Existence, uniqueness and computation of solutions to dynamic models with occasionally binding constraints.

Tom Holden

School of Economics, University of Surrey

Motivation

- Since the financial crisis, many central banks around the world have set their nominal interest rates near zero.
- Additionally, during the crisis many households, firms and banks were pushed up against their borrowing constraints.

- The zero lower bound on nominal interest rates and borrowing constraints are prominent examples of occasionally binding constraints (OBCs).

- At present though, little is known about the behaviour of dynamic models featuring such constraints, and there is no robust, accurate and scalable algorithm for simulating them.

This paper (now actually two papers!)

- “Blanchard and Kahn (1980) with occasionally binding constraints.”

- Two elements to this:
  - Paper 1: Existence and uniqueness results.
  - Paper 2: A new computational algorithm designed to be robust, accurate and scalable.

- We also present the DynareOBC toolkit, which is one stop shop for all things OBC.
  - And designed to be super easy to use!

Existence and uniqueness results: This paper provides… (1/2)

- Necessary and sufficient conditions for existence of a unique solution *about a given steady-state* to models with OBCs.
  - I.e., I assume that agents believe that if no shocks arrived in the future, the economy would eventually return to the original steady-state.
  - Accords with a belief in the credibility of the long-run inflation target.
  - In line with the approach of Brendon, Paustian, and Yates (2015).
  - Contrary to the approach of, e.g. Benhabib, Schmitt-Grohe, and Uribe (2001a,b), Schmitt-Grohe and Uribe (2012), Mertens and Ravn (2014), Aruoba, Cuba-Borda and Schorfheide (2013).

Existence and uniqueness results: This paper provides… (2/2)

- Necessary and sufficient conditions for existence of a unique solution when away from the bound.
  - E.g. suppose that the impulse response to some shock does not hit the bound. Must it be the unique solution?

- Some necessary conditions, and some sufficient conditions for existence of any solution.

- Sufficient conditions for the polynomial time computability of the solution.

A preview of the application of our results to Smets and Wouters (2003; 2007)

- Augment the Smets and Wouters (2003) and (2007) models, with a zero lower bound, at their estimated modes.

- For both, there are combinations of predicted future shocks for which:
  - There are multiple solutions to the model, including combinations for which one solution features strictly positive nominal interest rates.
  - There are zero solutions to the model.
  - There is no known algorithm capable of finding a solution in time polynomial in the simulation horizon.

- Both models are determinate under price level targeting.

New computational algorithm

- The base of the algorithm is a very efficient perfect foresight solver for models with occasionally binding constraints.
  - Unique in being guaranteed to find a solution in finite time if one exists.
  - If no solution exists, will also report this in finite time too.
  - Also applies to high order pruned perturbation solutions to the model.

- Restore consistency with rational expectations by integrating over future uncertainty.
  - Exploits the convenient properties of pruned perturbation solutions to very efficiently integrate over a large number of periods of future uncertainty.

Outline

- Theoretical results:
  - Problem set-up.
  - Introduction to linear complementarity problems and the associated matrix classes.
  - Existence/uniqueness/other results.
  - Results from dynamic programming.
- Applications to New Keynesian models.
- The computational algorithm:
  - Perfect foresight problems.
  - Application of the perfect foresight solver to non-linear models.
  - Integrating over future uncertainty.
- Preliminary accuracy results.

The set-up without bounds (1/3)

- Suppose for $t \in \mathbb{N}^+$:

$$\left(\hat{A} + \hat{B} + \hat{C}\right)\hat{\mu} = \hat{A}\hat{x}_{t-1} + B\hat{x}_t + \hat{C}\mathbb{E}_t\hat{x}_{t+1} + \widehat{D}\varepsilon_t,$$

  - where $\mathbb{E}_{t-1}\varepsilon_t = 0$ for all $t \in \mathbb{N}^+$,
  - $\varepsilon_t = 0$ for $t > 1$, (impulse response/perfect foresight simulation).

- $\hat{x}_0$ is given as an initial condition.
- Terminal condition: $\hat{x}_t \to \hat{\mu}$ as $t \to \infty$.

The set-up without bounds (2/3)

- For $t \in \mathbb{N}^+$, define:

$$x_t \coloneqq \begin{bmatrix} \hat{x}_t \\ \varepsilon_{t+1} \end{bmatrix}, \qquad \mu \coloneqq \begin{bmatrix} \hat{\mu} \\ 0 \end{bmatrix}, \qquad A \coloneqq \begin{bmatrix} \hat{A} & \widehat{D} \\ 0 & 0 \end{bmatrix}, \qquad B \coloneqq \begin{bmatrix} \hat{B} & 0 \\ 0 & I \end{bmatrix}, \qquad C \coloneqq \begin{bmatrix} \hat{C} & 0 \\ 0 & 0 \end{bmatrix}$$

- then, for $t \in \mathbb{N}^+$:

$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1},$$

- and $x_0 = \begin{bmatrix} \hat{x}_0 \\ \varepsilon_1 \end{bmatrix}$, $x_t \to \mu$ as $t \to \infty$.
- Take this as the form of our problem without bounds in the following.

The set-up without bounds (3/3)

- **Problem 1**
- Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:
$$(A + B + C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1}.$$

- **Assumption:** For any given $x_0 \in \mathbb{R}^n$, Problem 1 has a unique solution, of the form $x_t = (I - F)\mu + Fx_{t-1}$, for $t \in \mathbb{N}^+$, where $F = -(B + CF)^{-1}A$, and all of the eigenvalues of F are weakly inside the unit circle.
- **Assumption:** $\det(A + B + C) \neq 0$.

The set-up with bounds

- **Problem 2**
- Suppose that $x_0 \in \mathbb{R}^n$ is given. Find $T \in \mathbb{N}$ and $x_t \in \mathbb{R}^n$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$ as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$x_{1,t} = \max\left\{0, I_{1,\cdot}\mu + A_{1,\cdot}(x_{t-1} - \mu) + \left(B_{1,\cdot} + I_{1,\cdot}\right)(x_t - \mu) + C_{1,\cdot}(x_{t+1} - \mu)\right\},$$

$$\left(A_{-1,\cdot} + B_{-1,\cdot} + C_{-1,\cdot}\right)\mu = A_{-1,\cdot}x_{t-1} + B_{-1,\cdot}x_t + C_{-1,\cdot}x_{t+1},$$

- and such that $x_{1,t} > 0$ for $t > T$.
- Ruling out solutions that get stuck at another steady-state by assumption.

The news shock set-up

- **Problem 3**
- Suppose that $T \in \mathbb{N}$, $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^T$ is given. Find $x_t \in \mathbb{R}^n, y_t \in \mathbb{R}^T$ for $t \in \mathbb{N}^+$ such that $x_t \to \mu$, $y_t \to 0$, as $t \to \infty$, and such that for all $t \in \mathbb{N}^+$:

$$
\begin{gathered}
(A+B+C)\mu = Ax_{t-1} + Bx_t + Cx_{t+1} + I_{\cdot,1} y_{1,t-1}, \\
\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t} = y_{i+1,t-1}, \\
y_{T,t} = 0.
\end{gathered}
$$

- A version of Problem 1 with news shocks up to horizon T added to the first equation.
  - The value of $y_{t,0}$ gives the news shock that hits in period t.
  - I.e. $y_{1,t-1} = y_{t,0}$ for $t \leq T$, and $y_{1,t-1} = 0$ for $t > T$.

A representation of solutions to Problem 3

- **Lemma**: There is a unique solution to Problem 3 that is linear in x_0 and y_0.

- Let $x_t^{(3,k)}$ be the solution to Problem 3 when $x_0 = \mu$, $y_0 = I_{\cdot,k}$.
- Let $M \in \mathbb{R}^{T\times T}$ satisfy:

$$M_{t,k} = x_{1,t}^{(3,k)} - \mu_1, \qquad \forall t, k \in \{1, .., T\},$$

  - i.e. M horizontally stacks the (column-vector) relative impulse responses to the news shocks.

