Existence, uniqueness and computation of solutions to DSGE models with occasionally binding constraints.

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Motivation

- Since the financial crisis, many central banks around the world have set their nominal interest rates near zero.
- Additionally, during the crisis many households, firms and banks were pushed up against their borrowing constraints.

- The zero lower bound on nominal interest rates and borrowing constraints are prominent examples of occasionally binding constraints (OBCs).

- At present though, little is known about the behaviour of DSGE models featuring such constraints, and there is no robust, accurate and scalable algorithm for simulating such paper.

This paper

- “Blanchard and Kahn (1980) with occasionally binding constraints.”

- Two elements to this:

    - Existence and uniqueness results.

    - A new computational algorithm designed to be robust, accurate and scalable.

- We also present the DynareOBC toolkit, which is one stop shop for all things OBC.
    - And designed to be super easy to use!

Existence and uniqueness results (1/2)

- I give necessary and sufficient conditions for existence of a unique solution *about a given steady-state* to models with OBCs.
  - I.e., we assume that agents believe that if no shocks arrived in the future, the economy would eventually return to the original steady-state, rather than getting stuck in an alternative state in which the constraint always bounds.
  - In line with the approach of Brendon, Paustian, and Yates (2015).
  - Contrary to the approach of, e.g. Benhabib, Schmitt-Grohe, and Uribe (2001a,b), Schmitt-Grohe and Uribe (2012), Mertens and Ravn (2014), Aruoba, Cuba-Borda and Schorfheide (2013).
  - I will omit saying “about a given steady-state” in the following.

Existence and uniqueness results (2/2)

- Furthermore, for models with OBCs, I give:
  - Necessary and sufficient conditions for existence of a unique solution when away from the bound.
    - E.g. suppose that the impulse response to some shock does not hit the bound. Must it be the unique solution?
  - Some necessary conditions, and some sufficient conditions for existence of any solution.
  - Necessary and sufficient conditions for having a finite/convex set of solutions.
  - Sufficient conditions for polynomial time computability of the solution.

A preview of the application of our results to Smets and Wouters (2003; 2007)

- For both the Smets and Wouters (2003) and the Smets and Wouters (2007) models, when they are augmented with a zero lower bound, at their estimated modes:
  - There are combinations of predicted future shocks for which there are multiple solutions to the model.
  - There are combinations of predicted future shocks for which in the absence of a bound, nominal interest rates would always be positive, and yet when the bound is included in the model, there are multiple solutions.
  - There are combinations of predicted future shocks for which there are zero solutions to the model.
  - No known algorithm can solve for a perfect foresight solution to either model for all possible sequences of predicted future shocks in time polynomial in the simulation horizon.
- It is easier (in a sense to be made clear) to generate indeterminacy in Smets and Wouters (2003) than in Smets and Wouters (2007).

New computational algorithm: Three “layers” of increasing accuracy

- Layer 1 is a very efficient perfect foresight solver for models with occasionally binding constraints.
  - Unique in being guaranteed to find a solution in finite time if one exists.
  - If no solution exists, will also report this in finite time too.
  - Also applies to high order pruned perturbation solutions to the model.

- Layer 2 integrates over future uncertainty.
  - Exploits the convenient properties of pruned perturbation solutions to very efficiently integrate over a moderate number of periods of future uncertainty.

- Layer 3 captures the long run effects of uncertainty through a hybrid local/global algorithm.
  - Calculates global solutions with no numerical integration, and no simulation.

Outline

- Theoretical results:
  - Problem set-up.
  - Introduction to linear complementarity problems and the associated matrix classes.
  - Existence results.
  - Uniqueness results.
  - Other theoretical results.
  - Application to select models.
- The computational algorithm:
  - Perfect foresight problems.
  - Application of the perfect foresight solver to non-linear models.
  - Integrating over future uncertainty.
  - The hybrid local/global algorithm.
- Select accuracy results.

The set-up without bounds (1/2)

- Suppose for $t \in \mathbb{N}^+$:

$$\left(\hat{A} + \hat{B} + \hat{C}\right)\hat{\mu} = \hat{A}\hat{x}_{t-1} + B\hat{x}_t + \hat{C}\mathbb{E}_t\hat{x}_{t+1} + \widehat{D}\varepsilon_t,$$

  - where $\mathbb{E}_{t-1}\varepsilon_t = 0$ for all $t \in \mathbb{N}^+$,
  - $\hat{x}_0$ is given as an initial condition,
  - $\varepsilon_t = 0$ for $t > 1$, (impulse response/perfect foresight simulation),
  - terminal condition: $\hat{x}_t \to \hat{\mu}$ as $t \to \infty$.

- For $t \in \mathbb{N}^+$, define:

$$x_t \coloneqq \begin{bmatrix} \hat{x}_t \\ \varepsilon_{t+1} \end{bmatrix}, \qquad \mu \coloneqq \begin{bmatrix} \hat{\mu} \\ 0 \end{bmatrix}, \qquad A \coloneqq \begin{bmatrix} \hat{A} & \widehat{D} \\ 0 & 0 \end{bmatrix}, \qquad B \coloneqq \begin{bmatrix} \widehat{B} & 0 \\ 0 & I \end{bmatrix}, \qquad C \coloneqq \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix}$$

- then, for $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1},$$

- and $x_0 = \begin{bmatrix} \hat{x}_0 \\ \varepsilon_1 \end{bmatrix}$, $x_t \to \mu$ as $t \to \infty$.

- Take this as the form of our problem without bounds in the following.
  - I.e. we can forget about shocks at least.

The set-up without bounds (2/2)

- **Problem 1**
- Suppose that $x_0 \in R^n$ is given. Find $x_t \in R^n$ for $t \in N^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in N^+$:
$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1}.$$

- **Assumption 1**
- For any given $x_0 \in R^n$, Problem 1 has a unique solution, which takes the form $x_t = (I - F)\mu + Fx_{t-1}$, for $t \in N^+$, where $F = -(B + CF)^{-1}A$.

