_{\pi}, \tag{5}$$

then with $T = 200$, we have that M is an S-matrix, and the lower bound from Proposition 4 implies that $\varsigma > 0.003$, and hence that for all sufficiently large T, M is an S-matrix (by Corollary 1), so there is always a feasible solution. Furthermore, with $T = 20$, M is a P-matrix, and even with $T = 200$, M has no contiguous sub-matrices with negative determinant. This is strongly suggestive of uniqueness even for arbitrarily large T, given the relatively short lived dynamics of the model.

3.4. The Smets and Wouters (2003) and Smets and Wouters (2007) models

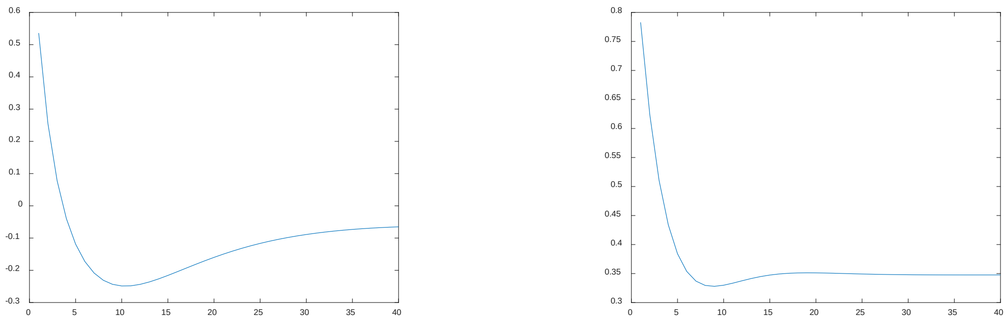
Smets and Wouters (2003) and Smets and Wouters (2007) are the canonical medium-scale linear DSGE models, featuring assorted shocks, habits, price and wage indexation, capital (with adjustment costs), (costly) variable utilisation and quite general monetary policy reaction functions. The former model is estimated on Euro area data, while the latter is estimated on US data. The latter model also contains trend growth (permitting its estimation on non-detrended data), and a slightly more general aggregator across industries. However, overall, they are quite similar models, and any differences in their behaviour chiefly stems from differences in the estimated parameters. Since both models are incredibly well known in the literature, we omit their equations here, referring the reader to the original papers for further details.

To assess the likelihood of multiple equilibria at or away from the zero lower bound, we augment each model with a ZLB on nominal interest rates, and evaluate the properties of each model's M matrix at the estimated posterior-modes from the

[21] Prior to linearization, we first transform the model's variables so that the transformed variables may take values on the entire real line. I.e. we work with the logarithms of labour supply, price dispersion and the auxiliary variable. For inflation, we note that inflation is always less than $\theta^{\frac{1}{1-\varepsilon}}$ (in the notation of Fernández-Villaverde et al. (2015)). Thus we work with a logit transformation of inflation over $\theta^{\frac{1}{1-\varepsilon}}$. This is generally more accurate than working with the logarithm of inflation.

original papers. Note that in order to minimise the deviation from the original papers, we do not introduce an auxiliary for shadow nominal interest rates, so the monetary rules take the form of $x_{r,t} = \max\{0, (1-\rho_r)(\cdots) + \rho_r x_{r,t-1} + \cdots\}$, in both cases. In any case, this seems more natural than the alternative, which can result in implausibly fast exits from the ZLB. However, our results would be essentially identical with a shadow nominal interest rate.

As shown in Lemma 4, if the diagonal of the M matrix ever goes negative, then the M matrix cannot be general positive semi-definite, semi-monotone, sufficient, P_0 or copositive, and hence the model will sometimes have multiple solutions even when away from the zero lower bound (i.e. for some strictly positive q), by Proposition 9. In Figure 6, we plot the diagonal of the M matrix for each model in turn,[22] i.e. the impact on nominal interest rates in period t of news in period 1 that a positive, magnitude one shock will hit nominal interest rates in period t. Immediately, we see that while in the US model, these impacts remain positive at all horizons, in the Euro area model, these impacts turn negative after just a few periods, and remain so at least up to period 40. Therefore, in the ZLB augmented Smets and Wouters (2003) model, there is not always a unique equilibrium. Furthermore, if a run of future shocks was drawn from a distribution with unbounded support, then the value of these shocks was revealed to the model's agents (as in the stochastic extended path), then there would be a positive probability that the model without the ZLB would always feature positive interest rates, but that the model with the ZLB could hit zero.



The Smets and Wouters (2003) model — The Smets and Wouters (2007) model

Figure 6: The diagonals of the M matrices for the Smets and Wouters (2003; 2007) models

It remains for us to assess whether M is a $P_{(0)}$-matrix or (strictly) semi-monotone for the Smets and Wouters (2007) model. Numerical calculations reveal that for $T < 9$, M is a P-matrix, and hence is strictly semi-monotone. However, with $T \geq 9$, the top-left 9×9 sub-matrix of M has negative determinant and is not an S or $S_{(0)}$ matrix. Thus, for $T \geq 9$, M is not a $P_{(0)}$-matrix or (strictly) semi-monotone, and hence this model also

[22] The MOD files for the Smets and Wouters (2003) model were derived from the Macro Model Database (Wieland et al. 2012). The MOD files for the Smets and Wouters (2007) model were derived from files provided by Johannes Pfeifer here: http://goo.gl/CP53x5

has multiple equilibria, even when away from the bound. Given that the US has been at the ZLB for over eight years, that T ought to be greater than eight quarters seems uncontroversial. While placing a larger coefficient on inflation in the Taylor rule can make the Euro area picture more like the US one, with a strictly positive diagonal to the M matrix, even with incredibly large coefficients, M remains a non-P-matrix for both models. Hence, in both the Euro area and the US, we ought to take seriously the possibility that the existence of the ZLB produces non-uniqueness.

As an example of such non-uniqueness, in Figure 7 we plot two different solutions following the most likely combination of shocks to the Smets and Wouters (2007) model that would produce negative interest rates for a year in the absence of a ZLB.[23] In both cases, the dotted line shows the response in the absence of the ZLB. Particularly notable is the flip in sign, since the shocks most likely to take the model to the ZLB for a year are expansionary ones reducing prices (i.e. positive productivity and negative mark-up shocks). The following section shows an example of multiplicity in the Smets and Wouters (2003) model, and discusses the economic relevance of such multiplicity.

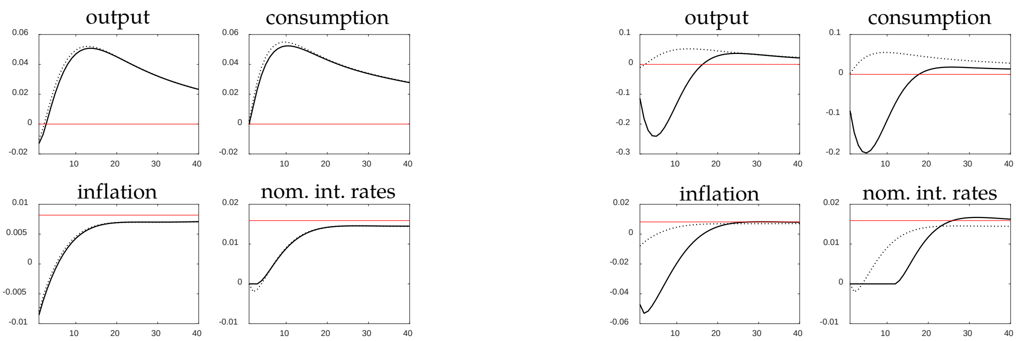


Figure 7: Two alternative solutions following a combination of shocks to the Smets and Wouters (2007) model
All variables are in logarithms. The precise combination of shocks is detailed in footnote 23.

In addition, it turns out that for neither model is M an S-matrix even with $T = 1000$, and thus for both models there are some $q \in \mathbb{R}^{1000}$ for which no solution exists. This is strongly suggestive of non-existence for some q even for arbitrarily large T, something that is reinforced by the fact that for the Smets and Wouters (2007) model, with $T = 1000$, Proposition 4 gives that $\varsigma \leq 0 +$ numerical error.

Alternatively, suppose we replace the monetary rule in both models by:

$$x_{r,t} = \max\{0, (1 - \rho_r)(x_{y,t} + x_{p,t}) + \rho_r x_{r,t-1}\}$$

where ρ_r is as in the respective original model, where the price level $x_{p,t}$ again evolves according to equation (16), and where $x_{y,t}$ is output relative to its linear trend. Then,

[23] I.e. the shock magnitudes are given by the vector w that minimises $w'w$ subject to $\bar{r} + Zw$, where $\bar{r}$ is the steady-state interest rate, and each column of Z gives the first four periods of the impulse response of interest rates to one of the model's shocks. This produces the following shock magnitudes: productivity, 3.56; risk premium, -2.70; government, -1.63; investment, -4.43; monetary, -2.81; price mark-up, -3.19; wage mark-up, -4.14.

for both models, with $T = 20$, M was a P-matrix, and there are no contiguous principal sub-matrices with negative determinant even for $T = 200$. Furthermore, from Proposition 4, with $T = 1000$, for the Euro area model we have that $\varsigma > 3 \times 10^{-7}$ and for the US model we have that $\varsigma > 0.002$, so Corollary 1 implies that a solution always exists to both models for sufficiently large T. As one would expect, this result is also robust to departures from equal, unit, coefficients. Thus, price level targeting again appears to be sufficient for determinacy in the presence of the ZLB.

3.5. Economic significance of multiplicity

There are two reasons why one might be sceptical about the economic significance of the multiple equilibria caused by the presence of the ZLB. Firstly, as with any non-fundamental equilibrium, the coordination of beliefs needed to sustain the equilibrium may be difficult. Secondly, as we have seen, self-fulfilling jumps to the zero lower bound may feature implausibly large falls in output and inflation. This reflects the implausibly large response to news about future policy innovations, a problem that has been termed the "forward guidance puzzle" in the literature (Carlstrom, Fuerst, and Paustian 2015; Del Negro, Giannoni, and Patterson 2015).[24]

However, if the economy is already in a recession, then both of these problems are substantially ameliorated. If interest rates are already low, then it does not seem too great a stretch to suggest that a drop in confidence may lead people to expect to hit the ZLB. Even more plausibly, if the economy is already at the ZLB, then small changes in confidence could easily select an equilibrium featuring a longer spell at the ZLB than in the equilibrium with the shortest time there. Indeed, there is no good reason why people should coordinate on the equilibrium with the shortest time at the ZLB. Furthermore, with interest rates already low, the size of the required self-fulfilling news shock is much smaller, meaning that the additional drop in output and inflation caused by a jump to the ZLB will be much more moderate.

As an example, in Figure 8 we plot the impulse response to a large magnitude preference shock (scaling utility), in the Smets and Wouters (2003) model.[25] The shock is not quite large enough to send the economy to the zero lower bound[26] in the

[24] McKay, Nakamura, and Steinsson (2016) point out that these implausibly large responses to news are muted in models with heterogeneous agents, and they give a simple "discounted Euler" approximation that produces similar results to a full heterogeneous agent model. However, while including a discounted Euler equation makes it harder to generate multiplicity (e.g. reducing the parameter space with multiplicity in the Brendon, Paustian, and Yates (2013) model), when there is multiplicity, the resulting responses are much larger, since the weaker response to news means that the required endogenous news shocks need to be much greater in order to drive the model to the bound.

[25] The shock is 22.5 standard deviations. While this is implausibly large, the economy could be driven to the bound with a run of much smaller shocks. It is also worth recalling that the model was estimated on the great moderation period, and so the estimated standard deviations may be too low. Finally, recent evidence (Cúrdia, del Negro, and Greenwald 2014) suggests that the shocks in DSGE models are better modelled as being fat tailed, making large shocks more likely.

[26] Since the Smets and Wouters (2003) model does not include trend growth, it is impossible to produce a steady-state value for nominal interest rates that is consistent with both the model and the data. We choose to follow the data, setting the steady-state

fundamental solution, shown with a dotted line. However, there is an alternative solution in which the economy jumps to the bound one period after the initial shock, remaining there for three periods. While the alternative solution features larger drops in output and inflation, they are less than twice the magnitude. Indeed, the falls are broadly in line with the magnitude of the crisis, with Eurozone GDP and consumption now being about 20% below a pre-crisis log-linear trend, and the largest drop in Eurozone consumption inflation from 2008q3 to 2008q4 being around 1%.[27] In light of this, we view it as plausible that multiplicity of equilibria could be a significant component of the explanation for the great recession.