- Let $x_t^{(1)}$ be the solution to Problem 1 for some given x_0.
- Then the solution to Problem 3 for given x_0, y_0 satisfies:

$$\left(x_{1,1\ldots T}\right)' = q + My_0,$$

  - where $q \coloneqq \left(x_{1,1\ldots T}^{(1)}\right)'$.

The links between the solutions to Problem 2 and the solution to Problem 3 (1/2)

- Let $x_t^{(2)}$ be a solution to Problem 2 given an arbitrary x_0.
- Define:

$$e_t \coloneqq \begin{cases} -\left[I_{1,\cdot}\mu + A_{1,\cdot}\left(x_{t-1}^{(2)} - \mu\right) + \left(B_{1,\cdot} + I_{1,\cdot}\right)\left(x_t^{(2)} - \mu\right) + C_{1,\cdot}\left(x_{t+1}^{(2)} - \mu\right)\right] & \text{if } x_{1,t}^{(2)} = 0 \\ 0 & \text{if } x_{1,t}^{(2)} > 0 \end{cases}$$

- **Lemma:** The following statements hold:
  - $e_{1\ldots T} \geq 0$, $x_{1,1\ldots T}^{(2)} \geq 0$ and $x_{1,1\ldots T}^{(2)} \circ e_{1\ldots T} = 0$,
  - $x_t^{(2)}$ is the unique solution to Problem 3 when started with $x_0 = x_0^{(2)}$ and with $y_0 = e_{1\ldots T}'$.
  - If $x_t^{(2)}$ solves Problem 3 when started with $x_0 = x_0^{(2)}$ and with some y_0, then $y_0 = e_{1\ldots T}'$.

The links between the solutions to Problem 2 and the solution to Problem 3 (2/2)

- **Proposition**: The following statements hold:

  - Let $x_t^{(3)}$ be the unique solution to Problem 3 when initialized with some x_0, y_0. Then $x_t^{(3)}$ is a solution to Problem 2 when initialized with x_0 if and only if $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$ and $x_{1,t}^{(3)} \geq 0$ for all $t \in \mathbb{N}$ with $t > T$.

  - Let $x_t^{(2)}$ be any solution to Problem 2 when initialized with x_0. Then there exists a $y_0 \in \mathbb{R}^T$ such that $y_0 \geq 0$, $y_0 \circ (q + My_0) = 0$, $q + My_0 \geq 0$, such that $x_t^{(2)}$ is the unique solution to Problem 3 when initialized with x_0, y_0.

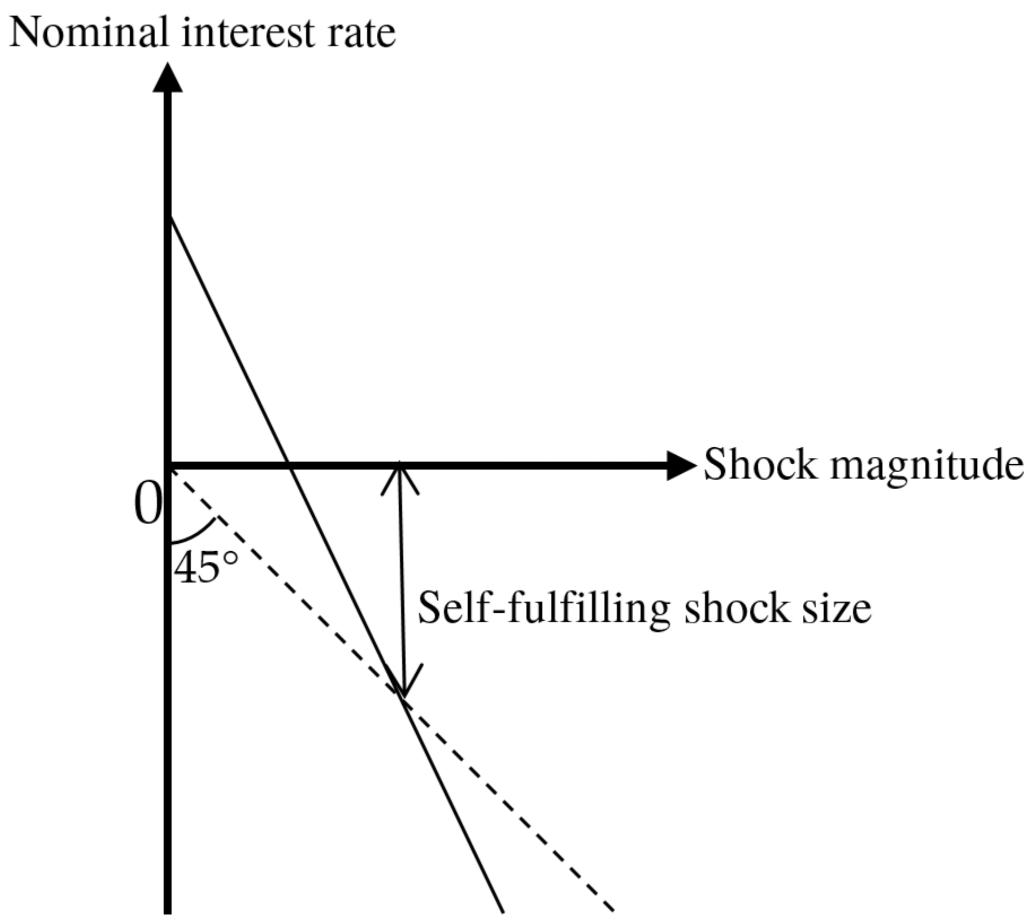
Linear complementarity problems (LCPs)

- The previous proposition establishes that solving the model with occasionally binding constraints is equivalent to solving the following "linear complementarity problem".

- **Problem 4**
- Suppose $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given. Find $y \in \mathbb{R}^T$ such that: $y \geq 0$, $y \circ (q + My) = 0$ and $q + My \geq 0$.

- We call this the linear complementarity problem (q, M).

Intuition for multiplicity: Self-fulfilling news shocks



Is our M matrix special?

- The properties of solutions to LCPs are determined by the properties of the M matrix.
  - One might think that ours would have “nice” properties because of where it came from.

- Unfortunately:

- **Proposition:** For any matrix $\mathcal{M} \in \mathbb{R}^{T \times T}$, there exists a model in the form of Problem 2 with a number of state variables given by a quadratic in T, such that $M = \mathcal{M}$ for that model.

Introduction to the relevant matrix classes for the LCP

- Properties of solutions of LCPs have been extensively studied in the linear algebra and optimization literatures.

- As previously mentioned, all existence and uniqueness results are given in terms of the properties of the matrix M.

- Unfortunately, the required properties are rather harder to state (and check) than just looking at a few eigenvalues.

- In the next few slides, I give definitions of the key properties.
  - The chief references for the below are Cottle, Pang, and Stone (2009) and Xu (1993).

Principal sub matrices and principal minors

- For a matrix $M \in \mathbb{R}^{T\times T}$, the principal sub-matrices of M are the matrices:

$$\left\{ \left[M_{i,j}\right]_{i,j=k_1,\dots,k_S} \middle| S, k_1, \dots, k_S \in \{1, \dots, T\}, k_1 < k_2 < \cdots < k_S \right\},$$

  - i.e. the principal sub-matrices of M are formed by deleting the same rows and columns.

- The principal minors of M are the collection of values:

$$\left\{ \det\left(\left[M_{i,j}\right]_{i,j=k_1,\dots,k_S} \right) \middle| S, k_1, \dots, k_S \in \{1, \dots, T\}, k_1 < k_2 < \cdots < k_S \right\},$$

  - i.e. the principal minors of M are the determinants of the principal sub-matrices of M.

$P_{(0)}$-matrices and General positive (semi-)definite matrices

- A matrix $M \in \mathbb{R}^{T \times T}$ is called a **P-matrix** if the principal minors of M are all strictly positive. M is called a **P_0-matrix** if the principal minors of M are all non-negative.
  - *Note: for symmetric M, M is a $P_{(0)}$-matrix if and only if all of its eigenvalues are strictly (weakly) positive.*

- A matrix $M \in \mathbb{R}^{T \times T}$ is called **general positive definite** if $M + M'$ is a P-matrix. If $M + M'$ is a P_0-matrix, then M is called **general positive semi-definite**.
  - *Note: that we do not require that M is symmetric in either case, but the definitions coincide with the standard ones for symmetric M.*

$S_{(0)}$-matrices, (Strictly) Semi-monotone matrices and (Strictly) Copositive matrices

- A matrix $M \in \mathbb{R}^{T\times T}$ is called an **S-matrix** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \gg 0$. M is called an **S_0-matrix** if there exists $y \in \mathbb{R}^T$ such that $y > 0$ and $My \geq 0$.