The set-up with bounds

- **Problem 2**
- Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $T \in \mathbb{N}$ and $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$x_{1,t} = \max\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + \left(B_{1,\cdot} + I_{1,\cdot}\right)(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu)\},$$
$$\left(A_{-1,\cdot} + B_{-1,\cdot} + C_{-1,\cdot}\right)\mu = A_{-1,\cdot}x_{t-1} + B_{-1,\cdot}x_t + C_{-1,\cdot}x_{t+1},$$

- and such that $x_{1,t} > 0$ for $t > T$.
- Ruling out solutions which get stuck at another steady-state by assumption.

The news shock set-up

- **Problem 3**
- Suppose that $T \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^T$ is given. Find $x_t \in \mathbb{R}^n, y_t \in \mathbb{R}^T$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$, $y_t \to 0$, as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1} + I_{\cdot,1} I_{1,\cdot} y_{t-1},$$
$$\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t} = y_{i+1,t-1},$$
$$y_{T,t} = 0.$$

- A version of Problem 1 with news shocks up to horizon T added to the first equation.
  - The value of $y_{t,0}$ gives the news shock that hits in period t, i.e. $I_{1,\cdot} y_{t-1} = y_{1,t-1} = y_{t,0}$ for $t \leq T$, and $I_{1,\cdot} y_{t-1} = y_{1,t-1} = 0$ for $t > T$.

A representation of solutions to Problem 3

- **Lemma 1**: There is a unique solution to Problem 3 that is linear in x_0 and y_0.

- Let $x_t^{(3,k)}$ be the solution to Problem 3 when $x_0 = \mu$, $y_0 = I_{\cdot,k}$.
- Let $M \in \mathbb{R}^{T \times T}$ satisfy:

$$M_{t,k} = x_{1,t}^{(3,k)} - \mu_1, \qquad \forall t, k \in \{1, .., T\},$$

  - i.e. M horizontally stacks the (column-vector) relative impulse responses to the news shocks.
- Let $x_t^{(1)}$ be the solution to Problem 1 for some given x_0.
- Then the solution to Problem 3 for given x_0, y_0 satisfies:

$$\left(x_{1,1 \ldots T}\right)' = q + M y_0,$$

- where $q \coloneqq \left(x_{1,1 \ldots T}^{(1)}\right)'$.

The links between the solutions to Problem 2 and the solution to Problem 3 (1/2)

- Let $x_t^{(2)}$ be a solution to Problem 2 given an arbitrary x_0.
- Define:

$$e_t \coloneqq \begin{cases} -\left[I_{1,\cdot}\mu + A_{1,\cdot}\left(x_{t-1}^{(2)} - \mu\right) + \left(B_{1,\cdot} + I_{1,\cdot}\right)\left(x_t^{(2)} - \mu\right) + C_{1,\cdot}\left(x_{t+1}^{(2)} - \mu\right)\right] & \text{if } x_{1,t}^{(2)} = 0 \\ 0 & \text{if } x_{1,t}^{(2)} > 0 \end{cases}$$

- **Lemma 2:** The following statements hold:
  - $e_{1\ldots T} \geq 0$, $x_{1,1\ldots T}^{(2)} \geq 0$ and $x_{1,1\ldots T}^{(2)} \circ e_{1\ldots T} = 0$,
  - $x_t^{(2)}$ is the unique solution to Problem 3 when started with $x_0 = x_0^{(2)}$ and with $y_0 = e_{1\ldots T}'$.
  - If $x_t^{(2)}$ solves Problem 3 when started with $x_0 = x_0^{(2)}$ and with some y_0, then $y_0 = e_{1\ldots T}'$.

The links between the solutions to Problem 2 and the solution to Problem 3 (2/2)

- **Proposition 1**: The following statements hold:
  - Let $x_t^{(3)}$ be the unique solution to Problem 3 when initialized with some x_0, y_0. Then $x_t^{(3)}$ is a solution to Problem 2 when initialized with x_0 if and only if $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$ and $x_{1,t}^{(3)} \geq 0$ for all $t \in \mathbb{N}$ with $t > T$.
  - Let $x_t^{(2)}$ be any solution to Problem 2 when initialized with x_0. Then there exists a $y_0 \in \mathbb{R}^T$ such that $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$, such that $x_t^{(2)}$ is the unique solution to Problem 3 when initialized with x_0, y_0.

Linear complementarity problems (LCPs)

- The previous proposition establishes that solving the model with occasionally binding constraints is equivalent to solving the following "linear complementarity problem".

- **Problem 4**
- Suppose $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given. Find $y \in \mathbb{R}^T$ such that: $y \geq 0$, $y \circ (q + My) = 0$ and $q + My \geq 0$.

- We call this the linear complementarity problem (q, M).

Is our M matrix special? (1/2)

- The properties of solutions to LCPs are determined by the properties of the M matrix.
  - One might think that ours would have "nice" properties because of where it came from.
  - Unfortunately this is false.
- In fact:

- **Proposition 2**
- For any matrix $\mathcal{M} \in \mathbb{R}^{T \times T}$, there exists a model in the form of Problem 2 with a number of state variables given by a quadratic in T, such that $M = \mathcal{M}$ for that model.

Is our M matrix special? (2/2)

- The proof is based on the following model, for some arbitrary matrix $\mathcal{M} \in \mathbb{R}^{T\times T}$:

$$a_t = \max\{0, \mathcal{b}_t\}, \qquad a_t = 1 + \sum_{j=1}^{T}\sum_{k=1}^{T} \mathcal{M}_{j,k}\left(c_{j-1,k-1,t} - c_{j,k,t}\right),$$

$$c_{0,0,t} = a_t - \mathcal{b}_t,$$

$$c_{0,k,t} = \mathbb{E}_t c_{0,k-1,t+1}, \qquad \forall k \in \{1, \dots, T\},$$

$$c_{j,k,t} = c_{j-1,k,t-1}, \qquad \forall j \in \{1, \dots, T\}, k \in \{0, \dots, T\},$$

  - with steady-state $a_{\cdot} = \mathcal{b}_{\cdot} = 1$, $c_{j,k,\cdot} = 0$ for all $j, k, \in \{0, \dots, T\}$.