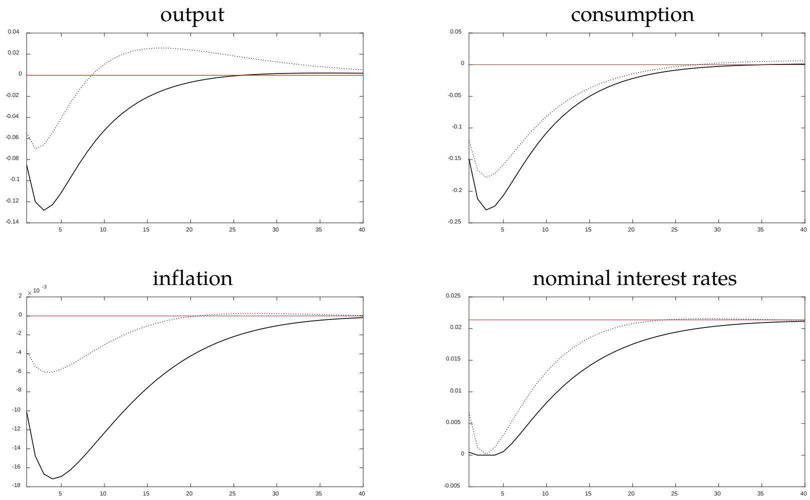


Figure 8: Two solutions following a preference shock in the Smets and Wouters (2003) model.
All variables are in logarithms.

4. Conclusion

This paper provides the first theoretical results on existence and uniqueness for otherwise linear models with occasionally binding constraints. As such, it may be thought of as doing for models with occasionally binding constraints what Blanchard and Kahn (1980) did for linear models.

We provided necessary and sufficient conditions for the existence of a unique equilibrium, as well as such conditions for uniqueness when away from the bound. In our application to New Keynesian models, we showed that these conditions were violated in entirely standard models, rather than just being a consequence of policy rules responding to growth rates. In the presence of multiplicity, there is the potential for additional endogenous volatility from sunspots, so the welfare benefits of avoiding multiplicity may be substantial. Additionally, as we saw in Figure 5, the additional

of nominal interest rates to its mean level over the same sample period used by Smets and Wouters (2003), using data from the same source (Fagan, Henry, and Mestre 2005).

[27] Data was again from the area-wide model database (Fagan, Henry, and Mestre 2005).

equilibria may feature huge drops in output, providing further welfare reasons for their avoidance. The possibility of self-fulfilling jumps and returns from the ZLB also gives an alternative rationale for the neo-Fisherian view that argues that raising interest rates may raise inflation at the ZLB.[28]

Luckily, our results suggest that a determinate equilibrium may be produced in standard New Keynesian models if the central bank switches to targeting the price level, rather than the inflation rate. There is of course a large literature advocating price level targeting already. Vestin (2006) made an important early contribution by showing that its history dependence mimics the optimal rule. This was built on by Eggertsson and Woodford (2003) who showed its particular desirability in the presence of the ZLB, since it produces inflation after the bound is escaped. A later contribution by Nakov (2008) showed that this result survived taking a fully global solution, and Coibion, Gorodnichenko, and Wieland (2012) showed that it still holds in a richer model. More recently, Basu and Bundick (2015) have argued that a response to the price level avoids the kinds of equilibrium non-existence problems stressed by Mendes (2011), while also solving the contractionary bias caused by the ZLB, drastically improving welfare. Our argument is distinct from all of these; we showed that in the presence of the ZLB, inflation targeting rules are indeterminate, whereas price level targeting rules produce determinacy. Consequently, if one believes the arguments for the Taylor principle in the absence of the ZLB, then one should advocate price level targeting if the ZLB constraint is inescapable.

In addition, we provided conditions for existence of any solution that converges to the "good" steady-state, and showed that under inflation targeting, standard New Keynesian models again failed to satisfy these conditions over all of the space of state variables and shocks. Whereas the literature started by Benhabib, Schmitt-Grohé, and Uribe (2001a; 2001b) showed that the existence of a "bad" steady-state may imply additional volatility if agents long-run beliefs are not pinned down by the inflation target, here we showed that under inflation targeting, there was positive probability of arriving in a state from which there was no way for the economy to converge to the "good" steady-state. This in turn implies that agents should not place prior certainty on converging to the "good" steady-state, thus rationalising the beliefs required to get the kind of global multiplicity at the zero lower bound that these and other authors have focussed on. Once again though, we showed that price level targeting is sufficient to restore existence and determinacy.

[28] Theoretical and empirical evidence for this view is presented in Cochrane (2015).

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6. Appendices

6.1. Other properties of the solution set

First, let us give one further definition:

Definition 9 ((Non-)Degenerate matrix) A matrix $M \in \mathbb{R}^{T \times T}$ is called a **non-degenerate matrix** if the principal minors of M are all non-zero. M is called a **degenerate matrix** if it is not a non-degenerate matrix.

Then, conditions for having a finite or convex set of solutions are given in the following propositions.

Proposition 12 The LCP (q, M) has a finite (possibly zero) number of solutions for all $q \in \mathbb{R}^T$ if and only if M is non-degenerate. (Cottle, Pang, and Stone 2009a)

Proposition 13 The LCP (q, M) has a convex (possibly empty) set of solutions for all $q \in \mathbb{R}^T$ if and only if M is column sufficient. (Cottle, Pang, and Stone 2009a)

6.2. Results from dynamic programming

Alternative existence and uniqueness results for the infinite T problem can be established via dynamic programming methods, under the assumption that Problem 2 comes from the first order conditions solution of a social planner problem. These have the advantage that their conditions are potentially much easier to evaluate, though they also have somewhat limited applicability. We focus here on uniqueness results, since these are generally of greater interest.

Suppose that the social planner in some economy solves the following problem:

Problem 5 Suppose $\mu \in \mathbb{R}^n$, $\Psi^{(0)} \in \mathbb{R}^{c\times 1}$ and $\Psi^{(1)} \in \mathbb{R}^{c\times 2n}$ are given, where $c \in \mathbb{N}$. Define $\tilde{\Gamma}: \mathbb{R}^n \to \mathbb{P}(\mathbb{R}^n)$ (where $\mathbb{P}$ denotes the power-set operator) by:

$$\tilde{\Gamma}(x) = \left\{ z \in \mathbb{R}^n \middle| \, 0 \le \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix} \right\}, \tag{6}$$

for all $x \in \mathbb{R}^n$. *(Note: $\tilde{\Gamma}(x)$ will give the set of feasible values for next period's state if the current state is x. Equality constraints may be included by including an identical lower bound and upper bound.)* Define:

$$\widetilde{X} := \{ x \in \mathbb{R}^n | \tilde{\Gamma}(x) \ne \emptyset \}, \tag{7}$$

and suppose without loss of generality that for all $x \in \mathbb{R}^n$, $\tilde{\Gamma}(x) \cap \widetilde{X} = \tilde{\Gamma}(x)$. *(Note: this means that the linear inequalities bounding $\widetilde{X}$ are already included in those in the definition of $\tilde{\Gamma}(x)$. It is without loss of generality as the planner will never choose an $\tilde{x} \in \tilde{\Gamma}(x)$ such that $\tilde{\Gamma}(\tilde{x}) = \emptyset$.)* Further define $\tilde{\mathcal{F}}: \widetilde{X} \times \widetilde{X} \to \mathbb{R}$ by:

$$\tilde{\mathcal{F}}(x,z) = u^{(0)} + u^{(1)} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix}' \tilde{u}^{(2)} \begin{bmatrix} x-\mu \\ z-\mu \end{bmatrix}, \tag{8}$$

for all $x, z \in \widetilde{X}$, where $u^{(0)} \in \mathbb{R}$, $u^{(1)} \in \mathbb{R}^{1\times 2n}$ and $\tilde{u}^{(2)} = \tilde{u}^{(2)\prime} \in \mathbb{R}^{2n\times 2n}$ are given. Finally, suppose $x_0 \in \widetilde{X}$ is given and $\beta \in (0,1)$, and choose $x_1, x_2, \dots$ to maximise:

$$\liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1} \tilde{\mathcal{F}}(x_{t-1}, x_t) \tag{9}$$

subject to the constraints that for all $t \in \mathbb{N}^+$, $x_t \in \tilde{\Gamma}(x_{t-1})$.

To ensure the problem is well behaved, we make the following assumption:

Assumption 3 $\tilde{u}^{(2)}$ is negative-definite.

In online appendix G, we establish the following (unsurprising) result:

Proposition 14 If either $\widetilde{X}$ is compact, or, $\tilde{\Gamma}(x)$ is compact valued and $x \in \tilde{\Gamma}(x)$ for all $x \in \widetilde{X}$, then for all $x_0 \in \widetilde{X}$, there is a unique path $(x_t)_{t=0}^{\infty}$ which solves Problem 5.

We wish to use this result to establish the uniqueness of the solution to the first order conditions. The Lagrangian for our problem is given by:

$$\sum_{t=1}^{\infty} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda'_{\Psi,t} \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1}-\mu \\ x_t-\mu \end{bmatrix} \right] \right], \tag{10}$$

for some KKT-multipliers $\lambda_t \in \mathbb{R}^c$ for all $t \in \mathbb{N}^+$. Taking the first order conditions leads to the following necessary KKT conditions, for all $t \in \mathbb{N}^+$:

$$0 = u^{(1)}_{\cdot,2} + \begin{bmatrix} x_{t-1}-\mu \\ x_t-\mu \end{bmatrix}' \tilde{u}^{(2)}_{\cdot,2} + \lambda'_t \Psi^{(1)}_{\cdot,2} + \beta \left[u^{(1)}_{\cdot,1} + \begin{bmatrix} x_t-\mu \\ x_{t+1}-\mu \end{bmatrix}' \tilde{u}^{(2)}_{\cdot,1} + \lambda'_{t+1} \Psi^{(1)}_{\cdot,1} \right], \tag{11}$$

$$0 \le \Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1}-\mu \\ x_t-\mu \end{bmatrix}, \qquad 0 \le \lambda_t, \qquad 0 = \lambda_t \circ \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1}-\mu \\ x_t-\mu \end{bmatrix} \right], \tag{12}$$

where subscripts 1 and 2 refer to blocks of rows or columns of length n. Additionally, for μ to be the steady-state of x_t and $\bar{\lambda}$ to be the steady-state of λ_t, we require:

$$0 = u_{\cdot,2}^{(1)} + \bar{\lambda}'\Psi_{\cdot,2}^{(1)} + \beta\big[u_{\cdot,1}^{(1)} + \bar{\lambda}'\Psi_{\cdot,1}^{(1)}\big], \tag{13}$$

$$0 \le \Psi^{(0)}, \qquad 0 \le \bar{\lambda}, \qquad 0 = \bar{\lambda} \circ \Psi^{(0)}. \tag{14}$$

In online appendix H we prove the following result:

Proposition 15 Suppose that for all $t \in \mathbb{N}$, $(x_t)_{t=1}^{\infty}$ and $(\lambda_t)_{t=1}^{\infty}$ satisfy the KKT conditions given in equations (10) and (11), and that as $t \to \infty$, $x_t \to \mu$ and $\lambda_t \to \bar{\lambda}$, where μ and λ satisfy the steady-state KKT conditions given in equations (12) and (13). Then $(x_t)_{t=1}^{\infty}$ solves Problem 5. If, further, either condition of Proposition 10 is satisfied, then $(x_t)_{t=1}^{\infty}$ is the unique solution to Problem 5, and there can be no other solutions to the KKT conditions given in equations (10) and (11) satisfying $x_t \to \mu$ and $\lambda_t \to \bar{\lambda}$ as $t \to \infty$.