- A matrix $M \in \mathbb{R}^{T\times T}$ is called **strictly semi-monotone** if each of its principal sub-matrices is an **S-matrix**. M is called **semi-monotone** if each of its principal sub-matrices is an **S_0-matrix**.

- A matrix $M \in \mathbb{R}^{T\times T}$ is called **strictly copositive** if $M + M'$ is strictly semi-monotone. If $M + M'$ is semi-monotone then M is called **copositive**.

Sufficient matrices

- Let $M \in \mathbb{R}^{T \times T}$. M is called **column sufficient** if M is a P_0-matrix, and the principal sub-matrices of M with zero determinant satisfy a further technical condition.

- M is called **row sufficient** if M' is column sufficient.

- M is called **sufficient** if it is column sufficient and row sufficient.

Relationships between the matrix classes (Cottle, Pang and Stone 2009)

- All general p.s.d. matrices are copositive and sufficient.
- P_0 includes skew-symmetric matrices, general p.s.d. matrices, sufficient matrices and P-matrices.
- All P_0-matrices, and all copositive matrices are semi-monotone.
- All P-matrices, and all strictly copositive matrices are strictly semi-monotone (and hence S-matrices).

- All general p.s.d., semi-monotone, sufficient, P_0 and copositive matrices have non-negative diagonals.
- All general p.d., strictly semi-monotone, P and strictly copositive matrices have strictly positive diagonals.

Uniqueness results (1/3)

- We would ideally like a unique solution to exist for all possible q, since there are predicted shocks which can bring about any such q.

- **Proposition:** The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$, if and only if M is a P-matrix.
- If M is not a P-matrix, then the LCP (q, M) has multiple solutions for some q.
- (Samelson, Thrall, and Wesler 1958; Cottle, Pang, and Stone 2009)

- This is the analogue of the Blanchard-Kahn conditions for models with occasionally binding constraints.
  - It tends to be satisfied in efficient models, but is rarely satisfied in New Keynesian models with zero lower bounds.

Uniqueness results (2/3)

- While for many models, the previous condition does not hold, we would hope that at least for $q \geq 0$ there ought to still be a unique solution.

- **Proposition:** The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$ with $q \gg 0$ if and only if M is semi-monotone. (Cottle, Pang, and Stone 2009)

- **Proposition:** The LCP (q, M) has a unique solution for all $q \in \mathbb{R}^T$ with $q \geq 0$ if and only if M is strictly semi-monotone. (Cottle, Pang, and Stone 2009)

Uniqueness results (3/3)

- Thus, if M is not semi-monotone, there are some $q \gg 0$ such that the LCP (q, M) has multiple solutions.

- I.e., if agents today got appropriate signals about future shocks, then the economy could jump to the bound, even though the bound would not have been violated had it not been there at all.

- This remains true even if shocks are arbitrarily small, and even if the steady-state is arbitrarily far away from the bound.

Finite T existence results (1/4)

- Suppose $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T \times T}$ are given. The LCP corresponding to M and q is called **feasible** if there exists $y \in \mathbb{R}^T$ such that $y \geq 0$ and $q + My \geq 0$.

- **Proposition:** The LCP (q, M) is feasible for all $q \in \mathbb{R}^T$ if and only if M is an S-matrix. (Cottle, Pang, and Stone 2009)

- If M is not an S-matrix, there are positive measure of q for which no solution exists.

Finite T existence results (2/4)

- **Proposition:** The LCP (q, M) is solvable if it is feasible and, either:
  - M is row-sufficient, or,
  - M is copositive and all the non-singular principal submatrices of M satisfy a further technical condition.
- (Cottle, Pang, and Stone 2009; Väliaho 1986)

- This gives sufficient conditions for existence for feasible q.
  - Checking feasibility just requires solving a linear programming problem, which is possible in time polynomial in T.

Finite T existence results (3/4)

- **Proposition**: The LCP (q, M) is solvable for all $q \in \mathbb{R}^T$, if at least one of the following conditions holds:
  - M is an S-matrix, and either of the conditions of the previous proposition are satisfied.
  - M is copositive with no zero principal minors.
  - M is a P-matrix, a strictly copositive matrix or a strictly semi-monotone matrix.
- (Cottle, Pang, and Stone 2009)

- This gives sufficient conditions for existence for all q.

Finite T existence results (4/4)

- In the special case in which M has nonnegative entries, we have both necessary and sufficient conditions:

- **Proposition:** If M is a matrix with nonnegative entries, then the LCP (q, M) is solvable for all $q \in \mathbb{R}^T$, if and only if M has a strictly positive diagonal. (Cottle, Pang, and Stone 2009)

Large T existence results

- Define:

$$\varsigma \coloneqq \sup_{\substack{y \in [0,1]^{\mathbb{N}^+} \\ \exists T \in \mathbb{N} \text{ s.t. } \forall t > T, y_t = 0}} \inf_{t \in \mathbb{N}^+} M_{t,1:\infty} y \,,$$

- Then M is an S-matrix for sufficiently large T if and only if $\varsigma > 0$.
- We show that there exists $\underline{\varsigma}_T, \bar{\varsigma}_T \geq 0$, both computable in time polynomial in T, such that $\underline{\varsigma}_T \leq \varsigma \leq \bar{\varsigma}_T$ and $\left|\underline{\varsigma}_T - \bar{\varsigma}_T\right| \to 0$ as $T \to \infty$.

- The proof relies on deriving constructive bounds on M.

Bounds on M

- **Lemma**
- The difference equation $A\hat{d}_{k+1} + B\hat{d}_k + C\hat{d}_{k-1} = 0$ for all $k \in \mathbb{N}^+$ has a unique solution satisfying the terminal condition $\hat{d}_k \to 0$ as $k \to \infty$, given by $\hat{d}_k = H\hat{d}_{k-1}$, for all $k \in \mathbb{N}^+$, for some H with eigenvalues in the unit circle.

- Define $d_0 := -(AH + B + CF)^{-1} I_{\cdot,1}$, $d_k = Hd_{k-1}$, for all $k \in \mathbb{N}^+$, and $d_{-t} = Fd_{-(t-1)}$, for all $t \in \mathbb{N}^+$.
- The rows and columns of M are converging to 0 (with constructive bounds).
- The k^{th} diagonal of the M matrix is converging to the value $d_{1,k}$.
  - Diagonals are indexed such that the principal diagonal is index 0, and indices increase as one moves up and to the right in the M matrix.

Generalisations

- For multiple bounds:
  - We stack the impulse responses of the bounded variables ignoring bounds into q.
  - We stack the vectors of news shocks to each variable into y.
  - M is a block matrix of each bounded variable's responses to each bounded variable's news shocks.
  - Then the stacked solution for the paths of the bounded variables is $q + My$, and we again have an LCP, so results go through as before.
- For bounds not at zero:
  - If $z_{1,t} = \max\{z_{2,t}, z_{3,t}\}$, then $z_{1,t} - z_{2,t} = \max\{0, z_{3,t} - z_{2,t}\}$.
- For minimums:
  - If $z_{1,t} = \min\{z_{2,t}, z_{3,t}\}$, then $-z_{1,t} = \max\{-z_{2,t}, -z_{3,t}\}$.

Results from dynamic programming (1/2)

- Here we introduce a class of problems that our algorithm will solve arbitrarily accurately, and give alternative uniqueness results.

- **Problem 5**
- Solve a concave quadratic dynamic programming problem subject to linear inequality constraints. (Full definition in the paper.)

- **Proposition:** If either:
  - $\tilde{\Gamma}(x)$ (the choice set) is compact valued and $x \in \tilde{\Gamma}(x)$ for all $x \in \tilde{X}$, or,
  - $\tilde{X}$ (the state space) is compact,
- then for all $x_0 \in \tilde{X}$, there is a unique path $(x_t)_{t=0}^{\infty}$ which solves Problem 5.