- For this model, the M matrix equals $\mathcal{M}$.

Introduction to the relevant matrix classes for the LCP

- Properties of solutions of LCPs have been extensively studied in the linear algebra and optimization literatures.

- As previously mentioned, all existence and uniqueness results are given in terms of the properties of the matrix M.

- Unfortunately, the required properties are rather harder to state (and check) than just looking at a few eigenvalues.

- In the next few slides, I give definitions of the key properties.
  - The chief references for the below are Cottle, Pang, and Stone (2009) and Xu (1993).

Principal sub matrices and principal minors

- For a matrix $M \in \mathbb{R}^{T\times T}$, the **principal sub-matrices** of M are the matrices:

$$\left\{ \left[M_{i,j}\right]_{i,j=k_1,\dots,k_S} \middle| S, k_1, \dots, k_S \in \{1,\dots,T\}, k_1 < k_2 < \cdots < k_S \right\},$$

  - i.e. the **principal sub-matrices** of M are formed by deleting the same rows and columns.

- The **principal minors** of M are the collection of values:

$$\left\{ \det\left(\left[M_{i,j}\right]_{i,j=k_1,\dots,k_S} \right) \middle| S, k_1, \dots, k_S \in \{1,\dots,T\}, k_1 < k_2 < \cdots < k_S \right\},$$

  - i.e. the **principal minors** of M are the determinants of the principal sub-matrices of M.

(Non-)Degenerate matrices, P($_0$)-matrices and General positive (semi-)definite matrices

- A matrix $M \in \mathbb{R}^{T\times T}$ is called a **non-degenerate matrix** if the principal minors of M are all non-zero. M is called a **degenerate matrix** if it is not a non-degenerate matrix.

- A matrix $M \in \mathbb{R}^{T\times T}$ is called a **P-matrix** if the principal minors of M are all strictly positive. M is called a **P_0-matrix** if the principal minors of M are all non-negative.
  - *Note: for symmetric M, M is a P-matrix if and only if all of its eigenvalues are strictly positive, and M is a P_0-matrix if and only if all of its eigenvalues are non-negative.*

- A matrix $M \in \mathbb{R}^{T\times T}$ is called **general positive definite** if $M + M'$ is a P-matrix. If $M + M'$ is a P_0-matrix, then M is called **general positive semi-definite**.
  - *Note: that we do not require that M is symmetric in either case, but that if M is symmetric, then, M is general positive-definite if and only if it is a P-matrix and M is a general positive semi-definite if and only if it is a P_0-matrix, so in this case the definitions coincide with the standard ones.*

S($_0$)-matrices, (Strictly) Semi-monotone matrices and (Strictly) Copositive matrices

- A matrix $M \in \mathbb{R}^{T\times T}$ is called an **S-matrix** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \gg 0$. M is called an **S$_0$-matrix** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \geq 0$.

- A matrix $M \in \mathbb{R}^{T\times T}$ is called **strictly semi-monotone** if each of its principal sub-matrices is an **S-matrix**. M is called **semi-monotone** if each of its principal sub-matrices is an **S$_0$-matrix**.

- A matrix $M \in \mathbb{R}^{T\times T}$ is called **strictly copositive** if $M + M'$ is strictly semi-monotone. If $M + M'$ is semi-monotone then M is called **copositive**.

Sufficient matrices

- Let $M \in \mathbb{R}^{T \times T}$. M is called **column sufficient** if M is a P_0-matrix, and for each principal sub-matrix $W := \left[M_{i,j}\right]_{i,j=k_1,\ldots,k_S}$ of M, with zero determinant, and for each proper principal sub-matrix $\left[W_{i,j}\right]_{i,j=l_1,\ldots,l_R}$ of W ($R < S$), with zero determinant, the columns of $\left[W_{i,j}\right]_{\substack{i=1,\ldots,S \\ j=l_1,\ldots,l_R}}$ do not form a basis for the column space of W.

- M is called **row sufficient** if M' is column sufficient.

- M is called **sufficient** if it is column sufficient and row sufficient.

Relationships between the matrix classes (Cottle, Pang and Stone 2009)

- All general positive semi-definite matrices are copositive and sufficient.
- P_0 includes skew-symmetric matrices, general positive semi-definite matrices, sufficient matrices and P-matrices.
- All P_0-matrices, and all copositive matrices are semi-monotone, and all P-matrices, and all strictly copositive matrices are strictly semi-monotone.

- All general positive semi-definite, semi-monotone, sufficient, P_0 and copositive matrices have non-negative diagonals, and all general positive definite, strictly semi-monotone, P and strictly copositive matrices have strictly positive diagonals.

Existence results (1/4)

- We would ideally like a solution to exist for all possible q, since there are predicted shocks which can bring about any such q.

- Suppose $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given. The LCP corresponding to M and q is called **feasible** if there exists $y \in \mathbb{R}^T$ such that $y \geq 0$ and $q + My \geq 0$.

- **Proposition 3:** The LCP (q, M) is feasible for all $q \in \mathbb{R}^T$ if and only if M is an S-matrix. (Cottle, Pang, and Stone 2009)

- Thus M being an S-matrix is a necessary condition for a solution to exist for all q.

Existence results (2/4)

- **Proposition 4:** The LCP (q, M) is solvable if it is feasible and, either:
  - M is row-sufficient, or,
  - M is copositive and for all non-singular principal sub-matrices $W \coloneqq \left[M_{i,j}\right]_{i,j=k_1,\dots,k_S}$ of M, all non-negative columns of W^{-1} possess a non-zero diagonal element.
- (Cottle, Pang, and Stone 2009; Väliaho 1986)

- This gives sufficient conditions for existence for feasible q.
  - Checking feasibility just requires solving a linear programming problem, which is possible in time polynomial in T.

Existence results (3/4)

- Proposition 5: The LCP (q, M) is solvable for all $q \in \mathbb{R}^T$, if at least one of the following conditions holds:
  - M is an S-matrix, and either condition 1 or condition 2 of Proposition 4 are satisfied.
  - M is copositive and non-degenerate.
  - M is a P-matrix, a strictly copositive matrix or a strictly semi-monotone matrix.
- (Cottle, Pang, and Stone 2009)

- This gives sufficient conditions for existence for all q.