Now, it is possible to convert the KKT conditions given in equations (10) and (11) into a problem in the form of the multiple-bound generalisation of Problem 2 quite generally. To see this, first note that we may rewrite equation (10) as:

$$\begin{aligned} 0 = {} & u_{\cdot,2}^{(1)\prime} + \tilde{u}_{2,1}^{(2)}(x_{t-1} - \mu) + \tilde{u}_{2,2}^{(2)}(x_t - \mu) + \Psi_{\cdot,2}^{(1)\prime}\lambda_t \\ & + \beta\big[u_{\cdot,1}^{(1)\prime} + \tilde{u}_{1,1}^{(2)}(x_t - \mu) + \tilde{u}_{1,2}^{(2)}(x_{t+1} - \mu) + \Psi_{\cdot,1}^{(1)\prime}\lambda_{t+1}\big]. \end{aligned}$$

Now, $\tilde{u}_{2,2}^{(2)} + \beta u_{1,1}^{(2)}$ is negative definite, hence it is valid to define:

$$\mathcal{V} := \Psi_{\cdot,2}^{(1)}\big[\tilde{u}_{2,2}^{(2)} + \beta\tilde{u}_{1,1}^{(2)}\big]^{-1}.$$

Then, equation (9) implies that:

$$\begin{aligned} & \Psi^{(0)} + \Psi^{(1)}\begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \\ & = \Psi^{(0)} + \big(\Psi_{\cdot,1}^{(1)} - \mathcal{V}\tilde{u}_{2,1}^{(2)}\big)(x_{t-1} - \mu) - \mathcal{V}\Big[u_{\cdot,2}^{(1)\prime} + \beta\Big[u_{\cdot,1}^{(1)\prime} + \tilde{u}_{1,2}^{(2)}(x_{t+1} - \mu) + \Psi_{\cdot,1}^{(1)\prime}\lambda_{t+1}\Big]\Big] \\ & -\Psi_{\cdot,2}^{(1)}\big[\tilde{u}_{2,2}^{(2)} + \beta\tilde{u}_{1,1}^{(2)}\big]^{-1}\Psi_{\cdot,2}^{(1)\prime}\lambda_t. \end{aligned} \tag{15}$$

Moreover, equation (11) implies that if the k^{th} element of $\Psi^{(0)} + \Psi^{(1)}\begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}$ is strictly positive, then the k^{th} element of λ_t is zero, so:

$$\Psi^{(0)} + \Psi^{(1)}\begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} = \max\{0, z_t\}, \tag{16}$$

where:

$$\begin{aligned} z_t := {} & \Psi^{(0)} + \big(\Psi_{\cdot,1}^{(1)} - \mathcal{V}\tilde{u}_{2,1}^{(2)}\big)(x_{t-1} - \mu) \\ & - \mathcal{V}\Big[u_{\cdot,2}^{(1)\prime} + \beta\big[u_{\cdot,1}^{(1)\prime} + \tilde{u}_{1,2}^{(2)}(x_{t+1} - \mu) + \Psi_{\cdot,1}^{(1)\prime}\lambda_{t+1}\big]\Big] \\ & - \Big[\Psi_{\cdot,2}^{(1)}\big[\tilde{u}_{2,2}^{(2)} + \beta\tilde{u}_{1,1}^{(2)}\big]^{-1}\Psi_{\cdot,2}^{(1)\prime} + \mathcal{W}\Big]\lambda_t, \end{aligned}$$

and $\mathcal{W} \in \mathbb{R}^{c \times c}$ is an arbitrary, strictly positive diagonal matrix. A natural choice is:

$$\mathcal{W} := -\operatorname{diag}\operatorname{diag}\Big[\Psi_{\cdot,2}^{(1)}\big[\tilde{u}_{2,2}^{(2)} + \beta\tilde{u}_{1,1}^{(2)}\big]^{-1}\Psi_{\cdot,2}^{(1)\prime}\Big],$$

providing this is strictly positive (it is weakly positive at least as $\tilde{u}_{2,2}^{(2)} + \beta\tilde{u}_{1,1}^{(2)}$ is negative definite), where the diag operator maps matrices to a vector containing their diagonal, and maps vectors to a matrix with the given vector on the diagonal, and zeros elsewhere.

We claim that we may replace equation (11) with equation (15) without changing the model. We have already shown that equation (11) implies equation (15), so we just have to prove the converse. We continue to suppose equation (9) holds, and thus, so too does equation (14). Then, from subtracting equation (14) from equation (15), we have that:

$$\mathcal{W}\lambda_t = \max\{-z_t, 0\}.$$

Hence, as $\mathcal{W}$ is a strictly positive diagonal matrix, and the right hand side is weakly positive, $\lambda_t \geq 0$. Furthermore, the kth element of λ_t is non-negative if and only if the kth element of z_t is non-positive (as $\mathcal{W}$ is a strictly positive diagonal matrix), which in turn holds if and only if the kth element of $\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}$ is equal to zero, by equation (15). Thus equation (11) is satisfied.

Combined with our previous results, this gives the following proposition:

Proposition 16 Suppose we are given a problem in the form of Problem 5. Then, the KKT conditions of that problem may be placed into the form of the multiple-bound generalisation of Problem 2. Let $\left(q_{x_0}, M\right)$ be the infinite LCP corresponding to this representation, given initial state $x_0 \in \widetilde{X}$. Then, if y is a solution to the LCP, $q_{x_0} + My$ gives the stacked paths of the bounded variables in a solution to Problem 5. If, further, either condition of Proposition 10 is satisfied, then this LCP has a unique solution for all $x_0 \in \widetilde{X}$, which gives the unique solution to Problem 5, and, for sufficiently large T^*, the finite LCP $\left(q_{x_0}^{(T^*)}, M^{(T^*)}\right)$ has a unique solution $y^{(T^*)}$ for all $x_0 \in \widetilde{X}$, where $q_{x_0}^{(T^*)} + M^{(T^*)} y^{(T^*)}$ gives the first T^* periods of the stacked paths of the bounded variables in a solution to Problem 5.

This proposition provides some evidence that the LCP will have a unique solution when it is generated from a dynamic programming problem with a unique solution. In online appendix I, we derive similar results for models with more general constraints and objective functions. The proof of this proposition also showed how one can convert KKT conditions into equations of the form handled by our methods.

Online appendices to: "Existence and uniqueness of solutions to dynamic models with occasionally binding constraints."

Tom D. Holden, School of Economics, University of Surrey

A. Relationship between multiplicity under perfect-foresight, and multiplicity under rational expectations

By augmenting the state-space appropriately, the first order conditions of a general, non-linear, rational expectations, DSGE model may always be placed in the form:

$$0 = \mathbb{E}_t \hat{f}(\hat{x}_{t-1}, \hat{x}_t, \hat{x}_{t+1}, \sigma\varepsilon_t),$$

for all $t \in \mathbb{Z}$, where $\sigma \in [0,1]$, $\hat{f}: (\mathbb{R}^{\hat{n}})^3 \times \mathbb{R}^m \to \mathbb{R}^{\hat{n}}$, and where for all $t \in \mathbb{Z}$, $\hat{x}_t \in \mathbb{R}^{\hat{n}}$, $\varepsilon_t \in \mathbb{R}^m$, $\mathbb{E}_{t-1}\varepsilon_t = 0$, and $\mathbb{E}_t \hat{x}_t = \hat{x}_t$. Since f is arbitrary, without loss of generality we may further assume that $\varepsilon_t \sim \text{NIID}(0, I)$. We further assume:

Assumption 4 $\hat{f}$ is everywhere continuous.

The continuity of $\hat{f}$ does rule out some models, but all models in which the only source of non-differentiability is a max or min operator (like those studied in this paper and its computational companion (Holden 2016)) will have a continuous $\hat{f}$.

Now, by further augmenting the state space, we can then find a continuous function $f: (\mathbb{R}^n)^3 \times \mathbb{R}^m \to \mathbb{R}^n$ such that for all $t \in \mathbb{Z}$:

$$0 = f(x_{t-1}, x_t, \mathbb{E}_t x_{t+1}, \sigma\varepsilon_t),$$

where for all $t \in \mathbb{Z}$, $x_t \in \mathbb{R}^n$ and $\mathbb{E}_t x_t = x_t$.[29] A solution to this model is given by a policy function. Given f is continuous, it is natural to restrict attention to continuous policy functions.[30] Furthermore, given the model's transversality conditions, we are usually only interested in stationary, Markov solutions, so the policy function will not be a function of t or of lags of the state. Additionally, in this paper we are only interested in solutions in which the deterministic model converges to some particular steady-state μ. Thus we make the following assumption:

Assumption 5 The policy function is given by a continuous function: $g: [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, such that for all $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f(x, g(\sigma, x, e), \mathbb{E}_\varepsilon g(\sigma, g(\sigma, x, e), \sigma\varepsilon), e),$$

where $\varepsilon \sim \text{N}(0, I)$ and $\mathbb{E}_\varepsilon$ denotes an expectation with respect to ε. Furthermore, for all $x_0 \in \mathbb{R}^n$, the recurrence $x_t = g(0, x_{t-1}, 0)$ satisfies $x_t \to \mu$ as $t \to \infty$.

[29] For example, we may use the equations: $\hat{x}_t^\circ = \hat{x}_{t-1}$, $\hat{\varepsilon}_t = \varepsilon_t$, $z_t = \hat{f}(\hat{x}_{t-1}^\circ, \hat{x}_{t-1}, \hat{x}_t, \sigma\hat{\varepsilon}_{t-1})$, $0 = \mathbb{E}_t z_{t+1}$, with $x_t := [\hat{x}_t' \quad \hat{x}_t^{\circ\prime} \quad \hat{\varepsilon}_t' \quad z_t']'$.

[30] Note also that in standard dynamic programming applications, the policy function will be continuous. See e.g. Theorem 9.8 of Stokey, Lucas, and Prescott (1989).

To produce a lower bound on the number of policy functions satisfying Assumption 5, we need two further assumptions. The first assumption just gives the existence of the "time iteration" (a.k.a. "policy function iteration") operator $\mathcal{T}$, and ensures that it maps fixed points to fixed points.

Assumption 6 Let $\mathcal{G}$ denote the space of all continuous functions $[0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. We assume there exists a function $\mathcal{T}: \mathcal{G} \to \mathcal{G}$ such that for all $(g, \sigma, x, e) \in \mathcal{G} \times [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f\big(x, \mathcal{T}(g)(\sigma, x, e), \mathbb{E}_\varepsilon g\big(\sigma, \mathcal{T}(g)(\sigma, x, e), \sigma\varepsilon\big), e\big).$$

We further assume that if there exists some $(g, \sigma) \in \mathcal{G} \times [0,1]$ such that for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f\big(x, g(\sigma, x, e), \mathbb{E}_\varepsilon g\big(\sigma, g(\sigma, x, e), \sigma\varepsilon\big), e\big),$$

then for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, $\mathcal{T}(g)(\sigma, x, e) = g(\sigma, x, e)$.

The second assumption ensures that time iteration always converges when started from a solution to the model with no uncertainty after the current period. This is a weak assumption since the policy functions under uncertainty are invariably close to the policy function in the absence of uncertainty.

Assumption 7 Let $h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function giving a solution to the model in which there is no future uncertainty, i.e. for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f(x, h(x, e), h(h(x, e), 0), e).$$

Further, define $g_{h,0} \in \mathcal{G}$ by $g_{h,0}(\sigma, x, e) = h(x, e)$ for all $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$, and define $g_{h,k} \in \mathcal{G}$ inductively by $g_{h,k+1} = \mathcal{T}(g_{h,k})$ for all $k \in \mathbb{N}$. Then there exists some $g_{h,\infty} \in \mathcal{G}$ such that $g_{h,\infty} = \mathcal{T}(g_{h,\infty})$ and for all $(\sigma, x, e) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$, $g_{h,k}(\sigma, x, e) \to g_{h,\infty}(\sigma, x, e)$ as $k \to \infty$.

Note, by construction, if h is as in Assumption 7, then for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$0 = f(x, g_{h,0}(0, x, e), \mathbb{E}_\varepsilon g_{h,0}(0, g_{h,0}(0, x, e), 0\varepsilon), e).$$

Hence, by Assumption 6, for all $k \in \mathbb{N}$, all $x(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, $g_{h,k}(0, x, e) = g_{h,0}(0, x, e)$. Consequently, for all $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, $g_{h,\infty}(0, x, e) = g_{h,0}(0, x, e) = h(x, e)$.

Now suppose that h_1 and h_2 were as in Assumption 7, and that there exists $(x, e) \in \mathbb{R}^n \times \mathbb{R}^m$, such that $h_1(x, e) \neq h_2(x, e)$. Then, by the continuity of $g_{h_1,\infty}$ and $g_{h_2,\infty}$, there is some $\mathcal{S} \subseteq [0,1] \times \mathbb{R}^n \times \mathbb{R}^m$ of positive measure, with $(0, x, e) \in \mathcal{S}$, such that for all $(\sigma, x, e) \in \mathcal{S}$, $g_{h_1,\infty}(\sigma, x, e) \neq g_{h_2,\infty}(\sigma, x, e)$. Hence, the rational expectations policy functions differ, at least for small σ. Thus, if Assumption 6 and Assumption 7 are satisfied, there are at least as many policy functions satisfying Assumption 5 as there are solutions to the model in which there is no future uncertainty.

B. Construction of a static model with no dynamic solution in some states

Consider the model:

$$a_t = \max\{0, \mathscr{b}_t\}, \qquad a_t = 1 - c_t, \qquad c_t = a_t - \mathscr{b}_t.$$

The model has steady-state $a = \mathscr{b} = 1$, $c = 0$. Furthermore, in the model's Problem 3 type equivalent, in which for $t \in \mathbb{N}^+$:

$$a_t = \begin{cases} \mathscr{b}_t + y_{t,0} & \text{if } t \le T \\ \mathscr{b}_t & \text{if } t > T \end{cases},$$

where $y_{\cdot,\cdot}$ is defined as in Problem 3, we have that:

$$c_t = \begin{cases} y_{t,0} & \text{if } t \le T \\ 0 & \text{if } t > T \end{cases},$$

so:

$$\mathscr{b}_t = \begin{cases} 1 - 2y_{t,0} & \text{if } t \le T \\ 1 & \text{if } t > T \end{cases},$$

implying:

$$a_t = \begin{cases} 1 - y_{t,0} & \text{if } t \le T \\ 1 & \text{if } t > T \end{cases}.$$

thus, $M = -I$ for this model.