Results from dynamic programming (2/2)

- **Proposition:**
- The KKT conditions of Problem 5 may be placed into the form of the multiple-bound generalisation of Problem 2.
- Let $\left(q_{x_0}, M\right)$ be the infinite LCP corresponding to this representation, given initial state $x_0 \in \tilde{X}$.
- If y is a solution to the LCP, $q_{x_0} + My$ gives the stacked paths of the bounded variables in a solution to Problem 5.
- If, further, either condition of the previous proposition holds, then:
  - Both Problem 5 and this LCP have a unique solution for all $x_0 \in \tilde{X}$.
  - For sufficiently large T, the finite LCP also has a unique solution.

Examples from New Keynesian models

- First look at a three equation NK model with a response to output growth, following Brendon, Paustian, and Yates (BPY) (2015) .
- Then assorted variants of this model.
- Then the Fernandez-Villaverde et al. (2012) model with price dispersion.
- Then Smets Wouters (2003) and (2007).

Simple Brendon, Paustian, and Yates (BPY) (2015) model (1/3)

- Equations:

$$x_{i,t} = \max\{0, 1-\beta + \alpha_{\Delta y}\left(x_{y,t} - x_{y,t-1}\right) + \alpha_\pi x_{\pi,t}\},$$
$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}\left(x_{i,t} + \beta - 1 - \mathbb{E}_t x_{\pi,t+1}\right),$$
$$x_{\pi,t} = \beta \mathbb{E}_t x_{\pi,t+1} + \gamma x_{y,t},$$

- $\beta \in (0,1), \gamma, \sigma, \alpha_{\Delta y} \in (0, \infty), \alpha_\pi \in (1, \infty)$.
- Unique stationary solution.
- If $T = 1$, then:

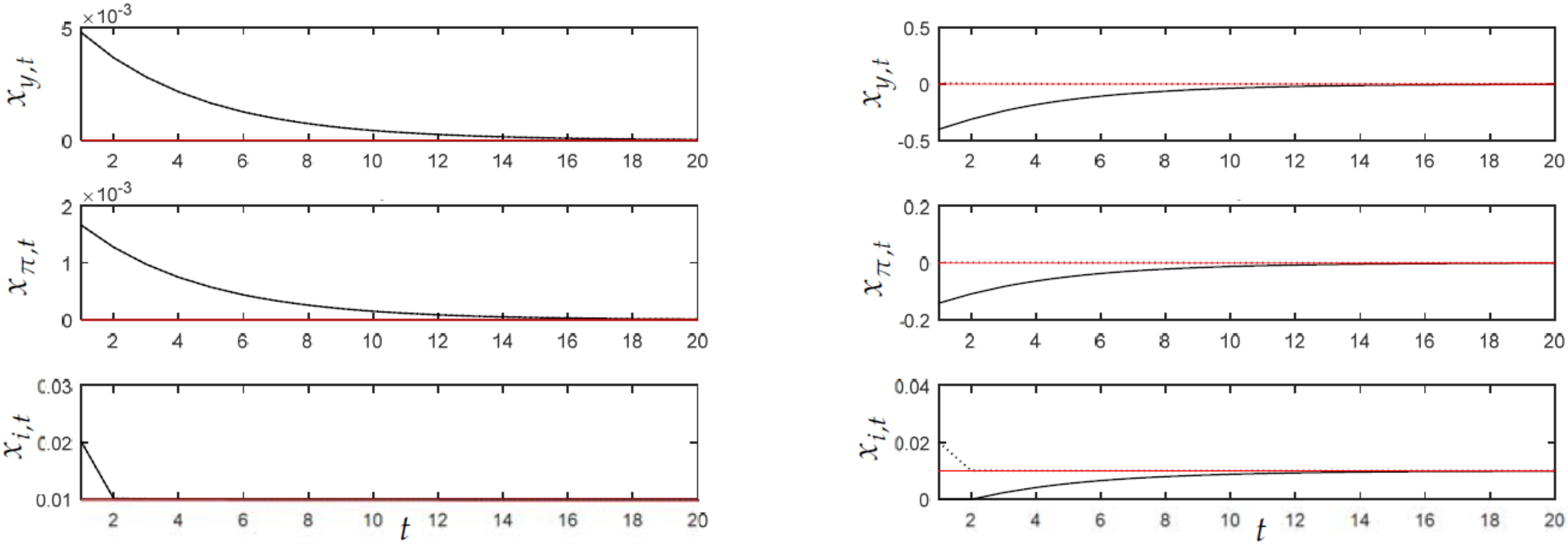
$$M = \frac{\beta\sigma f^2 - \left((1+\beta)\sigma + \gamma\right)f + \sigma}{\beta\sigma f^2 - \left((1+\beta)\sigma + \gamma + \beta\alpha_{\Delta y}\right)f + \sigma + \alpha_{\Delta y} + \gamma\alpha_\pi},$$

- M is negative if and only if $\alpha_{\Delta y} > \sigma\alpha_\pi$. M is zero if and only if $\alpha_{\Delta y} = \sigma\alpha_\pi$.

Simple BPY (2015) model (2/3)

- When $T = 1$:
  - If $\alpha_{\Delta y} < \sigma\alpha_\pi$ then the model has a unique solution for all q.
  - When $\alpha_{\Delta y} > \sigma\alpha_\pi$, for any positive q, there exists $y > 0$ such that $q + My = 0$, so the model has multiple solutions.
  - When $\alpha_{\Delta y} > \sigma\alpha_\pi$, for any negative q, there is no $y \geq 0$ such that $q + My \geq 0$, so the model has no solutions.
- When $T > 1$:
  - If $\alpha_{\Delta y} > \sigma\alpha_\pi$ then at least for some $q \gg 0$, the model has multiple solutions.

Simple BPY (2015) model (3/3)



Minimum $\|y\|_\infty$ solution[12]   **Minimum $\|q + My\|_\infty$ solution**[13]

Figure 1: Alternative solutions following a magnitude 1 impulse to ε_t

BPY (2015) model with shadow interest rate persistence (1/3)

- Equations:

$$x_{i,t} = \max\{0, x_{d,t}\},$$
$$x_{d,t} = (1-\rho)\big(1-\beta+\alpha_{\Delta y}(x_{y,t}-x_{y,t-1})+\alpha_\pi x_{\pi,t}\big)+\rho x_{d,t-1},$$
$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}\big(x_{i,t}+\beta-1-\mathbb{E}_t x_{\pi,t+1}\big),$$
$$x_{\pi,t} = \beta\mathbb{E}_t x_{\pi,t+1} + \gamma x_{y,t}.$$

- Set $\sigma = 1$, $\beta = 0.99$, $\gamma = \frac{(1-0.85)\big(1-\beta(0.85)\big)}{0.85}(2+\sigma)$, $\rho = 0.5$.

BPY (2015) model with shadow interest rate persistence (2/3)