Existence results (4/4)

- In the special case in which M has nonnegative entries, we have both necessary and sufficient conditions:

- **Proposition 6**
- If M is a matrix with nonnegative entries, then the LCP (q, M) is solvable for all $q \in \mathbb{R}^T$, if and only if M has a strictly positive diagonal. (Cottle, Pang, and Stone 2009)

Uniqueness results (1/3)

- General necessary and sufficient conditions do exist for uniqueness:

- **Proposition 7**
- The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$, if and only if M is a P-matrix.
- (Samelson, Thrall, and Wesler 1958; Cottle, Pang, and Stone 2009)

- This is the analogue of the Blanchard-Kahn conditions for models with occasionally binding constraints.
  - It appears that this is often satisfied in models of irreversible investment, but rarely satisfied in New Keynesian models with zero lower bounds.

Uniqueness results (2/3)

- While for many models, the previous condition does not hold, we would hope that at least for $q \geq 0$ there ought to still be a unique solution.
- Conditions for this follow:

- **Proposition 8**
- The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$ with $q \gg 0$ if and only if M is semi-monotone. (Cottle, Pang, and Stone 2009)

- **Proposition 9**
- The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$ with $q \geq 0$ if and only if M is strictly semi-monotone. (Cottle, Pang, and Stone 2009)

Uniqueness results (3/3)

- Thus, if M is not semi-monotone, there are some $q \gg 0$ such that the LCP (q, M) has multiple solutions.

- I.e., if agents today got appropriate signals about future shocks, then the economy could jump to the bound, even though the bound would not have been violated had it not been there at all.

- This remains true even if shocks are arbitrarily small, and even if the steady-state is arbitrarily far away from the bound.

Further properties of the solution set

- **Proposition 10**
- The LCP (q, M) has a finite (possibly zero) number of solutions for all $q \in \mathbb{R}^T$ if and only if M is non-degenerate. (Cottle, Pang, and Stone 2009)

- **Proposition 11**
- The LCP (q, M) has a convex (possibly empty) set of solutions for all $q \in \mathbb{R}^T$ if and only if M is column sufficient. (Cottle, Pang, and Stone 2009)

Generalisations

- For multiple bounds:
  - We stack the impulse responses of the bounded variables ignoring bounds into q.
  - We stack the vectors of news shocks to each variable into y.
  - M is a block matrix of each bounded variable's responses to each bounded variable's news shocks.
  - Then the stacked solution for the paths of the bounded variables is $q + My$, and we again have an LCP, so results go through as before.
- For bounds not at zero:
  - If $z_{1,t} = \max\{z_{2,t}, z_{3,t}\}$, then $z_{1,t} - z_{2,t} = \max\{0, z_{3,t} - z_{2,t}\}$.
- For minimums:
  - If $z_{1,t} = \min\{z_{2,t}, z_{3,t}\}$, then $-z_{1,t} = \max\{-z_{2,t}, -z_{3,t}\}$.

Example 1: Simple Brendon, Paustian, and Yates (2015) model (1/2)

- Equations:

$$x_{i,t} = \max\left\{0, 1 - \beta + \alpha_{\Delta y}\left(x_{y,t} - x_{y,t-1}\right) + \alpha_\pi x_{\pi,t}\right\},$$
$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}\left(x_{i,t} + \beta - 1 - \mathbb{E}_t x_{\pi,t+1}\right),$$
$$x_{\pi,t} = \beta \mathbb{E}_t x_{\pi,t+1} + \gamma x_{y,t},$$

- $\beta \in (0,1)$, $\gamma, \sigma, \alpha_{\Delta y} \in (0, \infty)$, $\alpha_\pi \in (1, \infty)$.
- Unique stationary solution.
- If $T = 1$, then:

$$M = \frac{\beta\sigma f^2 - \left((1+\beta)\sigma + \gamma\right) f + \sigma}{\beta\sigma f^2 - \left((1+\beta)\sigma + \gamma + \beta\alpha_{\Delta y}\right) f + \sigma + \alpha_{\Delta y} + \gamma\alpha_\pi},$$

- Tedious calculation gives that M is negative if and only if $\alpha_{\Delta y} > \sigma\alpha_\pi$, and M is zero if and only if $\alpha_{\Delta y} = \sigma\alpha_\pi$.

Example 1: Simple Brendon, Paustian, and Yates (2015) model (2/2)

- When $T = 1$:
  - If $\alpha_{\Delta y} < \sigma\alpha_{\pi}$ then the model has a unique solution for all q.
  - If $\alpha_{\Delta y} \leq \sigma\alpha_{\pi}$ then the model has a convex set of solutions for all q, a unique solution whenever $q > 0$, and at least one solution when $q = 0$.
  - When $\alpha_{\Delta y} > \sigma\alpha_{\pi}$, for any positive q, there exists $y > 0$ such that $q + My = 0$, so the model has multiple solutions.
    - I.e. there are solutions which jump to the bound, even when the nominal interest rate would always be positive were there no bound at all.
  - When $\alpha_{\Delta y} > \sigma\alpha_{\pi}$, for any negative q, there is no $y \geq 0$ such that $q + My \geq 0$, so the model has no solutions.
- When $T > 1$:
  - If $\alpha_{\Delta y} > \sigma\alpha_{\pi}$ then at least for some $q \gg 0$, the model has multiple solutions.