C. Proof of sufficient conditions for feasibility with $T = \infty$

First, define $G := -C(B + CF)^{-1}$, and note that if L is the lag (right-shift) operator, the model from Problem 1 can be written as:

$$L^{-1}(ALL + BL + C)(x - \mu) = 0.$$

Furthermore, by the definitions of F and G:

$$(L - G)(B + CF)(I - FL) = ALL + BL + C,$$

so the stability of the model from Problem 1 is determined by the solutions for $z \in \mathbb{C}$ of the polynomial:

$$0 = \det(Az^2 + Bz + C) = \det(Iz - G)\det(B + CF)\det(I - Fz).$$

Now by Assumption 1, all of the roots of $\det(I - Fz)$ are strictly outside of the unit circle, and all of the other roots of $\det(Az^2 + Bz + C)$ are weakly inside the unit circle (else there would be indeterminacy), thus, all of the roots of $\det(Iz - G)$ are weakly inside the unit circle. Therefore, if we write $\rho_{\mathcal{M}}$ for the spectral radius of some matrix $\mathcal{M}$, then, by this discussion and Assumption 2, $\rho_G < 1$.

Next, let $s_t^*, x_t^* \in \mathbb{R}^{n \times \mathbb{N}^+}$ be such that for any $y \in \mathbb{R}^{\mathbb{N}^+}$, the k^{th} columns of $s_t^* y$ and $x_t^* y$ give the value of s_t and x_t following a magnitude 1 news shock at horizon k, i.e. when $x_0 = \mu$ and y_0 is the k^{th} row of $I_{\mathbb{N}^+ \times \mathbb{N}^+}$. Then:

$$\begin{aligned} s_t^* &= -(B + CF)^{-1}[I_{\cdot,1}I_{t,1:\infty} + GI_{\cdot,1}I_{t+1,1:\infty} + G^2 I_{\cdot,1}I_{t+2,1:\infty} + \cdots] \\ &= -(B + CF)^{-1}\sum_{k=0}^{\infty}(GL)^k I_{\cdot,1}I_{t,1:\infty} \\ &= -(B + CF)^{-1}(I - GL)^{-1}I_{\cdot,1}I_{t,1:\infty}, \end{aligned}$$

where the infinite sums are well defined as $\rho_G < 1$, and where $I_{t,1:\infty} \in \mathbb{R}^{1\times\mathbb{N}^+}$ is a row vector with zeros everywhere except position t where there is a 1. Thus:

$$s_t^* = [0_{n\times(t-1)} \quad s_1^*] = L^{t-1}s_1^*.$$

Furthermore,

$$(x_t^* - \mu^*) = \sum_{j=1}^{t} F^{t-j}s_k^* = \sum_{j=1}^{t} F^{t-j}L^{j-1}s_1^*,$$

i.e.:

$$(x_t^* - \mu^*)_{\cdot,k} = \sum_{j=1}^{t} F^{t-j}s_{1,\cdot,k+1-j}^* = -\sum_{j=1}^{\min\{t,k\}} F^{t-j}(B+CF)^{-1}G^{k-j}I_{\cdot,1},$$

and so the k^{th} offset diagonal of M ($k \in \mathbb{Z}$) is given by the first row of the k^{th} column of:

$$L^{-t}(x_t^* - \mu^*) = L^{-1}\sum_{j=1}^{t}(FL^{-1})^{t-j}s_1^* = L^{-1}\sum_{j=0}^{t-1}(FL^{-1})^{j}s_1^*,$$

where we abuse notation slightly by allowing L^{-1} to give a result with indices in $\mathbb{Z}$ rather than $\mathbb{N}^+$, with padding by zeros. Consequently, for all $k \in \mathbb{N}^+$, $M_{t,k} = \mathrm{O}(t^n\rho_F^t)$, as $t \to \infty$, for all $t \in \mathbb{N}^+$, $M_{t,k} = \mathrm{O}\left(t^n\rho_G^k\right)$, as $k \to \infty$, and for all $k \in \mathbb{Z}$, $M_{t,t+k} - \lim_{\tau\to\infty} M_{\tau,\tau+k} = \mathrm{O}(t^{n-1}(\rho_F\rho_G)^t)$, as $t \to \infty$. Hence,

$$\sup_{y\in[0,1]^{\mathbb{N}^+}} \inf_{t\in\mathbb{N}^+} M_{t,1:\infty}y$$

exists and is well defined, and so:

$$\varsigma = \sup_{\substack{y\in[0,1]^{\mathbb{N}^+} \\ \exists T\in\mathbb{N} \text{ s.t. } \forall t>T, y_t=0}} \inf_{t\in\mathbb{N}^+} M_{t,1:\infty}y = \sup_{y\in[0,1]^{\mathbb{N}^+}} \inf_{t\in\mathbb{N}^+} M_{t,1:\infty}y,$$

since every point in $[0,1]^{\mathbb{N}^+}$ is a limit (under the supremum norm) of a sequence of points in the set:

$$\left\{y \in [0,1]^{\mathbb{N}^+} \middle| \exists T \in \mathbb{N} \text{ s.t. } \forall t > T, y_t = 0\right\}.$$

Thus, we just need to provide conditions under which $\sup_{y\in[0,1]^{\mathbb{N}^+}} \inf_{t\in\mathbb{N}^+} M_{t,1:\infty}y > 0$.

To produce such conditions, we need constructive bounds on M, even if they have slightly worse convergence rates. For any matrix, $\mathcal{M} \in \mathbb{R}^{n\times n}$ with $\rho_{\mathcal{M}} < 1$, and any $\phi \in (\rho_{\mathcal{M}}, 1)$, let:

$$C_{\mathcal{M},\phi} := \sup_{k\in\mathbb{N}} \left\|(\mathcal{M}\phi^{-1})^k\right\|_2.$$

Furthermore, for any matrix, $\mathcal{M} \in \mathbb{R}^{n\times n}$ with $\rho_{\mathcal{M}} < 1$, and any $\epsilon > 0$, let:

$$\rho_{\mathcal{M},\epsilon} := \max\{|z| | z \in \mathbb{C}, \sigma_{\min}(\mathcal{M} - zI) = \epsilon\},$$

where $\sigma_{\min}(\mathcal{M} - zI)$ is the minimum singular value of $\mathcal{M} - zI$, and let $\epsilon^*(\mathcal{M}) \in (0,\infty]$ solve:

$$\rho_{\mathcal{M},\epsilon} = 1.$$

(This has a solution in $(0,\infty]$ by continuity as $\rho_{\mathcal{M}} < 1$.)

Then, by Theorem 16.2 of Trefethen and Embree (2005), for any $K \in \mathbb{N}$ and $k > K$:

$$\left\|(\mathcal{M}\phi^{-1})^k\right\|_2 \le \left\|(\mathcal{M}\phi^{-1})^K\right\|_2 \left\|(\mathcal{M}\phi^{-1})^{k-K}\right\|_2 \le \frac{\left\|(\mathcal{M}\phi^{-1})^K\right\|_2}{\epsilon^*(\mathcal{M}\phi^{-1})}.$$

Now, $\left\|(\mathcal{M}\phi^{-1})^K\right\|_2 \to 0$ as $K \to \infty$, hence, there exists some $K \in \mathbb{N}$ such that:

$$\sup_{k=0,\dots,K} \left\|(\mathcal{M}\phi^{-1})^k\right\|_2 \ge \frac{\left\|(\mathcal{M}\phi^{-1})^K\right\|_2}{\epsilon^*(\mathcal{M}\phi^{-1})} \ge \sup_{k>K} \left\|(\mathcal{M}\phi^{-1})^k\right\|_2,$$

meaning $C_{\mathcal{M},\phi} = \sup_{k=0,\dots,K} \left\|(\mathcal{M}\phi^{-1})^k\right\|_2$. The quantity $\rho_{\mathcal{M},\epsilon}$ (and hence $\epsilon^*(\mathcal{M})$) may be efficiently computed using the methods described by Wright and Trefethen (2001), and implemented in their EigTool toolkit[31]. Thus, $C_{\mathcal{M},\phi}$ may be calculated in finitely many operations by iterating over $K \in \mathbb{N}$ until a K is found which satisfies:

$$\sup_{k=0,\dots,K} \left\|(\mathcal{M}\phi^{-1})^k\right\|_2 \ge \frac{\left\|(\mathcal{M}\phi^{-1})^K\right\|_2}{\epsilon^*(\mathcal{M}\phi^{-1})}.$$

From the definition of $C_{\mathcal{M},\phi}$, we have that for any $k \in \mathbb{N}$ and any $\phi \in (\rho_{\mathcal{M}}, 1)$:

$$\left\|\mathcal{M}^k\right\|_2 \le C_{\mathcal{M},\phi}\phi^k.$$

Now, fix $\phi_F \in (\rho_F, 1)$ and $\phi_G \in (\rho_G, 1)$,[32] and define:

$$\mathcal{D}_{\phi_F,\phi_G} := C_{F,\phi_F} C_{G,\phi_F} \left\|(B + CF)^{-1}\right\|_2,$$

then, for all $t, k \in \mathbb{N}^+$:

$$\begin{aligned}\left|M_{t,k}\right| = \left|(x_t^* - \mu^*)_{1,k}\right| \le \left\|(x_t^* - \mu^*)_{\cdot,k}\right\|_2 &\le \sum_{j=1}^{\min\{t,k\}} \left\|F^{t-j}\right\|_2 \left\|(B + CF)^{-1}\right\|_2 \left\|G^{k-j}\right\|_2 \\ \le \mathcal{D}_{\phi_F,\phi_G} \sum_{j=1}^{\min\{t,k\}} \phi_F^{t-j}\phi_G^{k-j} &= \mathcal{D}_{\phi_F,\phi_G} \phi_F^t \phi_G^k \frac{(\phi_F\phi_G)^{-\min\{t,k\}} - 1}{1 - \phi_F\phi_G}.\end{aligned}$$

Additionally, for all $t \in \mathbb{N}^+$, $k \in \mathbb{Z}$:

$$\begin{aligned}\left|M_{t,t+k} - \lim_{\tau\to\infty} M_{\tau,\tau+k}\right| &= \left|\left(L^{-t}(x_t^* - \mu^*)\right)_{1,k} - \left(\lim_{\tau\to\infty} L^{-t}(x_t^* - \mu^*)\right)_{1,k}\right| \\ &\le \left\|\left(L^{-1}\sum_{j=0}^{t-1}(FL^{-1})^j s_1^* - L^{-1}\sum_{j=0}^{\infty}(FL^{-1})^j s_1^*\right)_{\cdot,k}\right\|_2 \\ &= \left\|\left(\sum_{j=\max\{t,-k\}}^{\infty} F^j s_{1,\cdot,j+k+1}^*\right)_{\cdot,0}\right\|_2 \\ &= \left\|\sum_{j=\max\{t,-k\}}^{\infty} F^j (B + CF)^{-1} G^{j+k} I_{\cdot,1}\right\|_2 \\ &\le \sum_{j=\max\{t,-k\}}^{\infty} \left\|F^j\right\|_2 \left\|(B + CF)^{-1}\right\|_2 \left\|G^{j+k}\right\|_2 \\ &\le \mathcal{D}_{\phi_F,\phi_G} \sum_{j=\max\{t,-k\}}^{\infty} \phi_F^j \phi_G^{j+k} = \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^{\max\{t,-k\}}\phi_G^{\max\{0,t+k\}}}{1 - \phi_F\phi_G},\end{aligned}$$

[31] This toolkit is available from https://github.com/eigtool/eigtool, and is included in DynareOBC.

[32] In practice, we try a grid of values, as it is problem dependent whether high ϕ_F and low $\mathcal{K}(\mathcal{M}\phi^{-1})$ is preferable to low ϕ_F and high $\mathcal{K}(\mathcal{M}\phi^{-1})$.

so, for all $t, k \in \mathbb{N}^{+}$:

$$\left|M_{t,k} - \lim_{\tau\to\infty} M_{\tau,\tau+k-t}\right| \le \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G}.$$

To evaluate $\lim_{\tau\to\infty} M_{\tau,\tau+k-t}$, note that this limit is the top element from the $(k-t)^{\text{th}}$ column of:

$$\begin{aligned} d &:= \lim_{\tau\to\infty} L^{-\tau}(x_\tau^* - \mu^*) = L^{-1}(I - FL^{-1})^{-1} s_1^* \\ &= -(I - FL^{-1})^{-1}(B + CF)^{-1}(I - GL)^{-1} I_{\cdot,1} I_{0,-\infty:\infty}, \end{aligned}$$

where $I_{0,-\infty:\infty} \in \mathbb{R}^{1\times\mathbb{Z}}$ is zero everywhere apart from index 0 where it equals 1. Hence, by the definitions of F and G:

$$AL^{-1}d + Bd + CLd = -I_{\cdot,1} I_{0,-\infty:\infty}.$$

In other words, if we write d_k in place of $d_{\cdot,k}$ for convenience, then, for all $k \in \mathbb{Z}$:

$$Ad_{k+1} + Bd_k + Cd_{k-1} = -\begin{cases} I_{\cdot,1} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

I.e. the homogeneous part of the difference equation for d_{-t} is identical to that of $x_t - \mu$. The time reversal here is intuitive since we are indexing diagonals such that indices increase as we move up and to the right in M, but time is increasing as we move down in M.