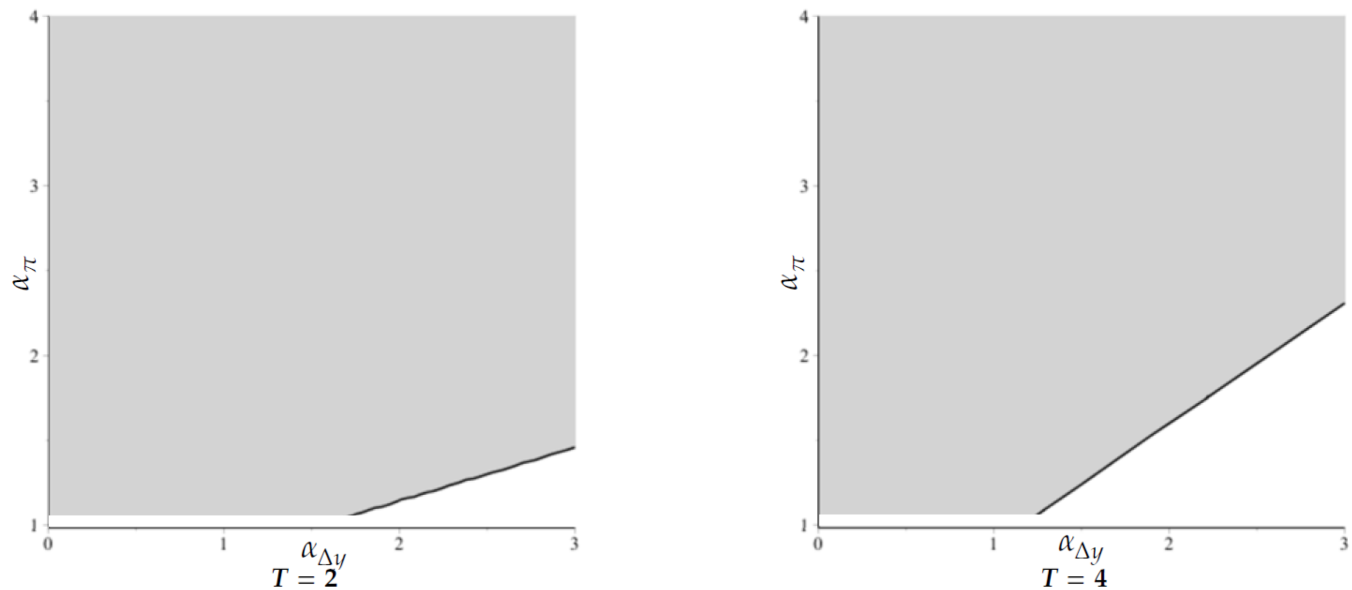


Figure 1: Regions in which M is a P-matrix (shaded grey) or a P_0-matrix (shaded grey, plus the black line), when $T = 2$ (left) or $T = 4$ (right).

BPY (2015) model with shadow interest rate persistence (3/3)

- With large T, it appears that the determinacy region agrees with that in the model without persistence in the shadow rate.

- Evidence for this is provided by the fact that with $\alpha_\pi = 1.5$:

  - If $\alpha_{\Delta y} = 1.51$, and $T = 200$ then 1) M is not an S-matrix, so it is not a P-matrix for higher T and 2) $\varsigma \leq 0 +$ num. err., providing numerical evidence that for sufficiently large T, the LCP (q, M) is not feasible for some q.

  - If $\alpha_{\Delta y} = 1.05$, then M is a P-matrix, and $\varsigma > 6.1319 \times 10^{-8}$, so M is an S-matrix for all sufficiently large T.

BPY (2015) model with price level targeting (1/2)

- Equations:

$$x_{i,t} = \max\{0,1-\beta+\alpha_{\Delta y}x_{y,t}+\alpha_{\pi}x_{p,t}\},$$
$$x_{y,t} = \mathbb{E}_t x_{y,t+1} - \frac{1}{\sigma}\left(x_{i,t}+\beta-1-\mathbb{E}_t x_{p,t+1}+x_{p,t}\right),$$
$$x_{p,t} - x_{p,t-1} = \beta\mathbb{E}_t x_{p,t+1} - \beta x_{p,t} + \gamma x_{y,t},$$

- Determinacy in the absence of the ZLB requires $\alpha_{\pi} \in (0,\infty)$, $\alpha_{\Delta y} \in [0,\infty)$.

BPY (2015) model with price level targeting (2/2)

- With $T = 1$, M is strictly positive for all $\alpha_{\Delta y}, \alpha_\pi \in (0, \infty)$, so price level targeting cures the multiplicity found by BPY.

- With parameters as before, and $\alpha_{\Delta y} = 1$, $\alpha_\pi = 1$:

  - From our lower bound on ς with $T = 20$, we find that $\varsigma \geq 0.042659$. Hence, this model is always feasible for any sufficiently large T.

  - Given that $d_0 > 0$ for this model, and that for $T = 20$, M is a P-matrix, this is strongly suggestive of the existence of a unique solution for any q and T.

Linearized Fernandez-Villaverde et al. (2012) model

- A basic NK model without investment, but with positive steady-state inflation, and hence price dispersion.

- With $T \leq 14$, M is a P-matrix, but with $T \geq 15$, M is not a P-matrix.
  - Thus this model always has multiple solutions.

- With $T = 1000$, from our upper bound on ς, we have that: $\varsigma \leq 0 +$ numerical error.
  - Provides strong evidence that M is not an S-matrix for large T either.

Linearized Fernandez-Villaverde et al. (2012) model with price level targetting

- With nominal GDP targeting (unit coefficients), with $T = 200$, our lower bound gives $\varsigma > 0.0048175$.
  - Hence, for all sufficiently large T, M is an S-matrix, so there is always a feasible solution.

- For all T tested, M is a P-matrix.

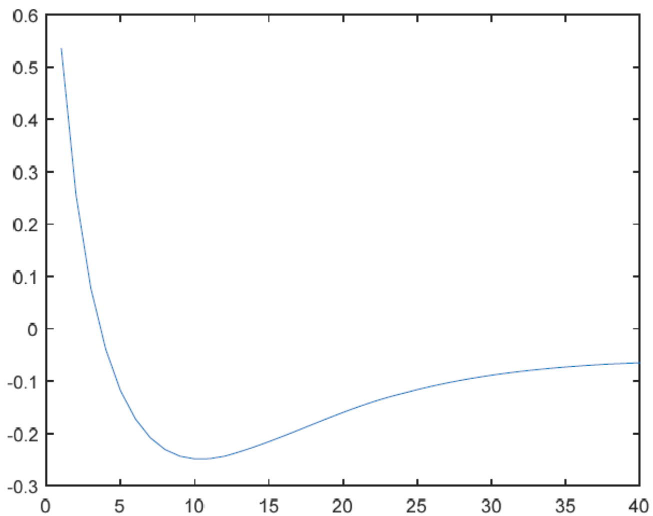
Smets & Wouters (2003; 2007) (1/3)

- Both models have:
  - assorted shocks, habits, price and wage indexation, capital (with adjustment costs), (costly) variable utilisation, general monetary policy reaction functions
- We augment both models with nominal interest rate rules of the form:

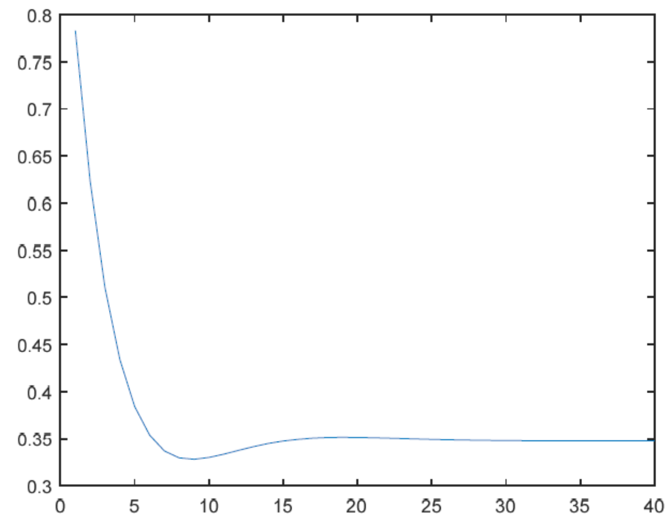
$$r_t = \max\{0, (1-\rho)(\cdots) + \rho r_{t-1} + \cdots\}$$

- Recall that the 2003 model is estimated on Euro area data, and the 2007 one is estimated on US data.
  - We use the posterior modes.
- Fairly similar models, except that the 2007 one:
  - Contains trend growth (permitting its estimation on non-detrended data),
  - Has a slightly more general aggregator across industries.

Smets & Wouters (2003; 2007) (2/3)



The Smets and Wouters (2003) model



The Smets and Wouters (2007) model

Figure 2: The diagonals of the M matrices for the Smets and Wouters (2003) and Smets and Wouters (2007) models

Smets & Wouters (2003; 2007) (3/3)

- Perhaps surprising that these graphs are so different.
- Negative diagonal for the Euro area model implies that the model does not always have a unique solution, even when $q \gg 0$.
- In fact, providing $T \geq 9$, the US model also does not always have a unique solution, even when $q \gg 0$.
- Furthermore, for both countries, there are some q for which the model has no solution.
  - Suggests that only solutions converging to the “bad” steady-state exist in those cases.
- Increasing the coefficient on inflation results in a positive diagonal for M even for the Euro area model, but does not result in a P-matrix.
- However, replacing inflation by the price-level minus a linear trend in the Taylor rule produces a P-matrix.