Example 2: Brendon, Paustian, and Yates (2015) model with shadow interest rate persistence (1/2)

- Equations:

$$x_{i,t} = \max\{0, x_{d,t}\},$$
$$x_{d,t} = (1-\rho)\left(1 - \beta + \alpha_{\Delta y}\left(x_{y,t} - x_{y,t-1}\right) + \alpha_\pi x_{\pi,t}\right) + \rho x_{d,t-1},$$
$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}\left(x_{i,t} + \beta - 1 - \mathbb{E}_t x_{\pi,t+1}\right),$$
$$x_{\pi,t} = \beta \mathbb{E}_t x_{\pi,t+1} + \gamma x_{y,t}.$$

Example 2: Brendon, Paustian, and Yates (2015) model with shadow interest rate persistence (2/2)

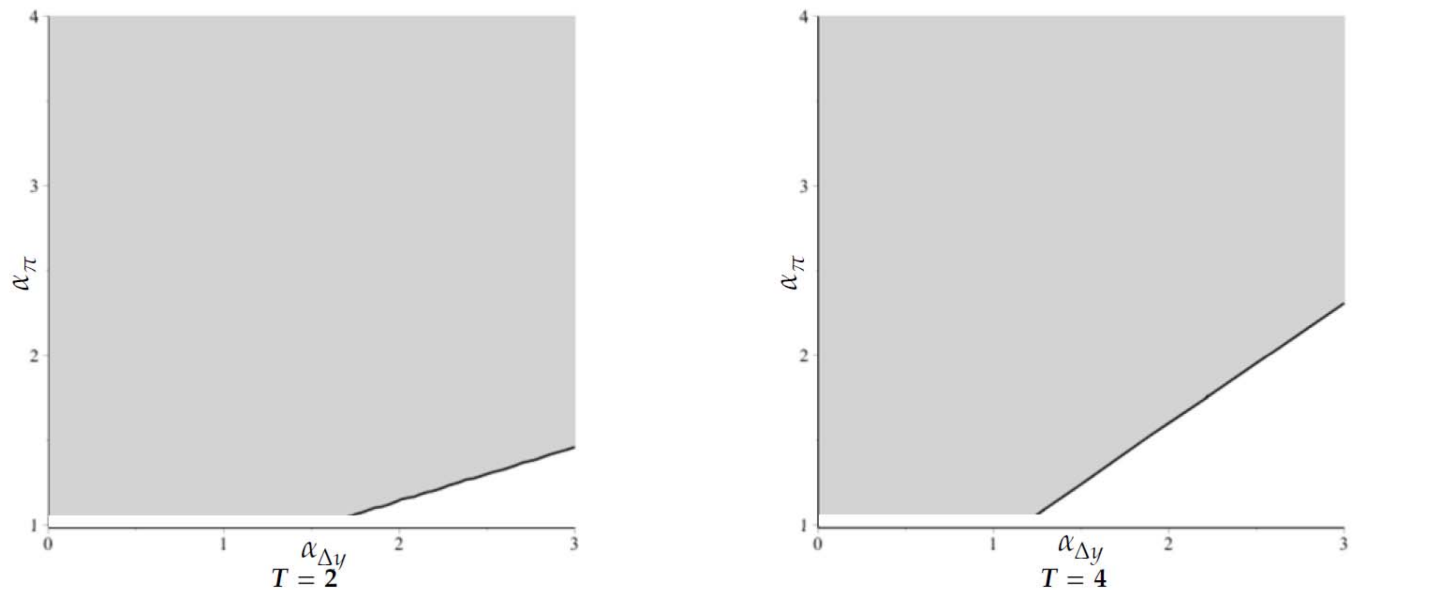


Figure 1: Regions in which M is a P-matrix (shaded grey) or a P_0-matrix (shaded grey, plus the black line), when $T = 2$ (left) or $T = 4$ (right).

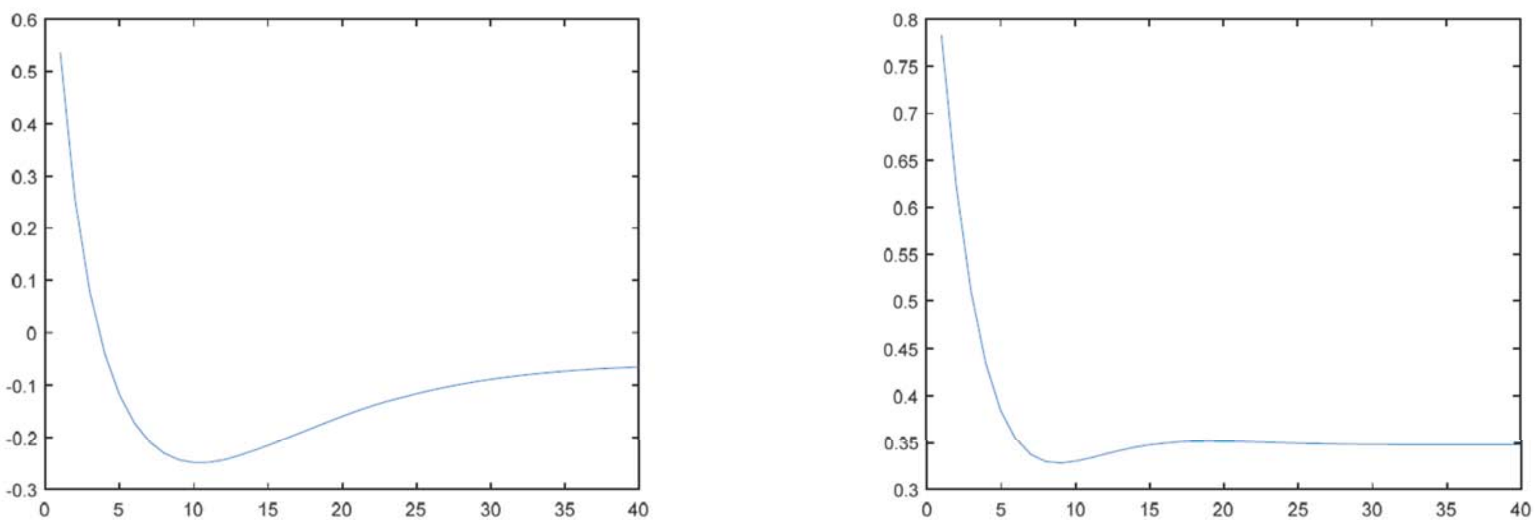
Example 3&4: Smets & Wouters (2003; 2007) (1/3)

- Both models have:
  - assorted shocks, habits, price and wage indexation, capital (with adjustment costs), (costly) variable utilisation, general monetary policy reaction functions
- We augment both models with nominal interest rate rules of the form:

$$r_t = \max\{0, (1-\rho)(\cdots) + \rho r_{t-1} + \cdots\}$$

- Recall that the 2003 model is estimated on Euro area data, and the 2007 one is estimated on US data.
  - We use the posterior modes.
- Fairly similar models, except that the 2007 one:
  - Contains trend growth (permitting its estimation on non-detrended data),
  - Has a slightly more general aggregator across industries.