It turns out that exploiting the possibility of reversing time is the key to easy evaluating d_k. First, note that for $k < 0$, it must be the case that $d_k = Fd_{k+1}$, since the shock has already "occurred" (remember, that we are going forwards in "time" when we reduce k). Now consider the model in which we are going forwards time when we increase k, i.e. the model with:

$$L(AL^{-1}L^{-1} + BL^{-1} + C)d = 0,$$

subject to the terminal condition that $d_k \to 0$ as $k \to \infty$, which must hold as we have already proved that the first row of M converges to zero. Now, let $z \in \mathbb{C}$, $z \neq 0$ be a solution to:

$$0 = \det(Az^2 + Bz + C),$$

and define $\tilde{z} = z^{-1}$, so:

$$\begin{aligned} 0 = \det(A + B\tilde{z} + C\tilde{z}^2) &= z^{-2}\det(Az^2 + Bz + C) \\ &= \det(I - G\tilde{z})\det(B + CF)\det(I\tilde{z} - F). \end{aligned}$$

By Assumption 1, all of the roots of $\det(I\tilde{z} - F)$ are inside the unit circle, thus they cannot contribute to the dynamics of the time reversed process, else the terminal condition would be violated. Thus, the time reversed model has a unique solution satisfying the terminal condition with a transition matrix with the same eigenvalues as G. Consequently, this solution can be calculated via standard methods for solving linear DSGE models, and it will be given by $d_k = Hd_{k-1}$, for all $k > 0$, where $H = -(B + AH)^{-1}C$, and $\phi_H = \phi_G < 1$, by Assumption 2.

It just remains to determine the value of d_0. By the previous results, this must satisfy:

$$-I_{\cdot,1} = Ad_1 + Bd_0 + Cd_{-1} = (AH + B + CF)d_0.$$

Hence:

$$d_0 = -(AH + B + CF)^{-1} I_{\cdot,1}.$$

This gives a readily computed solution for the limits of the diagonals of M. Lastly, note that:

$$|d_{-t,1}| \le \|d_{-t}\|_2 = \|F^t d_0\|_2 \le \|F^t\|_2 \|d_0\|_2 \le \mathcal{C}_{F,\phi_F} \phi_F^t \|d_0\|_2,$$

and:

$$|d_{t,1}| \le \|d_t\|_2 = \|H^t d_0\|_2 \le \|H^t\|_2 \|d_0\|_2 \le \mathcal{C}_{H,\phi_H} \phi_H^t \|d_0\|_2.$$

We will use these results in producing our bounds on ς.

First, fix $T \in \mathbb{N}^+$, and define a new matrix $\underline{M}^{(T)} \in \mathbb{R}^{\mathbb{N}^+ \times \mathbb{N}^+}$ by $\underline{M}^{(T)}_{1:T,1:T} = M_{1:T,1:T}$, and for all $t, k \in \mathbb{N}^+$, with $\min\{t,k\} > T$, $\underline{M}^{(T)}_{t,k} = d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G}$, then:

$$
\begin{aligned}
\varsigma &\ge \max_{\substack{y\in[0,1]^T\\ y_\infty\in[0,1]}} \inf_{t\in\mathbb{N}^+} M_{t,1:\infty} \begin{bmatrix} y \\ y_\infty 1_{\infty\times 1} \end{bmatrix} \ge \max_{\substack{y\in[0,1]^T\\ y_\infty\in[0,1]}} \inf_{t\in\mathbb{N}^+} \underline{M}^{(T)}_{t,1:\infty} \begin{bmatrix} y \\ y_\infty 1_{\infty\times 1} \end{bmatrix} \\
&= \max_{\substack{y\in[0,1]^T\\ y_\infty\in[0,1]}} \min \left\{ \begin{array}{c} \min\limits_{t=1,\dots,T} \left[M_{t,1:T} y + \sum\limits_{k=T+1}^{\infty} \left(d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G} \right) y_\infty \right], \\ \inf\limits_{t\in\mathbb{N}^+, t>T} \left[\sum\limits_{k=1}^{T} \left(d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G} \right) y_k + \sum\limits_{k=T+1}^{\infty} \left(d_{k-t,1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G} \right) y_\infty \right] \end{array} \right\} \\
&\ge \max_{\substack{y\in[0,1]^T\\ y_\infty\in[0,1]}} \min \left\{ \begin{array}{c} \min\limits_{t=1,\dots,T} \left[M_{t,1:T} y + \left((I-H)^{-1} d_{T+1-t}\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} y_\infty \right], \\ \min\limits_{t=T+1,\dots,2T} \left[\begin{array}{c} \sum\limits_{k=1}^{T} \left(d_{-(t-k),1} - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^k}{1-\phi_F\phi_G} \right) y_k + \left((I-F)^{-1}(d_{-1} - d_{-(t-T)})\right)_1 y_\infty \\ + \left((I-H)^{-1} d_0\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^t \phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)} y_\infty \end{array} \right], \\ \inf\limits_{t\in\mathbb{N}^+, t>2T} \left[\begin{array}{c} \sum\limits_{k=1}^{T} d_{-(t-k),1} y_k + \left((I-F)^{-1}(d_{-1} - d_{-(t-T)})\right)_1 y_\infty \\ + \left((I-H)^{-1} d_0\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G} \frac{\phi_F^{2T+1} \phi_G}{(1-\phi_F\phi_G)(1-\phi_G)} \end{array} \right] \end{array} \right\}.
\end{aligned}
$$

Now, for $t \ge T$:

$$
\begin{aligned}
\left| \left((I-F)^{-1} d_{-(t-T)}\right)_1 \right| &\le \left\| (I-F)^{-1} d_{-(t-T)} \right\|_2 \le \left\| (I-F)^{-1} \right\|_2 \left\| d_{-(t-T)} \right\|_2 \\
&\le \mathcal{C}_{F,\phi_F} \phi_F^{t-T} \left\| (I-F)^{-1} \right\|_2 \|d_0\|_2,
\end{aligned}
$$

so:

$$
\begin{aligned}
&\sum_{k=1}^{T} d_{-(t-k),1} y_k - \left((I-F)^{-1} d_{-(t-T)}\right)_1 y_\infty \\
&\qquad \ge -\sum_{k=1}^{T} \mathcal{C}_{F,\phi_F} \phi_F^{t-k} \|d_0\|_2 - \mathcal{C}_{F,\phi_F} \phi_F^{t-T} \left\| (I-F)^{-1} \right\|_2 \|d_0\|_2 y_\infty \\
&\qquad = -\mathcal{C}_{F,\phi_F} \frac{\phi_F^t \left(\phi_F^{-T} - 1 \right)}{1-\phi_F} \|d_0\|_2 - \mathcal{C}_{F,\phi_F} \phi_F^{t-T} \left\| (I-F)^{-1} \right\|_2 \|d_0\|_2 y_\infty,
\end{aligned}
$$

thus $\varsigma \geq \underline{\varsigma}$, where:

$$\underline{\varsigma}_T := \max_{\substack{y\in[0,1]^T\\ y_\infty\in[0,1]}} \min \left\{ \begin{array}{c} \min_{t=1,\dots,T}\left[M_{t,1:T}y + \left((I-H)^{-1}d_{T+1-t}\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G}\frac{\phi_F^t\phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)}y_\infty\right], \\ \min_{t=T+1,\dots,2T}\left[\begin{array}{c}\sum_{k=1}^{T}\left(d_{-(t-k),1} - \mathcal{D}_{\phi_F,\phi_G}\frac{\phi_F^t\phi_G^k}{1-\phi_F\phi_G}\right)y_k + \left((I-F)^{-1}(d_{-1}-d_{-(t-T)})\right)_1 y_\infty \\ +\left((I-H)^{-1}d_0\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G}\frac{\phi_F^t\phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)}y_\infty\end{array}\right], \\ \left[\begin{array}{c}-C_{F,\phi_F}\frac{\phi_F^{2T+1}(\phi_F^{-T}-1)}{1-\phi_F}\|d_0\|_2 - C_{F,\phi_F}\phi_F^{T+1}\|(I-F)^{-1}\|_2\|d_0\|_2 y_\infty + \left((I-F)^{-1}d_{-1}\right)_1 y_\infty \\ +\left((I-H)^{-1}d_0\right)_1 y_\infty - \mathcal{D}_{\phi_F,\phi_G}\frac{\phi_F^{2T+1}\phi_G}{(1-\phi_F\phi_G)(1-\phi_G)}\end{array}\right] \end{array} \right\}.$$

It is worth noting that as $T\to\infty$, the final minimand in this expression tends to:

$$((I-F)^{-1}d_{-1})_1 y_\infty + ((I-H)^{-1}d_0)_1 y_\infty,$$

i.e. a positive multiple of the doubly infinite sum of $d_{1,k}$ over all $k\in\mathbb{Z}$. If this expression is negative, then our lower bound on ς will be negative as well, and hence uninformative.

To construct an upper bound on ς, fix $T\in\mathbb{N}^+$, and define a new matrix $\overline{M}^{(T)}\in\mathbb{R}^{\mathbb{N}^+\times\mathbb{N}^+}$ by $\overline{M}^{(T)}_{1:T,1:T} = M_{1:T,1:T}$, and for all $t,k\in\mathbb{N}^+$, with $\min\{t,k\} > T$, $\overline{M}^{(T)}_{t,k} = |d_{k-t,1}| + \mathcal{D}_{\phi_F,\phi_G}\frac{\phi_F^t\phi_G^k}{1-\phi_F\phi_G}$. Then:

$$\begin{aligned}\varsigma &= \sup_{y\in[0,1]^{\mathbb{N}^+}}\inf_{t\in\mathbb{N}^+} M_{t,1:\infty}y \leq \sup_{y\in[0,1]^{\mathbb{N}^+}}\inf_{t\in\mathbb{N}^+}\overline{M}_{t,1:\infty}y \leq \sup_{y\in[0,1]^{\mathbb{N}^+}}\min_{t=1,\dots,T}\overline{M}_{t,1:\infty}y \\ &\leq \max_{y\in[0,1]^T}\min_{t=1,\dots,T}\overline{M}_{t,1:\infty}\begin{bmatrix} y \\ 1_{\infty\times 1}\end{bmatrix} \\ &\leq \max_{y\in[0,1]^T}\min_{t=1,\dots,T}\left[M_{t,1:T}y + \sum_{k=T+1}^{\infty}|d_{k-t,1}| + \sum_{k=T+1}^{\infty}\mathcal{D}_{\phi_F,\phi_G}\frac{\phi_F^t\phi_G^k}{1-\phi_F\phi_G}\right] \\ &\leq \max_{y\in[0,1]^T}\min_{t=1,\dots,T}\left[M_{t,1:T}y + \sum_{k=T+1-t}^{\infty}|d_{k,1}| + \mathcal{D}_{\phi_F,\phi_G}\frac{\phi_F^t\phi_G^{T+1}}{1-\phi_F\phi_G}\sum_{k=0}^{\infty}\phi_G^k\right] \\ &\leq \max_{y\in[0,1]^T}\min_{t=1,\dots,T}\left[M_{t,1:T}y + C_{H,\phi_H}\|d_0\|_2\phi_H^{T+1-t}\sum_{k=0}^{\infty}\phi_H^k + \mathcal{D}_{\phi_F,\phi_G}\frac{\phi_F^t\phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)}\right] \\ &= \overline{\varsigma}_T := \max_{y\in[0,1]^T}\min_{t=1,\dots,T}\left[M_{t,1:T}y + \frac{C_{H,\phi_H}\|d_0\|_2\phi_H^{T+1-t}}{1-\phi_H} + \mathcal{D}_{\phi_F,\phi_G}\frac{\phi_F^t\phi_G^{T+1}}{(1-\phi_F\phi_G)(1-\phi_G)}\right].\end{aligned}$$

D. Proof of the properties of the BPY model

Defining $x_t = [x_{i,t} \quad x_{y,t} \quad x_{\pi,t}]'$, the BPY model is in the form of Problem 2, with:

$$A := \begin{bmatrix} 0 & -\alpha_{\Delta y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B := \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_\pi \\ -\frac{1}{\sigma} & -1 & 0 \\ 0 & \gamma & -1 \end{bmatrix}, \qquad C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & \beta \end{bmatrix}.$$

Assumption 2 is satisfied for this model as:

$$\det(A+B+C) = \det\begin{bmatrix} -1 & 0 & \alpha_\pi \\ -\frac{1}{\sigma} & 0 & \frac{1}{\sigma} \\ 0 & \gamma & -1 \end{bmatrix} \neq 0$$

as $\alpha_\pi \neq 1$ and $\gamma \neq 0$. Let $f := F_{2,2}$, where F is as in Assumption 1. Then:

$$F = \begin{bmatrix} 0 & \alpha_{\Delta y}(f-1) + \alpha_\pi \dfrac{\gamma f}{1-\beta f} & 0 \\ 0 & f & 0 \\ 0 & \dfrac{\gamma f}{1-\beta f} & 0 \end{bmatrix}.$$

Hence:

$$f = f^2 - \frac{1}{\sigma}\left(\alpha_{\Delta y}(f-1) + \alpha_\pi \frac{\gamma f}{1-\beta f} - \frac{\gamma f^2}{1-\beta f}\right),$$

i.e.:

$$\beta\sigma f^3 - \Big(\big(\alpha_{\Delta y} + \sigma\big)\beta + \gamma + \sigma\Big)f^2 + \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big)f - \alpha_{\Delta y} = 0. \tag{17}$$

When $f \leq 0$, the left hand side is negative, and when $f = 1$, the left hand side equals $(\alpha_\pi - 1)\gamma > 0$ (by assumption on α_π), hence equation (3) has either one or three solutions in $(0,1)$, and no solutions in $(-\infty, 0]$. We wish to prove there is a unique solution in $(-1,1)$. First note that when $\alpha_\pi = 1$, the discriminant of the polynomial is:

$$\Big((1-\beta)\big(\alpha_{\Delta y} - \sigma\big) - \gamma\Big)^2\Big(\big(\beta\alpha_{\Delta y}\big)^2 + 2\beta(\gamma - \sigma)\alpha_{\Delta y} + (\gamma+\sigma)^2\Big).$$

The first multiplicand is positive. The second is minimised when $\sigma = \beta\alpha_{\Delta y} - \gamma$, at the value $4\beta\gamma\alpha_{\Delta y} > 0$, hence this multiplicand is positive too. Consequently, at least for small α_π, there are three real solutions for f, so there may be multiple solutions in $(0,1)$.