The algorithm

- At its core is a very efficient perfect foresight solver that also applies to pruned perturbation solutions to the non-linear model.
  - We will discuss this first.

- To ensure consistency with rational expectations, we use this core algorithm within a variant of the stochastic extended path algorithm of Adjemian and Juillard (2013) that further exploits the properties of pruned perturbation solutions.

Efficient computation of solutions (1/2)

- If M is unrestricted, or M is a P_0-matrix, then finding a single solution to the LCP (q, M) is "strongly NP complete".
- If we could do this efficiently (i.e. in polynomial time), we could also solve in polynomial time any problem whose solution could be efficiently verified.
  - This includes, for example, breaking all standard forms of cryptography.

- Since there is a model corresponding to any M matrix, with quadratic in T states, this means that if there were a solution algorithm for DSGE models with occasionally binding constraints that worked in time polynomial in the number of states, then it could also be used to defeat all known forms of cryptography.

Efficient computation of solutions (2/2)

- Polynomial time algorithms exist for special cases, but checking whether the relevant ones apply is not possible in polynomial time.

- This means that there cannot be an algorithm for checking if a model e.g. has a unique solution, that runs in time polynomial in the number of states.

Our computational approach to the perfect foresight problem (1/2)

- There is no way of escaping solving an NP-complete problem if we wish to simulate DSGE models with OBCs.
- Any algorithm we invented for the problem is likely to be inefficient, and possibly even non-finite.
- A better approach is to map our problem into another to which smart computer scientists have devoted a lot of time.
- It turns out that the solution to an LCP can be represented as a mixed integer linear programming problem.
  - One of the best studied problems in computer science.
  - Extremely well optimised, fully global, solvers exist.

Our computational approach to the perfect foresight problem (2/2)

- **Problem 7**
- Suppose $\widetilde{\omega} > 0$, $q \in \mathbb{R}^T$ and $M \in \mathbb{R}^{T\times T}$ are given.
- Find $\alpha \in \mathbb{R}$, $\hat{y} \in \mathbb{R}^T$, $z \in \{0,1\}^T$ to maximise α subject to the following constraints: $\alpha \geq 0$, $0 \leq \hat{y} \leq z$, $0 \leq \alpha q + M\hat{y} \leq \widetilde{\omega}(1_{T\times 1} - z)$.

- **Proposition**: If α, $\hat{y}$, z solve Problem 7, then if $\alpha = 0$, the LCP (q, M) has no solution, and if $\alpha > 0$, then $y \coloneqq \frac{\hat{y}}{\alpha}$ solves it. (Partial converse in paper.)

- As $\widetilde{\omega} \to 0$, the solution to Problem 7 is the solution to the LCP which minimises $\|q + My\|_\infty$.
- As $\widetilde{\omega} \to \infty$, the solution to Problem 7 is the solution to the LCP which minimises $\|y\|_\infty$.

Application to models with uncertainty

- To convert the perfect foresight solver into a solver for stochastic models, we use a variant of the extended path algorithm of Fair and Taylor (1983).
    - Each period we draw a shock, and then solve for the expected future path of the model, ignoring the impact of the OBC on expectations (for now).
    - From this expected path, we can solve for the news shocks necessary to impose the bound.
    - We then add those news shocks to today’s variables, and step the model forward using the model’s transition matrix.

The non-linear problem (1/2)

- **Problem 6**
- Suppose that $x_0 \in \mathbb{R}^n$ is given and that $f: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^m \to \mathbb{R}^n$, $g, h: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^m \to \mathbb{R}^c$ are given continuously $d \in \mathbb{N}^+$ times differentiable functions.

- Find $x_t \in \mathbb{R}^n$ and $v_t \in \mathbb{R}^c$ for $t \in \mathbb{N}^+$ such that for all $t \in \mathbb{N}^+$:

$$\begin{aligned} 0 &= \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t), \\ v_t &= \mathbb{E}_t \max\{h(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t), g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\} \end{aligned}$$

- where $\varepsilon_t \sim \text{NIID}(0, \Sigma)$, where the $\max$ operator acts elementwise on vectors, and where the information set is such that for all $t \in \mathbb{N}^+$, $\mathbb{E}_{t-1}\varepsilon_t = 0$ and $\mathbb{E}_t \varepsilon_t = \varepsilon_t$.

The non-linear problem (2/2)

- **Assumption:** There exists $\mu_x \in \mathbb{R}^n$ and $\mu_v \in \mathbb{R}^c$ such that:

$$0 = f(\mu_x, \mu_x, \mu_x, \mu_v, 0),$$

$$\mu_v = \max\{h(\mu_x, \mu_x, \mu_x, \mu_v, 0), g(\mu_x, \mu_x, \mu_x, \mu_v, 0)\},$$

- and such that for all $a \in \{1, \ldots, c\}$:

$$\big(h(\mu_x, \mu_x, \mu_x, \mu_v, 0)\big)_a \neq \big(g(\mu_x, \mu_x, \mu_x, \mu_v, 0)\big)_a.$$

Application via linearization (1/2)

- Without loss of generality, suppose our model is:

$$0 = \mathbb{E}_t f(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t),$$
$$v_t = \mathbb{E}_t \max\{0, g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\},$$

  - where $g(\mu_x, \mu_x, \mu_x, \mu_v, 0) \gg 0$.

- Linearizing around the steady-state gives:

$$v_t = \mu_v + g_1(x_{t-1} - \mu_x) + g_2(x_t - \mu_x) + g_3\mathbb{E}_t(x_{t+1} - \mu_x) + g_4(v_t - \mu_v) + g_5\varepsilon_t.$$

- We replace this with the more accurate:

$$v_t = \max\{0, \mu_v + g_1(x_{t-1} - \mu_x) + g_2(x_t - \mu_x) + g_3\mathbb{E}_t(x_{t+1} - \mu_x) + g_4(v_t - \mu_v) + g_5\varepsilon_t\}.$$

Application via linearization (2/2)

- For our algorithm, we replace this in turn with:

$$v_{a,t} = \mathbb{E}_t\big(g(x_{t-1}, x_t, x_{t+1}, v_t, \varepsilon_t)\big)_a + I_{1,\cdot} y_t^{(a)},$$

- for all $a \in \{1, \dots, c\}$, where, for all $a \in \{1, \dots, c\}$:

$$\forall i \in \{1, \dots, T-1\}, \qquad y_{i,t}^{(a)} = y_{i+1,t-1}^{(a)} + \eta_{i,t}^{(a)}$$
$$y_{T,t}^{(a)} = \eta_{T,t}^{(a)}.$$

Application via higher order pruned perturbation

- We first take a pruned perturbation approximation to the source non-linear model.

- A convenient property of pruned perturbation solutions of order d is that they are linear in additive shocks of the form η_t^d.
  - So using shocks of this form preserves the tractable linearity.
  - In fact the M matrix we get at second or higher order is equal to the M matrix at first order, (at least in the limit as the variance of the news shocks goes to zero).

- While we still treat the bound in a perfect-foresight manner (for now), by taking a higher order approximation we at least capture other risk channels.

Integrating over future uncertainty (1/3)

- Adjemian and Juillard (2013) showed how a perfect foresight solver could account for future uncertainty by integrating over future shocks.
  - I.e. draw shocks for a certain number of future periods, $t+1,\dots,t+S$.
  - Solve for the perfect foresight path assuming they were known at t.
  - Repeat many times to get expectations.

- In their very general non-linear set-up, doing this integration requires p^{mS} solutions of the perfect foresight problem,
  - for some $p>1$, m is the number of shocks, S is the integration horizon.

- Solving their general perfect foresight problem is also orders of magnitude slower than solving our LCP.

Integrating over future uncertainty (2/3)

- Let $w_{t,s}$ be the value the bounded variables would take at s if the constraints did not apply from period t onwards.