Example 3&4: Smets & Wouters (2003; 2007) (2/3)



The Smets and Wouters (2003) model

The Smets and Wouters (2007) model

Figure 2: The diagonals of the M matrices for the Smets and Wouters (2003) and Smets and Wouters (2007) models

Example 3&4: Smets & Wouters (2003; 2007) (3/3)

- Perhaps surprising that these graphs are so different.
- Negative diagonal for the Euro area model implies that the model does not always have a unique solution, even when $q \gg 0$.

- In fact, providing $T \geq 9$, the US model also does not always have a unique solution, even when $q \gg 0$.

- Furthermore, for both countries, there are some q for which the model has no solution.
  - What happens in this situation?

The algorithm

- Three “layers” of increasing accuracy.
- First layer applies the perfect foresight solver to a pruned perturbation solution to the non-linear model.
  - We will start by discussing the best approach to solving the perfect foresight problem.
- Second layer integrates over a certain number of periods of future uncertainty.
- Third layer takes a hybrid local/global approach to capture long run uncertainty.

Efficient computation of solutions (1/2)

- If M is unrestricted, or M is a P_0-matrix, then finding a single solution to the LCP (q, M) is “strongly NP complete”.
- If we could do this efficiently (i.e. in polynomial time), we could also solve in polynomial time any problem whose solution could be efficiently verified.
  - This includes, for example, breaking all standard forms of cryptography.

- Since there is a model corresponding to any M matrix, with quadratic in T states, this means that if there were a solution algorithm for DSGE models with occasionally binding constraints that worked in time polynomial in the number of states, then it could also be used to defeat all known forms of cryptography.
  - It is extremely unlikely that such an algorithm exists!

Efficient computation of solutions (2/2)

- Polynomial time algorithms exist for special cases:
  - If M is general positive semi-definite then we can find a solution in polynomial time.
  - If M is row sufficient then we can check if a solution exists in polynomial time.
  - For "most" (conjectured: all) sufficient matrices, a solution may be found in polynomial time.

- Checking sufficiency is also NP-complete though, and it is frequently violated in DSGE models.
  - Checking most of the other matrix classes is NP-complete too.
  - Intuitively, this is because of the need to consider exponentially many principal sub-matrices.

Our computational approach (1/2)

- There is no way of escaping solving an NP-complete (i.e. computationally very difficult) problem if we wish to simulate DSGE models with OBCs.
  - Aside: Are rational expectations really plausible here??
- Any algorithm we invented for the problem is likely to be inefficient, and possibly even non-finite.
- A better approach is to map our problem into another to which smart computer scientists have devoted a lot of time.
- It turns out that the solution to an LCP can be represented as a mixed integer linear programming problem.
  - One of the best studied problems in computer science.
  - Extremely well optimised, fully global, solvers exist (commercial and open source.)

Our computational approach (2/2)

- **Problem 5**
- Suppose $\widetilde{\omega} > 0$, $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given. Find $\alpha \in \mathbb{R}$, $\hat{y} \in \mathbb{R}^T$, $z \in \{0,1\}^T$ to maximise α subject to the following constraints: $\alpha \geq 0$, $0 \leq \hat{y} \leq z$, $0 \leq \alpha q + M\hat{y} \leq \widetilde{\omega}(1_{T\times 1} - z)$.

- **Proposition 12**
- If α, $\hat{y}$, z solve Problem 5, then if $\alpha = 0$, the LCP (q, M) has no solution, and if $\alpha > 0$, then $y \coloneqq \frac{\hat{y}}{\alpha}$ solves that LCP.
- A partial converse is given in the paper.

- As $\widetilde{\omega} \to 0$, the solution to Problem 5 is the solution to the LCP which minimises $\|q + My\|_\infty$.
- As $\widetilde{\omega} \to \infty$, the solution to Problem 5 is the solution to the LCP which minimises $\|y\|_\infty$.

Application to models with uncertainty

- To convert the perfect foresight solver into a solver for stochastic models, we use a variant of the extended path algorithm of Fair and Taylor (1983).
  - Each period we draw a shock, and then solve for the expected future path of the model, ignoring the impact of the OBC on expectations (for now).
  - From this expected path, we can solve for the news shocks necessary to impose the bound.
  - We then add those news shocks to today's variables, and step the model forward using the model's transition matrix.

The non-linear problem

- **Problem 6**
- Suppose that $x_0 \in \mathbb{R}^n$ is given and that $f: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^m \to \mathbb{R}^n$, $g, h: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^m \to \mathbb{R}^c$ are given continuously $d \in \mathbb{N}^+$ times differentiable functions. Find $x_t \in \mathbb{R}^n$ and $v_t \in \mathbb{R}^c$ for $t \in \mathbb{N}^+$ such that for all $t \in \mathbb{N}^+$:

$$0 = \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t),$$
$$v_t = \mathbb{E}_t \max\{h(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t), g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\}$$

  - where $\varepsilon_t \sim \text{NIID}(0, \Sigma)$, where the max operator acts elementwise on vectors, and where the information set is such that for all $t \in \mathbb{N}^+$, $\mathbb{E}_{t-1}\varepsilon_t = 0$ and $\mathbb{E}_t \varepsilon_t = \varepsilon_t$.