Suppose for a contradiction that there were at least three solutions to equation (3) in $(0,1)$ (double counting repeated roots), even for arbitrary large $\beta \in (0,1)$. Let $f_1, f_2, f_3 \in (0,1)$ be the three roots. Then, by Vieta's formulas:

$$3 > f_1 + f_2 + f_3 = \frac{\big(\alpha_{\Delta y} + \sigma\big)\beta + \gamma + \sigma}{\beta\sigma},$$

$$3 > f_1 f_2 + f_1 f_3 + f_2 f_3 = \frac{(1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma}{\beta\sigma},$$

$$1 > f_1 f_2 f_3 = \frac{\alpha_{\Delta y}}{\beta\sigma},$$

so:

$$(2\beta - 1)\sigma > \beta\alpha_{\Delta y} + \gamma > \gamma > 0$$

$$\beta > \frac{1}{2}, \qquad (2\beta-1)\sigma > \gamma,$$

$$\beta\sigma > \beta\alpha_{\Delta y} + \gamma + \sigma(1-\beta),$$

$$2\beta\sigma > (1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma(1-\beta),$$

$$\beta\sigma > \alpha_{\Delta y}.$$

Also, the first derivative of equation (3) must be strictly positive at $f = 1$, so:

$$(1-\beta)\big(\alpha_{\Delta y} - \sigma\big) + (\alpha_\pi - 2)\gamma > 0.$$

Combining all of these inequalities gives the bounds:

$$0 < \alpha_{\Delta y} < 2\sigma - \frac{\gamma+\sigma}{\beta},$$

$$2 + \frac{(1-\beta)(\sigma - \alpha_{\Delta y})}{\gamma} < \alpha_\pi < \frac{(3\beta-1)\sigma - (1+\beta)\alpha_{\Delta y}}{\gamma}.$$

Furthermore, if there are multiple solutions to equation (3), then the discriminant of its first derivative must be weakly positive, i.e.:

$$\Big((\alpha_{\Delta y}+\sigma)\beta + \gamma + \sigma\Big)^2 - 3\beta\sigma\Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi + \sigma\Big) \geq 0.$$

Therefore, we have the following bounds on α_π:

$$2 + \frac{(1-\beta)(\sigma - \alpha_{\Delta y})}{\gamma} < \alpha_\pi \leq \frac{\Big((\alpha_{\Delta y}+\sigma)\beta + \gamma + \sigma\Big)^2 - 3\beta\sigma\Big((1+\beta)\alpha_{\Delta y} + \sigma\Big)}{3\beta\sigma\gamma}$$

since,

$$\frac{(3\beta-1)\sigma - (1+\beta)\alpha_{\Delta y}}{\gamma} - \frac{\Big((\alpha_{\Delta y}+\sigma)\beta + \gamma + \sigma\Big)^2 - 3\beta\sigma\Big((1+\beta)\alpha_{\Delta y} + \sigma\Big)}{3\beta\sigma\gamma}$$
$$= \frac{\Big((2\sigma - \alpha_{\Delta y})\beta - \gamma - \sigma\Big)\Big((4\sigma + \alpha_{\Delta y})\beta + \gamma + \sigma\Big)}{3\beta\gamma\sigma} > 0$$

as $\alpha_{\Delta y} < 2\sigma - \frac{\gamma+\sigma}{\beta}$. Consequently, there exists $\lambda, \mu, \kappa \in [0,1]$ such that:

$$\begin{aligned}
\alpha_\pi = (1-\lambda)&\left[2 + \frac{(1-\beta)(\sigma - \alpha_{\Delta y})}{\gamma}\right] \\
&+ \lambda\left[\frac{\Big((\alpha_{\Delta y}+\sigma)\beta + \gamma + \sigma\Big)^2 - 3\beta\sigma\Big((1+\beta)\alpha_{\Delta y} + \sigma\Big)}{3\beta\sigma\gamma}\right], \\
\alpha_{\Delta y} &= (1-\mu)[0] + \mu\left[2\sigma - \frac{\gamma+\sigma}{\beta}\right], \\
\gamma &= (1-\kappa)[0] + \kappa[(2\beta-1)\sigma]
\end{aligned}$$

These simultaneous equations have unique solutions for α_π, $\alpha_{\Delta y}$ and γ in terms of λ, μ and κ. Substituting these solutions into the discriminant of equation (3) gives a polynomial in $\lambda, \mu, \kappa, \beta, \sigma$. As such, an exact global maximum of the discriminant may be found subject to the constraints $\lambda, \mu, \kappa \in [0,1]$, $\beta \in \left[\frac{1}{2}, 1\right]$, $\sigma \in [0, \infty)$, by using an exact compact polynomial optimisation solver, such as that in the Maple computer algebra package. Doing this gives a maximum of 0 when $\beta \in \left\{\frac{1}{2}, 1\right\}$, $\kappa = 1$ and $\sigma = 0$. But of course, we actually require that $\beta \in \left(\frac{1}{2}, 1\right)$, $\kappa < 1$, $\sigma > 0$. Thus, by continuity, the discriminant is strictly negative over the entire possible domain. This gives the required contradiction to our assumption of three roots to the polynomial, establishing that Assumption 1 holds for this model.

Now, when $T = 1$, M is equal to the top left element of the matrix $-(B + CF)^{-1}$, i.e.:

$$M = \frac{\beta\sigma f^2 - ((1+\beta)\sigma + \gamma)f + \sigma}{\beta\sigma f^2 - \Big((1+\beta)\sigma + \gamma + \beta\alpha_{\Delta y}\Big)f + \sigma + \alpha_{\Delta y} + \gamma\alpha_\pi}.$$

Now, multiplying the denominator by f gives:

$$\begin{aligned}&\beta\sigma f^3-\Big((1+\beta)\sigma+\gamma+\beta\alpha_{\Delta y}\Big)f^2+(\sigma+\alpha_{\Delta y}+\gamma\alpha_\pi)f\\&\qquad=\Big[\beta\sigma f^3-\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f^2+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big)f-\alpha_{\Delta y}\Big]\\&\qquad-\big[\beta\alpha_{\Delta y}f-\alpha_{\Delta y}\big]=(1-\beta f)\alpha_{\Delta y}>0,\end{aligned}$$

by equation (19). Hence, the sign of M is that of $\beta\sigma f^2-\big((1+\beta)\sigma+\gamma\big)f+\sigma$. I.e., M is negative if and only if:

$$\begin{aligned}&\frac{\big((1+\beta)\sigma+\gamma\big)-\sqrt{\big((1+\beta)\sigma+\gamma\big)^2-4\beta\sigma^2}}{2\beta\sigma}<f\\&\qquad<\frac{\big((1+\beta)\sigma+\gamma\big)+\sqrt{\big((1+\beta)\sigma+\gamma\big)^2-4\beta\sigma^2}}{2\beta\sigma}.\end{aligned}$$

The upper limit is greater than 1, so only the lower is relevant. To translate this bound on f into a bound on $\alpha_{\Delta y}$, we first need to establish that f is monotonic in $\alpha_{\Delta y}$.

Totally differentiating equation (19) gives:

$$\begin{aligned}&\Big[3\beta\sigma f^2-2\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big)\Big]\frac{df}{d\alpha_{\Delta y}}=(1-\beta f)(1-f)\\&\qquad>0.\end{aligned}$$

Thus, the sign of $\frac{df}{d\alpha_{\Delta y}}$ is equal to that of:

$$3\beta\sigma f^2-2\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big).$$

Note, however, that this expression is just the derivative of the left hand side of equation (19) with respect to f.

To establish the sign of $\frac{df}{d\alpha_{\Delta y}}$, we consider two cases. First, suppose that equation (19) has three real solutions. Then, the unique solution to equation (19) in (0,1) is its lowest solution. Hence, this solution must be below the first local maximum of the left hand side of equation (19). Consequently, at the $f\in(0,1)$, which solves equation (19), $3\beta\sigma f^2-2\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big)>0$. Alternatively, suppose that equation (19) has a unique real solution. Then the left hand side of this equation cannot change sign in between its local maximum and its local minimum (if it has any). Thus, at the $f\in(0,1)$ at which it changes sign, we must have that $3\beta\sigma f^2-2\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big)>0$. Therefore, in either case $\frac{df}{d\alpha_{\Delta y}}>0$, meaning that f is monotonic increasing in $\alpha_{\Delta y}$.

Consequently, to find the critical $(f,\alpha_{\Delta y})$ at which M changes sign, it is sufficient to find the lowest solution with respect to both f and $\alpha_{\Delta y}$ of the pair of equations:

$$\begin{gathered}\beta\sigma f^2-\big((1+\beta)\sigma+\gamma\big)f+\sigma=0,\\\beta\sigma f^3-\Big((\alpha_{\Delta y}+\sigma)\beta+\gamma+\sigma\Big)f^2+\Big((1+\beta)\alpha_{\Delta y}+\gamma\alpha_\pi+\sigma\Big)f-\alpha_{\Delta y}=0.\end{gathered}$$

The former implies that:

$$\beta\sigma f^3 - \big((1+\beta)\sigma + \gamma\big)f^2 + \sigma f = 0,$$

so, by the latter:

$$\alpha_{\Delta y}\beta f^2 - \Big((1+\beta)\alpha_{\Delta y} + \gamma\alpha_\pi\Big)f + \alpha_{\Delta y} = 0.$$

If $\alpha_{\Delta y} = \sigma\alpha_\pi$, then this equation holds if and only if:

$$\sigma\beta f^2 - \big((1+\beta)\sigma + \gamma\big)f + \sigma = 0.$$

Therefore, the critical $(f, \alpha_{\Delta y})$ at which M changes sign are given by:

$$\alpha_{\Delta y} = \sigma\alpha_\pi,$$

$$f = \frac{\big((1+\beta)\sigma + \gamma\big) - \sqrt{\big((1+\beta)\sigma + \gamma\big)^2 - 4\beta\sigma^2}}{2\beta\sigma}.$$

Thus, M is negative if and only if $\alpha_{\Delta y} > \sigma\alpha_\pi$, and M is zero if and only if $\alpha_{\Delta y} = \sigma\alpha_\pi$.

E. **The BPY model with shadow interest rate persistence**

Following BPY (2013), we introduce persistence in the shadow interest rate by replacing the previous Taylor rule with $x_{i,t} = \max\{0, x_{d,t}\}$, where $x_{d,t}$, the shadow nominal interest rate is given by:

$$x_{d,t} = (1-\rho)\big(1 - \beta + \alpha_{\Delta y}(x_{y,t} - x_{y,t-1}) + \alpha_\pi x_{\pi,t}\big) + \rho x_{d,t-1}.$$

It is easy to verify that this may be put in the form of Problem 2, and that with $\beta \in (0,1)$, $\gamma, \sigma, \alpha_{\Delta y} \in (0, \infty)$, $\alpha_\pi \in (1, \infty)$, $\rho \in (-1,1)$, Assumption 2 is satisfied. For our numerical exercise, we again set $\sigma = 1$, $\beta = 0.99$, $\gamma = \frac{(1-0.85)(1-\beta(0.85))}{0.85}(2+\sigma)$, $\rho = 0.5$, following BPY.

In Figure 9, we plot the regions in $(\alpha_{\Delta y}, \alpha_\pi)$ space in which M is a P-matrix (P_0-matrix) when $T = 2$ or $T = 4$. For this model, these correspond to the regions in which M is strictly semi-monotone (semi-monotone). As may be seen, in the smaller T case, the P-matrix region is much larger. This relationship appears to continue to hold for both larger and smaller T, with the equivalent $T = 1$ plot being almost entirely shaded, and the large T plot apparently tending to the equivalent plot from the model without monetary policy persistence. Intuitively, the persistence in the shadow nominal interest rate dampens the immediate response of nominal interest rates to inflation and output growth, making it harder to induce a zero lower bound episode over short-horizons.