- By the properties of pruned perturbation solutions, we can evaluate $\operatorname{cov}_t\left(w_{t,t+i}, w_{t,t+j}\right)$, for $t, i, j \in \mathbb{N}$ in closed form.
  - So we can take a Gaussian approximation to the joint distribution of $w_{t,t}, w_{t,t+1}, \ldots$, and efficiently integrate over these variables via Gaussian cubature techniques.
  - Rather than exponential in both m and S evaluations, we just need polynomial in S evaluations.

- For each draw of $w_{t,t}, w_{t,t+1}, \ldots$, we solve the bounds problem to get the cumulated news shocks (i.e. y).

Integrating over future uncertainty (3/3)

- Unlike Adjemian and Juillard (2013) we do not just consider full variance shocks up to some horizon, and then nothing beyond.

- Instead, we apply a windowing function to the shock variances, to ensure that the covariance is a smooth function of time.
  - This reduces artefacts caused by the sudden change at horizon S.

- In particular, we scale the shock variance at horizon k by:

$$\frac{1}{2}\left(1+\cos\left(\pi\frac{k-1}{S}\right)\right).$$

- The cosine form has some desirable frequency domain properties.

Three alternative Gaussian cubature methods

- With $\hat{S} \leq S$ the integration dimension, these are:
  - A degree 3 monomial rule with $2\hat{S}+1$ nodes and positive weights.
    - Positive weights give robustness. Evaluates far from steady-state though.
  - The Genz and Keister (1996) Gaussian cubature rules with $\mathrm{O}\left(\hat{S}^K\right)$ nodes.
    - $2K+1$ is the degree of monomial integrated exactly.
    - Since the rules are nested, adaptive degree is possible.
  - Quasi-Monte Carlo.
    - Much less efficient than the others on well behaved functions, but is much better behaved on non-differentiable ones.

The DynareOBC toolkit (1/2)

- Complete code implementing all of these algorithms is available under an open source license from:

https://github.com/tholden/dynareOBC

- To use DynareOBC, just include a max, min or abs in your mod file, then type “dynareOBC modfilename.mod”.

- Assorted command line options are documented on the home page.

- Also implements an experimental hybrid local/global algorithm.

The DynareOBC toolkit (2/2)

- Even if you do not have OBCs in your model, DynareOBC may be useful since it can:

  - Simulate MLVs, including integrating over ones with +1 terms, which makes checking accuracy very easy.

  - Perform exact, faster, average IRF calculation without Monte Carlo.

  - Estimate non-linear models at 3rd order using the cubature Kalman filter.

Accuracy results

- By way of conclusion, we give some accuracy results.

- We examine three simple models, to ensure accuracy tests are reliable.

- These are:
  - A very simple model with an analytic solution.
  - A model for which log-linearization gives the exact answer in the absence of bounds.
  - An otherwise linear open-economy model.

A simple model

- Closed economy, no capital, inelastic unit labour supply.
- Households maximise:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma} - 1}{1 - \gamma}$$

- Subject to the budget constraint:

$$A_t + R_{t-1} B_{t-1} = C_t + B_t$$

  - A_t is productivity. R_t is the real interest rate.
  - B_t is the household's holdings of zero net supply bonds.

- Define $g_t \coloneqq \log A_t - \log A_{t-1}$. Evolves according to:

$$g_t = \max\{0, (1 - \rho)\bar{g} + \rho g_{t-1} + \sigma \varepsilon_t\},$$

- $\varepsilon_t \sim \mathrm{NIID}(0,1)$, $\beta \coloneqq 0.99$, $\gamma \coloneqq 5$, $\bar{g} \coloneqq 0.05$, $\rho \coloneqq 0.95$, $\sigma \coloneqq 0.07$.

Accuracy in the simple model, along simulated paths of length 1000 (after 100 periods dropped)

Bound in Model	Order	Cubature	Seconds	Mean Abs Error	Root Mean Squared Error	Max Abs Error	Mean Abs Error at Bound
No	1	N/A	66	6.13E-04	6.13E-04	6.13E-04	
No	2	N/A	62	1.52E-17	2.36E-17	1.67E-16	
No	3	N/A	53	1.99E-17	2.72E-17	1.11E-16	
Yes	1	No	141	3.67E-03	6.05E-03	1.31E-02	1.31E-02
Yes	2	No	139	3.76E-03	6.39E-03	1.37E-02	1.37E-02
Yes	3	No	140	3.76E-03	6.39E-03	1.37E-02	1.37E-02
Yes	1	Monomial, Degree 3	274	7.32E-04	8.45E-04	1.88E-03	7.40E-04
Yes	2	Monomial, Degree 3	1537	4.18E-04	6.73E-04	1.97E-03	1.28E-04
Yes	3	Monomial, Degree 3	1397	4.18E-04	6.73E-04	1.97E-03	1.28E-04
Yes	2	Sparse, Degree 3	1794	9.65E-04	1.67E-03	3.85E-03	3.85E-03
Yes	2	Sparse, Degree 5	1840	9.65E-04	1.67E-03	3.85E-03	3.85E-03
Yes	2	Sparse, Degree 7	2009	5.25E-04	9.30E-04	2.17E-03	2.17E-03
Yes	2	QMC, 15 Points	1965	9.12E-04	1.27E-03	2.17E-03	2.17E-03
Yes	2	QMC, 31 Points	2214	5.98E-04	8.17E-04	1.39E-03	1.39E-03
Yes	2	QMC, 63 Points	3184	4.04E-04	5.49E-04	9.55E-04	9.55E-04
Yes	2	QMC, 1023 Points	5197	1.57E-04	2.30E-04	4.45E-04	4.45E-04

[1] All timings are "wall" time, and include time spent starting the parallel pool, time spent compiling code (although written in MATLAB, DynareOBC generates and compiles C code for key routines), and time spent calculating accuracy. Code was run on one of the following (very similar) twenty core machines: 2x E5-2670 v2 2.5GHz, 64GB RAM; 2x E5-2660 v3 2.6GHz, 128GB RAM. Use of machines with network attached storage means that there may be some additional variance in these timings.

[2] Errors conditional on the bounded variable being less than *0.0001*. The numbers for this column would be identical had we used root mean squared errors or maximum absolute errors, conditional on being at the bound.