- **Assumption 2**
- There exists $\mu_x \in \mathbb{R}^n$ and $\mu_v \in \mathbb{R}^c$ such that:

$$0 = f(\mu_x, \mu_x, \mu_x, \mu_v, 0),$$
$$\mu_v = \max\{h(\mu_x, \mu_x, \mu_x, \mu_v, 0), g(\mu_x, \mu_x, \mu_x, \mu_v, 0)\},$$

- and such that for all $a \in \{1, \dots, c\}$:

$$\big(h(\mu_x, \mu_x, \mu_x, \mu_v, 0)\big)_a \neq \big(\mathrm{g}(\mu_x, \mu_x, \mu_x, \mu_v, 0)\big)_a.$$

Application via linearization

- Without loss of generality, suppose our model is:

$$0 = \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t),$$
$$v_t = \mathbb{E}_t \max\{0, g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\},$$

  - where $g(\mu_x, \mu_x, \mu_x, \mu_v, 0) \gg 0$.
- Linearizing around the steady-state gives:

$$v_t = \mu_v + g_1(x_{t-1} - \mu_x) + g_2(x_t - \mu_x) + g_3 \mathbb{E}_t(x_{t+1} - \mu_x) + g_4(v_t - \mu_v) + g_5 \varepsilon_t.$$

- We replace this with the more accurate:

$$v_t = \max\{0, \mu_v + g_1(x_{t-1} - \mu_x) + g_2(x_t - \mu_x) + g_3 \mathbb{E}_t(x_{t+1} - \mu_x) + g_4(v_t - \mu_v) + g_5 \varepsilon_t\}.$$

- For our algorithm, we replace this in turn with:

$$v_{a,t} = \mathbb{E}_t\big(g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\big)_a + I_{1,\cdot} y_t^{(a)},$$

- for all $a \in \{1, \dots, c\}$, where, for all $a \in \{1, \dots, c\}$:

$$\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t}^{(a)} = y_{i+1,t-1}^{(a)} + \eta_{i,t}^{(a)},$$
$$y_{T,t}^{(a)} = \eta_{T,t}^{(a)}.$$

- The M matrix stacks impulse responses to the ηs.

Application via higher order pruned perturbation

- We first take a pruned perturbation approximation to the source non-linear model.
- A convenient property of pruned perturbation solutions of order d is that they are linear in additive shocks of the form η_t^d.
- Thus, if we use news shocks which hit the bounded equation raised to the power of d, then the linearity which gave tractability before will be preserved.
- In fact the M matrix we get at second or higher order is equal to the M matrix at first order, at least in the limit as the variance of the news shocks goes to zero.
  - Since these news shocks represent probability 0 events, at the point around which the perturbation is taken, this is the correct approach.
- While we still treat the bound in a perfect-foresight manner, by taking a higher order approximation we at least capture other risk channels.

Integrating over future uncertainty (1/4)

- Adjemian and Juillard (2013) showed how a perfect foresight solver could account for future uncertainty by integrating over future shocks.
  - I.e. draw shocks for a certain number of future periods, $t+1, \dots, t+S$.
  - Assume that they were known at t.
  - Solve for the perfect foresight path in that case.
  - Repeat many times to get expectations.

- In their very general non-linear set-up, doing this integration requires p^{mS} solutions of the perfect foresight problem,
  - for some $p > 1$, m is the number of shocks, S is the integration horizon.
  - Makes integrating over more than a few periods computationally intractable.

- Solving their general perfect foresight problem is also orders of magnitude slower than solving our LCP.

Integrating over future uncertainty (2/4)

- Let $w_{t,s}$ be the value the bounded variables would take at s if the constraints did not apply from period t onwards.
- By the nice properties of pruned perturbation solutions, we can evaluate $\mathrm{cov}_t\left(w_{t,t+i}, w_{t,t+j}\right)$, for $t, i, j \in \mathbb{N}$ in closed form, without Monte Carlo, or other numerical integration.
  - Thus, if we take a Gaussian approximation to the joint distribution of $w_{t,t}, w_{t,t+1}, \ldots$, we can efficiently integrate over these variables via Gaussian cubature techniques.
  - Rather than exponential in both m and S evaluations, we just need polynomial in S evaluations.
- For each draw of $w_{t,t}, w_{t,t+1}, \ldots$, we solve the bounds problem to get the cumulated news shocks (i.e. y).
  - Combining the results gives an approximation to the expected value of the cumulated news shocks, which we then apply to other equations.

Integrating over future uncertainty (3/4)

- Unlike Adjemian and Juillard (2013) we do not just consider full variance shocks up to some horizon, and then nothing beyond.
- Instead, we apply a windowing function to the covariance matrix, to ensure that the covariance is a smooth function of time.
  - This reduces artefacts caused by the sudden change at horizon S.
- In particular, we work with a covariance matrix with i, j^{th} element:

$$\frac{1}{4}\left(1+\cos\left(\pi\frac{i-1}{S}\right)\right)\left(1+\cos\left(\pi\frac{j-1}{S}\right)\right)\operatorname{cov}_t\left(w_{t,t+i}, w_{t,t+j}\right).$$

- The cosine form has some desirable frequency domain properties.

Integrating over future uncertainty (4/4)

- In the paper (and in DynareOBC) I discuss (implement) three alternative cubature methods.
  - A degree 3 monomial rule with $2\hat{S}+1$ nodes and positive weights.
    - Where $\hat{S} \leq S$ is the integration dimension.
    - Positive weights give robustness.
    - Evaluates far from steady-state though.
  - The Genz and Keister (1996) nested Gaussian cubature rules which use $\mathrm{O}\left(\hat{S}^K\right)$ nodes.
    - Where $2K+1$ is the degree of monomial integrated exactly.
    - Since the rules are nested, adaptive degree is possible.
    - In the paper I discuss a way of smoothing the results of integrals at different degrees, to reduce the oscillations caused by the non-differentiabilities.
  - Quasi-Monte Carlo.
    - Much less efficient than the others on well behaved functions, but is much better behaved on non-differentiable ones.