Further evidence that the long-horizon behaviour is the same as in the model without persistence is provided by the fact that with $\alpha_\pi = 1.5$ and $\alpha_{\Delta y} = 1.05$, [33] then M is a P-matrix with $T = 20$. Moreover, from Proposition 4 with $T = 50$, we have that $\varsigma > 6.385 \times 10^{-8}$, so M is an S-matrix for all sufficiently large T, by Corollary 1.

[33] Results for larger $\alpha_{\Delta y}$ were impossible due to numerical errors.

On the other hand, with $\alpha_\pi = 1.5$ and $\alpha_{\Delta y} = 1.51$, then with $T = 200$, M is not an S-matrix, [34] meaning that for all sufficiently large T, M is not a P-matrix, so there are sometimes multiple solutions. Additionally, from Proposition 4 with $T = 200$, $\varsigma \le 0 +$ numerical error, meaning that it is likely that the model does not always possess a solution, no matter how high is T.

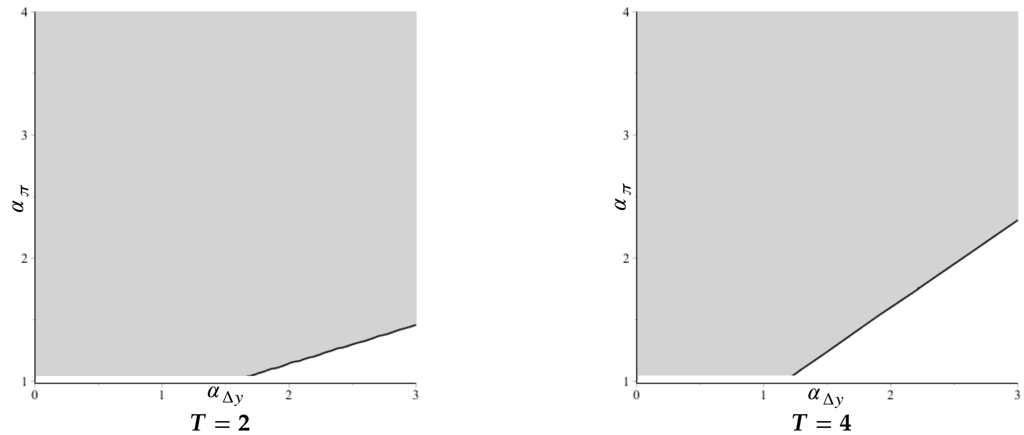

Figure 9: Regions in which M is a P-matrix (shaded grey) or a P$_0$-matrix (shaded grey, plus the black line), when $T = 2$ (left) or $T = 4$ (right).

F. Proof of the properties of the BPY model with level targeting

Defining $x_t = [x_{i,t} \quad x_{y,t} \quad x_{p,t}]'$, the model of section 3.3 is in the form of Problem 2, with:

$$A := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B := \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_\pi \\ -\frac{1}{\sigma} & -1 & -\frac{1}{\sigma} \\ 0 & \gamma & -1-\beta \end{bmatrix}, \qquad C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & \beta \end{bmatrix}.$$

Assumption 2 is satisfied for this model as:

$$\det(A + B + C) = \det \begin{bmatrix} -1 & \alpha_{\Delta y} & \alpha_\pi \\ -\frac{1}{\sigma} & 0 & 0 \\ 0 & \gamma & -1 \end{bmatrix} \neq 0$$

as $\alpha_{\Delta y} \neq 0$ and $\alpha_\pi \neq 0$. Let $f := F_{3,3}$, where F is as in Assumption 1. Then:

$$F = \begin{bmatrix} 0 & 0 & \dfrac{f(1-f)(\sigma\alpha_\pi - \alpha_{\Delta y})}{\alpha_{\Delta y} + (1-f)\sigma} \\ 0 & 0 & \dfrac{f(1-f-\alpha_\pi)}{\alpha_{\Delta y} + (1-f)\sigma} \\ 0 & 0 & f \end{bmatrix},$$

[34] This was verified a second way by checking that $-M'$ was an S$_0$-matrix, as discussed in footnote 14.

and so:

$$\beta\sigma f^3 - \Big((1+2\beta)\sigma + \beta\alpha_{\Delta y} + \gamma\Big)f^2 + \Big((2+\beta)\sigma + (1+\beta)\alpha_{\Delta y} + (1+\alpha_\pi)\gamma\Big)f \\ - \left(\sigma + \alpha_{\Delta y}\right) = 0.$$

Now define:

$$\hat{\alpha}_{\Delta y} := \sigma + \alpha_{\Delta y}, \qquad \hat{\alpha}_\pi := 1 + \alpha_\pi$$

so:

$$\beta\sigma f^3 - \Big((\hat{\alpha}_{\Delta y} + \sigma)\beta + \gamma + \sigma\Big)f^2 + \Big((1+\beta)\hat{\alpha}_{\Delta y} + \gamma\hat{\alpha}_\pi + \sigma\Big)f - \hat{\alpha}_{\Delta y} = 0.$$

This is identical to the equation for f in the previous section, apart from the fact that $\hat{\alpha}_{\Delta y}$ has replaced $\alpha_{\Delta y}$ and $\hat{\alpha}_\pi$ has replaced α_π. Hence, by the results of the previous section, Assumption 1 holds for this model as well.

Finally, for this model, with $T = 1$, we have that:

$$M = \frac{(1-f)(1+(1-f)\beta)\sigma^2 + \Big((1+(1-f)\beta)\alpha_{\Delta y} + \big((1-f) + \alpha_\pi f\big)\gamma\Big)\sigma + (1-f)\gamma\alpha_{\Delta y}}{\Big((1-f)(1+(1-f)\beta)\sigma + (1+(1-f)\beta)\alpha_{\Delta y} + \big((1-f)+\alpha_\pi\big)\gamma\Big)\left(\sigma + \alpha_{\Delta y}\right)} > 0.$$

G. Proof of the sufficient conditions for the existence of a unique solution to the dynamic programming problem

Results when $\tilde{X}$ is possibly non-compact, but $\tilde{\Gamma}(x)$ is compact valued and $x \in \tilde{\Gamma}(x)$ for all $x \in \tilde{X}$ We first note that for all $x, z \in \tilde{X}$:

$$\tilde{\mathcal{F}}(x,z) \le u^{(0)} - \frac{1}{2}u^{(1)}\tilde{u}^{(2)^{-1}}u^{(1)\prime},$$

thus our objective function is bounded above without additional assumptions. For a lower bound, we assume that for all $x \in \tilde{X}$, $x \in \tilde{\Gamma}(x)$, so holding the state fixed is always feasible. This is true in very many standard applications. Then, the value of setting $x_t = x_0$ for all $t \in \mathbb{N}^+$ provides a lower bound for our objective function.

More precisely, we define $\mathbb{V} := \{v | v: \tilde{X} \to [-\infty, \infty)\}$ and $\underline{v}, \overline{v} \in \mathbb{V}$ by:

$$\underline{v}(x) = \frac{1}{1-\beta}\tilde{\mathcal{F}}(x_0, x_0),$$
$$\overline{v}(x) = \frac{1}{1-\beta}\left[u^{(0)} - \frac{1}{2}u^{(1)}\tilde{u}^{(2)^{-1}}u^{(1)\prime}\right],$$

for all $x \in \tilde{X}$.

Finally, define $\mathcal{B}: \mathbb{V} \to \mathbb{V}$ by:

$$\mathcal{B}(v)(x) = \sup_{z \in \tilde{\Gamma}(x)}\left[\tilde{\mathcal{F}}(x,z) + \beta v(z)\right] \tag{18}$$

for all $v \in \mathbb{V}$ and for all $x \in \tilde{X}$. Then $\mathcal{B}(\underline{v}) \ge \underline{v}$ and $\mathcal{B}(\overline{v}) \le \overline{v}$. Furthermore, if some sequence $(x_t)_{t=1}^\infty$ satisfies the constraint that for all $t \in \mathbb{N}^+$, $x_t \in \tilde{\Gamma}(x_{t-1})$, and the objective in (8) is finite for that sequence, then it must be the case that $\|x_t\|_\infty t\beta^{\frac{t}{2}} \to 0$ as $t \to \infty$ (by the comparison test), so:

$$\liminf_{t\to\infty} \beta^t \underline{v}(x_t) = 0.$$

Additionally, for any sequence $(x_t)_{t=1}^{\infty}$:

$$\limsup_{t\to\infty} \beta^t \overline{v}(x_t) = 0.$$

Thus, our dynamic programming problem satisfies the assumptions of Theorem 2.1 of Kamihigashi (2014), and so $\mathcal{B}$ has a unique fixed point in $[\underline{v}, \overline{v}]$ to which $\mathcal{B}^k(\underline{v})$ converges pointwise, monotonically, as $k \to \infty$, and which is equal to the function $v^*: \widetilde{X} \to \mathbb{R}$ defined by:

$$v^*(x_0) = \sup\left\{ \sum_{t=1}^{\infty} \beta^{t-1} \widetilde{\mathcal{F}}(x_{t-1}, x_t) \,\middle|\, \forall t \in \mathbb{N}^+, x_t \in \Gamma(x_{t-1}) \right\}, \tag{19}$$

for all $x_0 \in \widetilde{X}$.

Furthermore, if we define:

$$\mathbb{W} := \{ v \in V | v \text{ is continuous on } \widetilde{X}, v \text{ is concave on } \widetilde{X} \},$$

then as $\tilde{u}^{(2)}$ is negative-definite, $\underline{v} \in \mathbb{W}$. Additionally, under the assumption that $\tilde{\Gamma}(x)$ is compact valued, if $v \in \mathbb{W}$, then $\mathcal{B}(v) \in \mathbb{W}$, by the theorem of the maximum,[35] and, furthermore, there is a unique policy function which attains the supremum in the definition of $\mathcal{B}(v)$. Moreover, $v^* = \lim_{k\to\infty} \mathcal{B}^k(\underline{v})$ is concave and lower semi-continuous on $\widetilde{X}$.[36] We just need to prove that v^* is upper semi-continuous.[37] Suppose for a contradiction that it is not, so there exists $x^* \in \widetilde{X}$ such that:

$$\limsup_{x\to x^*} v^*(x) > \lim_{k\to\infty} v^*(x^*).$$

Then, there exists $\delta > 0$ such that for all $\epsilon > 0$, there exists $x_0^{(\epsilon)} \in \widetilde{X}$ with $\|x^* - x_0^{(\epsilon)}\|_\infty < \epsilon$ such that:

$$v^*\left(x_0^{(\epsilon)}\right) > \delta + v^*(x^*).$$

Now, by the definition of a supremum, for all $\epsilon > 0$, there exists $\left(x_t^{(\epsilon)}\right)_{t=1}^{\infty}$ such that for all $t \in \mathbb{N}^+$, $x_t^{(\epsilon)} \in \Gamma\left(x_{t-1}^{(\epsilon)}\right)$ and:

$$v^*\left(x_0^{(\epsilon)}\right) < \delta + \sum_{t=1}^{\infty} \beta^{t-1} \widetilde{\mathcal{F}}\left(x_{t-1}^{(\epsilon)}, x_t^{(\epsilon)}\right).$$

Hence:

$$\sum_{t=1}^{\infty} \beta^{t-1} \widetilde{\mathcal{F}}\left(x_{t-1}^{(\epsilon)}, x_t^{(\epsilon)}\right) > v^*\left(x_0^{(\epsilon)}\right) - \delta > v^*(x^*).$$

Now, let $\mathcal{S}_0 := \{ x \in \widetilde{X} | \|x^* - x\|_\infty \le 1 \}$, and for $t \in \mathbb{N}^+$, let $\mathcal{S}_t := \Gamma(\mathcal{S}_{t-1})$. Then, since we are assuming Γ is compact valued, for all $t \in \mathbb{N}$, $\mathcal{S}_t$ is compact by the continuity of Γ. Furthermore, for all $t \in \mathbb{N}$ and $\epsilon \in (0,1)$, $x_t^{(\epsilon)} \in \mathcal{S}_t$. Hence, $\prod_{t=0}^{\infty} \mathcal{S}_t$ is sequentially compact in the product topology. Thus, there exists a sequence $(\epsilon_k)_{k=1}^{\infty}$ with $\epsilon_k \to 0$ as $k \to \infty$ and such that $x_t^{(\epsilon_k)}$ converges for all $t \in \mathbb{N}$. Let $x_t := \lim_{k\to\infty} x_t^{(\epsilon_k)}$, and note that

[35] See e.g. Theorem 3.6 and following of Stokey, Lucas, and Prescott (1989).

[36] See e.g. Lemma 2.41 of Aliprantis and Border (2013).