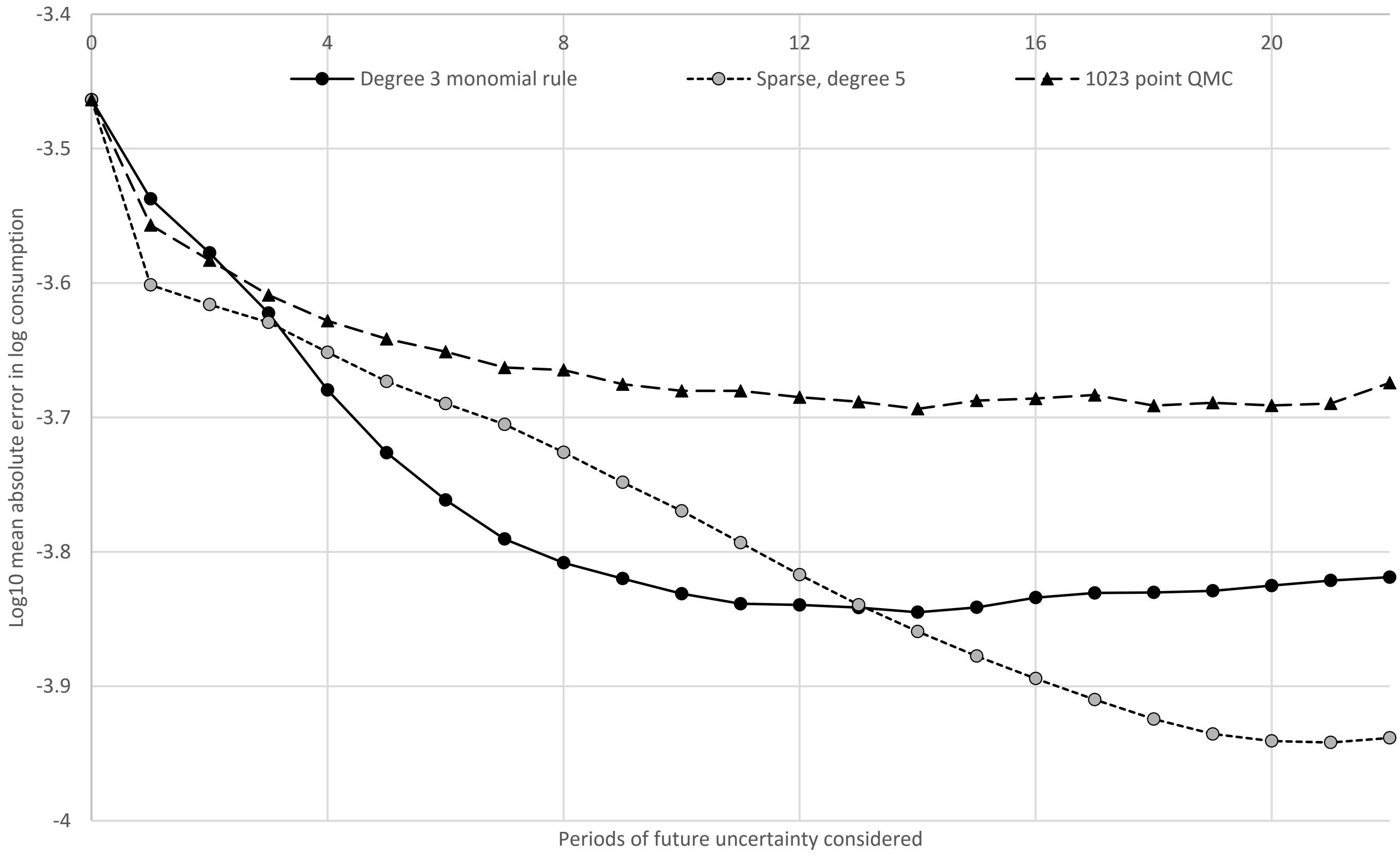
A model for which log-linearization is exact without bounds

- The social planner chooses consumption, C_t, L_t, and K_t, to maximise:

$$\mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \left[\log C_{t+k} - \frac{L_{t+k}^{1+\nu}}{1+\nu} \right],$$

- subject to the capital constraint: $K_t \geq \theta K_{t-1}$,
- and the budget constraint $C_t + K_t = Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha}$.
- Productivity, A_t, evolves according to $A_t = A_{t-1}^{\rho} \exp \varepsilon_t$, where $\varepsilon_t \sim \mathrm{N}(0, \sigma^2)$.
- In the following, we set $\alpha = 0.3$, $\beta = 0.99$, $\nu = 2$, $\theta = 0.99$, $\rho = 0.95$ and $\sigma = 0.01$.
- Compare to a full global solution.

Effect of increasing periods of uncertainty on accuracy, along simulated paths (as before)



An otherwise linear open-economy model

- The social planner chooses C_t, D_t and B_t to maximise:

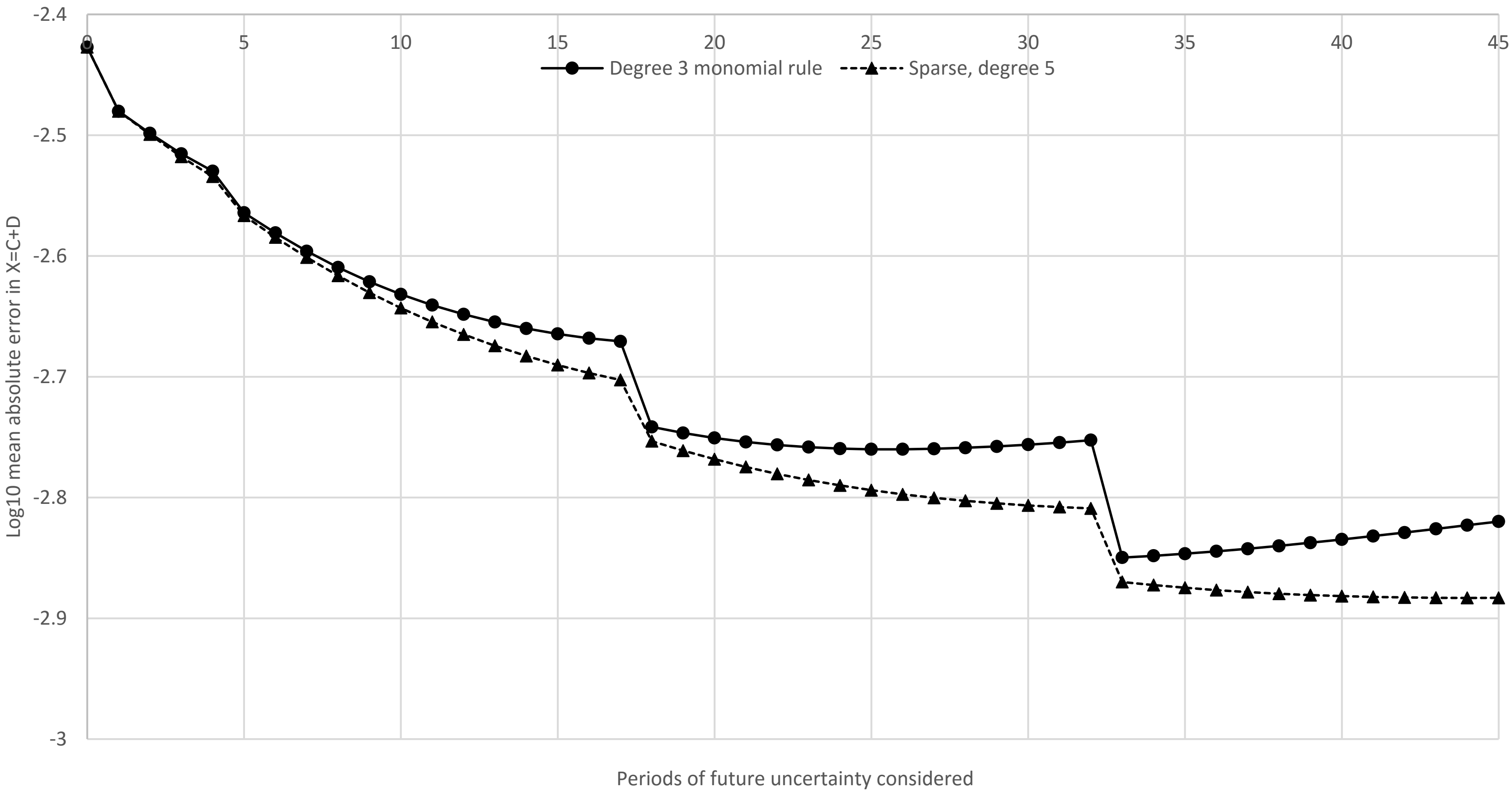
$$\mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \left[-\frac{1}{2}(1 - C_t)^2 - \frac{\phi}{2} B_t^2 \right],$$

- subject to the budget constraint: $C_t + D_t + B_t - RB_{t-1} = Y_t = \max\{\underline{Y}, A_t\}$,
- the positivity constraints: $0 \leq C_t,\ 0 \leq D_t$,
- and the certain repayment of interest constraint:

$$\forall k \in \mathbb{N}^+, \qquad \Pr_t\big((R - 1)B_t \leq Y_{t+k}\big) = 1.$$

- Productivity evolves according to $A_t = (1 - \rho)\mu + \rho A_{t-1} + \sigma \varepsilon_t$, where $\varepsilon_t \sim \text{NIID}(0,1)$.
- We set $\beta = 0.99$, $\mu = 0.5$, $\rho = 0.95$, $\sigma = 0.05$, $\underline{Y} = 0.25$, $R = \beta^{-1}$ and $\phi = R - 1$.

Effect of increasing periods of uncertainty on accuracy, along simulated paths (as before)



Conclusion

- We proved "Blanchard Kahn conditions" for models with occasionally binding constraints.
  - Applicable to all models.

- We showed that multiplicity of equilibria is to be expected in models with zero lower bounds on nominal interest rates, but that price targeting solves it.

- We developed efficient algorithms for solving models with OBCs, and extensions for improving their accuracy.