Hybrid local/global approach (1/3)

- To capture long-run risk we need some kind of global solution.
- We now replace the v_t equations with:

$$v_{a,t} = \mathbb{E}_t\big(g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\big)_a + I_{1,\cdot}y_t^{(a)} + \theta_{\cdot,a}^{'} \mathscr{s}_t,$$
$$\tilde{v}_{a,t} = \mathbb{E}_t\big(g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\big)_a + I_{1,\cdot}y_t^{(a)},$$

- for all $a \in \{1, \dots, c\}$, where $\theta = [\theta_{\cdot,1} \quad \cdots \quad \theta_{\cdot,c}]$ is a matrix of unknown parameters,

$$\mathscr{s}_t := \begin{bmatrix} 1 \\ x_{t-1} \\ \varepsilon_t \end{bmatrix}^{\otimes d},$$

- where d is the degree of perturbation approximation being taken, and where for all $a \in \{1, \dots, c\}$:

$$\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t}^{(a)} = y_{i+1,t-1}^{(a)} + \kappa\left(\eta_{i,t}^{(a)}\right)^d,$$
$$y_{T,t}^{(a)} = \kappa\left(\eta_{T,t}^{(a)}\right)^d.$$

Hybrid local/global approach (2/3)

- First: set $\eta_{i,t}^{(a)} = 0$ for all i, t, a and attempt to solve for θ such that:

$$0 = \mathbb{E}\mathcal{s}_t(\max\{0, \tilde{v}_t\} - v_t)' = \mathbb{E}\mathcal{s}_t(\max\{0, \tilde{v}_t\} - \tilde{v}_t - \theta'\mathcal{s}_t)'$$
$$= \mathbb{E}\mathcal{s}_t(\max\{-\tilde{v}_t, 0\} - \theta'\mathcal{s}_t)'.$$

- Implies:

$$\theta = (\mathbb{E}\mathcal{s}_t\mathcal{s}_t')^{-1}\mathbb{E}\mathcal{s}_t \max\{-\tilde{v}_t, 0\}'$$

- By approximating the joint distribution of $\mathcal{s}_t$ and v_t by a Gaussian (and, again by the nice properties of pruned perturbation solutions), we can evaluate this expression without any Monte Carlo or other numerical integration.
  - Incredibly fast.
- Solve for the fixed point θ.
  - I.e. guess θ, solve the model, evaluate a new θ, repeat.
  - In practice we use a non-linear equation solver.
  - Takes less than a minute even for large models.

Hybrid local/global approach (3/3)

- After this, the required news shocks should be orthogonal to the states and shocks.
- But v_t is unlikely to be a good approximation at short horizons.
  - Polynomials do not well approximate kinks!
- So we combine expectations produced from $\tilde{v}_t$ (integrating over future uncertainty) at short horizons and expectations produced from v_t at long horizons.
- This is done via a convex quadratic programming problem which is solvable in polynomial time.

The DynareOBC toolkit

- Complete code implementing all of these algorithms is available under an open source license from:

  https://github.com/tholden/dynareOBC

- To use DynareOBC, just include a max, min or abs in your mod file, then type "dynareOBC modfilename.mod".
- Assorted command line options are documented on the home page.

- Even if you do not have OBCs in your model, DynareOBC may be useful since it can:
  - Simulate MLVs, including integrating over ones with +1 terms, which makes checking accuracy very easy.
  - Perform exact, faster, average IRF calculation without Monte Carlo.
  - Estimate non-linear models at 3rd order using the cubature Kalman filter.
    - To be documented in a future paper.

Accuracy results

- By way of conclusion, we give some early accuracy results.
- Obviously there will be many more in the final version of the paper.
- We examine two models:
  - A very simple model with an analytic solution.
  - The model of Fernández-Villaverde et al. (2012).
- In each case, we will report accuracy measures, along the simulation path.
  - Simulation paths are all length 1000. (A 100 period initial burn-in was discarded.)

A simple model

- Closed economy, no capital, inelastic unit labour supply.
- Households maximise:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1-\gamma}$$

- Subject to the budget constraint:

$$A_t + R_{t-1}B_{t-1} = C_t + B_t$$

  - A_t is productivity.
  - R_t is the real interest rate.
  - B_t is the household's holdings of zero net supply bonds.
- Define $g_t \coloneqq \log A_t - \log A_{t-1}$.
- Productivity evolves according to:

$$g_t = \max\{0, (1-\rho)\bar{g} + \rho g_{t-1} + \sigma\varepsilon_t\},$$

- $\varepsilon_t \sim \mathrm{NIID}(0,1)$, $\beta \coloneqq 0.99$, $\gamma \coloneqq 5$, $\bar{g} \coloneqq 0.05$, $\rho \coloneqq 0.95$, $\sigma \coloneqq 0.07$.

Accuracy in the simple model

Order	1	2	3	2	2	2	2	2	2
Integration	None	None	None	$2\hat{S}+1$ rule	Nested, no smooth.	Nested, Smooth, No prior	Nested, Smooth, Prior	QMC 31 points	QMC 255 points
Mean abs err	0.00367	0.00376	0.00376	0.00030	0.00232	0.00232	0.00232	0.00058	0.00015
Root mean squared err	0.00605	0.00639	0.00639	0.00045	0.00399	0.00399	0.00399	0.00081	0.00022
Max err	0.01313	0.01374	0.01374	0.00114	0.00904	0.00904	0.00904	0.00152	0.00046
Time	19	20	21	435	410	409	449	550	2457

All timings are in seconds for performing the full accuracy calculations.
On a 4 core 2.3ghz machine. (Dell XPS 15 laptop)

Errors are the gap between the exact real interest rate, and the approximated one.
The equation governing the evolution of g_t is essentially exact in each case.

Accuracy in the model of Fernández-Villaverde et al. (2012).

Order	1	2	3	2
Integration	None	None	None	Nested, Smooth, Prior
Sum of mean abs errs	0.001153	0.0004511	0.0004899	0.0010994
Sum of root mean squared errs	0.003122	0.0015266	0.0019431	0.0028926
Sum of max errs	0.029947	0.014881	0.019839	0.027644

Reported errors are the Jin and Judd (2002) errors summed over each of the model's four equations.
Expectations for the errors are evaluated using 2047 point quasi Monte Carlo integration.

Conclusion

- We proved “Blanchard Kahn conditions” for models with occasionally binding constraints.
  - Applicable to all models.

- We showed that multiplicity of equilibria is to be expected in models with zero lower bounds on nominal interest rates.

- We developed efficient algorithms for solving models with OBCs, and extensions for improving their accuracy.