Written work to obtain every formula in exercise 3 of Stat. meth. of Prof. TARDELLA

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• Likelihood $= \prod_{i=1}^{N=\text{length}(y)} \frac{1}{(2\pi\tau^2)^{1/2}} \cdot e^{\frac{-(y_i-\alpha+\beta\gamma^{x})^2}{2\tau^2}} = \frac{1}{(2\pi\tau^2)^{\frac{N}{2}}} e^{\left(\frac{-\sum_i (y_i-\alpha+\beta\gamma^{x_i})^2}{2\tau^2}\right)}$

• PRIOR = product of the priors ⇒

$$\frac{1}{\sqrt{2\pi\sigma_\alpha^2}} e^{\left(\frac{-\alpha^2}{2\sigma_\alpha^2}\right)} \frac{1}{\sqrt{2\pi\sigma_\beta^2}} e^{\left(\frac{-x^2}{2\sigma_\beta^2}\right)} \mathbb{I}_{[\gamma\in[0,1]]} \frac{b^a}{\Gamma(a)} \frac{1}{\tau^{2(a+1)}} e^{\left(-b/\tau^2\right)}$$

⇒ • POSTERIOR ~ Likelihood × Prior ⇒ given that $\sigma_\alpha^2, \sigma_\beta^2, a, b$ are fixed (by precedent point) ⇒

$$\simeq \frac{1}{(2\pi\tau^2)^{N/2}} e^{\left(\frac{-\sum(y_i-\alpha+\beta\gamma^{x})^2}{2\tau^2} - \frac{\alpha^2}{2\sigma_\alpha^2} - \frac{\beta^2}{2\sigma_\beta^2} - b/\tau^2\right)} \cdot \frac{1}{\tau^{2(a+1)}}$$

From here are asked the 4 full conditionals

⇒

1) $\pi(\alpha \mid \beta, \gamma, \tau^2) \simeq$ (take out constants)

$$\exp\left(-\sum_i \frac{(y_i-\alpha+\beta\gamma^{x_i})^2}{2\tau^2} - \frac{\alpha^2}{2\sigma_\alpha^2}\right) =$$

$$\exp\left(\frac{-\sigma_\alpha^2 N\alpha^2 + 2\sigma_\alpha^2\alpha\sum y_i + 2\sigma_\alpha^2\alpha\beta\gamma^{x_i}N - \tau^2\alpha^2}{2\tau^2\sigma_\alpha^2}\right)$$

=

$$\exp\left(-\frac{1}{2}\alpha^2\underbrace{\left(\frac{N\sigma_\alpha^2+\tau^2}{\tau^2\sigma_\alpha^2}\right)}_{a}+\alpha\underbrace{\left(\frac{2\sigma_\alpha^2\sum y_i+2\sigma_\alpha^2\beta\gamma^{x_i}N}{2\tau^2\sigma_\alpha^2}\right)}_{b}\right)$$

IS A NORMAL!

$$\sim N\left(\frac{b}{a},\frac{1}{a^2}\right)=N\left(\frac{2\sigma_\alpha^2\sum y_i+2\sigma_\alpha^2\beta\sum\gamma^{x_i}}{N\sigma_\alpha^2+\tau^2},\left(\frac{\tau^2\sigma_\alpha^2}{N\sigma_\alpha^2+\tau^2}\right)^2\right)$$

2) $$\pi(\beta\mid\alpha,\gamma,\tau^2)\sim\exp\left(-\frac{\sum(y_i-\alpha+\beta\gamma^{x_i})^2}{2\tau^2}-\frac{\beta^2}{2\sigma_\beta^2}\right)$$

$$\sim\exp\left(\frac{-\sigma_\beta^2\beta^2\sum_i\gamma^{2x_i}+2\sigma_\beta^2\sum y_i\gamma^{x_i}-2\sigma_\beta^2\alpha\beta\sum_i\gamma^{x_i}-\beta^2\tau^2}{2\sigma_\beta^2\tau^2}\right)$$

$$=\exp\left(-\frac{\beta^2}{2}\overbrace{\left(\frac{\sigma_\beta^2\sum\gamma^{2x_i}+\tau^2}{2\sigma_\beta^2\tau^2}\right)}^{a}+\beta\overbrace{\left(\frac{2\sigma_\beta^2\sum y_i\gamma^{x_i}-2\sigma_\beta^2\alpha\sum\gamma^{x_i}}{2\sigma_\beta^2\tau^2}\right)}^{b}\right)$$

$$\Rightarrow\sim N\left(\frac{b}{a},\frac{1}{a^2}\right)=$$

$$N\left(\frac{\left(2\sigma_\beta^2\sum_i y_i\gamma^{x_i}-2\sigma_\beta^2\alpha\sum\gamma^{x_i}\right)}{\sigma_\beta^2\sum\gamma^{2x_i}+\tau^2},\left(\frac{(\sigma_\beta\tau^2)}{(\sigma_\beta^2\sum\gamma^{2x_i}+\tau^2)}\right)^2\right)$$

IS A NORMAL!

3) $\pi(\gamma \mid \alpha, \beta, \tau^2) \approx$

$$\exp\left(\frac{-\sum \beta^2 \gamma^{2x_i} - 2\alpha\beta\gamma^{x_i} + 2y_i \beta\gamma^{x_i}}{2\tau^2} \right) \mathbb{I}_{\{\gamma \in [0,1]\}}$$

NOT A KNOW DISTRIBUTION TO ME!

4) $\pi(\tau^2 \mid \alpha, \beta, \gamma) \approx$

$$\frac{1}{(\tau^2)^{\frac{N}{2}}} \frac{1}{\tau^{2(a+1)}} \exp\left(\frac{-\sum_i (y_i - \alpha + \beta\gamma^{x_i})^2}{2\tau^2} - \frac{b}{\tau^2} \right)$$

$\Rightarrow$ is an inverse Gamma!

$$\approx \text{IG}(\text{shape}, \text{rate}) = IG\left(\frac{N}{2} + a + 2, \frac{\sum (y_i - \alpha + \beta\gamma^{x_i})^2 + 2b}{2} \right)$$

I prefered to rewrite this section on tablet since my notebook is hard to comprehend.

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