[37] In the following, we broadly follow the proof of Lemma 3.3 of Kamihigashi and Roy (2003).

$x^* = x_0 \in \mathcal{S}_0 \subseteq \widetilde{X}$, and that for all $t, k \in \mathbb{N}^+$, $x_t^{(\epsilon_k)} \in \Gamma\left(x_{t-1}^{(\epsilon_k)}\right)$, so by the continuity of Γ, $x_t \in \Gamma(x_{t-1})$ for all $t \in \mathbb{N}^+$. Thus, by Fatou's Lemma:

$$v^*(x^*) \geq \sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}(x_{t-1}, x_t) \geq \limsup_{k \to \infty} \sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}\left(x_{t-1}^{(\epsilon,k)}, x_t^{(\epsilon,k)}\right) > v^*(x^*),$$

which gives the required contradiction. Thus v^* is continuous and concave, and there is a unique policy function which attains the supremum in the definition of $\mathcal{B}(v^*) = v^*$.

Results when $\tilde{X}$ is compact If $\widetilde{X}$ is compact, then Γ is compact valued. Furthermore, $\widetilde{X}$ is clearly convex, and Γ is continuous. Thus assumption 4.3 of Stokey, Lucas, and Prescott (1989) (henceforth: SLP) is satisfied. Since the continuous image of a compact set is compact, $\tilde{\mathcal{F}}$ is bounded above and below, so assumption 4.4 of SLP is satisfied as well. Furthermore, $\tilde{\mathcal{F}}$ is concave and Γ is convex, so assumptions 4.7 and 4.8 of SLP are satisfied too. Thus, by theorem 4.6 of SLP, with $\mathcal{B}$ defined as in equation (17) and v^* defined as in equation (18), $\mathcal{B}$ has a unique fixed point which is continuous and equal to v^*. Moreover, by theorem 4.8 of SLP, there is a unique policy function which attains the supremum in the definition of $\mathcal{B}(v^*) = v^*$.

H. Proof of the sufficiency of the KKT and limit conditions

Suppose that $(x_t)_{t=1}^{\infty}$, $(\lambda_t)_{t=1}^{\infty}$ satisfy the KKT conditions given in equations (10) and (11), and that $x_t \to \mu$ and $\lambda_t \to \overline{\lambda}$ as $t \to \infty$. Let $(z_t)_{t=0}^{\infty}$ satisfy $z_0 = x_0$ and $z_t \in \tilde{\Gamma}(z_{t-1})$ for all $t \in \mathbb{N}^+$. Then, by the KKT conditions and the concavity of:

$$(x_{t-1}, x_t) \mapsto \tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right],$$

we have that for all $T \in \mathbb{N}^+$:[38]

$$\begin{aligned}
&\sum_{t=1}^{T} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right] \\
&= \sum_{t=1}^{T} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right] - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right] \\
&\geq \sum_{t=1}^{T} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) + \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix} \right] - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right. \\
&\qquad\qquad \left. - \lambda_t' \left[\Psi^{(0)} + \Psi^{(1)} \begin{bmatrix} z_{t-1} - \mu \\ z_t - \mu \end{bmatrix} \right] \right] \\
&\geq \sum_{t=1}^{T} \beta^{t-1} \left[\left[u_{\cdot,2}^{(1)} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}_{\cdot,2}^{(2)} + \lambda_t' \Psi_{\cdot,2}^{(1)} \right] (x_t - z_t) \right. \\
&\qquad\qquad \left. + \left[u_{\cdot,1}^{(1)} + \begin{bmatrix} x_{t-1} - \mu \\ x_t - \mu \end{bmatrix}' \tilde{u}_{\cdot,1}^{(2)} + \lambda_t' \Psi_{\cdot,1}^{(1)} \right] (x_{t-1} - z_{t-1}) \right]
\end{aligned}$$

[38] Here, we broadly follow the proof of Theorem 4.15 of Stokey, Lucas, and Prescott (1989).

$$
\begin{aligned}
&= \sum_{t=1}^{T} \beta^{t-1} \Bigg[\Bigg[u^{(1)}_{\cdot,2} + \begin{bmatrix} x_{t-1}-\mu \\ x_t-\mu \end{bmatrix}' \tilde{u}^{(2)}_{\cdot,2} + \lambda_t' \Psi^{(1)}_{\cdot,2} \\
&\qquad\qquad + \beta \left[u^{(1)}_{\cdot,1} + \begin{bmatrix} x_t-\mu \\ x_{t+1}-\mu \end{bmatrix}' \tilde{u}^{(2)}_{\cdot,1} + \lambda_{t+1}' \Psi^{(1)}_{\cdot,1} \right] \Bigg] (x_t - z_t) \Bigg] \\
&\qquad\qquad + \beta^T \left[u^{(1)}_{\cdot,1} + \begin{bmatrix} x_T-\mu \\ x_{T+1}-\mu \end{bmatrix}' \tilde{u}^{(2)}_{\cdot,1} + \lambda_{T+1}' \Psi^{(1)}_{\cdot,1} \right] (z_T - x_T) \\
&= \beta^T \left[u^{(1)}_{\cdot,1} + \begin{bmatrix} x_T-\mu \\ x_{T+1}-\mu \end{bmatrix}' \tilde{u}^{(2)}_{\cdot,1} + \lambda_{T+1}' \Psi^{(1)}_{\cdot,1} \right] (z_T - x_T).
\end{aligned}
$$

Thus:

$$
\begin{aligned}
&\sum_{t=1}^{\infty} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right] \\
&\qquad \geq \lim_{T\to\infty} \beta^T \left[u^{(1)}_{\cdot,1} + \begin{bmatrix} x_T-\mu \\ x_{T+1}-\mu \end{bmatrix}' \tilde{u}^{(2)}_{\cdot,1} + \lambda_{T+1}' \Psi^{(1)}_{\cdot,1} \right] (z_T - x_T) \\
&\qquad = \lim_{T\to\infty} \beta^T \left[u^{(1)}_{\cdot,1} + \bar{\lambda}' \Psi^{(1)}_{\cdot,1} \right] (z_T - \mu) = \lim_{T\to\infty} \beta^T \left[u^{(1)}_{\cdot,1} + \bar{\lambda}' \Psi^{(1)}_{\cdot,1} \right] z_T.
\end{aligned}
$$

Now, suppose $\lim_{T\to\infty} \beta^T z_T \neq 0$, then since $\tilde{u}^{(2)}$ is negative definite:

$$
\sum_{t=1}^{\infty} \beta^{t-1} \tilde{\mathcal{F}}(z_{t-1}, z_t) = -\infty,
$$

so $(z_t)_{t=0}^{\infty}$ cannot be optimal. Hence, regardless of the value of $\lim_{T\to\infty} \beta^T z_T$:

$$
\sum_{t=1}^{\infty} \beta^{t-1} \left[\tilde{\mathcal{F}}(x_{t-1}, x_t) - \tilde{\mathcal{F}}(z_{t-1}, z_t) \right] \geq 0,
$$

which implies that $(x_t)_{t=1}^{\infty}$ solves Problem 5.

I. Results from and for general dynamic programming problems

Here we consider non-linear dynamic programming problems with general objective functions. Consider then the following generalisation of Problem 5:

Problem 6 Suppose $\Gamma: \mathbb{R}^n \to \mathbb{P}(\mathbb{R}^n)$ is a given compact, convex valued continuous function. Define $X := \{x \in \mathbb{R}^n | \Gamma(x) \neq \emptyset\}$, and suppose without loss of generality that for all $x \in \mathbb{R}^n$, $\Gamma(x) \cap X = \Gamma(x)$. Further suppose that $\mathcal{F}: X \times X \to \mathbb{R}$ is a given twice continuously differentiable, concave function, and that $x_0 \in X$ and $\beta \in (0,1)$ are given.

Choose $x_1, x_2, \ldots$ to maximise:

$$
\liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1} \mathcal{F}(x_{t-1}, x_t),
$$

subject to the constraints that for all $t \in \mathbb{N}^+$, $x_t \in \Gamma(x_{t-1})$.

For tractability, we make the following additional assumption, which enables us to uniformly approximate Γ by a finite number of inequalities:

Assumption 8 X is compact.

Then, by theorem 4.8 of Stokey, Lucas, and Prescott (1989), there is a unique solution to Problem 6 for any x_0. We further assume the following to ensure that there is a natural point to approximate around:[39]

Assumption 9 There exists $\mu \in X$ such that for any given $x_0 \in X$, in the solution to Problem 6 with that x_0, as $t \to \infty$, $x_t \to \mu$.

Having defined μ, we can let $\tilde{\mathcal{F}}$ be a second order Taylor approximation to $\mathcal{F}$ around μ, which will take the form of equation (7). Assumption 3 will be satisfied for this approximation thanks to the concavity of $\mathcal{F}$. To apply the previous results, we also then need to approximate the constraints.

Suppose first that the graph of Γ is convex, i.e. the set $\{(x, z) | x \in X, z \in \Gamma(x)\}$ is convex. Since it is also compact, by Assumption 4, for any $\epsilon > 0$, there exists $c \in \mathbb{N}$, $\Psi^{(0)} \in \mathbb{R}^{c\times 1}$ and $\Psi^{(1)} \in \mathbb{R}^{c\times 2n}$ such that with $\tilde{\Gamma}$ defined as in equation (5) and $\tilde{X}$ defined as in equation (6):

1) $\mu \in \tilde{X} \subseteq X$,
2) for all $x \in X$, there exists $\tilde{x} \in \tilde{X}$ such that $\|x - \tilde{x}\|_2 < \epsilon$,
3) for all $x \in \tilde{X}$, $\tilde{\Gamma}(x) \subseteq \Gamma(x)$,
4) for all $x \in \tilde{X}$, and for all $z \in \Gamma(x)$, there exists $\tilde{z} \in \tilde{\Gamma}(x)$ such that $\|z - \tilde{z}\|_2 < \epsilon$.

(This follows from standard properties of convex sets.) Then, by our previous results, the following proposition is immediate:

Proposition 17 Suppose we are given a problem in the form of Problem 6 (and which satisfies Assumption 4 and Assumption 5). If the graph of Γ is convex, then we can construct a problem in the form of the multiple-bound generalisation of Problem 2 which encodes a local approximation to the original dynamic programming problem around $x_t = \mu$. Furthermore, the LCP corresponding to this approximation will have a unique solution for all $x_0 \in \tilde{X}$. Moreover, the approximation is consistent for quadratic objectives in the sense that as the number of inequalities used to approximate Γ goes to infinity, the approximate value function converges uniformly to the true value function.

Unfortunately, if the graph of Γ is non-convex, then we will not be able to derive similar results. To see the best we could do along similar proof lines, here we merely sketch the construction of an approximation to the graph of Γ in this case. We will need to assume that there exists $z \in \operatorname{int}\Gamma(x)$ for all $x \in X$, which precludes the existence of equality constraints.[40] We first approximate the graph of Γ by a polytope (i.e. n dimensional polygon) contained in the graph of Γ such that all points in the

[39] If X is convex, then the existence of a fixed point of the policy function is a consequence of the Brouwer fixed point theorem, but there is no reason the fixed point guaranteed by Brouwer's theorem should be even locally attractive.
[40] This is often not too much of a restriction, since equality constraints may be substituted out.

graph of Γ are within $\frac{\epsilon}{2}$ of a point in the polytope. Then, providing ϵ is sufficiently small, for each simplicial surface element of the polytope, indexed by $k \in \{1, \dots, c\}$, we can find a quadratic function $q_k: X \times X \to \mathbb{R}$ with:

$$q_k = \Psi_k^{(0)} + \Psi_{k,\cdot}^{(1)} \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix} + \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix}' \Psi_k^{(2)} \begin{bmatrix} x - \mu \\ z - \mu \end{bmatrix}$$

for all $x, z \in X$ and such that q_k is zero at the corners of the simplicial surface element, such that q_k is weakly negative on its surface, such that $\Psi_k^{(2)}$ is symmetric positive definite, and such that all points in the polytope are within $\frac{\epsilon}{2}$ of a point in the set:

$$\{(x, z) \in X \times X | \forall k \in \{1, \dots, S\}, 0 \le q_k(x, z)\}.$$

This gives a set of quadratic constraints that approximate Γ. If we then define:

$$\tilde{u}^{(2)} := u^{(2)} + \sum_{k=1}^{c} \bar{\lambda}'_{\Psi,k} \Psi_k^{(2)},$$

where $u^{(2)}$ is the Hessian of $\mathcal{F}$, then the Lagrangian in equation (9) is the same as what would be obtained from taking a second order Taylor approximation to the Lagrangian of the problem of maximising our non-linear objective subject to the approximate quadratic constraints, suggesting it may perform acceptably well for x near μ, along similar lines to the results of Levine, Pearlman, and Pierse (2008) and Benigno and Woodford (2012). However, existence of a unique solution to the original problem cannot be used to establish even the existence of a solution of the approximated problem, since only linear approximations to the quadratic constraints would be imposed by our algorithm, giving a greatly reduced choice set (as the quadratic terms are positive definite).